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STEADY HYDRODYNAMIC MODEL OF SEMICONDUCTORS WITH SONIC BOUNDARY: (II) SUPERSONIC DOPING PROFILE*

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Abstract. This is the second part of our series of studies concerning the well-posedness/ill-posedness and regularity of stationary solutions to the hydrodynamic model of semiconductors represented by Euler–Poisson equations with sonic boundary condition. In this paper, we consider the case of a supersonic doping profile, and prove that the system does not hold any interior subsonic solution; furthermore, the system doesn't admit any interior supersonic solution and any transonic solution if such a supersonic doping profile is small enough or the relaxation time is small, but it has at least one interior supersonic solution and infinitely many transonic solutions if the supersonic doping profile is close to the sonic line and the relaxation time is large. The nonexistence of any type of solutions in the case of a small doping profile or small relaxation time indicates that the semiconductor effect for the system is remarkable and cannot be ignored.

Key words. Euler–Poisson equations, hydrodynamic model of semiconductors, sonic boundary, supersonic doping profile, subsonic solutions, supersonic solutions, transonic solutions with shock

AMS subject classifications. 35R35, 35Q35, 76N10, 35J70

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1. Introduction.

Subsequently to the first part of a stationary hydrodynamic model with sonic boundary [16], we consider the system in the case of a supersonic doping profile:

$$\begin{cases} J = \text{constant}, \\ \left(\dfrac{J^2}{\rho} + P(\rho)\right)_x = \rho E - \dfrac{J}{\tau}, \qquad x \in (0,1). \\ E_x = \rho - b(x), \end{cases} \tag{1.1}$$

Here ρ, J, and E represent the electron density, the current density, and the electric field, respectively. $P(\rho)$ is the pressure function of the electron density. When the system is isothermal, the pressure function is physically represented by

$$P(\rho) = T\rho \ \text{ with the constant temperature } T > 0. \tag{1.2}$$

The function $b(x) > 0$ is the doping profile standing for the density of impurities in a semiconductor device. The constant $\tau > 0$ denotes the momentum relaxation time.

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From fluid dynamics, we call $c := \sqrt{P'(\rho)} = \sqrt{T} > 0$ the sound speed for $P(\rho) = T\rho$ (see (1.2)). Thus, the stationary flow of (1.1) is said to be subsonic/sonic/supersonic, if the fluid velocity satisfies

$$\text{fluid velocity: } u = \frac{J}{\rho} \lesseqgtr c = \sqrt{P'(\rho)} = \sqrt{T}: \text{ sound speed.} \tag{1.3}$$

Without loss of generality, let us assume throughout the paper

$$T = J = 1.$$

Thus, (1.1) is transformed into

$$\begin{cases} \left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - b(x). \end{cases} \tag{1.4}$$

From (1.3), it can be identified that $\rho > 1$ is for the subsonic flow, $\rho = 1$ stands for the sonic flow, and $0 < \rho < 1$ represents the supersonic flow. Therefore, our sonic boundary conditions to (1.1) are proposed as follows:

$$\text{sonic boundary:} \quad \rho(0) = \rho(1) = 1. \tag{1.5}$$

Dividing the first equation of (1.4) by ρ and differentiating the resultant equation with respect to x, and substituting the second equation of (1.4) into this modified equation, we then have

$$\begin{cases} \left[\left(\dfrac{1}{\rho} - \dfrac{1}{\rho^3}\right)\rho_x\right]_x + \dfrac{1}{\tau}\left(\dfrac{1}{\rho}\right)_x - [\rho - b(x)] = 0, \quad x \in (0,1), \\ \rho(0) = \rho(1) = 1 \text{ (sonic boundary)}. \end{cases} \tag{1.6}$$

Throughout the paper we assume that the doping profile $b(x) \in L^\infty(0,1)$ is supersonic:

$$\text{supersonic doping profile:} \quad 0 < b(x) < 1 \quad \text{for all } x \in [0,1].$$

We denote $\underline{b} := \operatorname{essinf}_{x\in(0,1)} b(x)$ and $\overline{b} := \operatorname{esssup}_{x\in(0,1)} b(x)$.

Recalled from [16], we define the interior subsonic/supersonic/transonic solutions as follows.

Definition 1.1. *$\rho(x)$ is called an interior subsonic (correspondingly, interior supersonic) solution of (1.6) if $\rho(0) = \rho(1) = 1$ but $\rho(x) > 1$ (correspondingly, $0 < \rho(x) < 1$) for $x \in (0,1)$, and $(\rho(x) - 1)^2 \in H_0^1(0,1)$, and it holds that for any $\varphi \in H_0^1(0,1)$*

$$\int_0^1 \left(\frac{1}{\rho} - \frac{1}{\rho^3}\right)\rho_x \varphi_x dx + \frac{1}{\tau}\int_0^1 \frac{\varphi_x}{\rho} dx + \int_0^1 (\rho - b)\varphi dx = 0,$$

which is equivalent to

$$\frac{1}{2}\int_0^1 \frac{\rho+1}{\rho^3}\left((\rho-1)^2\right)_x \varphi_x dx + \frac{1}{\tau}\int_0^1 \frac{\varphi_x}{\rho} dx + \int_0^1 (\rho - b)\varphi dx = 0. \tag{1.7}$$

Once $\rho = \rho(x)$ is determined by (1.6), in view of the first equation of (1.4), the electric field $E(x)$ can be solved by

$$E(x) = \left(\frac{1}{\rho} - \frac{1}{\rho^3}\right)\rho_x + \frac{1}{\tau\rho} = \frac{(\rho+1)[(\rho-1)^2]_x}{2\rho^3} + \frac{1}{\tau\rho}.$$

In this way, we could obtain the interior subsonic/supersonic solutions to system (1.4)–(1.5).

