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ASYMPTOTIC CONVERGENCE TO STATIONARY WAVES FOR UNIPOLAR HYDRODYNAMIC MODEL OF SEMICONDUCTORS*

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Abstract. In this paper, we study the one-dimensional unipolar hydrodynamic model for semiconductors in the form of Euler–Poisson equations. In the case when the state constants on the current density and the electric field are nonzero (switch-on case), the stability of stationary waves of one-dimensional isentropic Euler–Poisson equations for the unipolar hydrodynamic model has been open. In order to overcome this difficulty, we first analyze the behaviors of the solutions at $x = \pm\infty$, and observe what are the exact gaps between the original solutions and the stationary solutions in L^2-space; then we technically construct some new correction functions to delete these gaps. Finally, based on the energy methods, we prove that the solutions of one-dimensional isentropic Euler–Poisson equations for the unipolar hydrodynamic model decay exponentially fast to the stationary solutions.

Key words. Euler–Poisson, unipolar hydrodynamic model, semiconductor, nonlinear damping, asymptotic behavior, convergence rates.

AMS subject classifications. 35L50, 35L60, 35L65, 76R50

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1. Introduction.

The dynamical phenomena of charged fluid particles, for example, the fluid dynamical system of electrons and holes in semiconductor devices or positively and negatively charged ions in a plasma, are usually described using the hydrodynamic models of Euler–Poisson equations. These models, which can be derived from kinetic models, take an important place in the fields of applied and computational mathematics. A standard approach for this derivation is the moment method. According to the different analysis for the phase space densities, introduced to prescribe the dependence on the velocity, we recover different limit models and, in particular, the drift-diffusion equations and the hydrodynamic (Euler–Poisson) systems. The hydrodynamic models are usually considered to describe high field phenomena of submicronic devices. For details on the applications in semiconductors and in plasma physics, see [16, 24, 30].

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In this paper, we study isentropic Euler–Poisson equations for the unipolar hydrodynamical model of a semiconductor device:

$$\begin{cases} n_t + J_x = 0, \\ J_t + (\frac{J^2}{n} + p(n))_x = nE - \frac{J}{\tau}, \\ E_x = n - b(x) \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}$ and $t > 0$. Here n, J, and E represent the electron density, the current electron densities, and the electric field, respectively. The coefficient τ denotes the relaxation time. Since our interest here is the long-time behavior of the solutions rather than the limit of relaxation times, without loss of generality, we assume throughout this paper that $\tau = 1$. The function $b(x)$ stands for the density of fixed, positively charged background ions, the so-called doping profile. $p(n)$ is the pressure-density relation satisfying

$$p'(n) > 0 \text{ and } n^2p'(n) \text{ is a strictly increasing function for } n > 0. \tag{1.2}$$

As explained in [17], condition (1.2) guarantees that system (1.1) is hyperbolic and fully subsonic under consideration. In this paper, we consider the Cauchy problem to system (1.1) with the initial conditions prescribed as

$$\begin{cases} n(x, 0) = n_0(x) > 0, \\ J(x, 0) = J_0(x), \end{cases} \tag{1.3}$$

where

$$\begin{cases} \lim_{x\to\pm\infty} n_0(x) = n_\pm, \\ \lim_{x\to\pm\infty} J_0(x) = J_\pm. \end{cases} \tag{1.4}$$

Furthermore, the "boundary" condition at far field $x = -\infty$ is also needed,

$$\lim_{x\to-\infty} E(x, t) = E_-, \tag{1.5}$$

where $n_\pm$, $J_\pm$, and E_- are given state constants. As we analyze below in section 3 for the behavior of the solutions $(n, J, E)(x, t)$ at the far fields $x = \pm\infty$, the boundary condition (1.5), or replaced by

$$\lim_{x\to+\infty} E(x, t) = E_+, \tag{1.6}$$

is necessary and natural. Otherwise, the far-field state functions $J(\pm\infty, t)$ and $E(\pm\infty, t)$ will be underdetermined, which will cause the system to be ill-posed. For details, we refer to section 3.

Integrating $(1.1)_3$ with respect to x over $(-\infty, x]$ and applying (1.5), we have

$$E(x, t) = E_- + \int_{-\infty}^{x} [n(y, t) - b(y)]dy.$$

So, the initial data $E_0(x)$ is given by

$$E_0(x) = E_- + \int_{-\infty}^{x} [n_0(y) - b(y)]dy. \tag{1.7}$$

The theoretical study and scientific computations on hydrodynamical systems of semiconductor devices has been one of the hot spots of research in mathematical physics. For the unipolar isentropic and nonisentropic hydrodynamical equations of semiconductors (one carrier type), Degond and Markowich [3, 4], Fang and Ito [5], and Gamba [6] investigated the existence and uniqueness of (subsonic) stationary solutions in the one-dimensional (1-D) case. Such stationary solutions are usually called the stationary waves to the original equations (1.1). Then Luo, Natalini, and Xin [21] proved that such stationary solutions for the Cauchy problem are time-asymptotically stable when the state constants of the current density are zero, i.e., $J_+ = J_- = E_- = 0$. They had required such a stiff condition due to a technical difficulty in reformulating the perturbed system in the L^2-sense. Later, Li, Markowich, and Mei [17] showed the stability of stationary solutions for the initial-boundary value problem within a bounded domain $[0, 1]$. Regarding relaxation limits, shock schemes, and entropy solutions, as well as the study in the multidimensional case, we refer the reader to the interesting works [1, 2, 6, 8, 11, 12, 13, 18, 19, 20, 22, 23, 29, 31] and the references therein. For the study on the bipolar hydrodynamic system of semiconductors, great progress has been made in [7, 9, 15, 14, 28] and the references therein.

Notice that when $J_- = J_+=0$, physically it stands for the switch-off case (no electric current). So, it is interesting but challenging to study the case $J_+ \neq J_-$ (the switch-on case) for the convergence to the stationary waves. As showed in [3, 17, 21], the stationary waves satisfy the corresponding steady-state equations

$$\begin{cases} \tilde{J} = \text{const}, \\ (\frac{\tilde{J}^2}{\tilde{n}} + p(\tilde{n}))_x = \tilde{n}\tilde{E} - \tilde{J}, \\ \tilde{E}_x = \tilde{n} - b(x), \end{cases}$$

with

$$(\tilde{n}, \tilde{J}, \tilde{E})(x) \to (n_\pm, \tilde{J}, \tilde{E}_\pm)$$

for $\tilde{J} = n_-E_-$, $\tilde{E}_- = E_-$, and $\tilde{E}_+ = \frac{n_-E_-}{n_+}$. However, when $J_- \neq J_+$, as analyzed in section 3, the original solutions at the far fields behave as

$$\begin{aligned} (n, J, E)|_{x=-\infty} &= (n_-, n_-E_- + O(1)e^{-\nu_0 t}, E_-), \\ (n, J, E)|_{x=+\infty} &= \left(n_+, n_-E_- + O(1)e^{-\nu_0 t}, \frac{n_-E_-}{n_+} + O(1)e^{-\nu_0 t}\right) \end{aligned}$$

for some $\nu_0 > 0$, which yields the gaps between the original solutions and the stationary waves, and causes the perturbations $J(x, t) - \tilde{J}(x)$ and $E(x, t) - \tilde{E}(x)$ not in $L^2(\mathbb{R})$. To delete these gaps, as Hsiao and Liu [10] showed, the correction functions need to be introduced. However, the technique for constructing the correction functions introduced first by Hsiao and Liu [10] cannot be applied to our case anymore due to the complexity and the nonlinearity of system (1.1). This causes the stability of the stationary waves to open for a long time. Inspired by our recent study on the bipolar hydrodynamical system of semiconductors [15], after carefully investigating the far-field states of the original solutions and understanding what the exact gaps will be between the original solutions and the stationary solutions in L^2-space, we then ingeniously construct the explicit correction functions in different cases due to the different eigenvalues. Then we can make a proper perturbation of the original solutions around the stationary waves, and by using the basic energy method, we can further prove the stability of the stationary waves with exponential decay rates. More precisely, when the perturbations around the stationary waves with suitable setup are small enough, we prove that the solutions of (1.1) converge exponentially to the corresponding stationary waves in the form

$$\begin{cases} \|(n-\tilde{n})(t)\|_{L^\infty} = O(1)e^{-\mu t} \\ \|(J-\tilde{J})(t)\|_{L^\infty} = O(1)e^{-\mu t} \\ \|(E-\tilde{E})(t)\|_{L^\infty} = O(1)e^{-\mu t} \end{cases} \quad \text{for some } \mu > 0. \tag{1.8}$$

