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# LARGE TIME BEHAVIOR OF SOLUTIONS TO $N$-DIMENSIONAL BIPOLAR HYDRODYNAMIC MODELS FOR SEMICONDUCTORS*

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*Dedicated to Professor Akitaka Matsumura on his 60th birthday*

**Abstract.** In this paper, we study the $n$-dimensional ($n \geq 1$) bipolar hydrodynamic model for semiconductors in the form of Euler–Poisson equations. In the 1-D case, when the difference between the initial electron mass and the initial hole mass is nonzero (switch-on case), the stability of nonlinear diffusion waves has been open for a long time. In order to overcome this difficulty, we ingeniously construct some new correction functions to delete the gaps between the original solutions and the diffusion waves in $L^2$-space, so that we can deal with the 1-D case for general perturbations, and prove the $L^\infty$-stability of diffusion waves in the 1-D case. The optimal convergence rates are also obtained. Furthermore, based on the 1-D results, we establish some crucial energy estimates and apply a new but key inequality to prove the stability of planar diffusion waves in $n$-D case. This is the first result for the multidimensional bipolar hydrodynamic model of semiconductors.

**Key words.** bipolar hydrodynamic model, semiconductor, nonlinear damping, (planar) nonlinear diffusion waves, asymptotic behavior, convergence rates

**AMS subject classifications.** 35L50, 35L60, 35L65, 76R50

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**1. Introduction.** In this paper, we study the $n$-D isentropic Euler–Poisson equations for the bipolar hydrodynamic model of the semiconductor device

$$
\begin{cases}
n_{1t} + \operatorname{div}(n_1 u_1) = 0, \\
(n_1 u_1)_t + \operatorname{div}(n_1 u_1 \otimes u_1) + \nabla p(n_1) = n_1 \nabla \Psi - n_1 u_1, \\
n_{2t} + \operatorname{div}(n_2 u_2) = 0, \\
(n_2 u_2)_t + \operatorname{div}(n_2 u_2 \otimes u_2) + \nabla q(n_2) = -n_2 \nabla \Psi - n_2 u_2, \\
\Delta \Psi = n_1 - n_2
\end{cases}
\tag{1.1}
$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 1$, $t > 0$. Here $n_1 = n_1(x,t)$, $n_2 = n_2(x,t)$, $u_1 = (u_{11}, \ldots, u_{1n})(x,t)$, $u_2 = (u_{21}, \ldots, u_{2n})(x,t)$, and $\Psi(x,t)$ represent the electron density, the hole density, the electron velocity, the hole velocity, and the electrostatic potential, respectively. $E := \nabla \Psi(x,t)$ is called the electric field. The nonlinear functions $p(s)$ and $q(s)$ denote the pressures of the electrons and the holes, respectively, which are smooth, strictly increasing, and nonnegative. Since both $p(s)$ and $q(s)$ possess the same characters, for simplicity, here and afterward we assume them to be

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identical, namely,

$$p(s) = q(s) \geq 0, \quad p'(s) = q'(s) > 0 \quad \text{for } s > 0. \tag{1.2}$$

If the pressures $p(s)$ and $q(s)$ are different, this will be a new story, and we will leave it for future study (see Remark 1 for details).

Hydrodynamic models of this type are generally used in the description of the charged fluid particles. Examples are electrons and holes in semiconductor devices, or positively and negatively charged ions in a plasma. These models, which can be derived from kinetic models, take an important place in the fields of applied and computational mathematics. A standard approach for this derivation is the moment method. According to the different analysis for the phase space densities, introduced to prescribe the dependence on the velocity, we recover different limit models and, in particular, the drift-diffusion equations and the hydrodynamic (Euler–Poisson) systems. The hydrodynamic models are usually considered to describe high field phenomena of submicronic devices. For details on the semiconductor applications, see [16, 26]. For the applications in plasma physics, see [16, 37].

For unipolar isentropic and nonisentropic hydrodynamic equations of semiconductors on the whole space or the spatial bounded domain, the main effort was made on the mathematical modellings [16, 26] and on the rigorous mathematical analysis, such as well-posedness of steady-state solutions [2, 4, 5], and their stability [8, 10, 15, 18, 22, 32], the global existence of classical and/or the entropy weak solutions [1, 19, 25, 36, 39], the large time behavior of solutions [10, 18, 20], and the zero relaxation limit problems [7, 11], etc.

For bipolar hydrodynamic semiconductor equations, however, the study is quite limited and far from being well, especially for the high-dimensional case. In the 1-D case, Natalini [31] and Hsiao and Zhang [11, 12] established the global entropic weak solutions in the framework of compensated compactness on the whole real line and spatial bounded domain, respectively. Hattori and Zhu [41] proved the stability of steady-state solutions for a recombined bipolar hydrodynamic model. Gasser, Hsiao, and Li [6] and Huang and Li [14] investigated the large time behavior of both small smooth and weak solutions, respectively. Furthermore, Li [21] studied the relaxation limit of a bipolar isentropic hydrodynamic model for semiconductors with small momentum relaxation time. But the study in the $n$-D case for the bipolar hydrodynamic system of semiconductors is never dealt with, to the best of our knowledge.

Physically, the frictional damping usually causes the dynamical system to possess the nonlinear diffusive phenomena. Such interesting phenomena for 1-D compressible Euler equations with damping was investigated first by Hsiao and Liu [9]. Since then, this problem has attracted considerable attention, for example, see [24, 30, 33, 34, 35, 38, 40] and the references therein; see also the new progress made by Mei [29] for the selection of the best asymptotic profiles with much faster convergence rates, recently. Here, our main interest is to investigate such diffusive phenomena for the Euler–Poisson equations (1.1).

For 1-D (1.1), we denote $J_1 = n_1u_1$ and $J_2 = n_2u_2$ as the current densities for the electrons and the holes, respectively, and $E = \Psi_{x_1}$ as the electric field for the sake of simplification. Thus, the 1-D isentropic Euler–Poisson equations of (1.1) are written as follows:

$$
\begin{cases}
n_{1t} + J_{1x_1} = 0, \\
J_{1t} + (\frac{J_1^2}{n_1} + p(n_1))_{x_1} = n_1 E - J_1, \\
n_{2t} + J_{2x} = 0, \\
J_{2t} + (\frac{J_2^2}{n_2} + p(n_2))_{x_1} = -n_2 E - J_2, \\
E_{x_1} = n_1 - n_2,
\end{cases} \tag{1.3}
$$

with the initial data

$$
(n_1, n_2, J_1, J_2)|_{t=0} = (n_{10}, n_{20}, J_{10}, J_{20})(x), \tag{1.4}
$$

where $(n_{10}, n_{20}, J_{10}, J_{20})(x) \to (n_\pm, n_\pm, J_{1\pm}, J_{2\pm})$ as $x \to \pm\infty$, and $(n_\pm, n_\pm, J_{1\pm}, J_{2\pm})$ are the state constants, and $J_{i\pm} = n_{i\pm} u_{i\pm}$.

The nonlinear diffusive phenomena both in smooth and weak senses were also observed for the bipolar hydrodynamic model (1.3) by Gasser, Hsiao, and Li [6] and Huang and Li [14], respectively. Namely, according to Darcy's law, it is expected that the solutions of (1.3) $(n_1, J_1, n_2, J_2, E)(x_1, t)$ converge in $L^\infty$-sense to the so-called *nonlinear diffusion waves* $(\bar{n}, \bar{J}, \bar{n}, \bar{J}, 0)(x_1, t)$, where $(\bar{n}, \bar{J}) = (\bar{n}, \bar{J})((x_1 + x_0)/\sqrt{1+t})$ ($x_0$ is a shift constant) are the self-similar solutions to the following equations:

$$
\begin{cases}
\bar{n}_t = p(\bar{n})_{x_1 x_1} & \text{porous medium equation,} \\
\bar{J} := \bar{n}\bar{u} = -p(\bar{n})_{x_1} & \text{Darcy's Law,} \\
(\bar{n}, \bar{J}) \to (n_\pm, 0), & \text{as } \ x_1 \to \pm\infty.
\end{cases} \tag{1.5}
$$

Such 1-D self-similar solutions $(\bar{n}, \bar{u})(x_1/\sqrt{1+t}) = (\bar{n}, \bar{u})(x \cdot \mathbf{e}/\sqrt{1+t})$ with $\mathbf{e} = (1, 0, \ldots, 0) \in \mathbb{R}^n$ are also called the *planar diffusion waves* to the $n$-D equations (1.1). In [6, 14], they need the difference of the initial mass of the electrons and the initial mass of the holes to be zero,

$$
\int_{\mathbb{R}} [n_{10}(x_1) - n_{20}(x_1)] dx_1 = 0 \quad \text{and} \quad J_{i+} = J_{i-}, i = 1, 2.
$$

This implies, from the last equation of (1.3), that the difference of the electric fields is zero,

$$
E(+\infty, t) - E(-\infty, t) = 0.
$$

However, such a condition is too stiff, because it looks like a switch-off situation for the device (no voltage). The most interesting but challenging case is

$$
E(+\infty, t) - E(-\infty, t) \neq 0, \quad \text{or, equivalently,} \int_{\mathbb{R}} [n_{10}(x_1) - n_{20}(x_1)] dx_1 \neq 0.
$$

The large time behavior of the solutions for this case has been open for a long time. In this case, the correction functions used in [6] for deleting the gaps between the original solutions and the diffusion waves at far fields (such an idea was first introduced by Hsiao and Liu in [9]) cannot be applied anymore because of the effect of the electric field, and still leave the perturbation equations with big gaps which are not in $L^2(\mathbb{R})$. In order to overcome such a difficulty, by a deep observation, we first make a heuristic analysis on the electric field and the current densities, i.e., $E$ and $J_i$ for $i = 1, 2$ at far fields, and then ingeniously construct some new correction functions to delete the gaps yielded by the original solutions and the corresponding diffusion waves. Then, we can prove the stability of nonlinear diffusion waves for the $5 \times 5$ bipolar hydrodynamic model of semiconductors (1.1) in the 1-D case. Precisely, for the diffusion waves $(\bar{n}, \bar{u})((x_1 + x_0)/\sqrt{1+t})$ with some shift constant $x_0$, when the initial perturbations around the waves are small enough, we can prove the stability of the shifted waves with the optimal rates in the form

$$\|\partial_x^k \partial_t^l (n_i - \bar{n})(t)\|_{L^\infty(\mathbb{R})} = O(1)(1+t)^{-1-\frac{k+2l}{2}}, \quad k, l \geq 0, \ i = 1, 2, \tag{1.6}$$

$$\|\partial_x^k \partial_t^l (u_i - \bar{u})(t)\|_{L^\infty(\mathbb{R})} = O(1)(1+t)^{-\frac{3}{2}-\frac{k+2l}{2}}, \quad k, l \geq 0, \ i = 1, 2. \tag{1.7}$$

And, particularly, we show the exponential decay for the electronic density $n_1$ toward the hole density $n_2$, and the electronic velocity $u_1$ toward the hole velocity $u_2$, as well as the electric field $E$ to 0 in the form of

$$\|\partial_x^k \partial_t^l (n_1 - n_2, u_1 - u_2, E)(t)\|_{L^\infty(\mathbb{R})} = O(1)e^{-\frac{\nu}{2}t}, \quad k, l \geq 0 \text{ for some } \nu > 0. \tag{1.8}$$

This is our first contribution of the present paper. Obviously, the results presented in [6] is a special case of ours. From the structure of the correction functions, we find also that it possesses much more interesting phenomena for the nonzero mass case and is consistent with physical phenomena.

For multidimensional (1.1), there is no relevant literature dealing with the stability of planar diffusion waves due to some particular difficulties. First, the difficulty comes from the complicated structure of the equations themselves. Second, the main difficulty in the study is that the strategy of antiderivative used in the 1-D case is no longer effective for the $n$-D case. For the 1-D problem, the antiderivative strategy was successfully used to remove the a priori assumption (cf. [9, 29]). However, for the multidimensional case, a direct generalization of the 1-D idea leads to the implicity and complexity of the defined shift function which depends on the solutions, so that it does not give a clear picture of the large time behavior of the solutions. To overcome this difficulty, instead of taking the antiderivative of the perturbation to the density function, we apply a new and technical inequality, which was contributed by Huang, Li, and Matsumura [13], to remove the a priori assumption, then we can establish some key energy estimates to prove the stability of planar diffusion waves for the system of the multidimensional bipolar hydrodynamic model for semiconductors. Namely, for the 1-D solutions of (1.5) $(\bar{n}, \bar{u})(x_1, t)$, the so-called planar diffusion waves to the $n$-D solutions $(n_1, u_1, n_2, u_2, \Psi)(x, t)$ with $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$(for simplicity, we just consider $n = 3$) of (1.1) converge to the planar diffusion waves in the form

$$\|(n_1 - \bar{n}, n_2 - \bar{n})(t)\|_{L^\infty(\mathbb{R}^3)} = O(1)(1+t)^{-\frac{3}{4}}, \tag{1.9}$$

$$\|(n_1 - n_2)(t)\|_{L^\infty(\mathbb{R}^3)} = O(1)(1+t)^{-\frac{9}{4}}, \tag{1.10}$$

$$\|((u_{i1}, u_{i2}, u_{i3}) - (\bar{u}, 0, 0))(t)\|_{L^\infty(\mathbb{R}^3)} = O(1)(1+t)^{-\frac{5}{4}}, \quad i = 1, 2, \tag{1.11}$$

$$\|\nabla\Psi(t)\|_{L^\infty(\mathbb{R}^3)} = O(1)(1+t)^{-\frac{7}{4}}. \tag{1.12}$$

This is our second contribution in the present paper. Note also that such a stability result of planar diffusion waves for the multidimensional isentropic Euler–Poisson equations is the first work as we know, so far.

The rest of this paper is arranged as follows. In section 2, we give some well-known results on the diffusion waves and one key inequality which will be used later for the stability proof in the $n$-D case. In section 3, we prove the stability of diffusion waves in the 1-D case. First, we artfully construct the correction functions to delete the gaps between the 1-D solutions and the 1-D diffusion waves at the far field, then we reformulate the original system of equations to a new one, and further prove the stability of diffusion waves by the energy method. In section 4, the main purpose is to study the $n$-D case. Since the technique used in the 1-D case is no longer working for the $n$-D case, we employ a key inequality to establish some crucial energy estimates, then prove the stability of planar diffusion waves.

**Notation.** Throughout this paper, the diffusion waves are always denoted by $(\bar{n}, \bar{J})(x_1/\sqrt{1+t})$, and the gap functions (or say, correction functions) are denoted by $(\hat{n}_1, \hat{J}_1, \hat{n}_2, \hat{J}_2, \hat{E})$. $C_0$, $\tilde{C}_i$, $c_*$ always denote some specific positive constants, and $C$ denotes the generic positive constant. $L^2(\mathbb{R}^n)$ is the space of square integrable real valued function defined on $\mathbb{R}^n$ with the norm $\|\cdot\|$, and $H^k(\mathbb{R}^n)$ ($H^k$ without any ambiguity) denotes the usual Sobolev space with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$. The nonnegative multi-index is $\alpha = (\alpha_1, \ldots, \alpha_n)$ with order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Given a multi-index $\alpha$, we define $\partial_x^\alpha f(x) := \partial_{x_1}^{\alpha_1}, \ldots, \partial_{x_n}^{\alpha_n} f(x)$. If $k$ is a nonnegative integer, we define $\nabla^k f(x) =: \{\partial^\alpha f(x)|\ |\alpha| = k\}$, and $|\nabla^k f| = (\sum_{|\alpha|=k} |\partial^\alpha f|^2)^{1/2}$.

**2. Diffusion waves and some preliminaries.** In this section, we are going to introduce some well-known results, i.e., the properties of the nonlinear diffusion waves and an important inequality which plays a fundamental role later in the higher-dimensional case to prove the stability of planar diffusion waves.

For the bipolar hydrodynamic model of semiconductors (1.1), its corresponding 1-D porous media equation is

$$\begin{cases} \bar{n}_t = p(\bar{n})_{x_1x_1}, \\ \bar{J} := \bar{n}\bar{u} = -p(\bar{n})_{x_1}, \end{cases} \quad \text{or, equivalently,} \quad \begin{cases} \bar{n}_t + \bar{J}_{x_1} = 0, \\ \bar{J} = -p(\bar{n})_{x_1}, \end{cases} \tag{2.1}$$

with

$$\lim_{x_1 \to \pm\infty} (\bar{n}, \bar{u})(x_1, t) = (n_\pm, 0). \tag{2.2}$$

Let $(\bar{n}, \bar{u})(\frac{x_1}{\sqrt{1+t}})$ be the self-similar solution of (2.1) satisfying the "boundary" condition (2.2). It has been proved in [3] (see also, for example, [9]) that the so-called nonlinear diffusion wave $(\bar{n}, \bar{u})(\frac{x_1}{\sqrt{1+t}})$ exists and behaves as follows. Such a 1-D self-similar solution $(\bar{n}, \bar{u})(\frac{x_1}{\sqrt{1+t}})$ is also called the nonlinear planar diffusion wave for the $n$-D system of (1.1).

It is noted that the solution $\bar{n}(\frac{x_1}{\sqrt{1+t}})$ is increasing if $n_- < n_+$ and decreasing if $n_- > n_+$ and satisfies the following lemma.