DEFINITION 1.2. *$\rho(x) > 0$ is called a transonic shock solution of system (1.4)–(1.5) if $\rho(0) = \rho(1) = 1$ and it is separated by a point $x_0 \in (0, 1)$ in the form*

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in (0, x_0), \\ \rho_{sub}(x), & x \in (x_0, 1), \end{cases}$$

where $0 < \rho_{sup}(x) < 1$ and $\rho_{sub}(x) > 1$ satisfy the entropy condition

$$0 < \rho_{sup}(x_0^-) < 1 < \rho_{sub}(x_0^+), \tag{1.8}$$

and the Rankine–Hugoniot condition at x_0

$$\begin{aligned} \rho_{sup}(x_0^-) + \frac{1}{\rho_{sup}(x_0^-)} &= \rho_{sub}(x_0^+) + \frac{1}{\rho_{sub}(x_0^+)}, \\ E_{sup}(x_0^-) &= E_{sub}(x_0^+). \end{aligned} \tag{1.9}$$

Set $\rho_l = \rho_{sup}(x_0^-)$ and $\rho_r = \rho_{sub}(x_0^+)$, a simple computation from (1.9) shows that

$$\rho_l \rho_r = 1. \tag{1.10}$$

The stationary hydrodynamic model with a different background setting and different boundary conditions has been extensively studied [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. When the doping profile is a supersonic constant, namely, $b(x) \equiv b < 1$, Ascher et al. [1] and Rosini [22] observed that there are some transonic solutions connected with shocks. The shock transonic solutions were also technically constructed by Gamba in [8] for the one-dimensional case and by Gamba and Morawetz in [9] for the two-dimensional case. Later, when the fixed boundary is also supersonic, and the current density has a strongly supersonic background, i.e., $J \gg 1$, Peng and Violet [21] further proved that there is a supersonic solution and it is unique. When the effect of a semiconductor is removed, namely, the term $-\frac{J}{\tau}$ in (1.1) is missing, Luo and Xin [18] showed the existence/nonexistence of transonic shock solutions when the boundaries are subsonic on one side and supersonic on the other side, where the doping profile $b(x)$ is fixed as a constant. A structural stability of transonic shocks is further studied by Luo et al. [17]. For the critical case of the sonic boundary, we investigated the structure of all possible solutions to (1.1) in [16] when the doping profile $b(x)$ is subsonic, that is, there are a unique subsonic solution, at least one supersonic solution, infinitely many transonic shock solutions when the relaxation time is big ($\tau \gg 1$), and infinitely many C^1-smooth transonic shock solutions when the relaxation time is small ($\tau \ll 1$). As a continuity of our first part, in this paper we focus on the case when the doping profile is supersonic or sonic. We prove that the stationary system (1.1) with the sonic boundary does not hold any subsonic solution; and the system also has no supersonic solution and no transonic solution if such a supersonic doping profile $b(x)$ is small enough or the relaxation time τ is small enough. The nonexistence of any type of solutions in the case of a small doping profile or small relaxation time indicates that the semiconductor effect for the system is remarkable and cannot be ignored. On the other hand, we further prove that the system (1.1) possesses at least one supersonic solution and infinitely many transonic shock solutions if the supersonic doping profile is close to the sonic line (namely, $b(x) \approx 1^-$) and the semiconductor effect is small ($\tau \gg 1$). As shown in the first part [16], the supersonic solution is globally $C^{\frac{1}{2}}$ Hölder continuous, and the $C^{\frac{1}{2}}$-regularity is optimal. These results are summarized in the following theorem.

THEOREM 1.3. *Let the doping profile be supersonic such that* $b(x) \in L^\infty(0,1)$ *and* $0 < b(x) \le \overline{b} \le 1$. *Then*

1. *there is no interior subsonic solution to* (1.4)–(1.5);
2. *there is no interior supersonic solution nor transonic solution to* (1.4)–(1.5) *if the doping profile is sufficiently small such that* $\overline{b}(1+\sqrt{2\overline{b}}) < 1$*;*
3. *there is no interior supersonic solution nor transonic solution to* (1.4)–(1.5) *if the relaxation time is small with* $\tau < \frac{1}{3}$*;*
4. *there exists at least one interior supersonic solution* $(\rho_{sup}, E_{sup})(x)$ *to* (1.4)–(1.5), *satisfying* $\rho_{sup} \in C^{\frac{1}{2}}[0,1]$ *and the optimal estimate*

$$\begin{cases} C_5 x^{\frac{1}{2}} \le 1 - \rho_{sup}(x) \le C_6 x^{\frac{1}{2}}, \\ -C_7 x^{-\frac{1}{2}} \le \rho'_{sup}(x) \le -C_8 x^{-\frac{1}{2}}, \end{cases} \quad \text{for } x \text{ near } 0, \tag{1.11}$$

   if the doping profile $b(x)$ *is close to the sonic state (i.e.,* $\|b-1\|_{L^\infty(0,1)} \ll 1$*) and the relaxation time is sufficiently large,* $\tau \gg 1$*;*
5. *there exist infinitely many transonic shock solutions* $(\rho_{trans}, E_{trans})(x)$ *to* (1.4)–(1.5) *jointly with some stationary shocks satisfying the entropy condition* (1.8) *and the Rankine–Hugoniot jump condition* (1.9) *at different jump locations* x_0*, if the doping profile* $b(x)$ *is close to the sonic state and the relaxation time is sufficiently large,* $\tau \gg 1$*, where* x_0 *can be uniquely determined when* ρ_r *satisfying* $\rho_r - \rho_l \ll 1$ *is fixed, but the choice of* ρ_l *can be infinitely many.*

Remark 1.

1. Theorem 1.3 indicates that when the doping profile is small enough, or the relaxation time is small enough, then the system has no solution. This also explains the physical phenomenon that the semiconductor device doesn't work efficiently when the background of the device is too pure.
2. In part 4 of Theorem 1.3, the estimates (1.11) imply that $C^{\frac{1}{2}}[0,1]$ is the optimal Hölder space for the global regularity of the supersonic solution $\rho_{sup}(x)$.
3. In part 5 of Theorem 1.3, when $\tau \gg 1$, namely, the semiconductor effect is small, then the steady hydrodynamic system possesses infinitely many transonic shock solutions. The similar results in [18, 17] can be regarded in some sense as our special example $\tau = \infty$.
4. If $b(x) \equiv 1$, then parts 1, 3, 4, and 5 of Theorem 1.3 still hold, and (1.4)–(1.5) also admit the sonic solution $(\rho_{sonic}, E_{sonic})(x) \equiv (1, \frac{1}{\tau})$.

The paper is organized as follows. In section 2, by using the technical energy method on specifically chosen domains, we prove our main Theorem 1.3. In section 3, when the pressure function is $P(\rho) = T\rho^\gamma$ for $\gamma > 1$, the hydrodynamic system of Euler–Poisson equations becomes isentropic, and we conclude that the results presented in [16, Theorem 1.3] and Theorem 1.3 above all hold for the isentropic system with $\gamma > 1$.