The interesting thing is that the current density J converges to a constant state which is independent of the initial current densities, but which is determined by the initial end state of electron density and the electric field at $x = -\infty$. Obviously, the results presented in [21] are a special case of ours.

To begin with, in this paper we assume that

$$\begin{cases} b(x) \in C^3(\mathbb{R}),\ \lim_{x\to\pm\infty} b(x) = n_\pm, \\ \int_{-\infty}^0 |b(x)-n_-|^2 dx + \int_0^{+\infty} |b(x)-n_+|^2 dx \le C_0, \end{cases} \tag{1.9}$$

where C_0 is a positive constant.

The rest of this paper is arranged as follows. In section 2, we give some well-known results on the stationary solutions. In section 3, we reformulate the original system (1.1). First of all, we construct the correction functions to delete the gaps between the 1-D solutions of (1.1) and the corresponding stationary waves at the far field, then we reformulate the original system of equations to a new one. In section 4, the main effort is contributed to prove Theorem 3.2.

Notation. Through out this paper, C_0, C_i, etc., always denote some specific positive constants, and C denotes the generic positive constant. $L^2(\mathbb{R})$ is the space of square integrable real-valued functions defined on $\mathbb{R}$ with the norm $\|\cdot\|$, and $H^k(\mathbb{R})$ (H^k without any ambiguity) denotes the usual Sobolev space with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$.

2. Stationary waves. In this section, we are going to introduce the well-known results on the stationary solutions to the corresponding steady-state equation of (1.1), the so-called nonlinear stationary waves. For the unipolar hydrodynamical model of semiconductors (1.1), its corresponding 1-D steady equation is

$$\begin{cases} \tilde{J} = \text{const}, \\ (\frac{\tilde{J}^2}{\tilde{n}} + p(\tilde{n}))_x = \tilde{n}\tilde{E} - \tilde{J}, \\ \tilde{E}_x = \tilde{n} - b(x), \end{cases} \tag{2.1}$$

with

$$\lim_{x\to-\infty} (\tilde{n},\tilde{E})(x) = (n_-, E_-) \quad \text{and} \quad \lim_{x\to+\infty} \tilde{n}(x) = n_+. \tag{2.2}$$

Let

$$b_* = \inf_x b(x) > 0 \quad \text{and} \quad b^* = \sup_x b(x) > 0. \tag{2.3}$$

Then we state the existence and uniqueness of the stationary wave for the steady-state equations (2.1) and (2.2) as follows.

LEMMA 2.1 (see [17, 21]). *Assume that $b'(x) \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$ and $b_*\sqrt{p'(b_*)} > |n_-E_-|$. Then there exists a unique smooth solution $(\tilde{n}, \tilde{J}, \tilde{E})(x)$ of* (2.1) *and* (2.2), *which satisfies*

$$\tilde{J} = n_-E_-, \tag{2.4}$$

$$\tilde{E}(+\infty) = \frac{n_-E_-}{n_+}, \tag{2.5}$$

$$b_* \le \tilde{n} \le b^*, \tag{2.6}$$

$$|\tilde{n} - b(x)| = O(1)e^{-n_\pm |x|}, \quad \text{as } x \to \pm\infty, \tag{2.7}$$

$$\|\tilde{n} - b\|_{H^3}^2 \le \bar{C}_1(\alpha_1 + \alpha_2 + \alpha_3), \tag{2.8}$$

$$|\tilde{n}_x| + |\tilde{n}_{xx}| + |\tilde{n}_{xxx}| \le \bar{C}_2\alpha_4^{\frac{1}{2}}, \tag{2.9}$$

$$|\tilde{E}| \le \bar{C}_3(|E_-| + \alpha_4^{\frac{1}{2}}), \tag{2.10}$$

$$|\tilde{E}_x| + |\tilde{E}_{xx}| \le \bar{C}_4(\alpha_4 + b^* - b_*), \tag{2.11}$$

where $\bar{C}_i$ $(i = 1, 2, 3, 4)$ are some positive constants dependent on n_-, E_-, b_*, and b^*, and α_i $(i = 1, 2, 3, 4)$ are defined as follows:

$$\alpha_1 = \|b'\|_{L^2}^2 + \|b'\|_{L^1} + |\log n_+ - \log n_-|, \tag{2.12}$$

$$\alpha_2 = \alpha_1 + \alpha_1^3 + \|b''\|_{L^2}^2 + \|b'\|_{L^4}^4, \tag{2.13}$$

$$\alpha_3 = \alpha_2^3 + \alpha_1^2\alpha_2 + \|b'''\|_{L^2} + \|b''\|_{L^4}^4 + \|b'\|_{L^6}^6, \tag{2.14}$$

$$\begin{aligned}\alpha_4 = {}& \|b'\|_{L^\infty}^2 + \|b'\|_{L^\infty}^6 + \|b''\|_{L^\infty}^2 + \|b'''\|_{L^\infty}^2 + \|b'\|_{L^\infty}^2\|b''\|_{L^\infty}^2 \\ & + \|b''\|_{L^\infty}^2\alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + \|b'\|_{L^\infty}^2\alpha_3^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + \alpha_1\alpha_2 + \alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + \alpha_3^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} \\ & + \alpha_1^{\frac{3}{2}}\alpha_2^{\frac{3}{2}} + \alpha_1^{\frac{1}{2}}\alpha_2\alpha_3^{\frac{1}{2}}.\end{aligned} \tag{2.15}$$

3. Stability of stationary waves. First of all, as in [26, 27], let us look into the behaviors of the solutions to (1.1)–(1.5) at the far fields $x = \pm\infty$. Then we may understand how big the gaps are between the solutions and the stationary solutions at the far fields. Let

$$\begin{cases} n^\pm(t) := n(\pm\infty, t), \\ J^\pm(t) := J(\pm\infty, t), \\ E^\pm(t) := E(\pm\infty, t). \end{cases} \tag{3.1}$$

From $(1.1)_1$, since $\partial_x J|_{x=\pm\infty} = 0$, it can be easily seen that

$$n^\pm(t) = n(\pm\infty, t) \equiv n_\pm. \tag{3.2}$$

Taking $x \to \pm\infty$ to $(1.1)_2$, we get two ODEs:

$$\frac{d}{dt}J^\pm(t) = n_\pm E^\pm(t) - J^\pm(t). \tag{3.3}$$

Then, differentiating $(1.1)_3$ with respect to t and using $(1.1)_1$, we have

$$E_{xt} = (n - b(x))_t = n_t = -J_x.$$

Integrating it with respect to x over $(-\infty, +\infty)$, we then have

$$\frac{d}{dt}E^+(t) - \frac{d}{dt}E^-(t) = -J^+(t) + J^-(t). \tag{3.4}$$

It is noted that these three equations (3.3) and (3.4) will underdetermine four unknowns $J^{\pm}(t)$ and $E^{\pm}(t)$, causing system (1.1) to be ill-posed. Thus, naturally we need an extra boundary condition for $E(x,t)$ either at $x=-\infty$ or $x=+\infty$. This indicates that the boundary condition (1.5) (or replaced by (1.6)) is necessary and proper.

