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CONVERGENCE RATES TO SUPERPOSITION OF TWO TRAVELLING WAVES OF THE SOLUTIONS TO A RELAXATION HYPERBOLIC SYSTEM WITH BOUNDARY EFFECTS

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This paper is to study the asymptotic behavior of solutions for an initial–boundary value problem to Jin–Xin's 2×2 relaxation hyperbolic system. When the initial data are small perturbation of the superposition of two travelling waves at $t = 0$, subsequent to the previous work,[6] we further show the convergence rates of the IBVP solutions to the superposition of two waves. Precisely, when the initial perturbations decay in the exponential or algebraic forms, we prove that the corresponding solutions tend to the superposition of two waves time-asymptotically in the exponential or algebraic forms, respectively. The method adopted is the weighted energy method. The use of a shift function for the forward travelling wave and the special choice of shift functions for backward travelling plays a key role to overcome the difficulties caused by the boundary and degeneration.

1. Introduction

The effect of relaxation is often taken into consideration when the physical situation of an investigated material is in nonequilibrium, such as gases in thermo-nonequilibrium states, kinetic theory of mono-atomic gases, water waves, viscoelasticity with memory, chromatography, etc.[5,36]

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In this paper, we consider the initial boundary value problems (IBVP) for the relaxation model on the quarter plane $(x,t) \in R_+ \times R_+$

$$\begin{cases} u_t + v_x = 0\,, \\ v_t + au_x = -\dfrac{1}{\varepsilon}(v - f(u))\,, \end{cases} \tag{1.1}$$

with the initial boundary values

$$\begin{cases} (u,v)|_{t=0} = (u_0,v_0)(x),\ x \geq 0\,, \quad (u_0,v_0)(+\infty) = (u_+,v_+)\,, \\ v(0,t) = v_-\,, \quad t \geq 0\,, \end{cases} \tag{1.2}$$

where the function $f(u)$ is smooth and nonconvex, $v_+ = f(u_+)$, $a > 0$ is a constant.

Equation (1.1) is the simplest form to the general conservation laws with relaxation proposed by Jin and Xin in Ref. 11, where the systems were used to numerically approximate a set of corresponding hyperbolic conservation laws with non-oscillation, which is exactly a local relaxation approximation. Equation (1.1) is also a simplified form to the general 2×2 conservation laws with relaxation considered by T.-P. Liu in Ref. 20.

As the relaxation time ε goes to 0^+, we formally obtain from (1.1) the following scalar conservation laws

$$u_t + f(u)_x = 0\,. \tag{1.3}$$

The relation between 2×2 conservation laws with relaxation and their corresponding equilibrium equation was studied theoretically by T.-P. Liu[20] first. Therein, he justified the nonlinear stability criteria (the sub-characteristic condition, see also Ref. 36) for elementary waves and showed the stability of them. For (1.1), the corresponding sub-characteristic condition is

$$-\sqrt{a} < f'(u) < \sqrt{a}\,. \tag{1.4}$$

For the general model, the stability of travelling waves with decay rates for the Cauchy problem and the stability theory but without decay rate for the initial boundary problem were studied by Zingano,[39] Nishibata,[31] and Nishibata and Yu,[33] respectively. The problem on the convergence to the diffusion waves was also given by Chern,[4] Yao and Zhu.[38] Related results on the relaxation time limit can be found in Chen and T.-P. Liu,[3] Chen, Levermore and T.-P. Liu,[2] Natalini,[30] Marcati and Rubino.[24]

For the simplest model (1.1), the stability of travelling wave solutions for the Cauchy problem were studied by H. L. Liu, Woo and Yang,[17] H. L. Liu, Wang and Yang,[16] Mascia and Natalini,[25] and finally Mei and Yang.[29] The authors[29] improved the algebraic decay rates obtained in Ref. 17 to the optimal one, and also contributed an exponential decay rate when the initial perturbation decays in a spatial exponential form. The convergence to the travelling wave solutions, as the relaxation time goes to zero, was recently considered by Jin and H. L. Liu.[10] The asymptotic relaxation limit for (1.1) with boundary effect was discussed by Wang and Xin.[35] Furthermore, the numerical computation and the properties of entropy solution for the model (1.1) were shown by Jin and Levermore,[9] Jin.[8] For Jin–Xin's model in higher space dimensions, the stability of front waves was shown by Luo and Xin.[23]

Boundary layer effect is always strong in some sense. Physically, such a phenomenon (see Ref. 34) occurs, for instance, in describing the interaction of fluid molecules with the molecules of the solid boundary, and has been modelled with the scalar viscous conservation laws observed by Xin[37] and the Boltzmann equations investigated by J.-G. Liu and Xin.[18,19] In addition, the appearance of boundary makes it impossible for a travelling wave to be an exact solution to the IBVP problem.[31] Regarding the stability of travelling waves to the IBVP for other model equations of hyperbolic conservation laws, we refer to those interesting works in Refs. 14, 21, 22 and 26.