2. Proof of Theorem 1.3. In this section, we consider the more general case of a nonsubsonic doping file, namely, we assume that $0 \le \underline{b} \le b(x) \le \overline{b} \le 1$, which allows $b(x)$ partially supersonic and partially sonic in the domain $[0,1]$. To observe the structure of the stationary solutions, let us first test the specially case with a constant supersonic doping profile $b(x) \equiv b < 1$, where the phase-plane analysis is

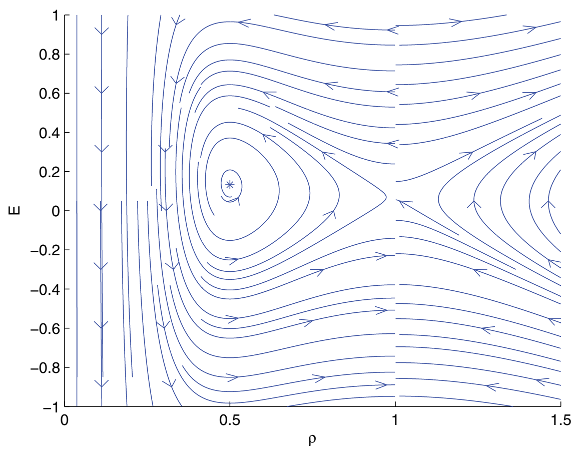

FIG. 1. *Phase plane of* (ρ, E) *with* $\tau = 15$ *and* $b = 0.5$; $*$ *is the focal point* $A = (0.5, 2/15)$.

helpful. In this case, the critical point $A = \left(b, \frac{1}{\tau b}\right)$ sits at the left-hand side of the sonic line $\rho = 1$. The Jacobian matrix of system (1.4) at A is

$$J(A) = \begin{bmatrix} \dfrac{b}{\tau(b^2-1)} & \dfrac{b^3}{b^2-1} \\ 1 & 0 \end{bmatrix}.$$

It is easy to see that the eigenvalues λ of matrix $J(A)$ satisfy the following characteristic equation

$$\lambda^2 - \frac{b\lambda}{\tau(b^2-1)} - \frac{b^3}{b^2-1} = 0. \tag{2.1}$$

By (2.1), the eigenvalues of the Jacobian matrix $J(A)$ satisfy

$$\lambda_1 + \lambda_2 < 0, \quad \lambda_1\lambda_2 > 0.$$

Thus, $\lambda_1, \lambda_2 < 0$, which indicates that A is a stable focal point. On the other hand, it follows from system (1.4) that

$$\frac{d\mathbf{E}}{d\rho} = \frac{(\rho - b)(1 - \frac{1}{\rho^2})}{\rho\mathbf{E} - \frac{1}{\tau}}, \tag{2.2}$$

which helps to determine the directions of all trajectories. Here, and in the following, to avoid confusion, we denote by $\mathbf{E} = \mathbf{E}(\rho)$ the function of the trajectory. In view of (2.2) and (1.4), we draw the phase plane of (ρ, E) in Figure 1 with $\tau = 15$ and $b = 0.5$. From Figure 1, we see that one outside curve starts from the sonic line, passes through the supersonic regime, then ends at the sonic line, so there is a possible interior supersonic solution. The other curve starts from the sonic line, but rotates in the supersonic regime, and never ends at the sonic line, thus such a curve is not a solution. Obviously, there is no interior subsonic solution.

Now, for the general case of a supersonic doping profile, we are going to prove that there is no interior subsonic solution, nor transonic solution, even no interior supersonic solution if the doping profile $b(x) \ll 1$ or $\tau \ll 1$, namely, when the semiconductor device is almost pure, or the relaxation time is really small (equivalently, the semiconductor effect is large). The supersonic solution and transonic solution exist only when the doping profile is close to the sonic line and τ is large enough. This is totally different from the previous studies [17, 18] for the case without a semiconductor effect.

Now we are going to prove each case stated in Theorem 1.3.

2.1. Nonexistence of interior subsonic/supersonic/transonic solutions. In this subsection, we are going to prove the nonexistence of interior subsonic/supersonic/transonic solutions when the doping profile is small or the relaxation time is small.

THEOREM 2.1. *No interior subsonic solution to* (1.6) *exists for the case of the nonsubsonic doping profile* $0 \le \underline{b} \le b(x) \le \overline{b} \le 1$.

Proof. Supposing there is an interior subsonic solution ρ_{sub} of (1.6) defined in Definition 1.1, let us take the test function $\varphi = (\rho-1)^2 \in H_0^1(0,1)$ in (1.7), then we have

$$\frac{1}{2}\int_0^1 \frac{\rho+1}{\rho^3}\cdot\left|\left[(\rho-1)^2\right]_x\right|^2 dx + \int_0^1 \frac{\left[(\rho-1)^2\right]_x}{\tau\rho}dx + \int_0^1 (\rho-b)(\rho-1)^2 dx = 0. \tag{2.3}$$

Noting that

$$\int_0^1 \frac{\left[(\rho-1)^2\right]_x}{\tau\rho}dx = \frac{2}{\tau}\int_0^1 (\rho-\ln\rho)_x dx = 0,$$

and that $\rho - b > 0$ on $(0,1)$, namely,

$$\int_0^1 (\rho-b)(\rho-1)^2 dx > 0,$$

then, from (2.3), we get a contradiction:

$$\frac{1}{2}\int_0^1 \frac{\rho+1}{\rho^3}\cdot\left|\left[(\rho-1)^2\right]_x\right|^2 dx < 0.$$

Therefore, there is no interior subsonic solution. ∎

THEOREM 2.2. *No interior supersonic solution to* (1.6) *exists when the doping profile* $b(x)$ *is small such that* $\bar{b}(1+\sqrt{2\bar{b}}) < 1$, *or the relaxation time* τ *is small such that* $\tau < \frac{1}{3}$.

Proof. Assume that $\rho(x)$ is an interior supersonic solution of (1.6) satisfying Definition 1.1. The velocity $u(x) = \frac{1}{\rho(x)}$ satisfies

$$\begin{cases} \left(u - \dfrac{1}{u}\right)u_x = E - \dfrac{u}{\tau}, \\ E_x = \dfrac{1}{u} - b(x). \end{cases} \tag{2.4}$$

Because $u \in C[0,1]$, there exists a maximal point denoted by $\hat{y} \in (0,1)$ such that $u(x) \le u(\hat{y})$ for any $x \in [0,1]$. Because $u_x(\hat{y}) = 0$, the first equation of (2.4) gives

$$E(\hat{y}) = \frac{u(\hat{y})}{\tau}. \tag{2.5}$$

Multiplying the first equation of (2.4) by $((u-1)^2)_x$, integrating the resultant equation over $(\hat{y},1)$, using the second equation of (2.4), and noting

$$u((u-1)^2)_x = \frac{1}{3}((u-1)^2(2u+1))_x,$$

we obtain

$$\begin{aligned}
&\int_{\hat{y}}^1 \frac{u(x)+1}{2u(x)} \left|\left[(u(x)-1)^2\right]_x\right|^2 dx \\
&= \int_{\hat{y}}^1 \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx - (u(\hat{y})-1)^2\left(E(\hat{y}) - \frac{2u(\hat{y})+1}{3\tau}\right) \\
&= \int_{\hat{y}}^1 \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx - \frac{(u(\hat{y})-1)^3}{3\tau},
\end{aligned} \tag{2.6}$$

where we have used (2.5) in the second equality.