LEMMA 2.1 (see [3]). *For the self-similar solution of* (2.1)–(2.2), *let* $\zeta = \frac{x_1}{\sqrt{1+t}}$, *which holds*

$$|\bar{n}(\zeta) - n_-|_{\zeta>0} + |\bar{n}(\zeta) - n_+|_{\zeta<0} \le C|n_+ - n_-|e^{-\mu\zeta^2}, \tag{2.3}$$

$$|\partial_{x_1}^k \partial_t^l \bar{n}| \le C|n_+ - n_-|(1+t)^{-\frac{k+2l}{2}} e^{-\frac{\mu x_1{}^2}{1+t}}, \quad k+l \ge 1,\ k,l \ge 0, \tag{2.4}$$

$$|\partial_{x_1}^k \partial_t^l \bar{u}| \le C|n_+ - n_-|(1+t)^{-\frac{k+2l+1}{2}} e^{-\frac{\mu x_1{}^2}{1+t}}, \quad k,l \ge 0, \tag{2.5}$$

*where* $\mu > 0$ *is a constant.*

LEMMA 2.2 (see [3]). *For the self-similar solution of* (2.1)–(2.2), *it holds that*

$$\int_{-\infty}^0 \left|\bar{n}\left(\frac{x_1}{\sqrt{1+t}}\right) - n_-\right|^2 dx_1 + \int_0^{+\infty} \left|\bar{n}\left(\frac{x_1}{\sqrt{1+t}}\right) - n_+\right|^2 dx_1$$

$$\leq C|n_+ - n_-|^2(1+t)^{\frac{1}{2}}, \tag{2.6}$$

$$\int_{\mathbb{R}} |\partial_{x_1}^k \partial_t^l \bar{n}|^2 dx_1 \leq C|n_+ - n_-|^2(1+t)^{\frac{1}{2}-2l-k}, \quad k+l \geq 1. \tag{2.7}$$

Next we introduce a useful lemma given in [13], which will plays a key role for us to build up the energy estimates in $n$-D case. Let $\mu > 0$ and

$$G(x_1,t) = e^{-\frac{\mu x_1^2}{2(1+t)}} \quad \text{and} \quad g(x_1,t) = \frac{1}{\sqrt{1+t}} \int_{-\infty}^{x_1} G(\xi,t)d\xi. \tag{2.8}$$

Then

$$g_t(x_1,t) = \frac{1}{2\mu\sqrt{1+t}} G_{x_1}(x_1,t) \quad \text{and} \quad \|g(\cdot,t)\|_{L^\infty(\mathbb{R})} = \sqrt{\frac{2\pi}{\mu}}.$$

LEMMA 2.3 (see [13]). *For $0 < T \leq \infty$, $x_1 \in \mathbb{R}$, suppose that $h(x_1,t)$ satisfies*

$$h \in L^\infty(0,T;L^2(\mathbb{R})), \quad h_{x_1} \in L^2(0,T;L^2(\mathbb{R})), \quad h_t \in L^2(0,T;H^{-1}(\mathbb{R}));$$

*then the following estimate holds for any $t \in (0,T]$:*

$$\begin{aligned}
&\int_0^t (1+\tau)^{-1} \int_{\mathbb{R}} \exp\Big(-\frac{\mu x_1^2}{1+\tau}\Big) h^2(x_1,\tau)dx_1 d\tau \\
&\leq C(\mu)\Big\{\|h(\cdot,0)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}} h_{x_1}^2(x_1,\tau)dx_1 d\tau + \int_0^t \langle h_t, hg^2\rangle_{H^{-1}\times H^1} d\tau\Big\}.
\end{aligned} \tag{2.9}$$

**3. 1-D case: Stability of diffusion waves.** In this section, we study the 1-D bipolar hydrodynamic model of semiconductors (1.3) with the initial value conditions (1.4) and the "boundary" condition at far field

$$E(-\infty,t) = E_-. \tag{3.1}$$

Here, for the sake of simplification, throughout this section, we still denote the 1-D spatial variable $x_1$ as $x \in \mathbb{R}$ without confusion.

Note that the boundary (3.1) at far field $x = -\infty$ (or replace it by $E(+\infty,t) = E_+$ on $x = +\infty$) is proper and necessary. Because, as we show below on the state functions $n_i(\pm\infty,t)$, $J_i(\pm\infty,t)$, and $E(\pm\infty,t)$, without such a boundary condition, these state functions will be underdetermined, which then will cause the system (1.3) and (1.4) to be ill-posed.

On the other hand, from $(1.3)_5$, we immediately have

$$E|_{t=0} = \int_{-\infty}^x (n_{10}(y) - n_{20}(y))dy + E_- =: E_0(x)$$

and

$$E_+ := E_0(+\infty) = \int_{-\infty}^{+\infty} (n_{10}(y) - n_{20}(y))dy + E_-.$$

The main target in this section is to prove that the solution $(n_1,J_1,n_2,J_2,E)(x,t)$ of (1.3) and (1.4) with the condition (3.1) converges to the nonlinear diffusion waves $(\bar{n},\bar{J},\bar{n},\bar{J},0)(\frac{x+x_0}{\sqrt{1+t}})$ for some shift constant $x_0$, even if

$$\int_{\mathbb{R}} [n_{10}(x) - n_{20}(x)]dx \neq 0 \quad (\text{or } J_{i+} \neq J_{i-} \text{ for } i = 1,2).$$

We will also derive the optimal convergence rates for the 1-D solutions to the corresponding diffusion waves, which are much better than what is shown in [6]. More interesting, we find that $n_1 - n_2$, and $J_1 - J_2$ and $E$ converge to zero time exponentially. To construct the correction functions to delete the gaps between the original solutions and the diffusion waves is very tricky and plays a crucial role in the proof.

**3.1. Reformulation of 1-D system.** First of all, as in [30], let us look into the behaviors of the solutions to (1.3) and (1.4) at the far fields $x = \pm\infty$. Then we may understand how big the gaps are between the solutions and the diffusion waves at the far fields. Let

$$\begin{cases} n_i^{\pm}(t) := n_i(\pm\infty, t), \\ J_i^{\pm}(t) := J_i(\pm\infty, t), \quad i = 1, 2. \\ E^{\pm}(t) := E(\pm\infty, t), \end{cases} \tag{3.2}$$

From $(1.3)_1$ and $(1.3)_3$, since $\partial_x J_i|_{x=\pm\infty} = 0$ for $i = 1, 2$, it can be easily seen that

$$n_i^{\pm}(t) = n_i(\pm\infty, t) \equiv n_{\pm}. \tag{3.3}$$

Differentiating $(1.3)_5$ with respect to $t$ and applying $(1.3)_1$ and $(1.3)_3$, we have

$$E_{xt} = (n_1 - n_2)_t = -(J_1 - J_2)_x,$$

and integrating it with respect to $x$ over $(-\infty, \infty)$, we obtain

$$\frac{d}{dt}E^+(t) - \frac{d}{dt}E^-(t) = -[J_1^+(t) - J_2^+(t)] + [J_1^-(t) - J_2^-(t)]. \tag{3.4}$$

Taking $x = \pm\infty$ to $(1.3)_2$ and $(1.3)_4$, we also formally have

$$\frac{d}{dt}J_1^{\pm}(t) = n_{\pm}E^{\pm}(t) - J_1^{\pm}(t), \tag{3.5}$$

$$\frac{d}{dt}J_2^{\pm}(t) = -n_{\pm}E^{\pm}(t) - J_2^{\pm}(t), \tag{3.6}$$

It can be easily seen that (3.4)–(3.6) cannot uniquely determine the six unknown state functions $J_i^{\pm}(t)$ $(i = 1, 2)$ and $E^{\pm}(t)$. So, we need to add one boundary condition at far field like (3.1). Otherwise, $J_i^{\pm}(t)$ $(i = 1, 2)$ and $E^{\pm}(t)$ will be underdetermined, and this will cause the system (1.3) to be ill-posed.

Therefore, without loss of generality, we may assume as before

$$E(-\infty, t) = E_- \equiv 0. \tag{3.7}$$

Then, from $(1.3)_2$ and $(1.3)_4$, we can easily get

$$J_i^-(t) = J_{i-}e^{-t}, \quad i = 1, 2. \tag{3.8}$$

From $(1.3)_2$, $(1.3)_4$, and (3.3), we get

$$\begin{cases} \frac{d}{dt}J_1^+(t) = n_+E^+(t) - J_1^+(t), \\ \frac{d}{dt}J_2^+(t) = -n_+E^+(t) - J_2^+(t), \\ J_i^+(0) = J_{i+}, \qquad i = 1, 2, \end{cases} \tag{3.9}$$

and from (3.4), (3.7), and (3.8), we have

$$\frac{d}{dt}E^+(t) = -[J_1^+(t) - J_2^+(t)] + (J_{1-} - J_{2-})e^{-t}. \tag{3.10}$$

Combining (3.9) and (3.10), we establish the following system of ODEs for $J_1^+(t)$, $J_2^+(t)$, and $E^+(t)$:

$$\begin{cases} \frac{d}{dt}J_1^+(t) = n_+E^+(t) - J_1^+(t), \\ \frac{d}{dt}J_2^+(t) = -n_+E^+(t) - J_2^+(t), \\ \frac{d}{dt}E^+(t) = -[J_1^+(t) - J_2^+(t)] + (J_{1-} - J_{2-})e^{-t}, \\ J_i^+(0) = J_{i+}, \quad i = 1, 2, \\ E^+(0) = E_+. \end{cases} \tag{3.11}$$

Adding $(3.11)_1$ and $(3.11)_2$, we get

$$\begin{cases} \frac{d}{dt}[J_1^+(t) + J_2^+(t)] = -[J_1^+(t) + J_2^+(t)], \\ [J_1^+(t) + J_2^+(t)]|_{t=0} = J_{1+} + J_{2+}, \end{cases} \tag{3.12}$$

which can be solved as

$$J_1^+(t) + J_2^+(t) = (J_{1+} + J_{2+})e^{-t}. \tag{3.13}$$

On the other hand, subtracting $(3.11)_2$ from $(3.11)_1$, we have

$$\begin{cases} \frac{d}{dt}[J_1^+(t) - J_2^+(t)] = 2n_+E^+(t) - [J_1^+(t) - J_2^+(t)], \\ \frac{d}{dt}E^+(t) = -[J_1^+(t) - J_2^+(t)] + (J_{1-} - J_{2-})e^{-t}, \\ [J_1^+(t) - J_2^+(t)]|_{t=0} = J_{1+} - J_{2+}, \\ E^+(t)|_{t=0} = E_+, \end{cases} \tag{3.14}$$

which, by substituting the second equation of (3.14), i.e.,

$$J_1^+(t) - J_2^+(t) = -\frac{d}{dt}E^+(t) + (J_{1-} - J_{2-})e^{-t}, \tag{3.15}$$

into the first equation of (3.14), can be reduced to

$$\begin{cases} \frac{d^2}{dt^2}E^+(t) + \frac{d}{dt}E^+(t) + 2n_+E^+(t) = 0, \\ E^+(0) = E_+, \\ \frac{d}{dt}E^+|_{t=0} = (J_{2+} - J_{2-}) - (J_{1+} - J_{1-}). \end{cases} \tag{3.16}$$

Notice that the eigenvalues of the second order ODE (3.16) are

$$\lambda_1 = \frac{-1 - \sqrt{1 - 8n_+}}{2} \quad \text{and} \quad \lambda_2 = \frac{-1 + \sqrt{1 - 8n_+}}{2}. \tag{3.17}$$

Thus, according to the signs of $1 - 8n_+$, we can directly but tediously solve (3.13), (3.15), and (3.16) to have the solutions $J_1^+(t)$, $J_2^+(t)$, and $E^+(t)$ as follows.

*Case* 1. When $1 - 8n_+ = 0$, then

$$J_1^+(t) = \frac{1}{2}(J_{1+} + J_{2+} + J_{1-} - J_{2-})e^{-t}$$

$$
\begin{aligned}
&+\frac{1}{2}e^{-\frac{t}{2}}\Big\{(J_{1+}-J_{1-})-(J_{2+}-J_{2-}) \\
&+\frac{1}{2}[(J_{2+}-J_{2-})-(J_{1+}-J_{1-})+\frac{1}{2}E_+]t\Big\}, \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
J_2^+(t) = &\frac{1}{2}(J_{1+}+J_{2+}-J_{1-}+J_{2-})e^{-t} \\
&-\frac{1}{2}e^{-\frac{t}{2}}\Big\{(J_{1+}-J_{1-})-(J_{2+}-J_{2-}) \\
&+\frac{1}{2}[(J_{2+}-J_{2-})-(J_{1+}-J_{1-})+\frac{1}{2}E_+]t\Big\}, \qquad (3.19)
\end{aligned}
$$

$$
E^+(t) = e^{-\frac{t}{2}}\Big\{E_+ + [(J_{2+}-J_{2-})-(J_{1+}-J_{1-})+\frac{1}{2}E_+]t\Big\}. \qquad (3.20)
$$

*Case* 2. When $1-8n_+<0$, then

$$
\begin{aligned}
J_1^+(t) = &\frac{1}{2}(J_{1+}+J_{2+}+J_{1-}-J_{2-})e^{-t} \\
&+\frac{1}{4}e^{-\frac{t}{2}}\left\{2[(J_{1+}-J_{1-})-(J_{2+}-J_{2-})]\cos\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right. \\
&+\Big(\frac{2((J_{2+}-J_{2-})-(J_{1+}-J_{1-}))+E_+}{\sqrt{8n_+-1}}+\sqrt{8n_+-1}E_+\Big) \\
&\left.\times\sin\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right\}, \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
J_2^+(t) = &\frac{1}{2}(J_{1+}+J_{2+}-J_{1-}+J_{2-})e^{-t} \\
&-\frac{1}{4}e^{-\frac{t}{2}}\left\{2[(J_{1+}-J_{1-})-(J_{2+}-J_{2-})]\cos\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right. \\
&+\Big(\frac{2[(J_{2+}-J_{2-})-(J_{1+}-J_{1-})]+E_+}{\sqrt{8n_+-1}}+\sqrt{8n_+-1}E_+\Big) \\
&\left.\times\sin\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right\}, \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
E^+(t) = e^{-\frac{t}{2}}&\left\{E_+\cos\frac{\sqrt{8n_+-1}}{2}t\right. \\
&\left.+\frac{2[(J_{2+}-J_{2-})-(J_{1+}-J_{1-})]+E_+}{\sqrt{8n_+-1}}\sin\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right\}. \qquad (3.23)
\end{aligned}
$$

*Case* 3. When $1-8n_+>0$, then

$$
J_1^+(t) = \frac{1}{2}(J_{1+}+J_{2+}+J_{1-}-J_{2-})e^{-t}-\frac{1}{2}\Big\{\lambda_1Ae^{\lambda_1t}+\lambda_2Be^{\lambda_2t}\Big\}, \qquad (3.24)
$$

$$
J_2^+(t) = \frac{1}{2}(J_{1+}+J_{2+}-J_{1-}+J_{2-})e^{-t}+\frac{1}{2}\Big\{\lambda_1Ae^{\lambda_1t}+\lambda_2Be^{\lambda_2t}\Big\}, \qquad (3.25)
$$

$$
E^+(t) = Ae^{\lambda_1t}+Be^{\lambda_2t}, \qquad (3.26)
$$

where $\lambda_1$ and $\lambda_2$ are the eigenvalues given in (3.17), and $A$ and $B$ are

$$
\begin{cases}
A = -\frac{(J_{2+}-J_{2-})-(J_{1+}-J_{1-})-\lambda_2E_+}{\sqrt{1-8n_+}} \\
B = \frac{(J_{2+}-J_{2-})-(J_{1+}-J_{1-})-\lambda_1E_+}{\sqrt{1-8n_+}}.
\end{cases} \qquad (3.27)
$$

From (3.3), (3.7), (3.8), and (3.18)–(3.26), we have

$$\begin{cases} n_i(\pm\infty,t)=n_\pm, & i=1,2,\\ |J_i(+\infty,t)|=O(1)e^{-\nu_0 t}, & i=1,2,\\ J_i(-\infty,t)=J_{i-}e^{-t}, & i=1,2,\\ |E(+\infty,t)|=O(1)e^{-\nu_0 t}, \\ E(-\infty,t)=E_-=0 \end{cases}$$

for some constant $0<\nu_0<\frac{1}{2}$, which combine with the diffusion waves $(\bar{n},\bar{J})(\pm\infty,t)=(n_\pm,0)$ to yield

$$\begin{cases} |n_i(\pm\infty,t)-\bar{n}(\pm\infty,t)|=0, & i=1,2,\\ |J_i(+\infty,t)-\bar{J}(+\infty,t)|=O(1)e^{-\nu_0 t}\neq 0, & i=1,2,\\ |J_i(-\infty,t)-\bar{J}(-\infty,t)|=|J_{i-}|e^{-t}\neq 0, & i=1,2,\\ |E(+\infty,t)-0|=O(1)e^{-\nu_0 t}\neq 0, \\ E(-\infty,t)=E_-=0. \end{cases}$$

Obviously, there are some gaps between $J_i(\pm\infty,t)$ and $\bar{J}(\pm\infty,t)$, and $E(+\infty,t)$ and $\bar{E}\equiv 0$, which lead to

$$J_i(x,t)-\bar{J}(x,t) \quad \text{and} \quad E(x,t)\notin L^2(\mathbb{R}).$$

To delete these gaps, we need to introduce some gap functions (also called the correction functions). However, the usual manner for constructing the correction functions in [9] for 1-D Euler equations with linear damping and in [30] with nonlinear damping/accumulating cannot be applied. So, we need to look for something else. Observing the structure of the solutions at far fields (see (3.11)), we ingeniously construct the correction functions $(\hat{n}_1,\hat{n}_2,\hat{J}_1,\hat{J}_2,\hat{E})(x,t)$ satisfying the following *linear* equations:

$$\begin{cases} \hat{n}_{1t}+\hat{J}_{1x}=0,\\ \hat{J}_{1t}=\breve{n}\hat{E}-\hat{J}_1,\\ \hat{n}_{2t}+\hat{J}_{2x}=0,\\ \hat{J}_{2t}=-\breve{n}\hat{E}-\hat{J}_2,\\ \hat{E}_x=\hat{n}_1-\hat{n}_2, \end{cases} \quad \text{with} \quad \begin{cases} \hat{J}_i(x,t)\to J_i^\pm(t) & \text{as } x\to\pm\infty,\\ \hat{E}(x,t)\to 0 & \text{as } x\to-\infty,\\ \hat{E}(x,t)\to E^+(t) & \text{as } x\to+\infty. \end{cases} \tag{3.28}$$

Here $\breve{n}=\breve{n}(x)$, $\hat{J}_i(x,0)$, and $\hat{E}(x,0)$ will be artfully constructed in (3.29) and (3.30) below such that

$$\breve{n}(x)\to n_\pm,\quad \hat{J}_i(x,0)\to J_{i\pm},\quad \text{and } \hat{E}(x,0)\to E_0^\pm,\quad \text{as } x\to\pm\infty. \tag{3.29}$$

In order to get $(\hat{n}_1,\hat{n}_2,\hat{J}_1,\hat{J}_2,\hat{E})(x,t)$ to (3.28), we consider the following linear system with some tricky selection on $\breve{n}=\breve{n}(x)$, $\hat{J}_i(x,0)$, and $\hat{E}(x,0)$:

$$\begin{cases} \hat{J}_{1t}=\breve{n}\hat{E}-\hat{J}_1,\\ \hat{J}_{2t}=-\breve{n}\hat{E}-\hat{J}_2,\\ \hat{E}_t=-(\hat{J}_1-\hat{J}_2)+(J_{1-}-J_{2-})e^{-t},\\ \hat{J}_i(x,0)=J_{i-}+(J_{i+}-J_{i-})\displaystyle\int_{-\infty}^x m_0(y)dy, \quad i=1,2,\\ \hat{E}(x,0)=E_+\displaystyle\int_{-\infty}^x m_0(y)dy, \end{cases} \tag{3.30}$$

where $m_0(x)$ and $\breve{n}(x)$ are also artfully selected as

$$\begin{cases} m_0(x) \geq 0,\ m_0 \in C_0^\infty(\mathbb{R}),\ \text{supp } m_0 \subseteq [-L_0, L_0],\ \displaystyle\int_{\mathbb{R}} m_0(y)dy = 1, \\ \breve{n}(x) = n_- + (n_+ - n_-) \displaystyle\int_{-\infty}^{x+2L_0} m_0(y)dy, \end{cases} \tag{3.31}$$

with some constant $L_0 > 0$.