Recently, the IBVP (1.1) and (1.2) was first considered by Mei and Rubino in Ref. 28 for the stability of travelling waves. Let u_- be the other state end piont for $u(x,t)$ and satisfy

$$v_- = f(u_-)\,. \tag{1.5}$$

If the Rankine–Hugoniot condition

$$s = \frac{v_+ - v_-}{u_+ - u_-} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}$$

and the Oleinik's entropy condition

$$f(u) - f(u_\pm) - s(u - u\pm)\begin{cases} < 0\,, & u_+ < u < u_- \\ > 0\,, & u_+ > u > u_- \end{cases}$$

hold, then there exists a unique (up to shift) travelling wave solution $(U_s, V_s)(x - st)$ for (1.1) with $(U_s, V_s)(\pm\infty) = (u_\pm, v_\pm)$. In the case of $s \geq 0$, the authors[28] showed the convergence with decay rates for the IBVP solutions to the corresponding travelling waves. However, in another case of $s < 0$, since there appears a really large boundary layer

$$|v(0,t) - V_s(0 - st)| = |v_- - V_s(-st)| \to |v_- - v_+|\,, \quad \text{as } \ t \to +\infty\,,$$

the stability of waves was not clear and proposed as an open question therein. Very recently, Hsiao and Li[6] answered the question. It is found that, the wave with the speed $s < 0$ alone is not the asymptotic state of the IBVP solutions, and some new front waves should be considered. By suitably extending the flux function $f(u)$ smoothly from $u = u_+$ to $u = u_-^1$, such that

$$v_- = f(u_-^1) \tag{1.6}$$

and (u_-^1, v_-) and (u_+, v_+) with the extended $f(u)$ (we still use the notation f) on $[u_-^1, u_+]$ satisfy the Rankine–Hugoniot condition and the Oleinik's entropy condition, see Fig. 1.1, one gets a new front wave $(U_{s_1}, V_{s_1})(x - s_1 t)$ with $s_1 > 0$

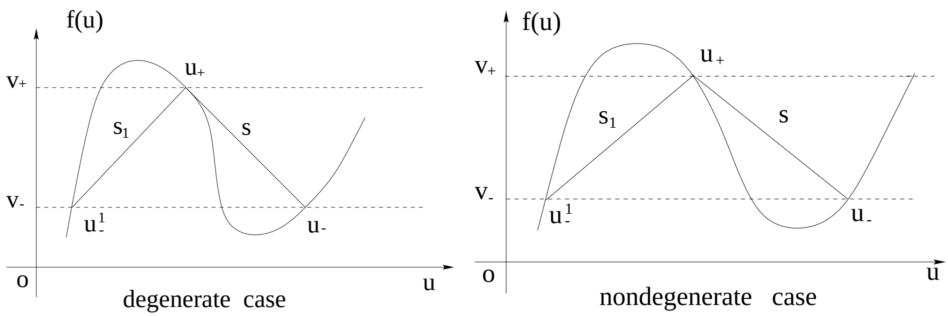

Fig. 1.1.

and $(U_{s_1}, V_{s_1})(\pm\infty) = (u_+/u_-^1, v_\pm)$. It is proved in Ref. 6 that the superposition of the two waves

$$(U_{s_1}, V_{s_1}) + (U_s, V_s) - (u_+, v_+)$$

can be the asymptotic state of the IBVP (1.1) and (1.2), when the corresponding initial perturbations are small. However, the decay rates are not discussed in Ref. 6, in particular, the degenerate case $f'(u_+) = s$ (see Fig. 1.1) is not studied.