In the case $b(x) \ll 1$, since $u(\hat{y}) > 1$, we get from (2.6) that

$$\begin{aligned}
&\int_{\hat{y}}^1 \frac{u(x)+1}{2u(x)} \left|\left[(u(x)-1)^2\right]_x\right|^2 dx \\
&\le \int_{\hat{y}}^1 \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx \\
&\le \int_{\hat{y}}^1 b(x)(u(x)-1)^2 dx \\
&\le \frac{1}{4}\int_{\hat{y}}^1 (u(x)-1)^4 dx + \int_{\hat{y}}^1 b^2(x) dx \\
&\le \frac{1}{4}\int_{\hat{y}}^1 \left|\left[(u(x)-1)^2\right]_x\right|^2 dx + \overline{b}^2.
\end{aligned} \tag{2.7}$$

Here we have used

$$\int_y^1 (u(x)-1)^4 dx \le \int_y^1 \left|\left[(u(x)-1)^2\right]_x\right|^2 dx \quad \text{for } y \in (0,1). \tag{2.8}$$

Then (2.7) gives

$$\int_{\hat{y}}^1 \left|\left[(u(x)-1)^2\right]_x\right|^2 dx \le 4\bar{b}^2,$$

and further

$$u(x) \le 1 + (2\overline{b})^{1/2} \text{ on } [\hat{y},1].$$

It then follows that

$$\left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 \le \left(\overline{b} - \frac{1}{1+(2\overline{b})^{1/2}}\right)(u(x)-1)^2 \quad \text{for any } x \in [\hat{y},1].$$

Thus, when $\overline{b}$ is small enough such that $\overline{b} - \frac{1}{1+(2\overline{b})^{1/2}} < 0$, we get from the first inequality of (2.7) that

$$0 \le \int_{\hat{y}}^1 \frac{u+1}{2u}\left|\left[(u-1)^2\right]_x\right|^2 dx \le \left(\overline{b} - \frac{1}{1+(2\overline{b})^{1/2}}\right)\int_{\hat{y}}^1 (u-1)^2 dx < 0, \tag{2.9}$$

which is a contradiction.

In the case $\tau \ll 1$, since $b \le 1$ and $1 \le u \le u(\hat{y})$, it follows from (2.6) that

$$\int_{\hat{y}}^{1} \frac{u+1}{2u} \left|\left[(u-1)^2\right]_x\right|^2 dx \le \int_{\hat{y}}^{1} \frac{(u-1)^3}{u} dx - \frac{(u(\hat{y})-1)^3}{3\tau} \le \left(1 - \frac{1}{3\tau}\right)(u(\hat{y})-1)^3.$$

Thus, when τ is small such that $\tau < \frac{1}{3}$, we get a contradiction. Therefore, no interior supersonic solution exists. The proof is complete. ∎

THEOREM 2.3. *No transonic solution to system* (1.4)–(1.5) *exists when the doping profile $b(x)$ is small such that $\bar{b}(1+\sqrt{2\bar{b}}) < 1$, or the relaxation time τ is small such that $\tau < \frac{1}{3}$.*

Proof. Suppose that (ρ, E) is a transonic solution separated by a point $y_0 \in (0,1)$ in the form

$$\rho(x) = \begin{cases} \rho_{sup}(x), & x \in (0, y_0), \\ \rho_{sub}(x), & x \in (y_0, 1), \end{cases}$$

and

$$\rho_l \rho_r = 1, \; E_l = E_r \text{ with } \rho_l < 1 \text{ and } \rho_r > 1.$$

We first claim

$$E_l = E_r < \frac{1}{\tau}. \tag{2.10}$$

In fact, if $E_r \ge \frac{1}{\tau}$, noting the second equation of (1.4) gives

$$(E_{sub})_x(x) = (\rho_{sub} - b) > (1-b) \ge 0 \quad \text{on } (y_0, 1),$$

i.e., E_{sub} is monotone increasing, we have

$$E_{sub}(x) > E_r \ge \frac{1}{\tau}, \quad \text{and} \quad \rho_{sub}(x)E_{sub}(x) - \frac{1}{\tau} > E_r - \frac{1}{\tau} \ge 0 \text{ on } (y_0, 1),$$

which in combination with the first equation of (1.4) further gives $(\rho_{sub})_x(x) > 0$ on $(y_0, 1)$. Thus, $1 < \rho_r < \rho_{sub}$ over $(y_0, 1)$, which contradicts $\rho_{sub}(1) = 1$. Hence (2.10) holds.

In the case $b(x) \ll 1$, multiplying the first equation of (2.4) by $((u-1)^2)_x$ and integrating the resultant equation over $(0, y_0)$, as in (2.6), we get

$$\begin{aligned}
&\int_0^{y_0} \frac{u(x)+1}{2u(x)} \left|\left[(u(x)-1)^2\right]_x\right|^2 dx \\
&\quad = \int_0^{y_0} \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx + (u_l-1)^2\left(E_l - \frac{2u_l+1}{3\tau}\right) \\
&\quad < \int_0^{y_0} \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx - \frac{2(u_l-1)^3}{3\tau} \\
&\quad < \int_0^{y_0} b(x)(u(x)-1)^2 dx,
\end{aligned}$$

where we have used (2.10) in the first inequality. Thus, as in (2.7)–(2.9), when $\bar{b}$ is small enough such that $\bar{b} - \frac{1}{1+(2\bar{b})^{1/2}} < 0$, we get the contradiction