When $x < -L_0$, we have $\hat{E}(x, 0) \equiv 0$. So, it can be easily seen that (3.30) possesses the particular solutions

$$\hat{J}_i(x,t) = J_{i-}e^{-t}, \quad i = 1, 2, \quad \hat{E}(x,t) = 0 \quad \text{for } -\infty < x < -L_0. \tag{3.32}$$

When $x \geq -L_0$, we have $\breve{n}(x) \equiv n_+$. Similarly to (3.11), by a straightforward but complicated calculation, we can solve (3.30) as (3.33)–(3.35), or (3.38)–(3.40), or (3.43)–(3.45) for $-L_0 \leq x < \infty$. However, we can verify that these solutions imply also the solutions given in (3.32) for $x < -L_0$. Therefore, we summarize them as follows.

*Case* 1. When $1 - 8n_+ = 0$, then, for $x \in \mathbb{R}$,

$$\begin{aligned} \hat{J}_1(x,t) = J_{1-}e^{-t} + \frac{1}{2}\Big\{&[(J_{2+} - J_{2-}) + (J_{1+} - J_{1-})]e^{-t} \\ &+ e^{-\frac{t}{2}}\Big[(J_{1+} - J_{1-}) - (J_{2+} - J_{2-}) \\ &+ \frac{1}{2}\Big[(J_{2+} - J_{2-}) - (J_{1+} - J_{1-}) + \frac{1}{2}E_+\Big]t\Big]\Big\} \int_{-\infty}^{x} m_0(y)dy, \end{aligned} \tag{3.33}$$

$$\begin{aligned} \hat{J}_2(x,t) = J_{2-}e^{-t} + \frac{1}{2}\Big\{&[(J_{2+} - J_{2-}) + (J_{1+} - J_{1-})]e^{-t} \\ &- e^{-\frac{t}{2}}\Big[(J_{1+} - J_{1-}) - (J_{2+} - J_{2-}) \\ &+ \frac{1}{2}\Big[(J_{2+} - J_{2-}) - (J_{1+} - J_{1-}) + \frac{1}{2}E_+\Big]t\Big]\Big\} \int_{-\infty}^{x} m_0(y)dy, \end{aligned} \tag{3.34}$$

$$\hat{E}(x,t) = e^{-\frac{t}{2}}\Big\{E_+ + \Big[J_{2+} - J_{2-} - (J_{1+} - J_{1-}) + \frac{E_+}{2}\Big]t\Big\} \int_{-\infty}^{x} m_0(y)dy. \tag{3.35}$$

Thus, let us artfully construct

$$\begin{aligned} \hat{n}_1(x,t) = \frac{1}{2}m_0(x)\Big\{&((J_{2+} - J_{2-}) + (J_{1+} - J_{1-}))e^{-t} \\ &+ e^{-\frac{t}{2}}\Big(E_+ + \Big((J_{2+} - J_{2-}) - (J_{1+} - J_{1-}) + \frac{1}{2}E_+\Big)t\Big)\Big\}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} \hat{n}_2(x,t) = \frac{1}{2}m_0(x)\Big\{&((J_{2+} - J_{2-}) + (J_{1+} - J_{1-}))e^{-t} \\ &- e^{-\frac{t}{2}}\Big(E_+ + \Big((J_{2+} - J_{2-}) - (J_{1+} - J_{1-}) + \frac{1}{2}E_+\Big)t\Big)\Big\}; \end{aligned} \tag{3.37}$$

then we can verify that $(\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})(x,t)$ satisfy (3.28) for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

*Case* 2. When $1 - 8n_+ < 0$, then the solutions of (3.30) are, for $x \in \mathbb{R}$,

$$\begin{aligned} \hat{J}_1(x,t) = J_{1-}e^{-t} + &\frac{(J_{2+} - J_{2-}) + (J_{1+} - J_{1-})}{2}e^{-t} \int_{-\infty}^{x} m_0(y)dy \\ &+ \frac{1}{4}e^{-\frac{t}{2}}\left\{2((J_{1+} - J_{1-}) - (J_{2+} - J_{2-}))\cos\left(\frac{\sqrt{8n_+ - 1}}{2}t\right)\right. \end{aligned}$$

$$+\Big(\frac{2((J_{2+}-J_{2-})-(J_{1+}-J_{1-}))+E_+}{\sqrt{8n_+-1}}$$

$$+\sqrt{8n_+-1}E_+\Big)\sin\left(\frac{\sqrt{8n_+-1}}{2}t\right)\Bigg\}\int_{-\infty}^x m_0(y)dy, \tag{3.38}$$

$$\hat J_2(x,t)=J_{2-}e^{-t}+\frac{(J_{2+}-J_{2-})+(J_{1+}-J_{1-})}{2}e^{-t}\int_{-\infty}^x m_0(y)dy$$

$$-\frac{1}{4}e^{-\frac{t}{2}}\left\{2((J_{1+}-J_{1-})-(J_{2+}-J_{2-}))\cos\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right.$$

$$+\Big(\frac{2((J_{2+}-J_{2-})-(J_{1+}-J_{1-}))+E_+}{\sqrt{8n_+-1}}$$

$$\left.+\sqrt{8n_+-1}E_+\Big)\sin\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right\}\int_{-\infty}^x m_0(y)dy, \tag{3.39}$$

$$\hat E^+(x,t)=e^{-\frac{t}{2}}\left\{E_+\cos\left(\frac{\sqrt{8n_+-1}}{2}t\right)+\frac{2(J_{2+}-J_{2-}+J_{1-}-J_{1+})+E_+}{\sqrt{8n_+-1}}\right.$$

$$\left.\times\sin\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right\}\int_{-\infty}^x m_0(y)dy. \tag{3.40}$$

So, let

$$\hat n_1(x,t)=\frac{(J_{2+}-J_{2-})+(J_{1+}-J_{1-})}{2}m_0(x)e^{-t}$$

$$+\frac{1}{16n_+}m_0(x)e^{-\frac{t}{2}}\left\{8n_+E_+\cos\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right.$$

$$+\Big[\sqrt{8n_+-1}(E_++2((J_{2+}-J_{2-})-(J_{1+}-J_{1-})))$$

$$\left.+\frac{2((J_{2+}-J_{2-})-(J_{1+}-J_{1-}))+E_+}{\sqrt{8n_+-1}}\Big]\sin\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right\}, \tag{3.41}$$

$$\hat n_2(x,t)=\frac{(J_{2+}-J_{2-})+(J_{1+}-J_{1-})}{2}m_0(x)e^{-t}$$

$$-\frac{1}{16n_+}m_0(x)e^{-\frac{t}{2}}\left\{8n_+E_+\cos\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right.$$

$$+\Big[\sqrt{8n_+-1}(E_++2((J_{2+}-J_{2-})-(J_{1+}-J_{1-})))$$

$$\left.+\frac{2((J_{2+}-J_{2-})-(J_{1+}-J_{1-}))+E_+}{\sqrt{8n_+-1}}\Big]\sin\left(\frac{\sqrt{8n_+-1}}{2}t\right)\right\}, \tag{3.42}$$

then the solutions $(\hat n_1,\hat n_2,\hat J_1,\hat J_2,\hat E)(x,t)$ satisfy (3.28) for $(x,t)\in\mathbb{R}\times\mathbb{R}_+$.

*Case* 3. When $1-8n_+>0$, then

$$\hat J_1(x,t)=J_{1-}e^{-t}+\frac{(J_{2+}-J_{2-})+(J_{1+}-J_{1-})}{2}e^{-t}\int_{-\infty}^x m_0(y)dy$$

$$-\frac{1}{2}\Big(\lambda_1Ae^{\lambda_1t}+\lambda_2Be^{\lambda_2t}\Big)\int_{-\infty}^x m_0(y)dy, \tag{3.43}$$

$$\hat J_2(x,t)=J_{2-}e^{-t}+\frac{(J_{2+}-J_{2-})+(J_{1+}-J_{1-})}{2}e^{-t}\int_{-\infty}^x m_0(y)dy$$

$$+\frac{1}{2}\Big(\lambda_1Ae^{\lambda_1t}+\lambda_2Be^{\lambda_2t}\Big)\int_{-\infty}^x m_0(y)dy, \tag{3.44}$$

$$\hat{E}(x,t) = \left(\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\right) \int_{-\infty}^{x} m_0(y)dy. \tag{3.45}$$

Similarly, we construct

$$\begin{aligned}\hat{n}_1(x,t) = &\frac{1}{2}((J_{2+} - J_{2-}) + (J_{1+} - J_{1-}))m_0(x)e^{-t}\\ &+\frac{1}{2}m_0(x)\left(\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\right),\end{aligned} \tag{3.46}$$

$$\begin{aligned}\hat{n}_2(x,t) = &\frac{1}{2}((J_{2+} - J_{2-}) + (J_{1+} - J_{1-}))m_0(x)e^{-t}\\ &-\frac{1}{2}m_0(x)\left(\lambda_1 A e^{\lambda_1 t} + \lambda_2 B e^{\lambda_2 t}\right);\end{aligned} \tag{3.47}$$

then we can check that $(\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})(x,t)$ are the solutions of (3.28) for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

Clearly, the correction functions $(\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})(x,t)$ expressed in each case have an exponential decay with respect to $t$, and $\hat{n}_1$ and $\hat{n}_2$ have the same compact support with $m_0(x)$. Namely, we proved the following lemma.

LEMMA 3.1. *There hold*

$$\|(\hat{n}_1, \hat{n}_2, \hat{J}_1, \hat{J}_2, \hat{E})(t)\|_{L^\infty(\mathbb{R})} \leq C\delta e^{-\nu_0 t} \tag{3.48}$$

*and*

$$\text{supp } \hat{n}_1 = \text{supp } \hat{n}_2 = \text{supp } m_0 \subseteq [-L_0, L_0]$$

*for* $\delta := |n_+ - n_-| + |J_{1+}| + |J_{1-}| + |J_{2+}| + |J_{2-}| + |E_+|$ *and* $0 < \nu_0 < \frac{1}{2}$.

Now we are going to make a perturbation of (1.3) to the diffusion waves (2.1). From (1.3), (3.28), and (2.1), we have

$$\begin{cases}(n_1 - \hat{n}_1 - \bar{n})_t + (J_1 - \hat{J}_1 - \bar{J})_x = 0,\\ (J_1 - \hat{J}_1 - \bar{J})_t + (\frac{J_1^2}{n_1} + p(n_1) - p(\bar{n}))_x\\ \qquad\qquad = n_1 E - \breve{n}\hat{E} - (J_1 - \hat{J}_1 - \bar{J}) + p(\bar{n})_{xt},\\ (n_2 - \hat{n}_2 - \bar{n})_t + (J_2 - \hat{J}_2 - \bar{J})_x = 0,\\ (J_2 - \hat{J}_2 - \bar{J})_t + (\frac{J_2^2}{n_2} + p(n_2) - p(\bar{n}))_x\\ \qquad\qquad = -n_2 E + \breve{n}\hat{E} - (J_2 - \hat{J}_2 - \bar{J}) + p(\bar{n})_{xt},\\ (E - \hat{E})_x = (n_1 - \hat{n}_1 - \bar{n}) - (n_2 - \hat{n}_2 - \bar{n}),\end{cases} \tag{3.49}$$

where $(\bar{n}, \bar{J}) = (\bar{n}, \bar{J})(x + x_0, t)$ are the shifted diffusion waves with some shift $x_0$, which will be specified later.

Let us integrate the first and third equations of (3.49) over $(-\infty, +\infty)$ with respect to $x$, and note that $J_i(\pm\infty, t) = \hat{J}_i(\pm\infty, t)$ for $i = 1, 2$ and $\bar{J}(\pm\infty, t) = 0$ as shown before. We then have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}[n_i(x,t) - \hat{n}_i(x,t) - \bar{n}(x + x_0, t)]dx\\ &= -[J_i(+\infty, t) - \hat{J}_i(+\infty, t) - \bar{J}(+\infty, t)]\\ &\quad + [J_i(-\infty, t) - \hat{J}_i(-\infty, t) - \bar{J}(-\infty, t)]\\ &= 0, \quad i = 1, 2,\end{aligned}$$

then integrate the above equation with respect to $t$ to obtain

$$\begin{aligned}&\int_{\mathbb{R}}[n_i(x,t)-\hat{n}_i(x,t)-\bar{n}(x+x_0,t)]dx\\&=\int_{\mathbb{R}}[n_i(x,0)-\hat{n}_i(x,0)-\bar{n}(x+x_0,0)]dx\\&=: I_i(x_0),\quad i=1,2.\end{aligned}\tag{3.50}$$

Now we are going to determine $x_0$ such that $I_i(x_0)=0$. Since

$$I_i'(x_0)=\frac{\partial}{\partial x_0}\Big(\int_{\mathbb{R}}[n_i(x,0)-\hat{n}_i(x,0)-\bar{n}(x+x_0,0)]dx\Big)=-(n_+-n_-),\quad i=1,2,$$

which gives, with $I_i(x_0)=0$, that

$$x_0:=\frac{1}{n_+-n_-}\int_{\mathbb{R}}[n_i(x,0)-\hat{n}_i(x,0)-\bar{n}(x,0)]dx,\quad i=1,2.\tag{3.51}$$

This implies that we need

$$\int_{\mathbb{R}}[n_1(x,0)-\hat{n}_1(x,0)-\bar{n}(x,0)]dx=\int_{\mathbb{R}}[n_2(x,0)-\hat{n}_2(x,0)-\bar{n}(x,0)]dx,$$

namely,

$$\int_{\mathbb{R}}[n_{10}(x)-n_{20}(x)]dx=\int_{\mathbb{R}}[\hat{n}_1(x,0)-\hat{n}_2(x,0)]dx.\tag{3.52}$$

However, such a condition is always true, and automatically guarantees by the system (1.3). In fact, from (3.36), (3.37), (3.41), (3.42), (3.46), and (3.47), for each case we always have

$$\begin{aligned}\hat{n}_1(x,0)&=\frac{1}{2}m_0(x)[(J_{2+}-J_{2-})+(J_{1+}-J_{1-})+E_+],\\ \hat{n}_2(x,0)&=\frac{1}{2}m_0(x)[(J_{2+}-J_{2-})+(J_{1+}-J_{1-})-E_+],\end{aligned}$$

which, with the fact $\int_{\mathbb{R}}m_0(x)dx=1$, gives

$$\int_{\mathbb{R}}[\hat{n}_1(x,0)-\hat{n}_2(x,0)]dx=E_+.$$

Substituting this to (3.52), we need

$$\int_{\mathbb{R}}[n_{10}(x)-n_{20}(x)]dx=E_+.\tag{3.53}$$

However, by integrating $(1.3)_5$ with respect to $x$ over $(-\infty,\infty)$ and taking $t=0$, as well as noting $E_0^-=0$, we immediately obtain (3.53). Hence, (3.52) automatically holds.

Thus, (3.50) with $I_i(x_0)=0$ for such selected $x_0$ in (3.51) implies that the integration of the perturbed equations (3.49) over $(-\infty,x]$ could be set in $L^2$ space. Therefore, by defining

$$\begin{cases}\phi_i(x,t):=\int_{-\infty}^{x}[n_i(\xi,t)-\hat{n}_i(\xi,t)-\bar{n}(\xi+x_0,t)]d\xi\\ \psi_i(x,t):=J_i(x,t)-\hat{J}_i(x,t)-\bar{J}(x+x_0,t),\quad i=1,2,\\ \mathcal{H}(x,t):=E(x,t)-\hat{E}(x,t),\end{cases}\tag{3.54}$$

namely, $\phi_{ix} = n_i - \hat{n}_i - \bar{n}$ and $-\phi_{it} = \psi_i = J_i - \hat{J}_i - \bar{J}$, we deduce (3.49) into

$$\begin{cases} \phi_{1t} + \psi_1 = 0, \\ \psi_{1t} + \left( \frac{(-\phi_{1t}+\hat{J}_1+\bar{J})^2}{\phi_{1x}+\hat{n}_1+\bar{n}} + p(\phi_{1x} + \hat{n}_1 + \bar{n}) - p(\bar{n}) \right)_x \\ \qquad = (\phi_{1x} + \hat{n}_1 + \bar{n})\mathcal{H} + (\phi_{1x} + \hat{n}_1 + \bar{n} - \breve{n})\hat{E} - \psi_1 + p(\bar{n})_{xt}, \\ \phi_{2t} + \psi_2 = 0, \\ \psi_{2t} + \left( \frac{(-\phi_{2t}+\hat{J}_2+\bar{J})^2}{\phi_{2x}+\hat{n}_2+\bar{n}} + p(\phi_{2x} + \hat{n}_2 + \bar{n}) - p(\bar{n}) \right)_x \\ \qquad = -(\phi_{2x} + \hat{n}_2 + \bar{n})\mathcal{H} - (\phi_{2x} + \hat{n}_2 + \bar{n} - \breve{n})\hat{E} - \psi_2 + p(\bar{n})_{xt}, \\ \mathcal{H} = \phi_1 - \phi_2, \end{cases} \tag{3.55}$$

with the initial data

$$\begin{cases} \phi_{i0}(x) := \phi_i(x, 0) = \int_{-\infty}^{x} [n_{i0}(\xi) - \hat{n}_i(\xi, 0) - \bar{n}(\xi + x_0, 0)] d\xi, \\ \psi_{i0}(x) := \psi_i(x, 0) = J_{i0}(x) - \hat{J}_i(x, 0) - \bar{J}(x + x_0, t), \quad i = 1, 2, \\ \mathcal{H}_0(x) := \phi_{10}(x) - \phi_{20}(x). \end{cases} \tag{3.56}$$

From (3.55), we obtain

$$\begin{cases} \phi_{1tt} + \phi_{1t} - \Big(p(\phi_{1x} + \hat{n}_1 + \bar{n}) - p(\bar{n})\Big)_x + (\phi_{1x} + \hat{n}_1 + \bar{n})\mathcal{H} \\ \qquad = -f_1 + g_{1x} - p(\bar{n})_{xt}, \\ \phi_{2tt} + \phi_{2t} - \Big(p(\phi_{2x} + \hat{n}_2 + \bar{n}) - p(\bar{n})\Big)_x - (\phi_{2x} + \hat{n}_2 + \bar{n})\mathcal{H} \\ \qquad = f_2 + g_{2x} - p(\bar{n})_{xt}, \end{cases} \tag{3.57}$$

where

$$f_i = (\phi_{ix} + \hat{n}_i + \bar{n} - \breve{n})\hat{E}, \quad g_i = \frac{(-\phi_{it} + \hat{J}_i + \bar{J})^2}{\phi_{ix} + \hat{n}_i + \bar{n}}, \quad i = 1, 2, \tag{3.58}$$

with the initial data

$$\phi_i(x, 0) = \phi_{i0}(x), \quad \phi_{it}(x, 0) = -\psi_{i0}(x), \quad i = 1, 2. \tag{3.59}$$

Furthermore, subtracting $(3.57)_2$ from $(3.57)_1$, we get the IVP for the following damped "Klein–Gordon" type equation:

$$\mathcal{H}_{tt} + \mathcal{H}_t + 2\bar{n}\mathcal{H} = h := h_{1x} - h_2 - h_3 + h_{4x}, \tag{3.60}$$

where

$$\begin{cases} h_1 = p(\phi_{1x} + \hat{n}_1 + \bar{n}) - p(\phi_{2x} + \hat{n}_2 + \bar{n}), \\ h_2 = (\phi_{1x} + \phi_{2x} + \hat{n}_1 + \hat{n}_2)\mathcal{H}, \\ h_3 = [\phi_{1x} + \phi_{2x} + \hat{n}_1 + \hat{n}_2 + 2(\bar{n} - \breve{n})]\hat{E}, \\ h_4 = \frac{(-\phi_{1t}+\hat{J}_1+\bar{J})^2}{\phi_{1x}+\hat{n}_1+\bar{n}} - \frac{(-\phi_{2t}+\hat{J}_2+\bar{J})^2}{\phi_{2x}+\hat{n}_2+\bar{n}}. \end{cases} \tag{3.61}$$

**3.2. Convergence theorem in the 1-D case.** In this subsection, we are going to state the convergence result of the bipolar hydrodynamic model for semiconductors in the 1-D case.