From (1.5), we have

$$E^{-}(t) = E_{-}. \tag{3.5}$$

Then, from $(1.1)_2$ and (3.5), we can easily get

$$J^{-}(t) = (J_{-} - n_{-}E_{-})e^{-t} + n_{-}E_{-}. \tag{3.6}$$

From $(1.1)_2$, we further have

$$\begin{cases} \frac{d}{dt}J^{+}(t) = n_{+}E^{+}(t) - J^{+}(t), \\ J^{+}(0) = J_{+}, \end{cases} \tag{3.7}$$

and from $(1.1)_3$ and (1.5), we obtain

$$E^{+}(t) = \lim_{x\to+\infty} E(x,t) = \int_{\mathbb{R}} (n(x,t) - b(x))dx + E_{-}. \tag{3.8}$$

Differentiating (3.8) with respect to t and using $(1.1)_1$ and (3.6), we obtain

$$\frac{d}{dt}E^{+}(t) = -\int_{\mathbb{R}} J_x(x,t)dx = -J^{+}(t) + (J_{-} - n_{-}E_{-})e^{-t} + n_{-}E_{-}. \tag{3.9}$$

From (3.8), we have

$$E^{+}(t)|_{t=0} = \int_{\mathbb{R}} (n_0 - b)(x)dx + E_{-} := E_{+}. \tag{3.10}$$

Combining (3.7), (3.9), and (3.10), we obtain

$$\begin{cases} \frac{d}{dt}J^{+}(t) = n_{+}E^{+}(t) - J^{+}(t), \\ \frac{d}{dt}E^{+}(t) = -J^{+}(t) + (J_{-} - n_{-}E_{-})e^{-t} + n_{-}E_{-}, \\ J^{+}(0) = J_{+}, \\ E^{+}(0) = E_{+}. \end{cases} \tag{3.11}$$

Differentiating $(3.11)_1$ with respect t, we obtain

$$\frac{d^2}{dt^2}J^{+}(t) = n_{+}\frac{d}{dt}E^{+}(t) - \frac{d}{dt}J^{+}(t), \tag{3.12}$$

and substituting $(3.11)_2$ into (3.12), then we can reach

$$\begin{cases} \frac{d^2}{dt^2}J^{+}(t) + \frac{d}{dt}J^{+}(t) + n_{+}J^{+}(t) = n_{+}n_{-}E_{-} + n_{+}(J_{-} - n_{-}E_{-})e^{-t}, \\ J^{+}(0) = J_{+}, \\ \frac{d}{dt}J^{+}(0) = n_{+}E_{+} - J_{+}. \end{cases} \tag{3.13}$$

Notice that the eigenvalues of the second order ODE of (3.13) are

$$\lambda_1 = \frac{-1-\sqrt{1-4n_{+}}}{2} \quad \text{and} \quad \lambda_2 = \frac{-1+\sqrt{1-4n_{+}}}{2}. \tag{3.14}$$

Thus, according to the signs of $1-4n_+$, we can directly but tediously solve (3.11) and (3.13) for $J^+(t)$ and $E^+(t)$ as follows.

Case 1. When $1-4n_+>0$, then

$$J^+(t)=A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t}+(J_- - n_-E_-)e^{-t}+n_-E_-, \tag{3.15}$$

$$E^+(t)=\frac{1}{n_+}\left[A_1(1+\lambda_1)e^{\lambda_1 t}+A_2(1+\lambda_2)e^{\lambda_2 t}+n_-E_-\right], \tag{3.16}$$

where

$$A_1=J_+-J_--A_2, \tag{3.17}$$

$$A_2=-\frac{1}{1-4n_+}\left[(1+\lambda_1)(J_+-J_-)-n_+E_++n_-E_-\right]. \tag{3.18}$$

Case 2. When $1-4n_+=0$, then

$$J^+(t)=A_3e^{-\frac{1}{2}t}+A_4te^{-\frac{1}{2}t}+(J_--n_-E_-)e^{-t}+n_-E_-, \tag{3.19}$$

$$E^+(t)=\frac{1}{n_+}\left[\left(A_4+\frac{1}{2}A_3\right)e^{-\frac{1}{2}t}+\frac{1}{2}A_4te^{-\frac{1}{2}t}+n_-E_-\right], \tag{3.20}$$

where

$$A_3=J_+-J_-, \tag{3.21}$$

$$A_4=n_+E_+-n_-E_--\frac{1}{2}(J_+-J_-). \tag{3.22}$$

Case 3. When $1-4n_+<0$, then

$$\begin{aligned} J^+(t)={}&\left(A_5\cos\left(\frac{\sqrt{4n_+-1}}{2}t\right)+A_6\sin\left(\frac{\sqrt{4n_+-1}}{2}t\right)\right)e^{-\frac{1}{2}t}\\ &+(J_--n_-E_-)e^{-t}+n_-E_-, \end{aligned} \tag{3.23}$$

$$\begin{aligned} E^+(t)={}&\frac{n_-E_-}{n_+}+\frac{1}{2n_+}\left[(A_5+\sqrt{4n_+-1}A_6)\cos\left(\frac{\sqrt{4n_+-1}}{2}t\right)\right.\\ &\left.+(A_6-\sqrt{4n_+-1}A_5)\sin\left(\frac{\sqrt{4n_+-1}}{2}t\right)\right]e^{-\frac{1}{2}t}, \end{aligned} \tag{3.24}$$

where

$$A_5=J_+-J_-, \tag{3.25}$$

$$A_6=\frac{2}{\sqrt{4n_+-1}}\left(n_+E_+-n_-E_--\frac{1}{2}(J_+-J_-)\right). \tag{3.26}$$

From (1.5), (3.2), (3.6), (3.15)–(3.26), and Lemma 2.1, we have

$$\begin{cases} |n(\pm\infty,t)-\tilde{n}(\pm\infty)|=0,\\ |J(+\infty,t)-\tilde{J}|=O(1)e^{-\mu t},\\ |J(-\infty,t)-\tilde{J}|=O(1)e^{-t},\\ E(-\infty,t)=E_-,\\ |E(+\infty,t)-\frac{n_-E_-}{n_+}|=O(1)e^{-\mu t} \end{cases} \tag{3.27}$$

for some constant $0<\mu<\frac{1}{2}$.