$$\int_0^{y_0} \frac{u+1}{2u} |[(u-1)^2]_x|^2 dx < 0.$$

In the case $\tau \ll 1$, since $\rho_l < 1$, by (2.10) we get $\rho_l E_l - 1/\tau < 0$. Thus, $\lim_{x\to y_0^-}(\rho_{sup})_x(x) = (1-1/\rho_l^2)^{-1}(\rho_l E_l - 1/\tau) > 0$. It is then easy to see that $\rho_{sup}(x)$ attains a local minimal point on $(0, y_0)$. Denote by $\breve{y}$ the last local minimal point of $\rho_{sup}(x)$ on $(0, y_0)$, then $(\rho_{sup})'_x(\breve{y}) = 0$. Set $u(x) := \frac{1}{\rho_{sup}(x)}$, then $u_x(\breve{y}) = 0$ and $u_l = \frac{1}{\rho_l} > 1$. Hence by the first equation of (2.4), we also get (2.5) at $\breve{y}$. Multiplying the first equation of (2.4) by $((u-1)^2)_x$ and integrating the resultant equation over $(\breve{y}, y_0)$, as shown in (2.6), using (2.5), we get

$$\begin{aligned}
&\int_{\breve{y}}^{y_0} \frac{u(x)+1}{2u(x)} \left|\left[(u(x)-1)^2\right]_x\right|^2 dx \\
&= \int_{\breve{y}}^{y_0} \left(b(x) - \frac{1}{u(x)}\right)[u(x)-1]^2 dx + (u_l-1)^2\left(E_l - \frac{2u_l+1}{3\tau}\right) \\
&\quad - (u(\breve{y})-1)^2\left(E(\hat{y}) - \frac{2u(\breve{y})+1}{3\tau}\right) \\
&= \int_{\breve{y}}^{y_0} \left(b(x) - \frac{1}{u(x)}\right)(u(x)-1)^2 dx + (u_l-1)^2\left(E_l - \frac{1}{\tau}\right) \\
&\quad - \frac{2(u_l-1)^3}{3\tau} - \frac{(u(\breve{y})-1)^3}{3\tau} \\
&\le \int_{\breve{y}}^{y_0} [u(x)-1]^3 dx - \frac{1}{3\tau}\left((u_l-1)^3 + (u(\breve{y})-1)^3\right),
\end{aligned}$$

where we have used $b \le 1$ and (2.10) in the inequality. Noting

$$\max_{x\in[\breve{y},y_0]}[u(x)-1]^3 = \max\left\{(u_l-1)^3, (u(\breve{y})-1)^3\right\} =: K,$$

we further have

$$\int_{\breve{y}}^{y_0} \frac{u+1}{2u}\left|\left[(u-1)^2\right]_x\right|^2 dx \le \left(1 - \frac{1}{3\tau}\right)K < 0 \quad \text{if} \quad \tau < \frac{1}{3}.$$

We thus get a contradiction. ∎

2.2. Existence of interior supersonic/transonic solutions. In this subsection, we prove the existence of supersonic/transonic solutions when the doping profile is close to the sonic line and the relaxation time is large. The approach adopted is the compactness technique.

THEOREM 2.4. *There exists at least one interior supersonic solution to system* (1.4)–(1.5) *satisfying* $\rho \in C^{\frac{1}{2}}[0,1]$ *and the optimal estimate* (1.11)*, when* $b(x)$ *is close to the sonic state* 1 *in the sense of* $\|b-1\|_{L^\infty(0,1)} \ll 1$ *and the relaxation time is large* $\tau \gg 1$.

Proof. The proof is long and technical, and we divide it into 5 steps.

Step 1. We first consider the Euler–Poisson equations without the semiconductor effect:

$$\begin{cases} \left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E, \\ E_x = \rho - 1, \\ \rho(0) = \rho(L) = 1-\delta \quad \text{(supersonic boundary)}, \end{cases} \tag{2.11}$$

where $L \geq \frac{1}{4}$ is the parameter of length and $\delta > 0$ is a small constant. Taking $\underline{b} = 1$ in [16, Lemma 4.1], one can see that (2.11) has a supersonic solution $(\rho_L, E_L)(x)$ satisfying

$$\beta(L) \leq \underline{\rho} \leq \gamma(L), \ E_L(0) \geq \sqrt{f(\gamma(L))} > 0, \tag{2.12}$$

where $\underline{\rho} := \min_{x\in[0,L]} \rho_L(x)$.

Step 2. Let η be a small number to be determined such that $\delta < \eta \ll 1$. Denote by $(\rho_1, E_1)(x)$ the solution of (2.11) with $L = \frac{1}{2}$. Now let us consider the ODE system with the semiconductor effect $-\frac{1}{\tau}$ and a small perturbation of the doping profile around the sonic line, i.e., $b(x) = 1 - \epsilon e(x)$:

$$\begin{cases} \left(1 - \dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - 1 + \epsilon e(x), \\ (\rho(0), E(0)) = (1-\delta, E_1(0)). \end{cases} \tag{2.13}$$

Here $\tau \gg 1$, $0 < \epsilon \ll 1$, $0 \leq e(x) \in L^\infty(\mathbb{R}^+)$, and we have extended periodically the doping profile b to $\mathbb{R}^+$. We claim that there exists a number $y_1 \leq C\eta$ such that $\rho(y_1) = 1 - \eta$, where $C > 0$ is a constant independent of τ, ϵ, δ, and η.

It is easy to see that if $\tau \geq \frac{4}{E_1(0)}$ and $\delta \leq \frac{1}{4}$, then the initial data of (2.13) satisfy

$$\rho(0)E(0) - \frac{1}{\tau} = (1-\delta)E_1(0) - \frac{1}{\tau} \geq \frac{E_1(0)}{2} > 0.$$

From the first equation of (2.13), we know that ρ is decreasing in a neighborhood of 0. If ρ keeps decreasing on $[0, x]$, then

$$E(x) = E_1(0) + \int_0^x (\rho - 1 + \epsilon e(s))ds \geq E_1(0) - x, \tag{2.14}$$

which indicates that if

$$x \leq \frac{E_1(0)}{4}, \tag{2.15}$$

then

$$E(x) \geq E_1(0) - \frac{E_1(0)}{4} = \frac{3E_1(0)}{4}.$$

We next prove that if ρ keeps decreasing, denoting by y_1 the first number that ρ attains $1 - \eta$, then $y_1 \leq C\eta^2$ for some constant $C > 0$. In fact, observe that

$$\rho_x = \frac{\rho E - \frac{1}{\tau}}{1 - \frac{1}{\rho^2}} = \frac{\rho^2(\rho E - \frac{1}{\tau})}{\rho^2 - 1} \leq \frac{\rho^3 E}{\rho^2 - 1} \leq -\frac{3(1-\eta)^3 E_1(0)}{4\eta(2-\eta)} \leq -\frac{E_1(0)}{16\eta} \text{ if } \eta \leq \frac{1}{2}.$$