THEOREM 3.2. *Let* $(\phi_{10},\phi_{20},\psi_{10},\psi_{20})(x)\in H^3(\mathbb{R})\times H^3(\mathbb{R})\times H^2(\mathbb{R})\times H^2(\mathbb{R})$, $\delta := |n_+-n_-|+|J_{1+}|+|J_{1-}|+|J_{2+}|+|J_{2-}|+|E_+|$, *and* $\Phi_0 := \|(\phi_{10},\phi_{20})\|_3+\|(\psi_{10},\psi_{20})\|_2$. *Then, there is a* $\delta_0>0$ *such that if* $\delta+\Phi_0\le\delta_0$, *the solutions* $(n_1,n_2,J_1,J_2,E)$ *of the IVP* (1.3), (1.4), *and* (3.1) *are unique and globally exist, and satisfy*

$$\begin{aligned}&\sum_{k=0}^{2}(1+t)^{k+1}\|\partial_x^k(n_1-\bar n,n_2-\bar n)(t)\|^2\\ &\quad+\sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k(J_1-\hat J_1-\bar J,J_2-\hat J_2-\bar J)(t)\|^2\le C(\delta+\Phi_0),\end{aligned}\tag{3.62}$$

$$\|(n_1-n_2)(t)\|_1^2+\|(J_1-\hat J_1-J_2+\hat J_2)(t)\|_1^2+\|(E-\hat E)(t)\|_2^2\le C(\delta^2+\Phi_0^2)e^{-\nu t}\tag{3.63}$$

*for some constant* $\nu>0$. *Furthermore, if* $(\phi_{10},\phi_{20})\in L^1(\mathbb{R})$, *then the optimal* $L^p(\mathbb{R})$ $(2\le p\le+\infty)$ *decay rates hold as follows:*

$$\|\partial_x^k(n_1-\bar n,n_2-\bar n)(t)\|_{L^p(\mathbb{R})}\le C(\delta+\Phi_0)^{\frac12}(1+t)^{-\frac12(1-\frac1p)-\frac{k+1}{2}},\tag{3.64}$$

$$\|\partial_x^k(J_1-\hat J_1-\bar J,J_2-\hat J_2-\bar J)(t)\|_{L^p(\mathbb{R})}\le C(\delta+\Phi_0)^{\frac12}(1+t)^{-\frac12(1-\frac1p)-\frac{k+2}{2}}\tag{3.65}$$

*for* $0\le k\le 2$ *if* $p=2$, *and* $0\le k\le 1$ *if* $p\in(2,+\infty]$.

COROLLARY 3.3. *Let* $(\phi_{10},\phi_{20})\in L^1(\mathbb{R})$. *Then, for* $i=1,2$,

$$\|\partial_x^k\partial_t^l(n_i-\bar n)(t)\|_{L^\infty(\mathbb{R})}\le C(\delta+\Phi_0)^{\frac12}(1+t)^{-1-\frac{k+2l}{2}},\quad k,l\ge0,\tag{3.66}$$

$$\|\partial_x^k\partial_t^l(u_i-\bar u)(t)\|_{L^\infty(\mathbb{R})}\le C(\delta+\Phi_0)^{\frac12}(1+t)^{-\frac32-\frac{k+2l}{2}},\quad k,l\ge0,\tag{3.67}$$

$$\|\partial_x^k\partial_t^l(n_1-n_2)(t)\|_{L^\infty(\mathbb{R})}\le C(\delta+\Phi_0)^{\frac12}e^{-\frac{\nu}{2}t},\quad k,l\ge0,\tag{3.68}$$

$$\|\partial_x^k\partial_t^l(u_1-u_2)(t)\|_{L^\infty(\mathbb{R})}\le C(\delta+\Phi_0)^{\frac12}e^{-\frac{\nu}{2}t},\quad k,l\ge0,\tag{3.69}$$

$$\|\partial_x^k\partial_t^l E(t)\|_{L^\infty(\mathbb{R})}\le C(\delta+\Phi_0)^{\frac12}e^{-\frac{\nu}{2}t},\quad k,l\ge0,\tag{3.70}$$

*where* $u_i=J_i/n_i$.

*Remark* 1.

1. Although the diffusion waves $(\bar n,\bar u)(x+x_0,t)$ are the asymptotic profiles of the original solutions $(n_i,u_i)(x,t)$, $i=1,2$, with algebraic decay, the much better asymptotic profiles of the original solutions $(n_1,u_1)(x,t)$ (or $(n_2,u_2)(x,t)$), in fact, are just their partner solutions $(n_2,u_2)(x,t)$ (or $(n_1,u_1)(x,t)$), because the corresponding decay as shown in (3.68) and (3.69) are exponential.
2. If the pressures $p(s)$ and $q(s)$ in (1.1) are different, the question will become more complicated and challenging, because the clarification of the asymptotic profiles (diffusion waves) in this case and the construction of the corresponding correction functions will be much harder, and totally different from the simple issue we study here. So, this will be something left for us in future research.
3. The new technique for constructing the correction functions can also be applied to solving the stability of diffusion waves for other hydrodynamic models of semiconductors, for example, the unipolar hydrodynamic model of semiconductors, the nonisentropic full Euler–Poisson system with fractional damping external-force, and so on.

**3.3. A priori estimates in 1-D case.** It is known that Theorem 3.2 can be proved by the classical energy method with the continuation argument based on the local existence and the a priori estimates; cf. [27, 28]. Since the local existence of the solutions of (3.57), (3.59), and (3.60) can be proved in the standard iteration method together with the energy estimates, so the main effort in this subsection is to establish the a priori estimates for the solutions, which is usually technical and crucial in the proof of stability.

Let $T \in (0, +\infty]$, we define the solution space for (3.57), (3.59), and (3.60) as follows:

$$X(T) = \Big\{(\phi_1, \phi_{1t}, \phi_2, \phi_{2t}, \mathcal{H})(x,t)\Big|\partial_t^j \phi_i \in C(0,T; H^{3-j}(\mathbb{R})), i = 1,2,\ j = 0,1,$$
$$\partial_t^j \mathcal{H} \in C(0,T; H^{2-j}(\mathbb{R})),\ j = 0,1,\ 0 \le t \le T\Big\}$$

with the norm

$$N(T)^2 = \sup_{0\le t\le T}\left\{\sum_{k=0}^{3}(1+t)^k\|\partial_x^k(\phi_1,\phi_2)(t)\|^2 + \sum_{k=0}^{2}(1+t)^{k+2}\|\partial_x^k(\phi_{1t},\phi_{2t})(t)\|^2 + \sum_{i+j=0}^{2} e^{\nu t}\|\partial_t^i\partial_x^j\mathcal{H}(t)\|^2\right\}.$$

Let $N(T)^2 \le \varepsilon^2$, where $\varepsilon$ is sufficiently small and will be determined later. Notice that, by Sobolev inequality $\|\partial_x^k f\|_{L^\infty(\mathbb{R})} \le C\|\partial_x^k f\|^{1/2}\|\partial_x^{k+1} f\|^{1/2}$, we have, $i = 1,2$,

$$(1+t)^{\frac14+\frac k2}\|\partial_x^k\phi_i(t)\|_{L^\infty(\mathbb{R})} + \sum_{k=0}^{1}(1+t)^{\frac54+\frac k2}\|\partial_x^k\phi_{it}(t)\|_{L^\infty(\mathbb{R})} \le C\varepsilon.$$

Clearly, there exists a positive constant $c_1$ such that

$$0 < \frac{1}{c_1} \le n_i = \phi_{ix} + \hat{n}_i + \bar{n} \le c_1, \quad i = 1,2. \tag{3.71}$$

Now we first establish the following basic energy estimate.

LEMMA 3.4. *It holds that*

$$\|(\mathcal{H}, \mathcal{H}_x, \mathcal{H}_t, \mathcal{H}_{xx}, \mathcal{H}_{xt}, H_{tt})(t)\|^2 \le C(\delta^2 + \Phi_0^2)e^{-\nu t} \tag{3.72}$$

*provided* $\varepsilon + \delta \ll 1$.

*Proof. Step* 1. Multiplying (3.60) by $\mathcal{H} + 2\mathcal{H}_t$ and integrating it over $(-\infty, +\infty)$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}}\Big(\mathcal{H}_t^2 + \Big[\frac12 + 2\bar{n}\Big]\mathcal{H}^2 + \mathcal{H}_t\mathcal{H}\Big)dx + \int_{\mathbb{R}}\Big(\mathcal{H}_t^2 + (2\bar{n} - 2\bar{n}_t)\mathcal{H}^2\Big)dx$$
$$= \int_{\mathbb{R}}(\mathcal{H} + 2\mathcal{H}_t)(h_{1x} - h_2 - h_3 + h_{4x})dx. \tag{3.73}$$

Applying Taylor's formula to (3.61), namely,

$$h_{1x} = p'(n_1)\mathcal{H}_{xx} + p'(n_1)(\hat{n}_{1x} - \hat{n}_{2x}) + O(1)(\phi_{2xx} + \hat{n}_{2x} + \bar{n}_x)(\mathcal{H}_x + \hat{n}_1 - \hat{n}_2),$$

then the first term of the right-hand side term of (3.73) can be estimated as follows:

$$\int_{\mathbb{R}} h_{1x}(\mathcal{H} + 2\mathcal{H}_t)dx := I_1 + I_2 + I_3$$

with

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} p'(n_1)\mathcal{H}_{xx}(\mathcal{H}+2\mathcal{H}_t)dx \\
&\le -\frac{d}{dt}\int_{\mathbb{R}} p'(n_1)\mathcal{H}_x^2dx - \int_{\mathbb{R}} p'(n_1)\mathcal{H}_x^2dx + C(\varepsilon+\delta)\|(\mathcal{H},\mathcal{H}_x,\mathcal{H}_t)(t)\|^2, \\
I_2 &= \int_{\mathbb{R}} p'(n_1)(\hat{n}_{1x}-\hat{n}_{2x})(\mathcal{H}+2\mathcal{H}_t)dx \\
&\le \kappa\|(\mathcal{H},\mathcal{H}_t)(t)\|^2 + C(\kappa)\delta^2 e^{-\nu_0 t} \quad \text{for some small constant } \kappa>0, \\
I_3 &= \int_{\mathbb{R}} O(1)(\phi_{2xx}+\hat{n}_{2x}+\bar{n}_x)(\mathcal{H}_x+\hat{n}_1-\hat{n}_2)(\mathcal{H}+2\mathcal{H}_t)dx \\
&\le C(\varepsilon+\delta)\|(\mathcal{H},\mathcal{H}_x,\mathcal{H}_t)(t)\|^2 + C\delta^2e^{-\nu_0 t},
\end{aligned}$$

so we have

$$\begin{aligned}
\int_{\mathbb{R}} h_{1x}(\mathcal{H}+2\mathcal{H}_t)dx \le{}& -\frac{d}{dt}\int_{\mathbb{R}} p'(n_1)\mathcal{H}_x^2dx - \int_{\mathbb{R}} p'(n_1)\mathcal{H}_x^2dx \\
&+ C(\varepsilon+\delta+\kappa)\|(\mathcal{H},\mathcal{H}_x,\mathcal{H}_t)(t)\|^2 + C\delta^2e^{-\nu_0 t},
\end{aligned} \tag{3.74}$$

where we used (3.48) and $\kappa$ is small and will be determined later.

Similarly, noticing (3.48) and (3.61), which imply $|\phi_{ix}| \le CN(t) \le C\varepsilon$ and $|\hat{n}_i| \le C\delta e^{-\nu_0 t}$, we can prove

$$-\int_{\mathbb{R}} h_2(\mathcal{H}+2\mathcal{H}_t)dx \le C(\varepsilon+\delta)\|(\mathcal{H},\mathcal{H}_t)(t)\|^2, \tag{3.75}$$

$$-\int_{\mathbb{R}} h_3(\mathcal{H}+2\mathcal{H}_t)dx \le C(\varepsilon+\delta)\|(\mathcal{H},\mathcal{H}_t)(t)\|^2 + C\delta^2e^{-\nu_0 t}, \tag{3.76}$$

where for (3.76), we also used the facts $\int_{\mathbb{R}}(\bar{n}-\breve{n})^2dx \le C\delta^2(1+t)^{\frac{1}{2}}$, which, as showed in (2.6) (Lemma 2.2), can be proved from the construction of $\breve{n}(x)\to n_\pm$ as $x\to\pm\infty$ and the property of the diffusion wave $\bar{n}(\frac{x+x_0}{\sqrt{1+t}})$. Noticing

$$\begin{aligned}
h_{4x} ={}& -\frac{J_1^2}{n_1^2}\mathcal{H}_{xx} - \frac{2J_1}{n_1}\mathcal{H}_{xt} + O(1)(\hat{n}_{1x}+\hat{n}_{2x}+\hat{J}_{1x}+\hat{J}_{2x}) \\
&+ O(1)(\phi_{2xx}+\phi_{2xt}+\bar{n}_x+\bar{J}_x+\hat{n}_{2x}+\hat{J}_{2x}) \\
&\times(\mathcal{H}_x+\mathcal{H}_t+\hat{n}_1+\hat{n}_2+\hat{J}_1+\hat{J}_2),
\end{aligned} \tag{3.77}$$

then, by integrating it by parts and using the Cauchy inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}} h_{4x}(\mathcal{H}+2\mathcal{H}_t)dx \le{}& \frac{d}{dt}\Big(\int_{\mathbb{R}}\frac{J_1^2}{n_1^2}\mathcal{H}_x^2dx\Big) \\
&+ C(\varepsilon+\delta+\kappa)\|(\mathcal{H},\mathcal{H}_x,\mathcal{H}_t)(t)\|^2 + C\delta^2e^{-\nu_0 t}.
\end{aligned} \tag{3.78}$$

Substituting (3.74)–(3.78) into (3.73), and noticing the smallness of $\varepsilon,\delta,\kappa$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_{\mathbb{R}}\mathcal{H}_t^2 + \Big(\frac{1}{2}+2\bar{n}\Big)\mathcal{H}^2 + \mathcal{H}_t\mathcal{H} + \Big(p'(n_1)-\frac{J_1^2}{n_1^2}\Big)\mathcal{H}_x^2dx\Big) \\
&+\frac{1}{2}\int_{\mathbb{R}}\Big(\mathcal{H}_t^2 + 2(\bar{n}-\bar{n}_t)\mathcal{H}^2 + p'(n_1)\mathcal{H}_x^2\Big)dx
\end{aligned}$$

$$\leq C\delta^2 e^{-\nu_0 t}.$$

Applying Gronwall's lemma to the above differential inequality, we obtain

$$\|(\mathcal{H}, \mathcal{H}_x, \mathcal{H}_t)(t)\|^2 \leq C(\delta^2 + \Phi_0^2)e^{-\nu_1 t}, \tag{3.79}$$

where $\nu_1$ is some positive constant.

*Step* 2. Differentiating (3.60) with respect to $x$, we obtain

$$\mathcal{H}_{xtt} + \mathcal{H}_{xt} + 2\bar{n}\mathcal{H}_x + 2\bar{n}_x\mathcal{H} = h_x = h_{1xx} - h_{2x} - h_{3x} + h_{4xx}. \tag{3.80}$$

Multiplying (3.80) by $\mathcal{H}_x + 2\mathcal{H}_{xt}$ and integrating it over $(-\infty, +\infty)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\int_{\mathbb{R}} \Big[\mathcal{H}_{xt}^2 + \Big(\frac{1}{2} + 2\bar{n}\Big)\mathcal{H}_x^2 + \mathcal{H}_{xt}\mathcal{H}_x\Big]dx\Big) + \int_{\mathbb{R}} \Big[\mathcal{H}_{xt}^2 + 2(\bar{n} - \bar{n}_t)\mathcal{H}_x^2\Big]dx \\ &= \int_{\mathbb{R}} 2\bar{n}_x\mathcal{H}(\mathcal{H}_x + 2\mathcal{H}_{xt})dx + \int_{\mathbb{R}} (\mathcal{H}_x + 2\mathcal{H}_{xt})(h_{1xx} - h_{2x} - h_{3x} + h_{4xx})dx. \end{aligned} \tag{3.81}$$

By using (3.48) and (3.79), the terms in the right-hand side of (3.81) can be similarly estimated as follows:

$$2\int_{\mathbb{R}} \bar{n}_x\mathcal{H}(\mathcal{H}_x + 2\mathcal{H}_{xt})dx \leq C\delta\|\mathcal{H}_{xt}(t)\|^2 + C(\delta^2 + \Phi_0^2)e^{-\nu_1 t}, \tag{3.82}$$

and

$$\begin{aligned} \int_{\mathbb{R}} (\mathcal{H}_x + 2\mathcal{H}_{xt})h_{1xx}dx \leq{}& -\frac{d}{dt}\Big(\int_{\mathbb{R}} p'(n_1)\mathcal{H}_{xx}^2 dx\Big) - \int_{\mathbb{R}} p'(n_1)\mathcal{H}_{xx}^2 dx \\ &+ C(\varepsilon + \delta + \kappa)\|(\mathcal{H}_{xx}, \mathcal{H}_{xt})(t)\|^2 + C\delta^2 e^{-\min(\nu_0,\nu_1)t}, \end{aligned} \tag{3.83}$$

and

$$-\int_{\mathbb{R}} (\mathcal{H}_x + 2\mathcal{H}_{xt})h_{2x}dx \leq C(\varepsilon + \delta)\|\mathcal{H}_{xt}(t)\|^2 + C(\delta^2 + \Phi_0^2)e^{-\min(\nu_0,\nu_1)t}, \tag{3.84}$$

and

$$-\int_{\mathbb{R}} (\mathcal{H}_x + 2\mathcal{H}_{xt})h_{3x}dx \leq \kappa\|\mathcal{H}_{xt}(t)\|^2 + C\delta^2 e^{-\min(\nu_0,\nu_1)t}, \tag{3.85}$$

and

$$\begin{aligned} \int_{\mathbb{R}} (\mathcal{H}_x + 2\mathcal{H}_{xt})h_{4xx}dx \leq{}& \frac{d}{dt}\Big(\int_{\mathbb{R}} \frac{J_1^2}{n_1^2}\mathcal{H}_{xx}^2 dx\Big) + C(\varepsilon + \delta + \kappa)\|(\mathcal{H}_{xx}, \mathcal{H}_{xt})(t)\|^2 \\ &+ C(\delta^2 + \Phi_0^2)e^{-\min(\nu_0,\nu_1)t}. \end{aligned} \tag{3.86}$$

Substituting (3.82)–(3.86) into (3.81), and noticing the smallness of $\varepsilon, \delta, \kappa$, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\int_{\mathbb{R}} \mathcal{H}_{xt}^2 + \Big(\frac{1}{2} + 2\bar{n}\Big)\mathcal{H}_x^2 + \mathcal{H}_{xt}\mathcal{H}_x + \Big(p'(n_1) - \frac{J_1^2}{n_1^2}\Big)\mathcal{H}_{xx}^2 dx\Big) \\ &+\frac{1}{2}\int_{\mathbb{R}} \Big(\mathcal{H}_{xt}^2 + 2(\bar{n} - \bar{n}_t)\mathcal{H}_x^2 + p'(n_1)\mathcal{H}_{xx}^2\Big)dx \\ &\leq C(\delta^2 + \Phi_0^2)e^{-\min(\nu_0,\nu_1)t}. \end{aligned} \tag{3.87}$$

Again, applying Gronwall's inequality to the above differential inequality, we obtain

$$\|(\mathcal{H}_x, \mathcal{H}_{xx}, \mathcal{H}_{xt})(t)\|^2 \le C(\delta^2 + \Phi_0^2)e^{-\nu_2 t} \tag{3.88}$$

for some constant $\nu_2 > 0$.