In this paper, we first establish the algebraic and exponential decay rates of the solutions, obtained in Ref. 6, to the superposition of two nondegenerate travelling waves. We prove that the rate is somehow like $O((1+t)^{-\alpha/2})$ or $O(e^{-\theta t/2})$ with some constants $\alpha, \theta > 0$. Here the flux function $f(u)$ can be allowed to be general nonconvex. To treat the nonconvexity, as in Refs. 6, 28 and 29 we will use two weight functions. In addition, to obtain the exponential decay rates, we need to study the structure relations of flux functions at the points $u = u_-^1$, u_+ and u_-. Then, we consider the same IBVP in degenerate case. For simplicity, we only consider the case $f'(u_+) = s$, to which new difficulty occurs since (U_s, V_s) tends to (u_+, v_+) in the algebraic form like $|x - st|^{-1}$ as $|x - st|$ goes to infinity, and this is not integrable in L^1-sense. To overcome this difficulty, we choose a suitable shift function like $e^{c(x-st)}$ or $(x-st)^k$ for the back wave (U_s, V_s). Thus, we can show the existence of global smooth solutions for general nonconvex $f(u)$, and obtain the algebraic and exponential decay rates by using the weighted energy method. Similarly to that used by Liu and Yu,[22] Liu and Nishihara,[21] Matsumura and Mei,[26] Mei and Rubino,[28] a shift function for the front travelling wave is used to overcome the difficulties caused by the boundary. To overcome the difficulties caused by the degeneration, we use two special kinds of shift functions for the backward travelling wave, where the α in the algebraic decay rate $O((1+t)^{-\alpha/2})$ is restrained by the degeneration and the choice of shift functions for the backward travelling wave.

This paper is organized as follows. In Sec. 2, we give some preliminaries on the travelling wave solutions to Cauchy problem for (1.1), then we discuss the nondegenerate case in Sec. 3. We will prove that the solutions $(u, v)(x, t)$ converge, with some algebraic and exponential decay rates, to the superposition of two travelling waves as t goes to infinity. In Sec. 4, we study the degenerate case and prove the existence of global smooth solutions. The exponential and algebraic decay rates like $O((1+t)^{-\alpha/2})$ and $O(e^{-\theta t/2})$ are also obtained.

Notations. L^2 denotes the space of measurable functions on R or R_+ which are square integrable with the norm $\|f\| = (\int |f(x)|^2 dx)^{1/2}$. $H^l (l \geq 0)$ denotes the Sobolev space of L^2-functions f on R or R_+ whose derivatives $\partial_x^j f, j = 1, \ldots, l$, are also L^2-functions, with the norm $\|f\|_l = (\sum_{j=0}^{l} \|\partial_x^j f\|^2)^{1/2}$. L_w^2 denotes the space of measurable functions on R or R_+ which satisfy $w(x)^{1/2} f \in L^2$, where $w(x) > 0$ is a weight function, with the norm $|f|_w = (\int w(x)|f(x)|^2 dx)^{1/2}$. H_w^l $(l \geq 0)$ denotes the weighted Sobolev space of L_w^2-functions f on R whose derivatives $\partial_x^j f, j = 1, \ldots, l$, are also L_w^2-functions, with the norm $|f|_{l,w} = (\sum_{j=0}^{l} |\partial_x^j f|_w^2)^{1/2}$. Denoting $\langle x \rangle = \sqrt{1+x^2}$ and

$$\langle x \rangle_+ = \begin{cases} \sqrt{1+x^2}\,, & \text{if} \quad x > 0\,, \\ 1\,, & \text{if} \quad x < 0\,, \end{cases}$$

we will make use of the spaces $L^2_{\langle x \rangle_+}$ and $H^l_{\langle x \rangle_+} (l = 1, 2)$ later. If $w(x) = \langle x \rangle^\alpha$, we denote $L_w^2 = L_\alpha^2$. The weighted space L_w^2 for such weight function $w = \langle x \rangle^\alpha \langle x \rangle_+$ is denoted as $L^2_{\alpha\langle x \rangle_+}$, and the corresponding norm is $|\cdot|_{\alpha\langle x \rangle_+}$. Since we consider $x \in R_+$, sometimes we mean $\langle x \rangle = \langle x \rangle_+$. We denote also $f(x) \sim g(x)$ as $x \to x_0$ when $C^{-1} g \leq f \leq Cg$ in a neighborhood of x_0, and $|(f_1, f_2, f_3)|_X \sim |f_1|_X + |f_2|_X + |f_3|_X$, where $|\cdot|_X$ is the norm of space X. Without any ambiguity, we denote several constants by C_i, or c_i, $i = 1, 2, \ldots$, or by C. When $C^{-1} \leq w(x) \leq C$ for $x \in R$, we note that $L^2 = H^0 = L_w^2 = H_w^0$ and $\|\cdot\| = \|\cdot\|_0 \sim |\cdot|_w = |\cdot|_{0,w}$.