Thus,

$$y_1 = \frac{\delta - \eta}{\int_0^1 \rho_x(sy_1)ds} \leq \frac{16\eta^2}{E_1(0)}.$$

Hence, if $\eta \leq \frac{E_1(0)}{8}$, then (2.15) holds, and ρ keeps decreasing and attains $1 - \eta$ at y_1 with $y_1 \leq \frac{16\eta^2}{E_1(0)}$. By (2.14),

$$E_1(0) - C\eta^2 \leq E(y_1) \leq E_1(0) + C\eta^2. \tag{2.16}$$

Step 3. Now let us reconsider the ODE system without the semiconductor effect,

$$\begin{cases} \left(1-\dfrac{1}{\hat\rho^2}\right)\hat\rho_x = \hat\rho\hat E, \\ \hat E_x = \hat\rho - 1, \\ (\hat\rho(0), \hat E(0)) = (1-\delta, \hat E_0). \end{cases} \tag{2.17}$$

Taking $b = 1$ in [16, step 2 in the proof of Theorem 4.2], we know that there exist $\hat E_0 \in (\frac{E_1(0)}{2}, 2E_1(0))$ and $y_2 \le C\eta^2$ such that (2.17) has a supersonic solution $(\hat\rho, \hat E)$ satisfying

$$\hat\rho(y_2) = 1-\eta,\ \hat E(y_2) = E(y_1),\ E_1(0) - C\eta^2 \le \hat E(y_2) \le E_1(0) + C\eta^2. \tag{2.18}$$

Here E and y_1 are given by step 2. Moreover, the length $\hat L$ of the solution of (2.17) with initial boundary data $(\hat\rho(0), \hat E(0)) = (1-\delta, \hat E_0)$, $\hat\rho(\hat L) = 1-\delta$ satisfies

$$\frac{1}{4} \le \hat L \le \frac{3}{4}.$$

Step 4. Set $(\bar\rho, \bar E)(x) := (\hat\rho, \hat E)(x - y_1 + y_2)$, then $(\bar\rho, \bar E)$ satisfies (2.11) with initial-boundary data

$$(\bar\rho, \bar E)(y_1) = (1-\eta, \hat E(y_2)) = (\rho, E)(y_1) \ \text{ and } \bar\rho(y_3) = 1-\eta$$

with $y_3 := \hat L + y_1 - 2y_2$. As in [16, step 3 of the proof of Theorem 4.2], when $\tau \gg 1$ and $0 < \epsilon \ll 1$ such that $C(\frac{1}{\tau^2} + \epsilon^2)e^{C/\eta^2} \le 1/4$, system (2.13) has a unique solution (ρ, E) on $[0, y_3]$ satisfying

$$\rho(y_3) \le 1 - \frac{\eta}{2}, \quad E(y_3) \le E_1(0) + C\eta. \tag{2.19}$$

Now taking y_3 as the initial data, as in [16, step 3 of the proof of Theorem 4.2], we can extend (ρ, E), the solution of (2.13), to the state $\rho = 1-\delta$. Denote by y_4 the number that $\rho(y_4) = 1-\delta$, then

$$\rho(0) = \rho(y_4) = 1-\delta,\ E(0) = E_1(0),\ E(y_4) \le E_1(0) + C\eta. \tag{2.20}$$

Moreover,

$$\frac{1}{4} - C\eta^2 \le y_4 \le \frac{3}{4} + C\eta^2.$$

Now we take $L = \frac{3}{2}$ in (2.11) and denote by (ρ_2, E_2) its solution. Applying a similar argument above, we know that there exists an interval $[0, y_5]$ with

$$\frac{5}{4} - C\eta^2 \le y_5 \le \frac{7}{4} + C\eta^2,$$

such that system (2.13) has a solution on $[0, y_5]$ satisfying

$$\rho(0) = \rho(y_5) = 1-\delta,\ E(0) = E_2(0),\ E(y_5) \le -E_2(0) + C\eta. \tag{2.21}$$

Without loss of generality, we assume that $E_1(0) < E_2(0)$; then when $\eta \ll 1$, for any initial data $E_L(0) \in (E_1(0), E_2(0))$, (2.13) has a solution. Noting the length parameter L is continuous with respect to the initial data, system (2.13) has a solution on $[0,1]$ satisfying $\rho(0) = \rho(1) = 1-\delta$ and $E(0) \in (E_1(0), E_2(0))$.

Step 5. For any $\delta > 0$, denote by (ρ^δ, E^δ) the solution of (2.13) with boundary data $\rho^\delta(0) = \rho^\delta(1) = 1-\delta$. The velocity $u^\delta = 1/\rho^\delta$ satisfies

$$\left(\left(u^\delta - \frac{1}{u^\delta}\right)(u^\delta)_x\right)_x + \frac{(u^\delta)_x}{\tau} - \left(\frac{1}{u^\delta} - b\right) = 0,\ u^\delta(0) = u^\delta(1) = \frac{1}{1-\delta}. \tag{2.22}$$

Multiplying (2.22) by $(u^\delta - \frac{1}{1-\delta})^2$, as in (2.15), we have

$$\begin{aligned}
&\frac{2\delta}{1-\delta}\int_0^1 \frac{(u^\delta+1)}{u^\delta}\left(u^\delta - \frac{1}{1-\delta}\right)|u_x^\delta|^2 dx + \int_0^1 \frac{(u^\delta+1)}{2u^\delta}\left|\left(\left(u^\delta - \frac{1}{1-\delta}\right)^2\right)_x\right|^2 \\
&= \int_0^1 \left(b - \frac{1}{u^\delta}\right)\left(u^\delta - \frac{1}{1-\delta}\right)^2 \\
&\le \frac{1}{2}\int_0^1 \left(u^\delta - \frac{1}{1-\delta}\right)^4 + \frac{1}{2}\int_0^1 \left(b - \frac{1}{u^\delta}\right)^2 \\
&\le \frac{1}{4}\int_0^1 \left|\left(\left(u^\delta - \frac{1}{1-\delta}\right)^2\right)_x\right|^2 + \frac{1}{2}\int_0^1 b^2,
\end{aligned}$$

which gives

$$\left\|\left(u^\delta - \frac{1}{1-\delta}\right)^2\right\|_{H^1} \le C$$

and, hence,

$$\|u^\delta\|_{L^\infty} \le C.$$

It then follows that

$$\rho^\delta = \frac{1}{u^\delta} \ge \frac{1}{\|u^\delta\|_{L^\infty}} \ge \frac{1}{C} \text{ and } \left\|\left(1-\delta-\rho^\delta\right)^2\right\|_{H^1} \le C.$$