From the above analysis, we find that there are some gaps between $J(\pm\infty, t)$ and $\tilde{J} = n_-E_-$, and $E(+\infty, t)$ and $\tilde{E}(\infty) = \frac{n_-E_-}{n_+}$, which lead to

$$J(x,t) - \tilde{J} \quad \text{and} \quad E(x,t) - \tilde{E}(x) \notin L^2(\mathbb{R}). \tag{3.28}$$

To delete these gaps, we need to introduce the correction functions, which plays a key role in the proof of convergence of the original solutions to the stationary waves. Now we are going to construct the correction functions. Let $(\hat{n}, \hat{J}, \hat{E})(x,t)$ be the solutions to the following linear equations:

$$\begin{cases} \hat{n}_t + \hat{J}_x = 0, \\ \hat{J}_t = \breve{n}\hat{E} - \hat{J}, \\ \hat{E}_x = \hat{n}, \\ \hat{J}(x,t) \to J^{\pm}(t) - n_-E_- & \text{as } x \to \pm\infty, \\ \hat{E}(x,t) \to 0 & \text{as } x \to -\infty, \\ \hat{E}(x,t) \to E^+(t) - \frac{n_-E_-}{n_+} & \text{as } x \to +\infty. \end{cases} \tag{3.29}$$

In order to get $(\hat{n}, \hat{J}, \hat{E})(x,t)$ to (3.29), we consider the following linear system with some tricky selection on $\breve{n} = \breve{n}(x)$, $\hat{J}(x,t)$, and $\hat{E}(x,0)$:

$$\begin{cases} \hat{J}_t(x,t) = \breve{n}(x)\hat{E}(x,t) - \hat{J}(x,t), \\ \hat{E}_t(x,t) = -\hat{J}(x,t) + (J_- - n_-E_-)e^{-t}, \\ \hat{J}(x,0) = (J^-(0) - n_-E_-) + (J^+ - J^-)(0) \displaystyle\int_{-\infty}^x m_0(y)dy, \\ \hat{E}(x,0) = \left(E^+(0) - \frac{n_-E_-}{n_+}\right) \displaystyle\int_{-\infty}^x m_0(y)dy, \end{cases} \tag{3.30}$$

where $m_0(x)$ and $\breve{n}(x)$ are also ingeniously selected as

$$\begin{cases} m_0(x) \geq 0, \ m_0 \in C_0^\infty(\mathbb{R}), \ \operatorname{supp} m_0 \subseteq [-L_0, L_0], \ \displaystyle\int_{\mathbb{R}} m_0(y)dy = 1, \\ \breve{n}(x) = n_- + (n_+ - n_-) \displaystyle\int_{-\infty}^{x+2L_0} m_0(y)dy \end{cases} \tag{3.31}$$

with some constant $L_0 > 0$.

When $x < -L_0$, we have $\hat{E}(x,0) \equiv 0$. So, it can be easily seen that (3.30) possesses the particular solutions

$$\hat{J}(x,t) = (J_- - n_-E_-)e^{-t}, \quad \hat{E}(x,t) = 0 \quad \text{for } -\infty < x < -L_0. \tag{3.32}$$

When $x \geq -L_0$, we have $\breve{n}(x) \equiv n_+$. Similarly to the previous but complicated calculation, we can solve (3.30) as the following. However, we can verify that these solutions imply also the solutions given in (3.32) for $x < -L_0$. Therefore, we summarize them as follows.

Case 1. When $1 - 4n_+ > 0$, then, for $x \in \mathbb{R}$,

$$\hat{J}(x,t) = \left(A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}\right) \int_{-\infty}^x m_0(y)dy + (J_- - n_-E_-)e^{-t}, \tag{3.33}$$

$$\hat{E}(x,t) = \frac{1}{n_+}\left(A_1(1+\lambda_1)e^{\lambda_1 t} + A_2(1+\lambda_2)e^{\lambda_2 t}\right) \int_{-\infty}^x m_0(y)dy, \tag{3.34}$$

and thus we define

$$\hat{n}(x,t) = \frac{1}{n_+}\left(A_1(1+\lambda_1)e^{\lambda_1 t} + A_2(1+\lambda_2)e^{\lambda_2 t}\right)m_0(x). \tag{3.35}$$

Then we can verify that $(\hat{n}, \hat{J}, \hat{E})(x,t)$ satisfy (3.29) for $(x,t) \in \mathbb{R}\times\mathbb{R}^+$.

Case 2. When $1-4n_+ = 0$, then, for $x \in \mathbb{R}$,

$$\hat{J}(x,t) = \left(A_3 e^{-\frac{1}{2}t} + A_4 t e^{-\frac{1}{2}t}\right)\int_{-\infty}^{x} m_0(y)dy + (J_- - n_- E_-)e^{-t}, \tag{3.36}$$

$$\hat{E}(x,t) = \frac{1}{n_+}\left(\left(A_4 + \frac{1}{2}A_3\right)e^{-\frac{1}{2}t} + \frac{1}{2}A_4 t e^{-\frac{1}{2}t}\right)\int_{-\infty}^{x} m_0(y)dy, \tag{3.37}$$

and thus we define

$$\hat{n}(x,t) = \frac{1}{n_+}\left(\left(A_4 + \frac{1}{2}A_3\right)e^{-\frac{1}{2}t} + \frac{1}{2}A_4 t e^{-\frac{1}{2}t}\right)m_0(x). \tag{3.38}$$

Then we can verify that $(\hat{n}, \hat{J}, \hat{E})(x,t)$ satisfy (3.29) for $(x,t) \in \mathbb{R}\times\mathbb{R}^+$.

Case 3. When $1-4n_+ < 0$, then, for $x \in \mathbb{R}$,

$$\begin{aligned}\hat{J}(x,t) = {} & \left(A_5 \cos\left(\frac{\sqrt{4n_+-1}}{2}t\right) + A_6 \sin\left(\frac{\sqrt{4n_+-1}}{2}t\right)\right)e^{-\frac{1}{2}t}\int_{-\infty}^{x} m_0(y)dy \\ & + (J_- - n_- E_-)e^{-t}, \end{aligned}\tag{3.39}$$

$$\begin{aligned}\hat{E}(x,t) = {} & \frac{1}{2n_+}\left((A_5 + \sqrt{4n_+-1}A_6)\cos\left(\frac{\sqrt{4n_+-1}}{2}t\right)\right. \\ & \left. + (A_6 - \sqrt{4n_+-1}A_5)\sin\left(\frac{\sqrt{4n_+-1}}{2}t\right)\right)e^{-\frac{1}{2}t}\int_{-\infty}^{x} m_0(y)dy, \end{aligned}\tag{3.40}$$

and thus we define

$$\begin{aligned}\hat{n}(x,t) = {} & \frac{1}{2n_+}\left((A_5 + \sqrt{4n_+-1}A_6)\cos\left(\frac{\sqrt{4n_+-1}}{2}t\right)\right. \\ & \left. + (A_6 - \sqrt{4n_+-1}A_5)\sin\left(\frac{\sqrt{4n_+-1}}{2}t\right)\right)e^{-\frac{1}{2}t}m_0(x). \end{aligned}\tag{3.41}$$

Then we can verify that $(\hat{n}, \hat{J}, \hat{E})(x,t)$ satisfy (3.29) for $(x,t) \in \mathbb{R}\times\mathbb{R}^+$.

LEMMA 3.1. *There hold*

$$\|(\hat{n}, \hat{J}, \hat{E})(t)\|_{L^\infty(\mathbb{R})} \le C\sigma e^{-\nu_0 t} \tag{3.42}$$

and

$$\text{supp } \hat{n} = \text{supp } m_0 \subseteq [-L_0, L_0] \tag{3.43}$$

for $\sigma := |J_+| + |J_-| + |E_-| + |E_+|$ *and* $0 < \nu_0 < \frac{1}{2}$.

Remark 1. From (3.10), we know that $E_+ = E^+(0)$ depends on the initial data $n_0(x)$, so the constructed correction functions are also dependent on the initial data $n_0(x)$. As constructed, these correction functions $(\hat{n}, \hat{J}, \hat{E})(x,t)$ delete the gaps such that

$$\begin{cases} \int_{-\infty}^{\infty}[n(x,t) - \hat{n}(x,t) - \tilde{n}(x,t)]dx \\ \quad = \int_{-\infty}^{\infty}[n_0(x) - \hat{n}(x,0) - \tilde{n}(x,0)]dx = 0, \\ (J - \hat{J} - \tilde{J})(\pm\infty, t) = 0, \\ (E - \hat{E} - \tilde{E})(\pm\infty, t) = 0. \end{cases} \tag{3.44}$$

Thus, we can reasonably make the following perturbation around the stationary waves.