Let T and B be a positive constant and a Banach space, respectively. We denote $C^k(0, T; B)$ $(k \geq 0)$ as the space of B-valued k-times continuously differentiable functions on $[0, T]$, and L^2 $(0, T; B)$ as the space of B-valued L^2-functions on $[0, T]$. The corresponding spaces of B-valued function on $[0, \infty)$ are defined similarly.

Finally, in this paper, we always assume the relaxation time $\varepsilon = 1$ without loss of generality, because we can rescale the variable (x, t) to a new one $(\varepsilon x, \varepsilon t)$, then we have Eq. (1.1) with $\varepsilon = 1$.

2. Preliminaries

A travelling wave solution to system (1.1) is a solution $(U_s, V_s)(\eta)$, $(\eta = x - st)$, satisfying (1.1) and $(U_s, V_s)(\pm\infty) = (u_\pm, v_\pm)$ with $v_\pm = f(u_\pm)$, namely,

$$\begin{cases} -sU_s' + V_s' = 0\,, \\ -sV_s' + aU_s' = f(U) - V\,, \\ (U_s, V_s)(\pm\infty) = (u_\pm, v_\pm)\,. \end{cases} \tag{2.1}$$

Here $' = d/d\eta$. Integrating the first equation of (2.1) over $(-\infty, \eta]$ and $[\eta, +\infty)$ respectively, and noticing $(U_s, V_s)(\pm\infty) = (u_\pm, v_\pm)$, we have

$$-sU_s + V_s = -su_\pm + v_\pm = -su_\pm + f(u_\pm)\,, \tag{2.2}$$

which shows that, the speed s and the two states $(u_\pm, v_\pm)$ satisfy the Rankine–Hugoniot condition

$$s = \frac{v_+ - v_-}{u_+ - u_-} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}\,. \tag{2.3}$$

Substituting (2.2) into the second equation of (2.1) we obtain

$$(a - s^2)U_s' = f(U_s) - f(u_\pm) - s(U_s - u_\pm) \equiv: h(U_s)\,. \tag{2.4}$$

It is well known that the ordinary differential equation (2.4) has a smooth solution if and only if the Rankine–Hugoniot condition (2.3) and the Oleinik's entropy condition

$$h(u) = f(u) - f(u_\pm) - s(u - u_\pm) \begin{cases} < 0\,, & u_+ < u < u_- \\ > 0\,, & u_+ > u > u_- \end{cases} \tag{2.5}$$

hold. This entropy condition implies

$$f'(u_+) < s < f'(u_-) \tag{2.6}$$

or

$$f'(u_+) = s < f'(u_-) \quad \text{or} \quad f'(u_+) < s = f'(u_-) \quad \text{or} \quad f'(u_\pm) = s\,. \tag{2.7}$$

Condition (2.6) is the well-known Laxian shock condition. Here we will call it the *nondegenerate* shock condition and we will refer to each of the possibilities in (2.7) as the *degenerate* shock condition, or the *contact* shock condition.

As shown in Fig. 1.1, there are two waves with different state points. One is the back wave ($s < 0$) with the state points $(u_\pm, v_\pm)$, another is the front wave ($s_1 > 0$) with the state points $(u_+/u_-^1, v_\pm)$. Throughout this paper, we mark the extended flux function on $u \in [u_-^1, u_+]$ still as $f(u)$. Now we first state some preliminaries on the back wave to (1.1) with $(u_\pm, v_\pm)$ which is given by (1.2), as shown in Fig. 1.1, and satisfies (2.3) and (2.5) with $u_+ < u_-$ and $f(u_+) > f(u_-)$, namely, $s < 0$.

Lemma 2.1.[16] *Assume $f \in C^2$, and the conditions* (2.3) *and* (2.5) *hold. There exists a unique solution $(U_s, V_s)(\eta)$ ($\eta = x - st$) up to a shift to* (1.1) *with* $0 < s^2 < a$ ($s < 0$), *which satisfies*

$$(a - s^2)U_s' = h(U_s) < 0\,, \quad \text{for} \quad u_+ < U_s < u_-\,.$$

In addition, as $\eta \to \pm\infty$, it holds that for $f'(u_\pm) \neq s$

$$|h(U_s)| \sim |(U_s - u_\pm, V_s - v_\pm)| \sim \exp\{-c_\pm|\eta|\}\,,$$

and that for $f(u) = su + (u - u_\pm)^{(n+1)}$, $n = 1, 2, 3, \ldots$,

$$|h(U_s)|^{1/(1+n)} \sim |(U_s - u_\pm, V_s - v_\pm)| \sim |\eta|^{-1/n}\,,$$

where $c_\pm = |f'(u_\pm) - s|/(a - s^2) > 0$.