Therefore, there exists a function ρ^0 such that, as $\delta \to 1^+$, up to a subsequence,

$$\begin{aligned}
&\left(1-\rho^\delta\right)^2 \rightharpoonup \left(1-\rho^0\right)^2 \quad \text{weakly in } H^1(0,1),\\
&\left(1-\rho^\delta\right)^2 \to \left(1-\rho^0\right)^2 \quad \text{strongly in } C^{\frac{1}{2}}[0,1].
\end{aligned} \tag{2.23}$$

Applying the same procedure as the proof of [16, Theorem 2.1], one can show that ρ^0 is the supersonic solution of (1.6). The proof of estimate (1.11) being similar to that of [16, Proposition 3.3], we thus omit it. ∎

THEOREM 2.5. *There exist infinitely many transonic solutions to* (1.4)–(1.5), *when* $b(x)$ *is close to* 1 *in the sense of* $\|b-1\|_{L^\infty(0,1)} \ll 1$ *and* $\tau \gg 1$.

Proof. The proof is also long and we do it in 3 steps.

Step 1. Consider the ODE system (2.13), in view of step 4 in the proof of Theorem 2.4, given small constants $\eta \ll 1$ $(\delta < \eta)$, $\epsilon \ll 1$, $\tau \gg 1$, (2.13) has a supersonic solution (ρ, E) on $[0, y_4]$ satisfying

$$\frac{1}{4} - C\eta^2 \le y_4 \le \frac{3}{4} + C\eta^2,\ \rho(0) = \rho(y_4) = 1-\delta,\ E(0) = E_1(0),\ E(y_4) \le -E_1(0) + C\eta,$$

where E_1 is the solution of (2.11) with $L = \frac{1}{2}$. Setting $\rho_l = 1-\eta$ and taking the jump location $\bar{y}_0 \in (0, y_4)$ by the last number when $\rho(\bar{y}_0) = \rho_l$, we focus this supersonic solution $(\rho_{sup}, E_{sup})(x)$ only on $[0, \bar{y}_0]$. As in [16, step 4 of the proof of Theorem 4.2], when $\eta \ll 1$ such that $(C + E_1(0))\eta \le \frac{E_1(0)}{2}$ and $C\eta^2 < \frac{E_1(0)}{4}$, then

$$\rho_r E_r - \frac{1}{\tau} \le -\frac{E_1(0)}{4} < 0.$$

From the first equation of (2.13), we know such an initial value problem has a decreasing subsonic solution in a neighborhood of $\bar{y}_0^+$. We denote this subsonic solution by $(\rho_{sub}, E_{sub})(x)$. If ρ_{sub} keeps decreasing, then

$$\begin{aligned} E_{sub}(x) &= E_r + \int_{\bar{y}_0}^{x} (\rho_{sub} - 1 + \epsilon e(x))dx \\ &\le -E_1(0) + C\eta + \int_{\bar{y}_0}^{x} (\rho_r - 1 + \epsilon e(x))dx \\ &\le -E_1(0) + C\eta + (x - \bar{y}_0)\left(\frac{\eta}{1-\eta} + \epsilon\|e\|_{L^\infty}\right), \end{aligned}$$

which implies that if $C\eta \le \min(\frac{E_1(0)}{2}, \frac{1}{2})$, $\epsilon \le 1$ and

$$x - \bar{y}_0 \le \frac{4}{(1 + \|e\|_{L^\infty})E_1(0)}, \tag{2.24}$$

then

$$E_{sub}(x) < -\frac{E_1(0)}{4} < 0.$$

We now claim that if ρ_{sub} keeps decreasing, denoting by y_6 the number that ρ_{sub} attains $1+\delta$, then $y_6 - \bar{y}_0 \le C\eta$.

In fact, observing that

$$(\rho_{sub})_x = \frac{\rho_{sub}E_{sub} - \frac{1}{\tau}}{1 - \frac{1}{\rho_{sub}^2}} \le -\frac{(1-\eta)^2 E_1(0)}{4\eta(2-\eta)} < -\frac{(1-\eta)^2 E_1(0)}{4\eta},$$

we get

$$\begin{aligned} y_6 - \bar{y}_0 &= \frac{\delta - \frac{\eta}{1-\eta}}{\int_0^1 (\rho_{sub})_x (sy_6 + (1-s)\bar{y}_0)ds} \\ &\le \frac{\eta}{1-\eta} \cdot \frac{4\eta}{E_1(0)(1-\eta)^2} \le 32\eta \text{ if } \eta < \min\left(E_1(0), \frac{1}{2}\right). \end{aligned}$$

Obviously, if $\eta < \frac{1}{16(1+\|e\|_{L^\infty})E_1(0)}$, then (2.24) holds and ρ_{sub} keeps decreasing and attains $1+\delta$ at y_6. Now we have constructed the transonic solution to (1.4) in $[0, y_6]$ with $\frac{1}{4} - C\eta \le y_6 \le \frac{3}{4} + C\eta$ as follows:

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \bar{y}_0), \\ (\rho_{sub}, E_{sub})(x), & x \in (\bar{y}_0, y_6], \end{cases}$$

which satisfies the boundary condition

$$\rho_{sup}(0) = 1-\delta,\ \rho_{sub}(y_6) = 1+\delta,$$

and the entropy condition at $\bar{y}_0$

$$0 < \rho_{sup}(\bar{y}_0^-) = 1-\eta < 1 < \rho_{sub}(\bar{y}_0^+),$$

and the Rankine–Hugoniot condition (1.9) at $\bar{y}_0$.