Now we are going to make the perturbation of (1.1) to the steady equations (2.1). Noticing (1.1), (2.1), and (3.29), we have

$$\begin{cases} (n-\hat{n}-\tilde{n})_t + (J-\hat{J}-\tilde{J})_x = 0, \\ (J-\hat{J}-\tilde{J})_t + (p(n)-p(\tilde{n}))_x \\ \qquad\qquad = nE - \tilde{n}\tilde{E} - \breve{n}\hat{E} - (J-\hat{J}-\tilde{J}) - (\frac{J^2}{n} - \frac{\tilde{J}^2}{\tilde{n}})_x, \\ (E-\hat{E}-\tilde{E})_x = n-\hat{n}-\tilde{n}. \end{cases} \tag{3.45}$$

Let

$$\begin{cases} \phi = n-\hat{n}-\tilde{n}, \\ \psi = J-\hat{J}-\tilde{J}, \\ e = E-\hat{E}-\tilde{E}. \end{cases} \tag{3.46}$$

Then we have

$$\begin{cases} e_x = n-\hat{n}-\tilde{n}, \\ -e_t = J-\hat{J}-\tilde{J}. \end{cases} \tag{3.47}$$

We deduce (3.45) into

$$\begin{aligned} &e_{tt} + e_t - (p(e_x+\tilde{n}+\hat{n}) - p(\tilde{n}))_x + ne \\ &\qquad = -(\tilde{E}+\hat{E})e_x - \hat{n}\tilde{E} - (\tilde{n}-\breve{n}+\hat{n})\hat{E} + \left(\frac{J^2}{n} - \frac{\tilde{J}^2}{\tilde{n}}\right)_x, \end{aligned} \tag{3.48}$$

with initial data

$$\begin{cases} e(x,0) = E_0(x) - \hat{E}(x,0) - \tilde{E}(x), \\ e_x(x,0) = n_0(x) - \hat{n}(x,0) - \tilde{n}(x), \\ e_t(x,0) = -J_0(x) + \hat{J}(x,0) + \tilde{J}(x), \end{cases} \tag{3.49}$$

where $E_0(x)$ is defined by

$$E_0(x) = \int_{-\infty}^{x} (n_0(y) - b(y))dy + E_-. \tag{3.50}$$

For convenience, we define

$$\begin{cases} f_1 = (\tilde{E}+\hat{E})e_x + \hat{n}\tilde{E} + (\tilde{n}-\breve{n}+\hat{n})\hat{E}, \\ f_2 = \frac{J^2}{n} - \frac{\tilde{J}^2}{\tilde{n}}. \end{cases} \tag{3.51}$$

THEOREM 3.2. *Let $\delta := |J_+| + |J_-| + |E_-| + |E_+| + \sum_{i=1}^{4} \alpha_i$, $\Phi_0 := \|e(0)\|_{H^3} + \|e_t(0)\|_{H^2}$. Then there is a $\delta_0 > 0$ such that when $\delta + \Phi_0 < \delta_0$, the solutions (n, J, E) of initial value problem* (1.1) *and* (1.3) *are unique and globally exist, and they satisfy*

$$\|(e, e_x, e_t, e_{xx}, e_{xt}, e_{xxx}, e_{xxt})(t)\|^2 \le C(\delta + \Phi_0)e^{-\nu t}, \tag{3.52}$$

where ν is a positive constant.

COROLLARY 3.3. *Under the conditions of Theorem* 3.2*, we have*

$$\begin{cases} \|(n-\tilde{n})(t)\|_{L^\infty} \le O(1)e^{-\mu t}, \\ \|(J-\tilde{J})(t)\|_{L^\infty} \le O(1)e^{-\mu t}, \\ \|(E-\tilde{E})(t)\|_{L^\infty} \le O(1)e^{-\mu t}, \end{cases} \tag{3.53}$$

where $\mu = \min\{\nu, \nu_0\} > 0$.

Remark 2. In [21], Luo, Natalini, and Xin proved that (1.1) converges to the stationary solutions decay exponentially under a stiff condition, i.e., $E_- = J_+ = J_- = 0$. In our paper, we remove such a condition and prove the stability of stationary waves for a very general perturbation. So [21] is a special case of ours.

4. A priori estimates. It is known that Theorem 3.2 can be proved by the classical energy method with the continuation argument based on the local existence and the a priori estimates (cf. [25]). Since the local existence of the solutions of (3.48), (3.49) can be proved using the standard iteration method together with the energy estimates, the main effort in this subsection is to establish the a priori estimates for the solutions, which is usually technical and crucial in the proof of stability.

Letting $T \in (0, +\infty]$, we define the solution space for

$$X(T) = \left\{ e(x,t) \middle| \partial_t^j e \in C(0,T; H^{3-j}(\mathbb{R})), j = 0,1,\ 0 \le t \le T \right\} \tag{4.1}$$

with the norm

$$N(T)^2 =: \sup_{0 \le t \le T} \left\{ \|e(t)\|_{H^3}^2 + \|e_t(t)\|_{H^2}^2 \right\}. \tag{4.2}$$

Let $N(T)^2 \le \varepsilon^2$, where ε is sufficiently small and will be determined later. It should be noted that (4.2) with Sobolev inequality $\|\partial_x^k f\|_{L^\infty(\mathbb{R})} \le C\|\partial_x^k f\|^{1/2}\|\partial_x^{k+1} f\|^{1/2}$ gives

$$\sum_{k=0}^{2} \|\partial_x^k e(t)\|_{L^\infty(\mathbb{R})} + \sum_{k=0}^{1} \|\partial_x^k e_t(t)\|_{L^\infty(\mathbb{R})} \le C\varepsilon. \tag{4.3}$$

It is easy to verify from (3.47) and the conditions of Theorem 3.2 that there exists a positive constant c such that

$$0 < \frac{1}{c} \le n = e_x + \hat{n} + \tilde{n} \le c. \tag{4.4}$$

Now we are going to establish the a priori estimates.

LEMMA 4.1. *It holds that*

$$\|(e, e_x, e_t, e_{xx}, e_{xt}, e_{tt})(t)\|^2 \le C(\delta + \Phi_0)e^{-\nu_1 t}, \tag{4.5}$$

provided $\varepsilon + \delta \ll 1$.