Similarly, we also have some results on the front travelling wave solution to (1.1), which connects the states $(u_+/u_-^1, v_\pm)$, satisfying the Rankine–Hugoniot (R-H) condition (2.8) and Oleinik's entropy condition (2.9), i.e.

$$s_1 = \frac{v_+ - v_-}{u_+ - u_-^1} = \frac{f(u_+) - f(u_-^1)}{u_+ - u_-^1} > 0\,, \tag{2.8}$$

$$h_1(u) = f(u) - f(u_+/u_-^1) - s(u - u_+/u_-^1) > 0\,, \quad u_+ > u > u_-^1\,, \tag{2.9}$$

where $u_-^1 < u_+$, $v_- = f(u_-^1) < f(u_+) = v_+$ such that $s_1 > 0$. The symbol u_+/u_-^1 implies "u_+ or u_-^1". Since the extended part of flux function $f(u)$ on $[u_-^1, u_+]$ is suitably chosen, it is possible to make the front wave $(U_{s_1}, V_{s_1})(x - s_1 t)$ nondegenerate, namely, the Laxian entropy condition holds

$$f'(u_+) < s_1 < f'(u_-^1)\,. \tag{2.10}$$

Lemma 2.2. *Assume $f \in C^2$, and the conditions* (2.8), (2.9) *and* (2.10) *hold. There exists a unique solution $(U_{s_1}, V_{s_1})(\eta)$ $(\eta = x - s_1 t)$ up to a shift to* (1.1) *with $0 < s_1^2 < a$ $(s_1 > 0)$, which satisfies $(U_{s_1}, V_{s_1})(\pm\infty) = (u_+/u_-^1, v_\pm)$ and*

$$(a - s_1^2)U'_{s_1} = h_1(U_{s_1}) > 0\,, \quad for \quad u_-^1 < U_{s_1} < u_+\,.$$

In addition, as $\eta \to \pm\infty$, it holds that

$$|h_1(U_{s_1})| \sim |(U_{s_1} - u_+/u_-^1, V_{s_1} - v_\pm)| \sim \exp\{-c'_\pm|\eta|\}\,,$$

where $c'_\pm = |f'(u_+/u_-^1) - s_1|/(a - s_1^2) > 0$.

Define the following weight functions (see Refs. 27 and 29),

$$Q_1(U_{s_1}) = \frac{(U_{s_1} - u_-^1)(u_+ - U_{s_1})}{h_1(U_{s_1})}\,, \quad Q_2(U_{s_1}) = \frac{(U_{s_1} - u_-^1)^{1/2}(u_+ - U_{s_1})^{1/2}}{h_1(U_{s_1})}\,, \tag{2.11}$$

for (U_{s_1}, V_{s_1}). There are properties for Q_1, Q_2 given in Refs. 27 and 29 as follows.

Lemma 2.3.[27,29] *Let $(U_{s_1}, V_{s_1})(\eta)$ be the travelling wave to* (1.1) *given by Lemma* 2.2, *then it holds, as $\eta \to \pm\infty$, that*

$$Q_1(U_{s_1}) \sim O(1)\,, \quad Q_2(U_{s_1}) \sim e^{c'_\pm|\eta|/2} \tag{2.12}$$

and

$$(Q_1 h_1)''(U_{s_1}) = -2\,, \quad \left|\frac{Q_i(U_{s_1})_\eta}{Q_i(U_{s_1})}\right| = O(1)\frac{u_+ - u_-^1}{a - s_1^2},\ i = 1, 2\,, \tag{2.13}$$

$$h_1(U_{s_1})(Q_2 h_1)''(U_{s_1}) = O(1)Q_2(U_{s_1})\,. \tag{2.14}$$

3. IBVP for Nondegenerate Case

In this section, we consider the IBVP (1.1) and (1.2) when the initial data are a perturbation of the superposition of the two nondegenerate travelling waves mentioned before, i.e.

$$f'(u_-^1) > s_1 > f'(u_+)\,, \quad f'(u_-) > s > f'(u_+)\,. \tag{ND}$$

The existence and large time behavior of global smooth solution to IBVP (1.1) and (1.2) under condition (ND) was proved in Ref. 6. What we expect is to obtain the exponential and algebraic decay rates for the convergence.