Furthermore, by applying (3.79) and (3.87) to (3.60), we can prove

$$\|\mathcal{H}_{tt}(t)\|^2 \le C(\delta^2 + \Phi_0^2)e^{-\nu_3 t} \tag{3.89}$$

for some constant $\nu_3 > 0$.

Finally, let $\nu = \min(\nu_0, \nu_1, \nu_2, \nu_3)$. Thus, (3.79), (3.88), and (3.89) imply (3.72). The proof of this lemma is complete. □

Beginning now, we will state more higher order energy estimates for the solutions $\phi_i$ $(i = 1, 2)$ to the wave equations (3.57) in different lemmas with sketchy proofs.

LEMMA 3.5. *It holds that, for $i = 1, 2$,*

$$\|(\phi_i, \phi_{ix}, \phi_{it})(t)\|^2 + \int_0^t \|(\phi_{ix}, \phi_{i\tau})(\tau)\|^2 d\tau \le C(\delta + \Phi_0) \tag{3.90}$$

*provided $\varepsilon + \delta \ll 1$.*

*Proof.* By taking $\int_{\mathbb{R}}[(3.57)_i \cdot (\phi_i + \lambda\phi_{it})]dx$ for some large number $\lambda > 0$, and applying Lemmas 2.1, 2.2, and 3.4, with a tedious calculation we complete the proof of Lemma 3.5. □

LEMMA 3.6. *It holds that, for $i = 1, 2$,*

$$(1+t)\|(\phi_{ix}, \phi_{it})(t)\|^2 + \int_0^t (1+\tau)\|\phi_{i\tau}(\tau)\|^2 d\tau \le C(\delta + \Phi_0) \tag{3.91}$$

*provided $\varepsilon + \delta \ll 1$.*

*Proof.* By taking $\int_0^t \int_{\mathbb{R}}(1+\tau)(3.57)_i \cdot \phi_{it} dx d\tau, i = 1, 2$, and integrating the resultant equation with respect to $t$ over $[0, t]$, and applying Lemma 3.5, we obtain

$$(1+t)\|(\phi_{ix}, \phi_{it})(t)\|^2 + \int_0^t (1+\tau)\|\phi_{i\tau}(\tau)\|^2 d\tau \le C(\delta + \Phi_0). \tag{3.92}$$

Hence, we complete the proof of Lemma 3.6. □

LEMMA 3.7. *It holds that, for $i = 1, 2$,*

$$\begin{aligned} &(1+t)^2\|(\phi_{ixx}, \phi_{ixt})(t)\|^2 + \int_0^t [(1+\tau)^2\|\phi_{ix\tau}(\tau)\|^2 + (1+\tau)\|\phi_{ixx}(\tau)\|^2]d\tau \\ &\le C(\delta + \Phi_0) \end{aligned} \tag{3.93}$$

*provided $\varepsilon + \delta \ll 1$.*

*Proof.* By taking $\int_0^t \int_{\mathbb{R}}(1+\tau)^2[\partial_x(3.57)_i \times \phi_{ix} + \partial_x(3.57)_i \times \phi_{ix\tau}]dx d\tau$, and applying Lemma 3.6, as shown before, we can similarly prove (3.93). □

LEMMA 3.8. *It holds that, for $i = 1, 2$,*

$$\begin{aligned} &(1+t)^3\|(\phi_{ixxx}, \phi_{ixxt})(t)\|^2 + \int_0^t [(1+\tau)^3\|\phi_{ixx\tau}(\tau)\|^2 + (1+\tau)^2\|\phi_{ixxx}(\tau)\|^2]d\tau \\ &\le C(\delta + \Phi_0) \end{aligned} \tag{3.94}$$

*provided $\varepsilon + \delta \ll 1$.*

*Proof.* By taking $\int_0^t\int_{\mathbb{R}}(1+\tau)^3[\partial_{xx\tau}(3.57)_1\times\phi_{1xx\tau}+\partial_{xx\tau}(3.57)_2\times\phi_{2xx\tau}]dxd\tau$, and applying Lemma 3.7, we can prove (3.94). □

LEMMA 3.9. *It holds that, for $i=1,2$,*

$$(1+t)^2\|(\phi_{it},\phi_{ixt},\phi_{itt})(t)\|^2+\int_0^t(1+\tau)^2\|(\phi_{ix\tau},\phi_{i\tau\tau})(\tau)\|^2d\tau\le C(\delta+\Phi_0) \tag{3.95}$$

*provided $\varepsilon+\delta\ll 1$.*

*Proof.* By taking $\int_0^t\int_{\mathbb{R}}(1+\tau)^2[\partial_\tau(3.57)_1\times(\phi_{1\tau}+2\phi_{1\tau\tau})+\partial_\tau(3.57)_2\times(\phi_{2\tau}+2\phi_{2\tau\tau})]dxd\tau$, and applying Lemma 3.8, we can prove (3.95). □

LEMMA 3.10. *It holds that*

$$(1+t)^3\|(\phi_{ixt},\phi_{itt})(t)\|^2+\int_0^t(1+\tau)^3\|\phi_{i\tau\tau}(\tau)\|^2d\tau\le C(\delta+\Phi_0) \tag{3.96}$$

*for $i=1,2$, provided $\varepsilon+\delta\ll 1$.*

*Proof.* By taking $\int_0^t\int_{\mathbb{R}}(1+\tau)^3[\partial_\tau(3.57)_1\times\phi_{1\tau\tau}+\partial_\tau(3.57)_2\times\phi_{2\tau\tau}]dxd\tau$, and applying Lemma 3.9, we can prove (3.96). □

LEMMA 3.11. *It holds that, for $i=1,2$,*

$$\begin{aligned}&(1+t)^3\|(\phi_{ixt},\phi_{ixxt},\phi_{ixtt})(t)\|^2+\int_0^t(1+\tau)^3\|(\phi_{ixx\tau},\phi_{ix\tau\tau})(\tau)\|^2d\tau\\ &\le C(\delta+\Phi_0)\end{aligned} \tag{3.97}$$

*provided $\varepsilon+\delta\ll 1$.*

*Proof.* By taking $\int_0^t\int_{\mathbb{R}}(1+\tau)^3[\partial_\tau(3.57)_1\times(\phi_{1x\tau}+2\phi_{1x\tau\tau})+\partial_\tau(3.57)_2\times(\phi_{2x\tau}+2\phi_{2x\tau\tau})]dxd\tau$, and applying Lemma 3.10, we can prove (3.97). □

LEMMA 3.12. *It holds that, for $i=1,2$,*

$$(1+t)^4\|(\phi_{ixxt},\phi_{ixtt})(t)\|^2+\int_0^t(1+\tau)^4\|\phi_{ix\tau\tau}(\tau)\|^2d\tau\le C(\delta+\Phi_0) \tag{3.98}$$

*provided $\varepsilon+\delta\ll 1$.*

*Proof.* By taking $\int_0^t\int_{\mathbb{R}}(1+\tau)^4[\partial_\tau(3.57)_1\times\phi_{1x\tau\tau}+\partial_\tau(3.57)_2\times\phi_{2x\tau\tau}]dxd\tau$, and applying Lemma 3.11, we can prove (3.98). □

Now we are going to prove the optimal decays (3.64) and (3.65), if the initial perturbation is further in $L^1(\mathbb{R})$. Let us rewrite (3.57) as follows:

$$\phi_{it}-(a_i(x,t)\phi_{ix})_x=F_i-\phi_{itt},\qquad i=1,2, \tag{3.99}$$

where

$$\begin{aligned}a_i(x,t):&=p'(\bar n_i(x,t))\ge C>0,\quad i=1,2,\\ F_1:&=-f_1+g_{1x}-p(\bar n_1)_{xt}-(\phi_{1x}+\hat n_1+\bar n_1)\mathcal{H}\\ &\quad(p(\phi_{1x}+\hat n_1+\bar n_1)-p(\bar n_1)-p'(\bar n_1)\phi_{1x})_x,\\ F_2:&=f_2+g_{2x}-p(\bar n_2)_{xt}+(\phi_{2x}+\hat n_2+\bar n_2)\mathcal{H}\\ &\quad(p(\phi_{2x}+\hat n_2+\bar n_2)-p(\bar n_2)-p'(\bar n_2)\phi_{2x})_x.\end{aligned}$$

As shown in [35], we can similarly construct the minimizing Green's functions as follows:

$$G_i(x,t;y,s)=\Big(\frac{1}{4\pi a_i(x,t)(t-s)}\Big)^{1/2}\exp\Big(-\frac{(x-y)^2}{4a_i(y,s)(t-s)}\Big),\qquad i=1,2,$$

and rewrite (3.99) in the integral form

$$
\begin{aligned}
\phi_i(x,t) = {} & \int_{\mathbb{R}} G_i(x,t;y,0)\phi_{i0}(y)dy \\
& + \int_0^t \int_{\mathbb{R}} G_i(x,t;y,s)[F_i(y,s) - \phi_{iss}(y,s)]dyds \\
& + \int_0^t \int_{\mathbb{R}} R_{G_i}(x,t;y,s)\phi_i(y,s)]dyds, \qquad i = 1,2,
\end{aligned} \tag{3.100}
$$

where

$$
R_{G_i}(x,t;y,s) := \partial_s G_i(x,t;y,s) + \partial_y\{a_i(y,s)\partial_y G_i(x,t;y,s)\}, \qquad i = 1,2.
$$

Differentiating (3.100) with respect to $x$ and $t$, we have, for $l \le 1$ and $k + l \le 3$,

$$
\begin{aligned}
\partial_t^l \partial_x^k \phi_i(x,t) = {} & \partial_t^l \partial_x^k \int_{\mathbb{R}} G_i(x,t;y,0)\phi_{i0}(y)dy \\
& + \partial_t^l \partial_x^k \int_0^t \int_{\mathbb{R}} G_i(x,t;y,s)[F_i(y,s) - \phi_{iss}(y,s)]dyds \\
& + \partial_t^l \partial_x^k \int_0^t \int_{\mathbb{R}} R_{G_i}(x,t;y,s)\phi_i(y,s)]dyds \\
=: {} & I_{i1}^{l,k} + I_{i2}^{l,k} + I_{i3}^{l,k}, \qquad i = 1,2.
\end{aligned} \tag{3.101}
$$

Based on the decay rates we obtained in Lemmas 3.4–3.12, and on the estimates of decay rates for the approximating Green's functions as shown in [35], by a similar but tedious calculation to [35], we can similarly prove

$$
\|I_{i1}^{l,k}\| + \|I_{i1}^{l,k}\| + \|I_{i1}^{l,k}\| = O(1)(1+t)^{-\frac{1}{4}-l-\frac{k}{2}} \quad \text{for } l \le 1,\ l + k \le 3,\ i = 1,2. \tag{3.102}
$$

Here, the details are omitted. Hence, applying (3.102) to (3.101), we obtain the optimal decay rates as follows.

Lemma 3.13. *Furthermore, if* $(\phi_{10}, \phi_{20}) \in L^1(\mathbb{R})$, *then*

$$
\|\partial_x^k \phi_i(t)\| = O(1)(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \quad k = 0,1,2,3,\ i = 1,2, \tag{3.103}
$$

$$
\|\partial_x^k \partial_t \phi_i(t)\| = O(1)(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \quad k = 0,1,\ i = 1,2. \tag{3.104}
$$

*Proof of Theorem* 3.2. Based on the local existence and the a priori estimates given in Lemmas 3.4–3.12, by using the continuity argument (cf. [27]), we can prove the global existence of the unique solutions of the IVP (1.3)–(1.4) with the desired decay rates (3.62) and (3.63). Furthermore, when the initial perturbation is in $L^1(\mathbb{R})$, from Lemma 3.13, we immediately obtain the optimal decay rates (3.64) and (3.65). ∎

**4. *n*-D case: Stability of planar diffusion waves.** In this section, we are going to study the multidimensional isentropic Euler–Poisson equations for the bipolar hydrodynamic model of semiconductors (1.1) with the initial data

$$
\begin{cases}
n_i(x,0) = n_{i0}(x) \to n_{\pm} \ \text{ as } x_1 \to \pm\infty, \\
u_i(x,0) = u_{i0}(x) \to (u_{i\pm}, 0, 0) \ \text{ as } x_1 \to \pm\infty,
\end{cases} \qquad i = 1,2, \tag{4.1}
$$

where $n_{i\pm}$ and $u_{i\pm}$ for $i = 1,2$ are constants.

We will prove that the solutions of the $n$-D equations (1.1) and (4.1) converge to the 1-D nonlinear diffusion waves of (2.1), the so-called planar diffusion waves to the $n$-D equations (1.1). It must be pointed out that the strategy of the antiderivatives for the problem setting up used in the 1-D case (see (3.54)) is no longer effective in the multidimensional case, because the shift $x_0$ defined in (3.51) in the 1-D case will be an implicit function in $n$-D case, and it depends on the solution $(n_1, u_1, n_2, u_2, \Psi)(x, t)$ of the system (1.1), rather than the initial data. Instead of this, we are going to apply the key lemma, Lemma 2.3, to establish some crucial energy estimates and then prove the stability of planar diffusion waves. For simplicity, we just consider the 3-D case, and denote $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Let $(\tilde{n}_1, \tilde{n}_2, \tilde{J}_1, \tilde{J}_2, \tilde{E})(x_1, t)$ be the solutions of 1-D isentropic Euler–Poisson equations (1.3) with small perturbations, i.e., $\Phi_0$ given in Theorem 3.2, and define

$$\begin{cases} \tilde{U}_i(x,t) = (\tilde{u}_i(x_1,t), 0, 0), \\ \bar{U}(x,t) = (\bar{u}(x_1,t), 0, 0), \\ \tilde{\mathcal{E}}(x,t) = (\tilde{E}(x_1,t), 0, 0) \end{cases} \qquad \text{for } i = 1, 2. \tag{4.2}$$

Based on the above preparation, we are going to make a perturbation of (1.1) to the 1-D solutions $(\tilde{n}_1, \tilde{n}_2, \tilde{J}_1, \tilde{J}_2, \tilde{E})(x_1, t)$ of (1.3) that we just stated above. We define $z_1 = n_1 - \tilde{n}_1$, $z_2 = n_2 - \tilde{n}_2$, $w_1 = u_1 - \tilde{U}_1$, $w_2 = u_2 - \tilde{U}_2$. Combining (1.1) with (1.3), we obtain

$$\begin{cases} z_{1t} + \operatorname{div}(z_1 w_1 + \tilde{n}_1 w_1 + \tilde{u}_1 z_1) = 0, \\ w_{1t} + w_1 + \theta_{10} \nabla z_1 - (\nabla \Psi - \tilde{\mathcal{E}}) = -L_1 - N_1, \\ z_{2t} + \operatorname{div}(z_2 w_2 + \tilde{n}_2 w_2 + \tilde{u}_2 z_2) = 0, \\ w_{2t} + w_2 + \theta_{20} \nabla z_2 + (\nabla \Psi - \tilde{\mathcal{E}}) = -L_2 - N_2, \\ \operatorname{div}(\nabla \Psi - \tilde{\mathcal{E}}) = z_1 - z_2, \end{cases} \tag{4.3}$$

where

$$\begin{cases} L_i = w_i \nabla \tilde{u}_i + \tilde{u}_i \nabla w_i, \quad N_i = w_i \nabla w_i + \theta_i \nabla z_i + \theta_i \nabla \tilde{n}_i, \\ \theta_{i0} = \frac{p'(\tilde{n}_i)}{\tilde{n}_i}, \quad \theta_i = \frac{p'(\tilde{n}_i + z_i)}{\tilde{n}_i + z_i} - \frac{p'(\tilde{n}_i)}{\tilde{n}_i}, \end{cases} \qquad i = 1, 2. \tag{4.4}$$

Noticing that $\operatorname{curl}(\nabla \Psi - \tilde{\mathcal{E}}) = 0$, there exists $\mathcal{E}$ such that $\nabla \mathcal{E} = \nabla \Psi - \tilde{\mathcal{E}}$. We can reduce (4.3) into

$$\begin{cases} z_{1t} + \operatorname{div}(z_1 w_1 + \tilde{n}_1 w_1 + \tilde{u}_1 z_1) = 0, \\ w_{1t} + w_1 + \theta_{10} \nabla z_1 - \nabla \mathcal{E} = -L_1 - N_1, \\ z_{2t} + \operatorname{div}(z_2 w_2 + \tilde{n}_2 w_2 + \tilde{u}_2 z_2) = 0, \\ w_{2t} + w_2 + \theta_{20} \nabla z_2 + \nabla \mathcal{E} = -L_2 - N_2, \\ \Delta \mathcal{E} = z_1 - z_2, \end{cases} \tag{4.5}$$

provided with the initial data

$$\begin{cases} z_{i0} := z_i(x, 0) = n_{i0}(x) - \tilde{n}_{i0}(x_1) = n_{i0}(x) - \bar{n}(x_1) \in H^4(\mathbb{R}^3), \\ w_{i0} := w_i(x, 0) = u_{i0}(x) - \tilde{U}_i(x, 0) = u_{i0}(x) - \bar{U}(x, 0) \in H^4(\mathbb{R}^3), \end{cases} \tag{4.6}$$

and the boundary condition

$$|\nabla \mathcal{E}| \to 0 \quad \text{as } |x| \to +\infty. \tag{4.7}$$

For later use, we define

$$\begin{cases} \Delta\mathcal{E}_0(x) := z_{10}(x) - z_{20}(x), \\ \mathcal{E}_0(x) \to 0 \quad \text{as } |x| \to +\infty, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\bar{\mathcal{E}}_0(x) := z_{1t}(x,0) - z_{2t}(x,0), \\ \bar{\mathcal{E}}_0(x) \to 0 \quad \text{as } |x| \to +\infty, \end{cases} \tag{4.8}$$

and

$$\eta := \sum_{i=1}^{2} \|(n_i - \tilde{n}_i, u_i - \tilde{U}_i)(0)\|_4 + \|(\nabla\Psi - \tilde{\mathcal{E}})(0)\|_4 + \|(\nabla\Psi_t - \tilde{\mathcal{E}}_t)(0)\|_3,$$

where $\nabla\Psi(x,0) = \nabla\mathcal{E}_0(x) + \tilde{\mathcal{E}}(x,0)$ and $\nabla\Psi_t(x,0) = \nabla\bar{\mathcal{E}}_0(x) + \tilde{\mathcal{E}}_t(x,0)$.