Proof. Multiplying (3.48) by e and integrating it over $(-\infty, +\infty)$, one obtains

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{1}{2}e^2 + e_t e\right)dx + \int_{\mathbb{R}} ne^2 dx + \int_{\mathbb{R}} (p(e_x + \tilde{n} + \hat{n}) - p(\tilde{n}))e_x dx \\ &= \|e_t\|^2 - \int_{\mathbb{R}} \left((\tilde{E} + \hat{E})e_x + \hat{n}\tilde{E} + (\tilde{n} - \breve{n} + \hat{n})\hat{E}\right) e dx \\ &\quad - \int_{\mathbb{R}} \left(\frac{J^2}{n} - \frac{\tilde{J}^2}{\tilde{n}}\right) e_x dx. \end{aligned} \tag{4.6}$$

By Taylor's formula, there exists a number ξ such that $p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}) = p'(\xi)(e_x+\tilde{n})$. Then

$$\begin{aligned}\int_{\mathbb{R}}(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))e_x dx &= \int_{\mathbb{R}} p'(\xi)(e_x+\hat{n})e_x dx \\ &\geq 2C_0\|e_x(t)\|^2 - C\delta\|e_x(t)\|^2 - C\delta e^{-\nu_0 t}.\end{aligned} \tag{4.7}$$

It can also be verified that

$$-\int_{\mathbb{R}}\left((\tilde{E}+\hat{E})e_x+\hat{n}\tilde{E}\right)e dx \leq C\delta\|(e,e_x)(t)\|^2 + C\delta e^{-\nu_0 t}. \tag{4.8}$$

From (1.9) and Lemma 2.1, one can prove

$$\int_{\mathbb{R}}(\tilde{n}-\breve{n})^2 dx \leq C. \tag{4.9}$$

Thus using (4.9), one then has

$$-\int_{\mathbb{R}}\left((\tilde{n}-\breve{n}+\hat{n})\hat{E}\right)e dx \leq C\delta e^{-\nu_0 t}, \tag{4.10}$$

$$\begin{aligned} &-\int_{\mathbb{R}}\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)e_x dx \\ &= -\int_{\mathbb{R}}\left(\frac{e_t^2-2e_t(\tilde{J}+\hat{J})}{n}+\frac{2\tilde{J}\hat{J}+\hat{J}^2}{n}-\tilde{J}^2\left(\frac{1}{n}-\frac{1}{\tilde{n}}\right)\right)e_x dx \\ &\leq C(\delta+\varepsilon)\|(e_t,e_x)(t)\|^2 - \int_{\mathbb{R}}\left(\frac{2\tilde{J}\hat{J}+\hat{J}^2}{n}\right)_x e dx + C\delta e^{-\nu_0 t} \\ &\leq C(\delta+\varepsilon)\|(e,e_t,e_x)(t)\|^2 + C\delta e^{-\nu_0 t}.\end{aligned} \tag{4.11}$$

Substituting (4.7), (4.8), (4.10), and (4.11) into (4.6), one finally obtains

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(ee_t+\frac{1}{2}e^2\right)dx + 2C_0\|(e,e_x)\|^2 \\ &\leq \|e_t(t)\|^2 + C(\delta+\varepsilon)\|(e,e_t,e_x)(t)\|^2 + C\delta e^{-\nu_0 t}.\end{aligned} \tag{4.12}$$

Multiplying (3.48) by e_t and integrating it over $(-\infty,+\infty)$, one obtains

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(e_t^2+ne^2\right)dx + \|e_t(t)\|^2 + \int_{\mathbb{R}}(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))e_{xt} dx \\ &= \int_{\mathbb{R}}\frac{1}{2}n_t e^2 dx - \int_{\mathbb{R}}\left((\tilde{E}+\hat{E})e_x+\hat{n}\tilde{E}+(\tilde{n}-\breve{n}+\hat{n})\hat{E}\right)e_t dx \\ &\quad + \int_{\mathbb{R}}\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x e_t dx.\end{aligned} \tag{4.13}$$

Noticing that

$$p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}) = p'(\tilde{n})e_x + O(1)(e_x+\hat{n})^2 + O(1)\hat{n}, \tag{4.14}$$

one can then estimate the third term on the left-hand side of (4.13) as

$$\begin{aligned}&\int_{\mathbb{R}} p(e_x+\tilde{n}+\hat{n})-p(\tilde{n})e_{xt} dx \\ &\geq \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} p'(\tilde{n})e_x^2 dx - C(\delta+\varepsilon)\|(e_x,e_{xt})(t)\|^2 - C\delta e^{-\nu_0 t},\end{aligned} \tag{4.15}$$

and the right-hand side of (4.13) can be estimates as

$$\begin{aligned}\int_{\mathbb{R}}\frac{1}{2}n_t e^2dx-\int_{\mathbb{R}}\Big((\tilde{E}+\hat{E})e_x+\hat{n}\tilde{E}+(\tilde{n}-\breve{n}+\hat{n})\hat{E}\Big)e_t dx\\ \le C(\delta+\varepsilon)\|(e,e_x)(t)\|^2+C\delta e^{-\nu_0 t},\end{aligned}\tag{4.16}$$

and by using the fact $\int_{\mathbb{R}}\tilde{n}_x^2dx\le C$, one has

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x e_t dx\\ &=\int_{\mathbb{R}}\left(\frac{2J}{n}(-e_{xt}+\hat{J}_x)-\frac{J^2}{n}(e_x+\hat{n}_x)+O(1)\tilde{n}_x(e_t+\hat{J}+e_x+\hat{n})\right)e_t dx\\ &\le C(\delta+\varepsilon)\|(e_t,e_x,e_{xt},e_{xx})(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}\tag{4.17}$$

Substituting (4.15), (4.16), and (4.17) into (4.13), one gets

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big(e_t^2+ne^2+p'(\tilde{n})e_x^2\Big)dx+\|e_t(t)\|^2\\ \le C(\delta+\varepsilon)\|(e,e_t,e_x,e_{xt},e_{xx})(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}\tag{4.18}$$

Differentiating (3.48) with respect x, one reaches

$$e_{xtt}+e_{xt}-(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_{xx}+ne_x=-n_xe-f_{1x}+f_{2xx}.\tag{4.19}$$

Multiplying (4.19) by e_x and integrating it over $(-\infty,+\infty)$, one obtains

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big(\frac{1}{2}e_x^2+e_{xt}e_x\Big)dx+\int_{\mathbb{R}}ne_x^2dx+\int_{\mathbb{R}}(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_x e_{xx}dx\\ &=\|e_{xt}(t)\|^2-\int_{\mathbb{R}}n_xee_xdx-\int_{\mathbb{R}}f_{1x}e_xdx+\int_{\mathbb{R}}f_{2xx}e_xdx.\end{aligned}\tag{4.20}$$

Since

$$\begin{aligned}\int_{\mathbb{R}}(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_x e_{xx}dx\\ \ge 2C_0\|e_{xx}(t)\|^2-C(\delta+\varepsilon)\|(e_x,e_{xx})(t)\|^2-C\delta e^{-\nu_0 t}\end{aligned}\tag{4.21}$$

and

$$-\int_{\mathbb{R}}n_xee_xdx\le C(\delta+\varepsilon)\|(e,e_x)(t)\|^2,\tag{4.22}$$

$$\begin{aligned}-\int_{\mathbb{R}}f_{1x}e_xdx&=\int_{\mathbb{R}}\Big((\tilde{E}+\hat{E})e_x+\hat{n}\tilde{E}+(\tilde{n}-\breve{n}+\hat{n})\hat{E}\Big)e_{xx}dx\\ &\le C(\delta+\varepsilon)\|(e_x,e_{xx})(t)\|^2+C\delta e^{-\nu_0 t},\end{aligned}\tag{4.23}$$

and also noticing that

$$\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x=-\frac{J^2}{n^2}(e_{xx}+\hat{n}_x)-\frac{J^2}{n}(e_{xt}-\hat{J}_x)-\tilde{n}_x\left(\frac{J^2}{n^2}-\frac{\tilde{J}^2}{\tilde{n}^2}\right),\tag{4.24}$$

one obtains

$$
\begin{aligned}
&\int_{\mathbb{R}} f_{2xx}e_x dx \\
&= -\int_{\mathbb{R}} \left(\frac{J^2}{n} - \frac{\tilde{J}^2}{\tilde{n}}\right)_x e_{xx}dx \\
&= -\int_{\mathbb{R}} \left(-\frac{J^2}{n^2}(e_{xx}+\hat{n}_x) - \frac{J^2}{n}(e_{xt}-\hat{J}_x) - \tilde{n}_x\left(\frac{J^2}{n^2} - \frac{\tilde{J}^2}{\tilde{n}^2}\right)\right)e_{xx}dx \\
&\le C(\delta+\varepsilon)\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2 + C\delta e^{-\nu_0 t}.
\end{aligned} \tag{4.25}
$$