3.1. *Main results*

For any given but fixed constants x_1 and d_{10} satisfying $0 < -d_{10} < x_1$, we first give the essential assumption on the initial data u_0 in this section

$$\int_0^\infty [u_0(x) - U_{s_1}(x + d_{10}) - U_s(x + x_1) + u_+]\, dx = 0\,. \tag{3.1}$$

Denote

$$\begin{aligned}(U_p, V_p)(x,t) = (U_{s_1}, V_{s_1})&(x - s_1 t + d_1(t) + d_{10})\\ &+ (U_s, V_s)(x - st + x_1) - (u_+, v_+)\,,\end{aligned} \tag{3.2}$$

where $d_1(t)$ is our desired shift function chosen as the solution of the following ordinary differential equation:

$$\begin{cases} d_1'(t)[u_+ - U_{s_1}(d_{10} - s_1 t + d_1(t))] \\ \quad = v_- - V_{s_1}(d_{10} - s_1 t + d_1(t)) + v_+ - V_s(x_1 - st)\,, \\ d_1(0) = 0\,. \end{cases} \tag{3.3}$$

It will be proved in Lemma 3.4 below that $d_1(t) \in C^1(0,+\infty)$, $d_1'(t) \in L^1(0,+\infty)$, and $d_1(t) \to d_{1\infty} < +\infty$ as $t \to +\infty$, where the value of $d_{1\infty}$ can be determined, by using the original idea of Matsumura and Mei to determine their shift,[26] as

$$d_{1\infty} = \frac{1}{u_+ - u_-^1}\left\{\int_0^{+\infty} (U_s(x + x_1) - u_+)dx + \int_{-\infty}^0 (u_-^1 - U_{s_1}(x + d_{10}))dx\right\}\,.$$

Set

$$w_0(x) = -\int_x^{+\infty} (u_0(y) - U_p(y,0))dy\,, \quad z_0(x) = v_0(x) - V_p(x,0)\,.$$

The authors in Ref. 6 proved the global existence and uniqueness of smooth solution for the IBVP (1.1) and (1.2) as follows.

Theorem 3.1. (Convergence[6]) *Suppose that* $f \in C^3$, *the conditions* (ND), (3.1), (2.3), (2.5), (2.8) *and* (2.9) *hold. Let* $a > 0$ *be a large constant,* $w_0 \in H^2$, *and*

$z_0 \in H^1$. *Then there exists a constant* $\varepsilon_1 > 0$, *such that if* $\|w_0\|_2 + \|z_0\|_2 + |d_{10}|^{-1} + x_1^{-1} < \varepsilon_1$, *a global smooth solution* $(u,v)(x,t)$ *to* (1.1) *and* (1.2) *exists uniquely and satisfies*

$$\sup_{x \in R_+} |(u,v)(x,t) - (U_p, V_p)(x,t)| \to 0\,, \quad \text{as } t \to +\infty\,.$$

Furthermore, we are going to state the theorems on decay rates. For simplicity, we define

$$Q_{0,2}(x) = \begin{cases} e^{c'_+(x+d_{10})/2}\,, & \text{if } x > -d_{10}\,, \\ 1\,, & \text{if } 0 \le x < -d_{10}\,. \end{cases} \tag{3.4}$$

Due to $d_{10} < 0$, we have $Q_{0,2}(x) \sim Q_2(U_{s_1}(x + d_{10}))$ on $x \ge 0$ by Lemma 2.3.

We now state the theorem on the algebraic decay rates.

Theorem 3.2. (Algebraic Rate) *Assume the hypotheses of Theorem* 3.1 *hold. Suppose* $w_0 \in L^2_\alpha \cap H^2$, $z_0 \in L^2_\alpha \cap H^1$ *for some* $\alpha > 0$. *Then, if* (w_0, z_0) *is small enough in* $(L^2_\alpha \cap H^2) \times (L^2_\alpha \cap H^1)$, *the IBVP* (1.1) *and* (1.2) *has a unique global solution* $(u,v)(x,t)$ *satisfying*

$$\sup_{x \in R_+} |(u,v)(x,t) - (U_p, V_p)(x,t)| \le C N_2 (1+t)^{-\alpha/2}\,, \tag{3.5}$$

where $N_2 = |(w_0, z_0)|_\alpha + \|w_0\|_2 + \|z_0\|_1 + e^{c'_- d_{10}/4} + e^{-c_+ x_1/4}$.