**4.1. Convergence theorem in the 3-D case.** We now state the stability results for the planar diffusion waves in the multidimensional case as follows.

THEOREM 4.1. *Let $\delta = |n_+ - n_-| + |u_{1+}| + |u_{1-}| + |u_{2+}| + |u_{2-}| + \Phi_0$. Then, if $\eta + \delta \ll 1$, there exists a unique global smooth solution $(n_1, n_2, u_1, u_2, \nabla\Psi)$ for the 3-D bipolar hydrodynamic model for semiconductors system* (1.1), (4.1), *and* (4.7) *and satisfies*

$$n_i - \tilde{n}_i, u_i - \tilde{U}_i, \nabla\Psi - \tilde{\mathcal{E}} \in C([0,\infty), H^4(\mathbb{R}^3)) \cap C^1([0,\infty), H^3(\mathbb{R}^3)) \quad \textit{for} \quad i = 1, 2,$$

*and*

$$\sum_{k=0}^{4}(1+t)^{k+1}\|\nabla^k(\nabla\Psi - \tilde{\mathcal{E}})(t)\|^2 + \sum_{i=1}^{2}\sum_{k=0}^{4}(1+t)^k\|(\nabla^k(n_i - \tilde{n}_i), \nabla^k(u_i - \tilde{U}_i))(t)\|^2_{L^2(\mathbb{R}^3)} \le C(\delta^{\frac{1}{2}} + \eta^2). \tag{4.9}$$

THEOREM 4.2. *Under the conditions in Theorem* 4.1, *we have*

$$\begin{cases} \|(n_1 - \tilde{n}_1, n_2 - \tilde{n}_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{3}{4}}, \\ \|(n_1 - n_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{9}{4}}, \\ \|(u_1 - \tilde{U}_1, u_2 - \tilde{U}_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{5}{4}}, \\ \|(\nabla\Psi - \tilde{\mathcal{E}})(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{7}{4}}. \end{cases} \tag{4.10}$$

Based on Corollary 3.3 and Theorem 4.2, we immediately obtain the convergence of the solution $(n_1, n_2, u_1, u_2, E)(x,t)$ for the 3-D equations (1.1) and (4.1) to the 1-D diffusion wave $(\bar{n}, \bar{u})(x_1/\sqrt{1+t})$ for (2.1), namely, the stability of the planar diffusion waves.

COROLLARY 4.3. *Under the conditions in Theorem* 4.1, *we have*

$$\begin{cases} \|(n_1 - \bar{n}, n_2 - \bar{n})(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{3}{4}}, \\ \|(u_1 - \bar{U}, u_2 - \bar{U})(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{5}{4}}, \\ \|\nabla\Psi(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{7}{4}}. \end{cases} \tag{4.11}$$

*Remark* 2. Notice that, for the 1-D case, the electron field decays exponentially fast, but here the decay rate of the electron field for the 3-D case is algebraic only.

**4.2. A priori estimates in the 3-D case.** Let $T \in (0, +\infty]$; we then define the solution space of (4.5) as, for $0 \le t \le T$,

$$\mathcal{X}(T) = \Big\{ (z_1, z_2, w_1, w_2, \nabla\mathcal{E})(x,t) \Big| z_i,\ w_i,\ \nabla\mathcal{E} \in C(0,T; H^4(\mathbb{R})), i = 1,2 \Big\}.$$

The local existence of (4.5) can be established by a standard contraction mapping argument; for example, see [17, 23]. The main duty of the rest of the present paper is to establish some crucial energy estimates.

We are going to establish the a priori estimates of $(z_1, z_2, w_1, w_2, \nabla\mathcal{E})$, which will be the main effort of this section. We define

$$\mathcal{N}(T)^2 = \sup_{0\le t\le T} \sum_{k=1}^{2}\sum_{k=0}^{4} \Big[ (1+t)^k \|(\nabla^k z_i, \nabla^k w_i)(t)\|^2 + (1+t)^{k+1} \|\nabla^{k+1}\mathcal{E}(t)\|^2 \Big].$$

Let $\mathcal{N}(T)^2 \le \varepsilon^2$, where $\varepsilon$ is sufficiently small and will be determined later. Then the Gagliardo–Nirenberg inequality guarantees, for $i = 1,2$, $k = 0,1,2$,

$$(1+t)^{\frac{3}{4}+\frac{k}{2}} \|(\nabla^k z_i, \nabla^k w_i)(t)\|_{L^\infty} + (1+t)^{\frac{5}{4}+\frac{k}{2}} \|\nabla^{k+1}\mathcal{E}(t)\|_{L^\infty} \le C\varepsilon. \tag{4.12}$$

*Remark* 3. Before we establish the a priori estimates for the solutions, we need to estimate $\mathcal{E}(x,t)$. Since $\Delta\mathcal{E} = z_1 - z_2$ and $\Delta\mathcal{E}, \nabla\mathcal{E} \in L^2(\mathbb{R}^3)$, we can formally solve it by using Green's functions and estimate it as

$$\begin{aligned} |\mathcal{E}(x,t)| &= \Big| \int_{\mathbb{R}^3} \frac{1}{|x-y|}(z_1 - z_2)(y,t)dy + C \Big| = \Big| \int_{\mathbb{R}^3} \frac{1}{|x-y|} \Delta\mathcal{E}(y,t)dy + C \Big| \\ &= \Big| \int_{\mathbb{R}^3} \nabla\Big(\frac{1}{|x-y|}\Big) \nabla\mathcal{E}(y,t)dy + C \Big| \le C \qquad \text{for all } x \in \mathbb{R}^3. \end{aligned} \tag{4.13}$$

Now we first establish the following useful estimate, which plays a fundamental role in the $n$-D case.

LEMMA 4.4. *It holds that*

$$\int_0^t B(\tau)d\tau \le C(\delta^{\frac{1}{2}} + \eta^2) + C\int_0^t \|(w_1, w_2, \nabla z_1, \nabla z_2)(\tau)\|^2 d\tau \tag{4.14}$$

*provided* $\delta + \varepsilon \ll 1$, *where*

$$B(t) := \int_{\mathbb{R}^3} \frac{1}{1+t} e^{-\frac{\mu x_1^2}{1+t}} [z_1^2(x,t) + z_2^2(x,t)]dx.$$

*Proof.* From Lemma 2.3, it holds that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \frac{e^{-\frac{\mu x_1^2}{1+\tau}}}{1+\tau}(z_1^2 + z_2^2)dxd\tau \le C(\mu)\Big\{ &\|(z_1, z_2)(0)\|^2 + \int_0^t \|(\nabla z_1, \nabla z_2)(\tau)\|^2 d\tau \\ &+ \sum_{i=1}^{2} \int_0^t \langle z_{it}, z_i g^2 \rangle_{H^{-1}\times H^1} d\tau \Big\}, \end{aligned} \tag{4.15}$$

where $z_i$ is the solution of (4.5). Notice also that

$$\sum_{i=1}^{2} \int_0^t \langle z_{it}, z_i g^2 \rangle_{H^{-1}\times H^1} d\tau$$

$$
\begin{aligned}
&= -\sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^3} z_i g^2 \mathrm{div}(z_i w_i + \tilde{n}_i w_i + \tilde{u}_i z_i)dxd\tau \\
&= \sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^3} \Big[(z_i+\tilde{n}_i)w_i + \tilde{u}_i z_i\Big]\nabla(z_i g^2)dxd\tau \\
&=: \tilde{I}_1 + \tilde{I}_2. \qquad (4.16)
\end{aligned}
$$

By using the Cauchy inequality and noticing (2.8), Corollary 3.3, and Lemma 2.1, we have

$$
\begin{aligned}
\tilde{I}_1 &= \sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^3}(z_i+\tilde{n}_i)w_i\nabla(z_i g^2)dxd\tau \\
&= \sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^3}(z_i+\tilde{n}_i)w_i(\nabla z_i g^2 + 2z_i g\nabla g)dxd\tau \\
&\le C\sum_{i=1}^{2}\int_0^t \|(w_i,\nabla z_i)(\tau)\|^2 d\tau \\
&\quad + \frac{1}{8C(\mu)}\int_0^t\int_{\mathbb{R}^3}\frac{e^{-\frac{\mu x_1^2}{1+\tau}}}{1+\tau}(z_1^2+z_2^2)(x,\tau)dxd\tau, \qquad (4.17)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{I}_2 &= \sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^3}\tilde{u}_i z_i\nabla(z_i g^2)dxd\tau \\
&\le \sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^3}(|\tilde{u}_i-\bar{u}|+|\bar{u}|)z_i(\nabla z_i g^2 + 2z_i g\nabla g)dxd\tau \\
&\le \sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}^3}\left(C\delta^{\frac{1}{2}}(1+t)^{-\frac{3}{2}} + C\delta\frac{e^{-\frac{\mu x_1^2}{2(1+t)}}}{\sqrt{1+t}}\right) z_i(\nabla z_i g^2 + 2z_i g\nabla g)dxd\tau \\
&\le C\delta^{\frac{1}{2}} + C\sum_{i=1}^{2}\int_0^t\|\nabla z_i(\tau)\|^2 d\tau \\
&\quad + C\delta\int_0^t\int_{\mathbb{R}^3}\frac{e^{-\frac{\mu x_1^2}{1+\tau}}}{1+\tau}(z_1^2+z_2^2)(x,\tau)dxd\tau. \qquad (4.18)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\sum_{i=1}^{2}\int_0^t\langle z_{it}, z_i g^2\rangle_{H^{-1}\times H^1}d\tau &\le C\delta^{\frac{1}{2}} + C\int_0^t\|(w_i,\nabla z_i)(\tau)\|^2 d\tau \\
&\quad + \left[C\delta + \frac{1}{8C(\mu)}\right]\int_0^t\int_{\mathbb{R}^3}\frac{e^{-\frac{\mu x_1^2}{1+\tau}}}{1+\tau}(z_1^2+z_2^2)dxd\tau. \qquad (4.19)
\end{aligned}
$$

Substituting (4.19) into (4.15) and using the smallness of $\delta$, we complete the proof of Lemma 4.4. ∎

LEMMA 4.5. *It holds that*

$$
\|(z_i,w_i)(t)\|^2 + \int_0^t\|w_i(\tau)\|^2 d\tau
$$

$$\leq C(\delta^{\frac{1}{2}}+\eta^2)+C\int_0^t\|\nabla\mathcal{E}(\tau)\|^2d\tau+C(\delta^{\frac{1}{2}}+\varepsilon)\int_0^t\|(\nabla w_i,\nabla z_i)(\tau)\|^2d\tau \tag{4.20}$$

*for* $i=1,2$, *provided* $\delta+\varepsilon\ll 1$.

*Proof.* By taking

$$(4.5)_1\times\frac{z_1}{\tilde n_1},\ (4.5)_2\times\frac{w_1}{\theta_{10}},\ (4.5)_3\times\frac{z_2}{\tilde n_2},\ \text{and}\ (4.5)_4\times\frac{w_2}{\theta_{20}},$$

we have

$$\begin{cases}\Big(\dfrac{z_1^2}{2\tilde n_1}\Big)_t-O(1)\tilde n_{1t}z_1^2+z_1\mathrm{div}w_1=-\dfrac{z_1}{\tilde n_1}\{\mathrm{div}(z_1w_1+\tilde u_1z_1)+w_1\nabla\tilde n_1\},\\ \Big(\dfrac{|w_1|^2}{2\theta_{10}}\Big)_t-O(1)\tilde n_{1t}|w_1|^2+\dfrac{|w_1|^2}{\theta_{10}}+\nabla z_1w_1-\nabla\mathcal{E}\dfrac{w_1}{\theta_{10}}=-\dfrac{w_1}{\theta_{10}}\{L_1+N_1\},\\ \Big(\dfrac{z_2^2}{2\tilde n_2}\Big)_t-O(1)\tilde n_{2t}z_2^2+z_2\mathrm{div}w_2=-\dfrac{z_2}{\tilde n_2}\{\mathrm{div}(z_2w_2+\tilde u_2z_2)+w_2\nabla\tilde n_2\},\\ \Big(\dfrac{|w_2|^2}{2\theta_{20}}\Big)_t-O(1)\tilde n_{2t}|w_2|^2+\dfrac{|w_2|^2}{\theta_{20}}+\nabla z_2w_2+\nabla\mathcal{E}\dfrac{w_2}{\theta_{20}}=-\dfrac{w_2}{\theta_{20}}\{L_2+N_2\}.\end{cases} \tag{4.21}$$

Combining them and applying the Cauchy inequality and the smallness of $\delta$ and $\varepsilon$, we obtain

$$\begin{aligned}&\Big(\frac{z_1^2}{2\tilde n_1}+\frac{z_2^2}{2\tilde n_1}+\frac{|w_1|^2}{2\theta_{10}}+\frac{|w_2|^2}{2\theta_{20}}\Big)_t+2C_0|(w_1,w_2)|^2+\mathrm{div}(z_1w_1+z_2w_2)\\&\leq C|\nabla\mathcal{E}|^2+O(1)\tilde n_{1t}z_1^2+O(1)\tilde n_{2t}z_2^2-\frac{w_1}{\theta_{10}}\{L_1+N_1\}-\frac{w_2}{\theta_{20}}\{L_2+N_2\}\\&\quad-\frac{z_1}{\tilde n_1}\{\mathrm{div}(z_1w_1+\tilde u_1z_1)+w_1\nabla\tilde n_1\}-\frac{z_2}{\tilde n_2}\{\mathrm{div}(z_2w_2+\tilde u_2z_2)+w_2\nabla\tilde n_2\}.\end{aligned} \tag{4.22}$$

Integrating (4.22) with respect to $x$ over $\mathbb{R}^3$ and using the smallness of $\varepsilon$ and $\delta$, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{z_1^2}{2\tilde n_1}+\frac{z_2^2}{2\tilde n_1}+\frac{|w_1|^2}{2\theta_{10}}+\frac{|w_2|^2}{2\theta_{20}}\Big)dx+C_0\|(w_1,w_2)(t)\|^2\\&\leq C(\delta^{\frac{1}{2}}+\varepsilon)\|(\nabla w_1,\nabla w_2,\nabla z_1,\nabla z_2)(t)\|^2\\&\quad+C\|\nabla\mathcal{E}(t)\|^2+C\delta^{\frac{1}{2}}(1+t)^{-2}+C\delta^{\frac{1}{2}}B(t),\end{aligned} \tag{4.23}$$

where we have used

$$\begin{aligned}&-\int_{\mathbb{R}^3}\Big(\frac{z_1}{\tilde n_1}\{\mathrm{div}(z_1w_1+\tilde u_1z_1)+w_1\nabla\tilde n_1\}+\frac{z_2}{\tilde n_2}\{\mathrm{div}(z_2w_2+\tilde u_2z_2)+w_2\nabla\tilde n_2\}\Big)dx\\&\leq C(\delta^{\frac{1}{2}}+\varepsilon)\|(w_1,w_2,\nabla w_1,\nabla w_2,\nabla z_1,\nabla z_2)(t)\|^2\\&\quad+C\delta^{\frac{1}{2}}(1+t)^{-2}+C\delta^{\frac{1}{2}}B(t)\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}&-\int_{\mathbb{R}^3}\Big(\frac{w_1}{\theta_{10}}\{L_1+N_1\}+\frac{w_2}{\theta_{20}}\{L_2+N_2\}\Big)dx\\&\leq C(\delta^{\frac{1}{2}}+\varepsilon)\|(w_1,w_2,\nabla w_1,\nabla w_2,\nabla z_1,\nabla z_2)(t)\|^2\\&\quad+C\delta^{\frac{1}{2}}(1+t)^{-2}+C\delta^{\frac{1}{2}}B(t).\end{aligned} \tag{4.25}$$

Integrating (4.23) over $[0,t]$, and using Lemma 4.4 and the smallness of $\delta$ and $\varepsilon$, we prove Lemma 4.5. $\square$

LEMMA 4.6. *It holds that*

$$\begin{aligned}
&\|(\nabla z_i,\nabla w_i)(t)\|^2+\int_0^t\|\nabla w_i(\tau)\|^2d\tau\\
&\le C(\delta^{\frac12}+\eta^2)+C(\delta^{\frac12}+\varepsilon)\int_0^t\|(\nabla z_i,w_i)(\tau)\|^2d\tau+C\int_0^t\|\nabla^2\mathcal{E}(\tau)\|^2d\tau
\end{aligned}\tag{4.26}$$

*for $i=1,2$, provided $\delta+\varepsilon\ll 1$.*

*Proof.* Let $\alpha$ be a multi-index, and let $|\alpha|=1$. Integrating $\partial_x^\alpha(4.5)_2\times\partial_x^\alpha w_1+\partial_x^\alpha(4.5)_4\times\partial_x^\alpha w_2$ with respect to $x$ over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac12\frac{d}{dt}\|(\nabla w_1,\nabla w_2)(t)\|^2+\|(\nabla w_1,\nabla w_2)(t)\|^2+\sum_{|\alpha|=1}\int_{\mathbb{R}^3}(\partial_x^\alpha w_2-\partial_x^\alpha w_1)\partial_x^\alpha\nabla\mathcal{E}dx\\
&\quad+\sum_{|\alpha|=1}\int_{\mathbb{R}^3}[\partial_x^\alpha(\theta_{10}\nabla z_1)\partial_x^\alpha w_1+\partial_x^\alpha(\theta_{20}\nabla z_2)\partial_x^\alpha w_2]dx\\
&=-\sum_{|\alpha|=1}\int_{\mathbb{R}^3}[\partial_x^\alpha(L_1+N_1)\partial_x^\alpha w_1+\partial_x^\alpha(L_2+N_2)\partial_x^\alpha w_2]dx.
\end{aligned}\tag{4.27}$$