Thus, by substituting (4.21), (4.22), (4.23), and (4.21) into (4.20), one further has

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \Big(\frac{1}{2}e_x^2 + e_{xt}e_x\Big)dx + 2C_0\|(e_x,e_{xx})(t)\|^2 \\
&\le \|e_{xt}(t)\|^2 + C(\delta+\varepsilon)\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2 + C\delta e^{-\nu_0 t}.
\end{aligned} \tag{4.26}
$$

Multiplying (4.19) by e_{xt} and integrating it over $(-\infty,+\infty)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left(e_{xt}^2 + ne_x^2\right)dx + \|e_{xt}(t)\|^2 + \int_{\mathbb{R}} (p(e_x+\tilde{n}+\hat{n}) - p(\tilde{n}))_x e_{xxt}dx \\
&= \frac{1}{2}\int_{\mathbb{R}} n_t e_x^2 dx - \int_{\mathbb{R}} n_x e e_{xt}dx - \int_{\mathbb{R}} f_{1x}e_{xt}dx + \int_{\mathbb{R}} f_{2xx}e_{xt}dx.
\end{aligned} \tag{4.27}
$$

It is easy to obtain the following estimates for the third term on the left-hand side of (4.27):

$$
\begin{aligned}
&\int_{\mathbb{R}} (p(e_x+\tilde{n}+\hat{n}) - p(\tilde{n}))_x e_{xxt}dx \\
&\ge \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} p'(n)e_{xx}^2 dx - C(\delta+\varepsilon)\|(e_x,e_{xx},e_{xt})(t)\|^2 - C\delta e^{-\nu_0 t}.
\end{aligned} \tag{4.28}
$$

For the right-hand side of (4.27), it can be estimated as follows:

$$
\int_{\mathbb{R}} \frac{1}{2}n_t e_x^2 dx \le C(\delta+\varepsilon)\|e_x(t)\|^2, \tag{4.29}
$$

$$
-\int_{\mathbb{R}} n_x e e_{xt}dx \le C(\delta+\varepsilon)\|(e,e_{xt})(t)\|^2. \tag{4.30}
$$

From Lemma 2.1, noticing that $\tilde{E}_x$ is bounded but has no smallness, one has

$$
\begin{aligned}
&-\int_{\mathbb{R}} f_{1x}e_{xt}dx \\
&= -\int_{\mathbb{R}} \left((\tilde{E}+\hat{E})e_x + \hat{n}\tilde{E} + (\tilde{n}-\breve{n}+\hat{n})\hat{E}\right)_x e_{xt}dx \\
&\le \frac{1}{32}\|e_{xt}(t)\|^2 + O(1)\|e_x(t)\|^2 + C(\delta+\varepsilon)\|e_{xx}(t)\|^2 + C\delta e^{-\nu_0 t}.
\end{aligned} \tag{4.31}
$$

Using (4.24), one gets

$$\begin{aligned}(4.32)\qquad &\int_{\mathbb{R}} f_{2xx}e_{xt}dx\\ &= -\int_{\mathbb{R}}\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_x e_{xxt}dx\\ &= -\int_{\mathbb{R}}\left(-\frac{J^2}{n^2}(e_{xx}+\hat{n}_x)-\frac{J^2}{n}(e_{xt}-\hat{J}_x)-\tilde{n}_x\left(\frac{J^2}{n^2}-\frac{\tilde{J}^2}{\tilde{n}^2}\right)\right)e_{xxt}dx\\ &\le \frac{d}{dt}\int_{\mathbb{R}}\frac{J^2}{2n^2}e_{xx}^2dx + C(\delta+\varepsilon)\|(e_x,e_t,e_{xx},e_{xt})(t)\|^2 + C\delta e^{-\nu_0 t}.\end{aligned}$$

Substituting (4.28)–(4.32) into (4.27), one obtains

$$\begin{aligned}(4.33)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(e_{xt}^2+ne_x^2+\left(p'(n)-\frac{J^2}{n^2}\right)e_{xx}^2\right)dx+\frac{7}{8}\|e_{xt}\|^2\\ &\le C(\delta+\varepsilon)\|(e,e_t,e_{xx},e_{xt})(t)\|^2+O(1)\|e_x(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}$$

Taking $\lambda(4.12)+2\lambda\times(4.18)+(4.26)+2(4.33)$, where λ is a large number, and noticing the smallness of δ,ε, one further obtains

$$\begin{aligned}(4.34)\qquad &\frac{d}{dt}\Bigg(\int_{\mathbb{R}}\lambda e e_t+\lambda\left(\frac{1}{2}+n\right)e^2+\lambda e_t^2+\left(\lambda p'(\tilde{n})+\frac{1}{2}+n\right)e_x^2\\ &\quad + e_xe_{xt}+e_{xt}^2+\left(p'(n)-\frac{J^2}{n^2}\right)e_{xx}^2dx\Bigg)\\ &\quad + C_1\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2\\ &\le C\delta e^{-\nu_0 t},\end{aligned}$$

where C_1 is a positive constant. Then Gronwall's inequality implies that

$$(4.35)\qquad \|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2\le C(\delta+\Phi_0)e^{-\nu_2 t}$$

for some $0<\nu_2<\nu_0$. Using (3.48) and (4.35), we have

$$(4.36)\qquad \|e_{tt}(t)\|^2\le C(\delta+\Phi_0)e^{-\nu_3 t}$$

for some $0<\nu_3<\nu_2$. Let $\nu_1=\min\{\nu_2,\nu_3\}$; we complete the proof of Lemma 4.1. ∎

LEMMA 4.2. *It holds that, for some $\nu_4>0$,*

$$(4.37)\qquad \|(e_{xx},e_{xxx},e_{xxt})(t)\|^2\le C(\delta^2+\Phi_0^2)e^{-\nu_4 t},$$

provided $\varepsilon+\delta\ll 1$.

Proof. Differentiating (3.48) with respect to x twice, we obtain

$$\begin{aligned}(4.38)\qquad &e_{xxtt}+e_{xxt}-(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_{xxx}+ne_{xx}\\ &=-n_{xx}e-2n_xe_x-f_{1xx}+f_{2xxx}.\end{aligned}$$

Multiplying (4.38) by e_{xx} and integrating it over $(-\infty,+\infty)$, we obtain

$$\begin{aligned}(4.39)\quad &\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{1}{2}e_{xx}^2+e_{xxt}e_{xx}\right)dx+\int_{\mathbb{R}}ne_{xx}^2dx\\ &+\int_{\mathbb{R}}(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_{xx}e_{xxx}dx\\ &=\|e_{xxt}(t)\|^2-\int_{\mathbb{R}}(n_{xx}e+2n_xe_x)e_{xx}dx-\int_{\mathbb{R}}f_{1xx}e_{xx}dx+\int_{\mathbb{R}}f_{2xxx}e_{xx}dx.\end{aligned}$$