To obtain the exponential decay rates, we need the following structure conditions for $f(u)$ at the points $u_\pm$ and u_-^1,

$$c_+ > \frac{1}{4}\max\{c'_+,\ c'_-\}\,, \quad 5c'_- s_1 + 16 c_+ s < 0\,, \tag{3.6}$$

where c_+ and $c'_\pm$ are defined in Lemmas 2.1 and 2.2, respectively.

Theorem 3.3. (Exponential Rate) *Assume that the hypotheses of Theorem* 3.1 *and condition* (3.6) *hold. Suppose that* $w_0 \in H^2_{Q_{0,2}}$, $z_0 \in H^1_{Q_{0,2}}$. *There exist* $\varepsilon_2 > 0$ *and* $\theta = \theta(|u_+ - u_-^1|, |u_+ - u_-|, a) > 0$ *such that if* $|w_0|_{2,Q_{0,2}} + |z_0|_{1,Q_{0,2}} + |d_{10}|^{-1} + x_1^{-1} \le \varepsilon_2$, *then the IBVP* (1.1) *and* (1.2) *has a unique global solution* $(u,v)(x,t)$ *satisfying*

$$u - U_p \in C^0(0,\infty; H^1_{Q_2}) \cap L^2(0,\infty; H^1_{Q_2})\,,$$
$$v - V_p \in C^0(0,\infty; H^1_{Q_2}) \cap L^2(0,\infty; H^1_{Q_2})$$

and

$$\sup_{x \in R_+} |(u,v)(x,t) - (U_p, V_p)(x,t)| \le C N_1 e^{-\theta t/2}\,, \quad t \ge 0\,, \tag{3.7}$$

where $N_1 = |w_0|_{2,Q_{0,2}} + |z_0|_{1,Q_{0,2}} + e^{c'_- d_{10}/4} + e^{-\tilde{c}_+ x_1/4}$ *with* $\tilde{c}_+ = \min\{4c_+ - c'_-,\ c_+\}$.

Remark 3.1. (a) The restriction of $a \gg 1$ means the requirement of a strong diffusion effect, which was used by H. Liu, Wang and Yang in Ref. 16, H. Liu, Woo and Yang in Ref. 17 and replaced with $|u_+ - u_-| \ll 1$ by Mei and Yang[29] for the Cauchy problem. For the IBVP (1.1) and (1.2), due to the resolutions on boundary terms, Mei and Rubino[28] also have to use $a \gg 1$. But, in this paper, it is not difficult to find in the proof that for Theorems 3.1–3.3 the condition $a \gg 1$ can be replaced by that, $|u_+ - u_-^1|$ is small enough such that

$$-a+(f'(U_p))^2+C_b\left\{\frac{|u_+-u_-^1|}{a-s_1^2}\left(f'(U_p)-\frac{3}{4}s_1\right)+\frac{|u_+-u_-^1|^2}{(a-s_1^2)^2}\left(a-\frac{9}{16}s_1^2\right)\right\}<0\,,$$

and $a - s_1^2 - C_b|u_+ - u_-^1| > 0$ with $C_b > 0$ a constant.
(b) For exponential decay rate, a simple example, satisfying R-H conditions (2.3) and (2.8), Oleinik's entropy conditions (2.5) and (2.9), (ND) and (3.6), is

$$\min\left\{f'(u_-^1),\ -\frac{3}{5}f'(u_+)\right\} > s_1 = -s > 0\,,$$

$$16f'(u_+) + 5f'(u_-^1) - 11s < 0,\ \ f'(u_-) > s\,.$$

(c) The algebraic decay rate seems to be optimal comparing with the corresponding Cauchy problem studied in Ref. 29 and the IBVP discussed in Ref. 28.
(d) By (3.2), it is not difficult to verify that $(U_p, V_p) \to (U_{s_1}, V_{s_1})$ as $t \to +\infty$ due to $|(U_s, V_s)(x - st + x_1) - (u_+, v_+)| = O(1)e^{-c_+(|s|t+x+x_1)}$. Thus, it holds

$$\sup_{x\in R_+} |(u,v)(x,t) - (U_{s_1}, V_{s_1})(x - s_1t + d_{10} + d_1(t))| \to 0,\ \text{as } t \to +\infty\,.$$