Step 2. Denoting by (ρ_2, E_2) the solution of (2.11) with $L = \frac{3}{2}$, by step 4 of the proof of Theorem 2.4, (2.13) has a supersonic solution (ρ, E) on $[0, y_7]$ with

$$\frac{5}{4} - C\eta^2 \le y_7 \le \frac{7}{4} + C\eta^2,\ \rho(0) = \rho(y_7) = 1-\delta,\ E(0) = E_2(0),\ E(x_{10}) \le -E_2(0) + C\eta.$$

As in step 1, we may construct another transonic solution for (1.4) in the form of

$$(\rho_{trans}, E_{trans})(x) = \begin{cases} (\rho_{sup}, E_{sup})(x), & x \in [0, \tilde{y}_0), \\ (\rho_{sub}, E_{sub})(x), & x \in (\tilde{y}_0, y_7], \end{cases}$$

where $\tilde{y}_0 \in (0, x_{10})$ and $\frac{5}{4} - C\eta^2 \le y_7 \le \frac{7}{4} + C\eta^2$ are some determined numbers. This transonic solution satisfies the boundary condition

$$\rho_{sup}(0) = 1-\delta,\ \rho_{sub}(y_7) = 1+\delta,$$

the entropy condition at $\tilde{y}_0$

$$0 < \rho_{sup}(\tilde{y}_0^-) = 1-\eta < 1 < \rho_{sub}(\tilde{y}_0^+),$$

and the Rankine–Hugoniot condition (1.9) at $\tilde{y}_0$.

Step 3. Without loss of generality, we assume that $E_1(0) < E_2(0)$. As in [16, step 6 in the proof of Theorem 4.2], for any $E_0 \in (E_1(0), E_2(0))$, (2.13) has a transonic solution on an interval $[0, y_8]$. Applying the continuation argument in the length of the interval, one can see that for any $\delta > 0$ (1.4)–(1.5) has a transonic solution denoted by $(\rho^\delta_{trans}, E^\delta_{trans})$ on $[0, 1]$, and it satisfies the boundary conditions

$$\rho^\delta_{sup}(0) = 1-\delta,\ \rho^\delta_{sub}(1) = 1+\delta,$$

the entropy condition

$$0 < \rho^\delta_{sup}(y^\delta_0) = 1-\eta < 1 < \rho^\delta_{sub}\left(y^\delta_0\right),$$

and the Rankine–Hugoniot condition (1.9) at a jump location y^δ_0 in $(0, 1)$. Letting $\delta \to 0^+$, applying the diagonal argument for $(\rho^\delta_{trans}, E^\delta_{trans})$, we know that (1.4)–(1.5) has a transonic solution $(\rho_{trans}, E_{trans})(x)$ for $x \in [0, 1]$ and satisfies the sonic boundary condition, the entropy condition, and the Rankine–Hugoniot condition at a jump location y_0 in $(0, 1)$.

Because τ and ϵ only depend on $(E_1(0), E_2(0), \eta)$, and η only depends on $(E_1(0), E_2(0))$, there exists an $\eta_0 > 0$ such that for any $\eta \in (0, \eta_0)$, there exists a transonic solution jump at $\rho_l = 1-\eta$. Thus, we obtain infinitely many transonic solutions due to the arbitrary choice of $0 < \eta < \eta_0$. ∎

Proof of Theorem 1.3. Combining Theorems 2.1–2.5, we immediately obtain Theorem 1.3. ∎

3. Concluding remarks. In this section, we note that [16, Theorem 1.3] and Theorem 1.3 above both hold for the isentropic hydrodynamic model. For isentropic flow, the pressure function satisfies $P(\rho) = T\rho^{\gamma}$ for some constants $T > 0$ and $\gamma > 1$. Then system (1.1) reduces to

$$\begin{cases} J = \text{constant}, \\ \left(\dfrac{J^2}{\rho} + T\rho^{\gamma}\right)_x = \rho E - \dfrac{J}{\tau}, \qquad x \in (0,1). \\ E_x = \rho - b(x), \end{cases} \tag{3.1}$$

The sonic flow means

$$\text{fluid velocity: } u = \frac{J}{\rho} = c = \sqrt{P'(\rho)} = \sqrt{T\gamma\rho^{\gamma-1}}: \text{ sound speed.}$$

Without loss of generality, we assume that

$$J = T\gamma = 1.$$

Then (3.1) is transformed to

$$\begin{cases} \left(\rho^{\gamma-1} - \dfrac{1}{\rho^2}\right)\rho_x = \rho E - \dfrac{1}{\tau}, \\ E_x = \rho - b(x), \end{cases} \tag{3.2}$$

and our sonic boundary conditions are proposed as

$$\rho(0) = \rho(1) = 1. \tag{3.3}$$

Now as in the isothermal fluid, we can also identify that, for system (3.2), $\rho > 1$ is for the subsonic flow, $\rho = 1$ stands for the sonic flow, and $0 < \rho < 1$ represents the supersonic flow.

Following the proofs of Theorem 1.3 in the first part [16] and Theorem 1.3 in this paper, one can easily obtain the following classification of solutions to system (3.2)–(3.3) for isentropic flow.

THEOREM 3.1.

1. *For subsonic doping profile:* $b(x) \in L^{\infty}(0,1)$ *and* $b(x) > 1$ *in* $[0,1]$, *then system* (3.2)–(3.3) *admit*
   (a) *a unique pair of interior subsonic solutions* $(\rho_{sub}, E_{sub})(x) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ *with* $\rho_{sub}(x) > 1$ *on* $(0,1)$*;*
   (b) *at least a pair of interior supersonic solutions* $(\rho_{sup}, E_{sup})(x) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ *with* $0 < \rho_{sup}(x) < 1$ *on* $(0,1)$*;*
   (c) *further that if* τ *is large and that* $\bar{b} - \underline{b} \ll 1$, *then system* (3.2)–(3.3) *has infinitely many transonic shock solutions* $(\rho_{trans}, E_{trans})(x) \in L^{\infty}(0,1) \times C^0(0,1)$*;*
   (d) *further that if* $b(x) = b > 1$ *is a constant, then when* τ *is small enough,* (3.2)–(3.3) *has infinitely many* C^1 *transonic solutions; moreover, in this case there is no transonic shock solution.*
2. *For supersonic doping profile:* $b(x) \in L^{\infty}(0,1)$ *and* $0 < b(x) \le 1$ *in* $[0,1]$, *then* (3.2)–(3.3) *admit*

(a) *a pair of interior supersonic solutions* $(\rho_{sup}, E_{sup})(x) \in C^{\frac{1}{2}}[0,1] \times H^1(0,1)$ *and infinitely many transonic shock solutions* $(\rho_{trans}, E_{trans})(x) \in L^\infty(0,1) \times C^0(0,1)$ *if* $b(x)$ *is close to 1 and* τ *is large enough;*
(b) *no interior subsonic solutions* $(\rho_{sub}, E_{sub})(x)$*;*
(c) *no interior supersonic solutions* $(\rho_{sup}, E_{sup})(x)$*, nor transonic shock solutions* $(\rho_{trans}, E_{trans})(x)$ *if* $b(x)$ *is small or* τ *is small.*

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