Noticing that

$$
\begin{aligned}
&(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_{xx}\\
&=p'(n)e_{xxx}+\tilde{n}_{xx}(p'(n)-p'(\tilde{n}))+\tilde{n}_x^2(p''(n)-p''(\tilde{n}))\\
&\quad+p''(n)(e_{xx}^2+\hat{n}_x^2+2\tilde{n}_xe_{xx}+2\hat{n}_xe_{xx}+2\tilde{n}_x\hat{n}),
\end{aligned}\tag{4.40}
$$

we have

$$
\int_{\mathbb{R}}(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_{xx}e_{xx}dx\geq 2C_0\|e_{xxx}(t)\|^2-C\|(e_{xx},e_x)(t)\|^2-C\delta e^{-\nu_0 t}.\tag{4.41}
$$

For the right-hand side terms, we can estimate them as follows:

$$
-\int_{\mathbb{R}}(n_{xx}e+2n_xe_x)e_{xx}dx\leq\frac{1}{32}C_0\|e_{xxx}(t)\|^2+C\|(e,e_x,e_{xx})(t)\|^2,\tag{4.42}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}}f_{1xx}e_{xx}dx&=\int_{\mathbb{R}}f_{1x}e_{xxx}dx\\
&\leq\frac{1}{32}C_0\|e_{xxx}(t)\|^2+C\|(e_x,e_{xx})(t)\|^2+C\delta e^{-\nu_0 t}.
\end{aligned}\tag{4.43}
$$

Noticing also

$$
\begin{aligned}
\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_{xx}&=-\frac{J^2}{n^2}e_{xxx}-\frac{2J}{n}e_{xxt}-\tilde{n}_{xx}\left(\frac{J^2}{n^2}-\frac{\tilde{J}^2}{\tilde{n}^2}\right)+2\tilde{n}_x^2\left(\frac{J^2}{n^3}-\frac{\tilde{J}^2}{\tilde{n}^3}\right)\\
&\quad+2\frac{J^2}{n^3}(e_{xx}^2+\hat{n}_x^2+2\tilde{n}_xe_{xx}+2\hat{n}_xe_{xx}+2\tilde{n}_x\hat{n})\\
&\quad-4\frac{J}{n^2}(-e_{xt}+\hat{J}_x)(e_{xx}+\tilde{n}_x+\hat{n}_x)+\frac{2}{n}(-e_{xt}+\hat{J}_x)^2\\
&\quad-\frac{J^2}{n^2}\hat{n}_{xx}-\frac{2J}{n}\hat{n}_{xx},
\end{aligned}\tag{4.44}
$$

we then have

$$
\begin{aligned}
\int_{\mathbb{R}}f_{2xxx}e_{xx}dx&=-\int_{\mathbb{R}}\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_{xx}e_{xxx}dx\\
&\leq\frac{1}{32}C_0\|e_{xxx}(t)\|^2+C\|(e_x,e_t,e_{xx},e_{xt},e_{xxt})(t)\|^2+C\delta e^{-\nu_0 t}.
\end{aligned}\tag{4.45}
$$

Substituting (4.41), (4.42), (4.43), and (4.45) into (4.39), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{1}{2}e_{xx}^2+e_{xxt}e_{xx}\right)dx+2C_0\|(e_{xx},e_{xxx})(t)\|^2\\
&\leq C\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2+C\|e_{xxt}(t)\|^2+C\delta e^{-\nu_0 t}.
\end{aligned}\tag{4.46}
$$

Multiplying (4.38) by e_{xxt} and integrating it over $(-\infty,+\infty)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(e_{xxt}^2+ne_{xx}^2\right)dx+\|e_{xxt}(t)\|^2\\
&\quad+\int_{\mathbb{R}}(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_{xx}e_{xxxt}dx\\
&=\frac{1}{2}\int_{\mathbb{R}}n_te_{xx}^2dx-\int_{\mathbb{R}}(n_{xx}e+2n_xe_x)e_{xxt}dx\\
&\quad-\int_{\mathbb{R}}f_{1xx}e_{xxt}dx+\int_{\mathbb{R}}f_{2xxx}e_{xxt}dx.
\end{aligned}\tag{4.47}
$$

Utilizing (4.40), we have

$$\begin{aligned}(4.48)\qquad &\int_{\mathbb{R}}(p(e_x+\tilde{n}+\hat{n})-p(\tilde{n}))_{xx}e_{xxxt}dx\\ &\geq \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}p'(n)e_{xxx}^2dx-\frac{1}{32}\|e_{xxt}(t)\|^2\\ &\quad -C(\delta+\varepsilon)\|e_{xxx}(t)\|^2-C\|(e_x,e_{xx})(t)\|^2-C\delta e^{-\nu_0 t}.\end{aligned}$$

Here we used the fact that $\tilde{n}_{xxx}$ is bounded, and that

$$(4.49)\qquad \frac{1}{2}\int_{\mathbb{R}}n_t e_{xx}^2dx\leq C\|e_{xx}(t)\|^2,$$

$$\begin{aligned}(4.50)\qquad &-\int_{\mathbb{R}}(n_{xx}e+2n_xe_x)e_{xxt}dx\\ &\leq \frac{1}{32}\|e_{xxt}(t)\|^2+C(\delta+\varepsilon)\|e_{xxx}(t)\|^2+C\|(e,e_x,e_{xx})(t)\|^2,\end{aligned}$$

and

$$(4.51)\qquad -\int_{\mathbb{R}}f_{1xx}e_{xxt}dx\leq \frac{1}{32}\|e_{xxt}(t)\|^2+C(\delta+\varepsilon)\|e_{xxx}(t)\|^2+C\|(e,e_x,e_{xx})(t)\|^2.$$

Here, in order to estimate (4.51), we used the fact that $\tilde{E}$ is bounded.

Utilizing (4.44), we have

$$\begin{aligned}(4.52)\qquad &\int_{\mathbb{R}}f_{2xxx}e_{xxt}dx=-\int_{\mathbb{R}}\left(\frac{J^2}{n}-\frac{\tilde{J}^2}{\tilde{n}}\right)_{xx}e_{xxxt}dx\\ &\leq \frac{d}{dt}\int_{\mathbb{R}}\frac{J^2}{2n^2}e_{xxx}^2dx+\frac{1}{32}\|e_{xxt}(t)\|^2+C(\delta+\varepsilon)\|e_{xxx}(t)\|^2\\ &\quad +C\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}$$

Substituting (4.48)–(4.52) into (4.47), we obtain

$$\begin{aligned}(4.53)\qquad &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(e_{xxt}^2+ne_{xx}^2+\left(p'(n)-\frac{J^2}{n^2}\right)e_{xxx}^2\right)dx+\frac{3}{4}\|e_{xxt}(t)\|^2\\ &\leq C(\delta+\varepsilon)\|e_{xxx}(t)\|^2+C\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}$$

Taking $(4.46)+2\times(4.53)$ and noticing the smallness of ε,δ, we have

$$\begin{aligned}(4.54)\qquad &\frac{d}{dt}\int_{\mathbb{R}}\left(e_{xxt}e_{xx}+e_{xxt}^2+\left(n+\frac{1}{2}\right)e_{xx}^2+\left(p'(n)-\frac{J^2}{n^2}\right)e_{xxx}^2\right)dx\\ &\quad +C_2\|(e_{xx},e_{xxx},e_{xxt})(t)\|^2\\ &\leq C\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2+C\delta e^{-\nu_0 t},\end{aligned}$$

where C_2 is a positive constant.

Using Gronwall's inequality and Lemma 4.1, we obtain

$$(4.55)\qquad \|(e_{xx},e_{xxx},e_{xxt})(t)\|^2\leq C(\delta+\Phi_0)e^{-\nu_4 t},$$

where ν_4 is a positive constant. □

Proof of Theorem 3.2. Let $\nu=\min\{\nu_1,\nu_4\}$; then Lemmas 4.1 and 4.2 imply Theorem 3.2.

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