Using the Cauchy inequality, we have

$$\sum_{|\alpha|=1}\int_{\mathbb{R}^3}(\partial_x^\alpha w_2-\partial_x^\alpha w_1)\partial_x^\alpha\nabla\mathcal{E}dx\ge-\frac{1}{32}\|(\nabla w_1,\nabla w_2)(t)\|^2-C\|\nabla^2\mathcal{E}(t)\|^2.\tag{4.28}$$

Let

$$H_3(t):=\int_{\mathbb{R}^3}\left(\frac{\theta_{10}}{2(z_1+\tilde n_1)}|\nabla z_1|^2+\frac{\theta_{20}}{2(z_2+\tilde n_2)}|\nabla z_2|^2\right)dx,\tag{4.29}$$

$$F_3(t):=\int_{\mathbb{R}^3}\left(\frac{\theta_1}{2(z_1+\tilde n_1)}|\nabla z_1|^2+\frac{\theta_2}{2(z_2+\tilde n_2)}|\nabla z_2|^2\right)dx.\tag{4.30}$$

We are going to estimate the other integral terms in (4.27) as follows:

$$\begin{aligned}
&\sum_{|\alpha|=1}\int_{\mathbb{R}^3}[\partial_x^\alpha(\theta_{10}\nabla z_1)\partial_x^\alpha w_1+\partial_x^\alpha(\theta_{20}\nabla z_2)\partial_x^\alpha w_2]dx\\
&=\sum_{i=1}^2\sum_{|\alpha|=1}\int_{\mathbb{R}^3}\sum_{|\beta|=0,\beta\le\alpha}^{1}C_\beta\partial^\beta\theta_{i0}\partial^{\alpha-\beta}\nabla z_i\partial^\alpha w_idx\\
&\ge-C\delta\|(\nabla w_1,\nabla w_2)\|^2-C\delta(1+t)^{-1}\|(\nabla z_1,\nabla z_2)\|^2\\
&\quad-\sum_{i=1}^2\sum_{|\alpha|=1}\int_{\mathbb{R}^3}\theta_{i0}\partial^\alpha z_i\partial^\alpha\,\mathrm{div}w_idx.
\end{aligned}\tag{4.31}$$

In order to control the last integral of (4.31), we first note that, by noting $(4.5)_1$ and $(4.5)_3$,

$$\partial^\alpha\,\mathrm{div}w_i=-\frac{1}{z_i+\tilde n_i}\left\{\partial^\alpha z_{it}+\sum_{|\beta|=1,\beta\le\alpha}^{|\alpha|}C_\beta\partial^\beta(\tilde n_i+z_i)\partial^{\alpha-\beta}\,\mathrm{div}w_i\right.$$

$$+\partial^\alpha[z_i \operatorname{div}\tilde{u}_i + \tilde{u}_i\nabla z_i + w_i\nabla\tilde{n}_i + w_i\nabla z_i]\Big\}, \quad i=1,2. \tag{4.32}$$

So, utilizing (4.32), we have

$$\begin{aligned}
&-\sum_{i=1}^{2}\sum_{|\alpha|=1}\int_{\mathbb{R}^3}\theta_{i0}\partial^\alpha z_i\partial^\alpha \operatorname{div}w_i dx\\
&=\sum_{|\alpha|=1}\int_{\mathbb{R}^3}\frac{\theta_{i0}}{z_i+\tilde{n}_i}\partial^\alpha z_i\Bigg\{\partial^\alpha z_{it}+\sum_{|\beta|=1,\beta\le\alpha}^{|\alpha|}C_\beta\partial^\beta\tilde{n}_i\partial^{\alpha-\beta}\operatorname{div}w_i dx\\
&\quad+\sum_{|\beta|=1,\beta\le\alpha}^{|\alpha|}C_\beta\partial^\beta z_i\partial^{\alpha-\beta}\operatorname{div}w_i+\partial^\alpha\Big[z_i \operatorname{div}\tilde{u}_i+\tilde{u}_i\nabla z_i+w_i\nabla\tilde{n}_i+w_i\nabla z_i\Big]\Bigg\}dx\\
&\ge -C(\delta^{\frac12}+\varepsilon)\|(\nabla w_1,\nabla w_2)(t)\|^2-C(\delta^{\frac12}+\varepsilon)(1+t)^{-1}\|(\nabla z_1,\nabla z_2,w_1,w_2)(t)\|^2\\
&\quad-C\delta^{\frac12}(1+t)^{-4}-C\delta^{\frac12}(1+t)^{-1}B(t)+\frac{d}{dt}H_3(t).
\end{aligned}\tag{4.33}$$

Here, we used the following estimates to complete (4.33):

$$\sum_{i=1}^{2}\sum_{|\alpha|=1}\int_{\mathbb{R}^3}\frac{\theta_{i0}}{z_i+\tilde{n}_i}\partial^\alpha z_i\partial^\alpha z_{it}\ge H_{3t}-C(\delta^{\frac12}+\varepsilon)(1+t)^{-1}\|(\nabla z_1,\nabla z_2)(t)\|^2, \tag{4.34}$$

and

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{|\alpha|=1}\int_{\mathbb{R}^3}\frac{\theta_{i0}}{z_i+\tilde{n}_i}\partial^\alpha z_i\sum_{|\beta|=1,\beta\le\alpha}^{1}C_\beta\partial^\beta\tilde{n}_i\partial^{\alpha-\beta}\operatorname{div}w_i dx\\
&\quad\ge -C(\delta^{\frac12}+\varepsilon)(1+t)^{-1}\|(\nabla z_1,\nabla z_2)(t)\|^2-C(\delta^{\frac12}+\varepsilon)\|(\nabla w_1,\nabla w_2)\|^2,
\end{aligned}\tag{4.35}$$

and

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{|\alpha|=1}\int_{\mathbb{R}^3}\frac{\theta_{i0}}{z_i+\tilde{n}_i}\partial^\alpha z_i\sum_{|\beta|=1,\beta\le\alpha}^{1}C_\beta\partial^\beta z_i\partial^{\alpha-\beta}\operatorname{div}w_i dx\\
&\quad\ge -C(\delta^{\frac12}+\varepsilon)(1+t)^{-1}\|(\nabla z_1,\nabla z_2)(t)\|^2-C(\delta^{\frac12}+\varepsilon)\|(\nabla w_1,\nabla w_2)\|^2,
\end{aligned}\tag{4.36}$$

and

$$\begin{aligned}
&\sum_{i=1}^{2}\sum_{|\alpha|=1}\int_{\mathbb{R}^3}\frac{\theta_{i0}}{z_i+\tilde{n}_i}\partial^\alpha z_i\partial^\alpha\Big[z \operatorname{div}\bar{u}+\bar{u}\nabla z+w\nabla\bar{\rho}+w\nabla z\Big]dx\\
&\quad\ge -C(\delta^{\frac12}+\varepsilon)\|(\nabla w_1,\nabla w_2)(t)\|^2-C(\delta^{\frac12}+\varepsilon)(1+t)^{-1}\|(\nabla z_1,\nabla z_2,w_1,w_2)(t)\|^2\\
&\quad-C\delta^{\frac12}(1+t)^{-4}-C\delta^{\frac12}(1+t)^{-1}B(t).
\end{aligned}\tag{4.37}$$

Applying (4.33) to (4.31), we obtain

$$\sum_{|\alpha|=1}\int_{\mathbb{R}^3}[\partial_x^\alpha(\theta_{10}\nabla z_1)\partial_x^\alpha w_1+\partial_x^\alpha(\theta_{20}\nabla z_2)\partial_x^\alpha w_2]dx$$

$$
\begin{aligned}
&\geq -C(\delta^{\frac{1}{2}}+\varepsilon)\|(\nabla w_1,\nabla w_2)(t)\|^2 - C(\delta^{\frac{1}{2}}+\varepsilon)(1+t)^{-1}\|(\nabla z_1,\nabla z_2,w_1,w_2)(t)\|^2 \\
&- C\delta^{\frac{1}{2}}(1+t)^{-4} - C\delta^{\frac{1}{2}}(1+t)^{-1}B(t) + \frac{d}{dt}H_3(t).
\end{aligned}\tag{4.38}
$$

Similarly, we can estimate the last term of (4.27) as

$$
\begin{aligned}
&-\sum_{|\alpha|=1}\int_{\mathbb{R}^3}[\partial_x^\alpha(L_1+N_1)\partial_x^\alpha w_1 + \partial_x^\alpha(L_2+N_2)\partial_x^\alpha w_2]dx \\
&\quad\leq C(\delta^{\frac{1}{2}}+\varepsilon)\|(\nabla w_1,\nabla w_2)(t)\|^2 + C(\delta^{\frac{1}{2}}+\varepsilon)(1+t)^{-1}\|(\nabla z_1,\nabla z_2,w_1,w_2)(t)\|^2 \\
&\quad + C\delta^{\frac{1}{2}}(1+t)^{-4} + C\delta^{\frac{1}{2}}(1+t)^{-1}B(t) - \frac{d}{dt}F_3(t).
\end{aligned}\tag{4.39}
$$

Let $M_3(t) = H_3(t) + F_3(t) + \frac{1}{2}\|(\nabla w_1,\nabla w_2)(t)\|^2$. Substituting (4.28)–(4.39) into (4.27) and noticing the smallness of $\varepsilon$ and $\delta$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}M_3(t) + \frac{4}{5}\|(\nabla w_1,\nabla w_2)(t)\|^2 \\
&\leq C\|\nabla^2\mathcal{E}(t)\|^2 + C(\delta^{\frac{1}{2}}+\varepsilon)(1+t)^{-1}\|(\nabla z_1,\nabla z_2,w_1,w_2)(t)\|^2 \\
&\quad + C\delta^{\frac{1}{2}}(1+t)^{-4} + C\delta^{\frac{1}{2}}(1+t)^{-1}B(t).
\end{aligned}\tag{4.40}
$$

Integrating (4.40) with respect to $\tau$ over $[0,t]$ and using Lemma 4.4 and the smallness of $\delta$ and $\varepsilon$, we then prove Lemma 4.6. ▯

By a tedious computation to $\int_{\mathbb{R}^3}[(4.5)_2\cdot\nabla z_1 + (4.5)_4\cdot\nabla z_2]dx$ and applying Lemma 4.4, we can prove the following lemma.

LEMMA 4.7. *It holds that*

$$
\int_0^t\|\nabla z_i(\tau)\|^2 d\tau \leq C(\delta^{\frac{1}{2}}+\eta^2) + C\|(w_i,\nabla z_i)(t)\|^2 + C\int_0^t\|w_i(\tau)\|_1^2 d\tau \tag{4.41}
$$

*for $i=1,2$, provided $\delta+\varepsilon\ll 1$.*

LEMMA 4.8. *It holds that*

$$
\begin{aligned}
&\|(\nabla\mathcal{E},\nabla\mathcal{E}_t,\nabla^2\mathcal{E})(t)\|^2 + \int_0^t\|(\nabla\mathcal{E},\nabla\mathcal{E}_t,\nabla^2\mathcal{E})(\tau)\|^2 d\tau \\
&\leq C(\delta^{\frac{1}{2}}+\eta^2) + C(\delta^{\frac{1}{2}}+\varepsilon)\int_0^t[\|(w_1,w_2)(\tau)\|_1^2 + \|(\nabla z_1,\nabla z_2)(\tau)\|^2]d\tau
\end{aligned}\tag{4.42}
$$

*provided $\delta+\varepsilon\ll 1$.*

*Proof.* By a tedious computation to

$$
\begin{aligned}
&\int_{\mathbb{R}^3}[(\tilde{n}_2+z_2)\times(4.5)_4 - (\tilde{n}_1+z_1)\times(4.5)_2]\nabla\mathcal{E}dx \\
&+4\int_{\mathbb{R}^3}[(\tilde{n}_2+z_2)\times(4.5)_4 - (\tilde{n}_1+z_1)\times(4.5)_2]\nabla\mathcal{E}_t dx,
\end{aligned}
$$

we can then obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\nabla\mathcal{E}_t\nabla\mathcal{E} + 2\Big(\tilde{n}_1+\tilde{n}_1+z_1+z_2+\frac{1}{4}\Big)|\nabla\mathcal{E}|^2 \\
&\quad + 2(\theta_{20}+\theta_2)(\tilde{n}_2+z_2)|\nabla^2\mathcal{E}|^2 + 2|\nabla\mathcal{E}_t|^2\Big)dx
\end{aligned}
$$

$$+\tilde{C}_0\|(\nabla\mathcal{E}_t,\nabla\mathcal{E},\nabla^2\mathcal{E})(t)\|^2\le Q_1 \tag{4.43}$$

for some constant $\tilde{C}_0>0$, where

$$\begin{aligned}Q_1&=C(\delta^{\frac12}+\varepsilon)(1+t)^{-1}\|(\nabla w_1,\nabla w_2)(t)\|^2\\&\quad+C(\delta^{\frac12}+\varepsilon)(1+t)^{-2}\|(\nabla z_1,\nabla z_2,w_1,w_2)(t)\|^2\\&\quad+C\,\delta^{\frac12}(1+t)^{-5}+C\delta^{\frac12}(1+t)^{-2}B(t).\end{aligned}\tag{4.44}$$

Integrating (4.43) with respect to $t$ over $[0,t]$, and applying the Cauchy inequality and Lemma 4.4 as well as noticing the smallness of $\delta$ and $\varepsilon$, we finally obtain (4.42). $\square$

Combining Lemmas 4.4–4.8, we obtain the following result.

LEMMA 4.9. *It holds that*

$$\begin{aligned}&\|(z_1,z_2,w_1,w_2,\nabla\mathcal{E})(t)\|_1^2+\|\nabla\mathcal{E}_t(t)\|^2\\&+\int_0^t[\|(w_1,w_2,\nabla\mathcal{E})(\tau)\|_1^2+\|(\nabla z_1,\nabla z_2,\nabla\mathcal{E}_\tau)(\tau)\|^2]d\tau\le C(\delta^{\frac12}+\eta^2)\end{aligned}\tag{4.45}$$

*provided* $\varepsilon+\delta\ll1$.

*Remark* 4. From Lemma 4.7, it is realized that

$$\int_0^t B(\tau)d\tau\le C(\delta^{\frac12}+\eta^2).\tag{4.46}$$

LEMMA 4.10. *There holds*

$$\begin{aligned}&(1+t)\|(\nabla z_i,\nabla w_i,z_{it})(t)\|^2+\int_0^t[(1+\tau)\|(\nabla w_i,z_{i\tau})(\tau)\|^2+\|\nabla z_i(\tau)\|^2]d\tau\\&\le C(\delta^{\frac12}+\eta^2),\end{aligned}\tag{4.47}$$

$$\begin{aligned}&(1+t)^2\|(\nabla\mathcal{E},\nabla\mathcal{E}_t,\nabla^2\mathcal{E})(t)\|^2+\int_0^t(1+\tau)^2\|(\nabla\mathcal{E},\nabla\mathcal{E}_\tau,\nabla^2\mathcal{E})(\tau)\|^2d\tau\\&\le C(\delta^{\frac12}+\eta^2)\end{aligned}\tag{4.48}$$

*for* $i=1,2$, *provided* $\delta+\varepsilon\ll1$.

*Proof.* Integrating $(1+\tau)\times(4.43)$ with respect to $\tau$ over $[0,t]$ and applying Lemma 4.9 and the Cauchy inequality, we obtain

$$(1+t)\|(\nabla\mathcal{E},\nabla\mathcal{E}_t,\nabla^2\mathcal{E})(t)\|^2+\int_0^t(1+\tau)\|(\nabla\mathcal{E},\nabla\mathcal{E}_\tau,\nabla^2\mathcal{E})(\tau)\|^2d\tau\le C(\delta^{\frac12}+\eta^2).\tag{4.49}$$

Again, integrating $(1+\tau)\times(4.40)$ with respect to $\tau$ over $[0,t]$ and using Lemma 4.9 and (4.49), we obtain

$$\begin{aligned}&(1+t)\|(\nabla z_1,\nabla z_2,\nabla w_1,\nabla w_1)(t)\|^2+\int_0^t(1+\tau)\|(\nabla w_1,\nabla w_2)(\tau)\|^2d\tau\\&\le C(\delta^{\frac12}+\eta^2).\end{aligned}\tag{4.50}$$

Thus, we can obtain

$$(1+t)\|(z_{1t},z_{1t})(t)\|^2+\int_0^t(1+\tau)\|(z_{1\tau},z_{1\tau})(\tau)\|^2d\tau\le C(\delta^{\frac12}+\eta^2).\tag{4.51}$$

Thus, (4.45), (4.50), and (4.51) imply (4.47).

Finally, integrating $(1+\tau)^2 \times (4.43)$ with respect to $\tau$ over $[0,t]$ and applying (4.47), (4.49), and the Cauchy inequality, we then prove (4.48). $\square$

As shown in Lemmas 4.6–4.10, by taking $\int_{\mathbb{R}^3}[\partial_x^\alpha(4.5)_2 \times \partial_x^\alpha w_1 + \partial_x^\alpha(4.5)_4 \times \partial_x^\alpha w_2]dx$ for multi-index $\alpha$ with $|\alpha|=2$, and $\int_{\mathbb{R}^3}[\partial_x^\beta(4.5)_2 \times \nabla\partial_x^\beta z_1 + \partial_x^\beta(4.5)_4 \times \nabla\partial_x^\beta z_1]dx$, as well as $\int_{\mathbb{R}^3}\partial_x^\beta((\tilde{n}_2+z_2)\times(4.5)_4 - (\tilde{n}_1+z_1)\times(4.5)_2)(\partial_x^\beta\nabla\mathcal{E} + 4\partial_x^\beta\nabla\mathcal{E}_t)dx$ for multi-index $\beta$ with $|\beta|=1$, and applying the above lemmas, we can similarly prove the following lemma.

Lemma 4.11. *There hold*

$$
\begin{aligned}
&(1+t)^2\|(\nabla^2 z_i, \nabla^2 w_i, \nabla z_{it})(t)\|^2 \\
&\quad + \int_0^t [(1+\tau)^2\|(\nabla^2 w_i, \nabla z_{i\tau})(\tau)\|^2 + (1+\tau)\|\nabla^2 z_i(\tau)\|^2]d\tau \\
&\leq C(\delta^{\frac{1}{2}} + \eta^2), \qquad (4.52)\\
&(1+t)^3\|(\nabla^2\mathcal{E}, \nabla^2\mathcal{E}_t, \nabla^3\mathcal{E})(t)\|^2 + \int_0^t (1+\tau)^3\|(\nabla^2\mathcal{E}, \nabla^2\mathcal{E}_\tau, \nabla^3\mathcal{E})(\tau)\|^2 d\tau \\
&\leq C(\delta^{\frac{1}{2}} + \eta^2) \qquad (4.53)
\end{aligned}
$$

*for $i=1,2$, provided $\delta+\varepsilon \ll 1$.*

Similarly, by taking $\int_{\mathbb{R}^3}[\partial_x^\alpha(4.5)_2 \times \partial_x^\alpha w_1 + \partial_x^\alpha(4.5)_4 \times \partial_x^\alpha w_2]dx$ for multi-index $\alpha$ with $|\alpha|=3$, and $\int_{\mathbb{R}^3}[\partial_x^\beta(4.5)_2 \times \nabla\partial_x^\beta z_1 + \partial_x^\beta(4.5)_4 \times \nabla\partial_x^\beta z_1]dx$, as well as $\int_{\mathbb{R}^3}\partial_x^\beta((\tilde{n}_2+z_2)\times(4.5)_4 - (\tilde{n}_1+z_1)\times(4.5)_2)(\partial_x^\beta\nabla\mathcal{E} + 4\partial_x^\beta\nabla\mathcal{E}_t)dx$ for multi-index $\beta$ with $|\beta|=2$, and applying the above lemmas, we can prove the following lemma.

Lemma 4.12. *There hold*

$$
\begin{aligned}
&(1+t)^3\sum_{i=1}^2\|(\nabla^3 z_i, \nabla^3 w_i, \nabla^2 z_{it})(t)\|^2 \\
&\quad + \sum_{i=1}^2\int_0^t [(1+\tau)^3\|(\nabla^3 w_i, \nabla^2 z_{i\tau})(\tau)\|^2 + (1+\tau)^2\|\nabla^3 z_i(\tau)\|^2]d\tau \\
&\leq C(\delta^{\frac{1}{2}} + \eta^2), \qquad (4.54)\\
&(1+t)^4\|(\nabla^3\mathcal{E}, \nabla^3\mathcal{E}_t, \nabla^4\mathcal{E})(t)\|^2 + \int_0^t (1+\tau)^4\|(\nabla^3\mathcal{E}, \nabla^3\mathcal{E}_\tau, \nabla^4\mathcal{E})(\tau)\|^2 d\tau \\
&\leq C(\delta^{\frac{1}{2}} + \eta^2) \qquad (4.55)
\end{aligned}
$$

*provided $\delta+\varepsilon \ll 1$.*

Similarly, by taking $\int_{\mathbb{R}^3}[\partial_x^\alpha(4.5)_2 \times \partial_x^\alpha w_1 + \partial_x^\alpha(4.5)_4 \times \partial_x^\alpha w_2]dx$ for multi-index $\alpha$ with $|\alpha|=4$, and $\int_{\mathbb{R}^3}[\partial_x^\beta(4.5)_2 \times \nabla\partial_x^\beta z_1 + \partial_x^\beta(4.5)_4 \times \nabla\partial_x^\beta z_1]dx$ for multi-index $\beta$ with $|\beta|=3$, and $\int_{\mathbb{R}^3}\partial_x^\beta((\tilde{n}_2+z_2)\times(4.5)_4 - (\tilde{n}_1+z_1)\times(4.5)_2)(\partial_x^\beta\nabla\mathcal{E} + 4\partial_x^\beta\nabla\mathcal{E}_t)dx$, and applying the above lemmas, we can further prove the following lemma.

Lemma 4.13. *There hold*

$$
\begin{aligned}
&(1+t)^4\sum_{i=1}^2\|(\nabla^4 z_i, \nabla^4 w_i, \nabla^3 z_{it})(t)\|^2 \\
&+ \sum_{i=1}^2\int_0^t (1+\tau)^4\|(\nabla^4 w_i, \nabla^3 z_{i\tau})(\tau)\|^2 + (1+\tau)^3\|\nabla^4 z_i(\tau)\|^2 d\tau
\end{aligned}
$$

$$\leq C(\delta^{\frac{1}{2}}+\eta^2), \tag{4.56}$$

$$(1+t)^5\|(\nabla^4\mathcal{E},\nabla^4\mathcal{E}_t,\nabla^5\mathcal{E})\|^2+\int_0^t(1+\tau)^5\|(\nabla^4\mathcal{E},\nabla^4\mathcal{E}_\tau,\nabla^5\mathcal{E})(\tau)\|^2d\tau$$
$$\leq C(\delta^{\frac{1}{2}}+\eta^2) \tag{4.57}$$

*provided* $\delta+\varepsilon\ll 1$.

*Proof of Theorem* 4.1. In view of Lemma 4.9–4.13, we have immediately proved Theorem 4.1. ▯

**4.3. More energy estimates.** In this subsection, we are going to prove Theorem 4.2 and Corollary 4.3. From the Gagliardo–Nirenberg inequality and the a priori estimates, we have

$$\begin{cases}\|(z_1,z_2,w_1,w_2)(t)\|_{L^\infty(\mathbb{R}^3)}\leq C(\delta^{\frac{1}{4}}+\eta)(1+t)^{-\frac{3}{4}},\\ \|(\nabla z_1,\nabla z_2,\nabla w_1,\nabla w_2)(t)\|_{L^\infty(\mathbb{R}^3)}\leq C(\delta^{\frac{1}{4}}+\eta)(1+t)^{-\frac{5}{4}},\\ \|\nabla\mathcal{E}(t)\|_{L^\infty(\mathbb{R}^3)}\leq C(\delta^{\frac{1}{4}}+\eta)(1+t)^{-\frac{7}{4}}.\end{cases} \tag{4.58}$$

But we may expect to have the high order decay rates for $w_1$ and $w_2$ because of the effect of damping.

COROLLARY 4.14. *Under the conditions of Theorem* 4.1, *it holds that*

$$(1+t)^2\|(w_{1t},w_{2t})(t)\|^2+\int_0^t(1+\tau)^2\|(w_{1\tau},w_{2\tau})(\tau)\|^2\leq C(\delta^{\frac{1}{2}}+\eta^2). \tag{4.59}$$

*Proof.* Differentiating $(4.5)_2$ and $(4.5)_4$ with respect to $t$, we obtain

$$\begin{cases}w_{1tt}+w_{1t}+\theta_{10t}\nabla z_1+\theta_{10}\nabla z_{1t}-\nabla\mathcal{E}_t=-L_{1t}-N_{1t},\\ w_{2tt}+w_{2t}+\theta_{20t}\nabla z_2+\theta_{20}\nabla z_{2t}+\nabla\mathcal{E}_t=-L_{2t}-N_{2t}.\end{cases} \tag{4.60}$$

Integrating $(4.60)_1\times w_{1t}+(4.60)_2\times w_{2t}$ with respect to $x$ over $\mathbb{R}^3$ and using the Cauchy inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(w_{1t},w_{2t})(t)\|^2+\frac{3}{4}\|(w_{1t},w_{2t})(t)\|^2\\ &\leq C\|(\nabla z_{1t},\nabla z_{2t},\nabla\mathcal{E}_t)(t)\|^2+C(1+t)^{-1}\|(z_{1t},z_{2t},\nabla w_1,\nabla w_2)(t)\|^2\\ &\quad+C(1+t)^{-2}\|(w_1,w_2,\nabla z_1,\nabla z_2)(t)\|^2+C(1+t)^{-1}\|(z_{1t},z_{2t})(t)\|^2\\ &\quad+C\delta^{\frac{1}{2}}(1+t)^{-5}+C\delta^{\frac{1}{2}}(1+t)^{-2}B(t).\end{aligned} \tag{4.61}$$

Applying a standard argument as in Lemma 4.10 to (4.61), we complete the proof of Corollary 4.14. ▯

Similarly, by taking $\int_0^t(1+\tau)^k\int_{\mathbb{R}^3}[\partial_x^\gamma(4.60)_1\times\partial_x^\gamma w_{1t}+\partial_x^\gamma(4.60)_2\times\partial_x^\gamma w_{2t}]dxd\tau$ with $k=0,1,2,3$, respectively, where $\gamma$ is a multi-index with $|\gamma|=1$, and applying Corollary 4.14, we have the following corollary.

COROLLARY 4.15. *Under the conditions of Theorem* 4.1, *it holds, for* $i=1,2$, *that*

$$(1+t)^3\|\nabla w_{it}(t)\|^2+\int_0^t(1+\tau)^3\|\nabla w_{i\tau}(\tau)\|^2\leq C(\delta^{\frac{1}{2}}+\eta^2). \tag{4.62}$$

Furthermore, by taking $\int_0^t (1+\tau)^k \int_{\mathbb{R}^3} [\partial_x^\gamma (4.60)_1 \times \partial_x^\gamma w_{1t} + \partial_x^\gamma (4.60)_2 \times \partial_x^\gamma w_{2t}] dx d\tau$ with $k = 0, 1, 2, 3, 4$, respectively, where $\gamma$ is a multi-index with $|\gamma| = 2$, and applying Corollary 4.15, we have the following corollary.

COROLLARY 4.16. *Under the conditions of Theorem* 4.1*, it holds, for* $i = 1, 2$, *that*

$$(1+t)^4 \|\nabla^2 w_{it}(t)\|^2 + \int_0^t (1+\tau)^4 \|\nabla^2 w_{i\tau}(\tau)\|^2 d\tau \le C(\delta^{\frac{1}{2}} + \eta^2). \tag{4.63}$$

Finally, we prove the decay rates for the solution in $L^\infty$ space.

COROLLARY 4.17. *Under the conditions of Theorem* 4.1*, it holds that*

$$\begin{cases} \|(z_1, z_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{3}{4}}, \\ \|(z_1 - z_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{9}{4}}, \\ \|(w_1, w_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{5}{4}}, \\ \|\nabla \mathcal{E}(t)\|_{L^\infty} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{7}{4}}. \end{cases} \tag{4.64}$$

*Proof.* From the Gagliardo–Nirenberg inequality, we have

$$\begin{cases} \|(z_1, z_2, w_1, w_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{3}{4}}, \\ \|(\nabla z_1, \nabla z_2, \nabla w_1, \nabla w_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{5}{4}}, \\ \|(z_1 - z_2)(t)\|_{L^\infty(\mathbb{R}^3)} = \|\nabla \mathcal{E}\|_{L^\infty} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{9}{4}}, \\ \|\nabla \mathcal{E}(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{7}{4}}. \end{cases} \tag{4.65}$$

We notice that, for $w_1$ and $w_2$, the decay rates shown in the above are not enough. In fact, we can estimate, for $i = 1, 2$,

$$\|w_{it}(t)\|_{L^\infty(\mathbb{R}^3)} \le C \|w_{it}(t)\|^{\frac{1}{4}} \|\nabla^2 w_{it}(t)\|^{\frac{3}{4}} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{7}{4}}.$$

Using $(4.5)_2$ and $(4.5)_4$, we then obtain

$$\|(w_1, w_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{5}{4}}.$$

Thus, the proof for Corollary 4.17 is complete. ∎

Next, from Lemma 2.1 and Corollary 4.17, we directly obtain the following result.

COROLLARY 4.18. *Under the conditions of Theorem* 4.1*, it holds that*

$$\begin{cases} \|(n_1 - \bar{n}_1, n_2 - \bar{n}_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{3}{4}}, \\ \|(n_1 - n_2)(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{9}{4}}, \\ \|(u_1 - \bar{U}, u_2 - \bar{U})(t)\|_{L^\infty(\mathbb{R}^3)} \le C(\delta^{\frac{1}{4}} + \eta)(1+t)^{-\frac{5}{4}}. \end{cases} \tag{4.66}$$

*Proof of Theorem* 4.2. From Corollaries 4.17 and 4.18, we have immediately proved Theorem 4.2 as well as Corollary 4.3. ∎

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REFERENCES

[1] G.-Q Chen and D. Wang, *Convergence of shock schemes for the compressible Euler-Poisson equations*, Comm. Math. Phys., 179 (1996), pp. 333–364.

[2] P. Degond and P. A. Markowich, *On a one-dimensional steady-state hydrodynamic model*, Appl. Math. Lett., 3 (1990), pp. 25–29.

[3] C. T. Duyn and L. A. Van Peletier, *A class of similarity solutions of the nonlinear diffusion equation*, Nonlinear Anal., 1 (1977), pp. 223–233.

[4] W. Fang and K. Ito, *Steady-state solutions of a one-dimensional hydrodynamic model for semiconductors*, J. Differential Equations, 133 (1997), pp. 224–244.

[5] I. M. Gamba, *Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductor*, Comm. Partial Differential Equations, 17 (1992), pp. 553–577.

[6] I. Gasser, L. Hsiao, and H.-L. Li, *Large time behavior of solutions of the bipolar hydrodynamic model for semiconductors*, J. Differential Equations, 192 (2003), pp. 326–359.

[7] I. Gasser and R. Natalini, *The energy transport and the drift diffusion equations as relaxation limits of the hydrodynamic model for semiconductors*, Quart. Appl. Math., 57 (1996), pp. 269–282.

[8] Y. Guo and W. Strauss, *Stability of semiconductor states with insulating and contact boundary conditions*, Arch. Ration. Mech. Anal., 179 (2006), pp. 1–30.

[9] L. Hsiao and T.-P. Liu, *Convergence to nonlinear diffusive waves for solutions of a system of hyperbolic conservation laws with damping*, Comm. Math. Phys., 143 (1992), pp. 599–605.

[10] L. Hsiao, P. A. Markowich, and S. Wang, *The asymptotic behavior of globally smooth solutions of the multidimensional isentropic hydrodynamic model for semiconductors*, J. Differential Equations, 192 (2003), pp. 111–133.

[11] L. Hsiao and K. Zhang, *The global weak solution and relaxation limits of the initial-boundary value problem to the bipolar hydrodynamic model for semiconductors*, Math. Models Methods Appl. Sci., 10 (2000), pp. 1333–1361.

[12] L. Hsiao and K. Zhang, *The relaxation of the hydrodynamic model for semiconductors to drift diffusion equations*, J. Differential Equations, 165 (2000), pp. 315–354.

[13] F.-M. Huang, J. Li, and A. Matsumura, *Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system*, Arch. Ration. Mech. Anal., 197 (2010), pp. 89–116.

[14] F.-M. Huang and Y.-P. Li, *Large time behavior and quasineutral limit of solutions to a bipolar hydrodynamic model with large data and vacuum*, Discrete Contin. Dyn. Syst., 24 (2009), pp. 455–470.

[15] F.-M. Huang, M. Mei, Y. Wang, and H.-M. Yu, *Asymptotic convergence to stationary waves for unipolar hydrodynamic model of semiconductors*, SIAM J. Math. Anal., 43 (2011), pp. 411–429.

[16] A. Jüngel, *Quasi-hydrodynamic Semiconductor Equations*, Progr. Nonlinear Differential Equations Appl. 41, Birkhäuser Verlag, Basel, 2001.

[17] T. Kato, *The Cauchy problem for quasilinear symmetric hyperbolic systems*, Arch. Rational Mech. Anal., 58 (1975), pp. 181–205.

[18] H.-L. Li, P. Markowich, and M. Mei, *Asymptotic behavior of solutions of the hydrodynamic model of semiconductors*, Proc. Royal Soc. Edinburgh Sect. A, 132 (2002), pp. 359–378.

[19] H.-L. Li, P. Markowich, and M. Mei, *Asymptotic behavior of subsonic shock solutions of the isentropic Euler-Poisson equations*, Quart. Appl. Math., 60 (2002), pp. 773–796.

[20] Y.-P. Li, *Global existence and asymptotic behavior for a multidimensional nonisentropic hydrodynamic semiconductor model with the heat source*, J. Differential Equations, 225 (2006), pp. 134–167.

[21] Y.-P. Li, *Diffusion relaxation limit of a nonisentropic hydrodynamic model for semiconductors*, Math. Methods Appl. Sci., 30 (2007), pp. 2247–2261.

[22] T. Luo, R. Natalini, and Z. Xin, *Large time behavior of the solutions to a hydrodynamic model for semiconductors*, SIAM J. Appl. Math., 59 (1998), pp. 810–830.

[23] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York, 1984.

[24] P. Marcati, M. Mei, and B. Rubino, *Optimal convergence rates to diffusion waves for solutions of the hyperbolic conservation laws with damping*, J. Math. Fluid Mech., 7 (2005), pp. S224–S240.

[25] P. Marcati and R. Natalini, *Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation*, Arch. Rat. Mech. Anal., 129 (1995), pp. 129–145.

[26] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, Berlin, 1990.
[27] A. Matsumura, *Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equation with the first-order dissipation*, Publ. Res. Inst. Math. Sci., 13 (1977), pp. 349–379.
[28] A. Matsumura and M. Mei, *Convergence to travelling fronts of solutions of the p-system with viscosity in the presence of a boundary*, Arch. Ration. Mech. Anal., 146 (1999), pp. 1–22.
[29] M. Mei, *Best asymptotic profile for hyperbolic p-system with damping*, SIAM J. Math. Anal., 42 (2010), pp. 1–23.
[30] M. Mei, *Nonlinear diffusion waves for hyperbolic p-system with nonlinear damping*, J. Differential Equations, 247 (2009), pp. 1275–1269.
[31] R. Natalini, *The bipolar hydrodynamic model for semiconductors and the drift-diffusion equations*, J. Math. Anal. Appl., 198 (1996), pp. 262–281.
[32] S. Nishibata and M. Suzuki, *Asymptotic stability of a stationary solution to a thermal hydrodynamic model for semiconductors*, Arch. Ration. Mech. Anal., 192 (2009), pp. 187–215.
[33] K. Nishihara, *Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping*, J. Differential Equations, 131 (1996), pp. 171–188.
[34] K. Nishihara, *Asymptotics toward the diffusion wave for a one-dimensional compressible flow through porous media*, Proc. Royal Soc. Edinburgh Sect. A, 133 (2003), pp. 177–196.
[35] K. Nishihara, W. Wang, and T. Yang, *$L_p$-convergence rate to nonlinear diffusion waves for p-system with damping*, J. Differential Equations, 161 (2000), pp. 191–218.
[36] F. Poupaud, M. Rascle, and J.-P. Vila, *Global solutions to the isothermal Euler-Poisson system with arbitrarily large data*, J. Differential Equations, 123 (1995), pp. 93–121.
[37] A. Sitenko and V. Malnev, *Plasma Physics Theory*, Appl. Math. Math. Comput. 10, Chapman & Hall, London, 1995.
[38] W. Wang and T. Yang, *Pointwise estimates and $L_p$ convergence rates to diffusion waves for p-system with damping*, J. Differential Equations, 187 (2003), pp. 310–336.
[39] B. Zhang, *Convergence of the Godunov scheme for a simplified one-dimensional hydrodynamic model for semiconductor devices*, Comm. Math. Phys., 157 (1993), pp. 1–22.
[40] H.-J. Zhao, *Convergence to strong nonlinear diffusion waves for solutions of p-system with damping*, J. Differential Equations, 174 (2001), pp. 200–236.
[41] C. Zhu and H. Hattori, *Stability of steady state solutions for an isentropic hydrodynamic model of semiconductors of two species*, J. Differential Equations, 166 (2000), pp. 1–32.