3.2. *Reformulation of original problems*

Let $(U_{s_1}, V_{s_1})(x-s_1t)$ and $(U_s, V_s)(x-st)$ be the front and back waves as mentioned before. Assume that the solution to (1.1) and (1.2) is $(u, v)(x, t)$. Since $(U_s, V_s)(x-st)$ satisfies (2.1) and $(U_{s_1}, V_{s_1})(x - s_1t + d_1(t) + d_{10})$ satisfies

$$\begin{cases} \partial_t U_{s_1} - d_1'(t)\partial_x U_{s_1} + \partial_x V_{s_1} = 0\,, \\ \partial_t V_{s_1} - d_1'(t)\partial_x V_{s_1} + a\partial_x U_{s_1} = f(U_{s_1}) - V_{s_1}\,, \end{cases} \tag{3.8}$$

it follows, by (3.8) and (2.1), that

$$\begin{cases} \partial_t(u - U_p) + d_1'(t)\partial_x U_{s_1} + \partial_x(v - V_p) = 0\,, \\ \partial_t(v - V_p) + d_1'(t)\partial_x V_{s_1} + a\partial_x(u - U_p) \\ \quad = f(u) - f(U_{s_1}) - f(U_s) + f(u_+) - (v - V_p)\,. \end{cases} \tag{3.9}$$

Integrating the first equation of (3.9) over $[0, +\infty)$ and noticing that $v(0, t) = v_-$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_0^{+\infty}(u - U_p)dx + d_1'(t)[u_+ - U_{s_1}(-s_1t + d_1(t) + d_{10})] \\ &\qquad - (v_- - V_{s_1}(-s_1t + d_1(t) + d_{10}) + v_+ - V_s(-st + x_1)) = 0\,. \end{aligned} \tag{3.10}$$

Let $d_1(t)$ be the solution of the ordinary differential equation (3.3), then the following holds:

$$\frac{d}{dt}\int_0^{+\infty}(u-U_p)(x,t)dx = 0\,, \quad t \geq 0\,. \tag{3.11}$$

Integrating it with respect to t and noting the essential assumption (3.1), it follows

$$\begin{aligned}\int_0^{\infty}(u-U_p)(x,t)dx &= \int_0^{\infty}[u_0(x)-U_p(x,0)]dx \\ &= \int_0^{\infty}[u_0(x)-U_{s_1}(x+d_{10})-U_s(x+x_1)+u_+]dx \\ &= 0\,.\end{aligned}$$

Thus, we may define

$$w(x,t) = -\int_x^{\infty}[u(y,t)-U_p(y,t)]dy\,, \quad z(x,t) = v(x,t)-V_p(x,t)\,, \tag{3.12}$$

to obtain

$$\begin{cases}\partial_t w + d_1'(t)[U_{s_1}(x+\eta_1)-u_+] + z = 0\,,\\ \partial_t z + d_1'(t)\partial_x V_{s_1}(x+\eta_1) + a\partial_x^2 w + z \\ \quad = f(U_p+w_x) - f(U_{s_1}(x+\eta_1)) - f(U_s(x+\eta_2)) + f(u_+)\,,\end{cases}$$

where $\eta_1 =: -s_1 t + d_1(t) + d_{10}$ and $\eta_2 =: -st + x_1$.

Therefore, $w(x,t)$ satisfies the following equation

$$w_{tt} + w_t - aw_{xx} + f'(U_p)w_x = g_1(x,t) + g_2(x,t)\,, \tag{3.13}$$

where

$$\begin{cases}g_1(x,t) = d_1'(t)V_{s_1}'(x+\eta_1) + (s_1 - d_1'(t))d_1'(t)U_{s_1}'(x+\eta_1) \\ \qquad\qquad + (d_1'(t)+d_1''(t))[u_+ - U_{s_1}(x+\eta_1)]\,,\\ g_2(x,t) = -\{f(U_p+w_x) - f(U_{s_1}(x+\eta_1)) \\ \qquad\qquad - f(U_s(x+\eta_2)) - f'(U_p)w_x + f(u_+)\}\,,\end{cases} \tag{3.14}$$

with the initial and boundary values

$$\begin{cases}w(x,0) = w_0(x)\,,\\ w_t(x,0) = -z_0(x) + z_0(0)\dfrac{u_+ - U_{s_1}(x+d_{10})}{u_+ - U_{s_1}(d_{10})} =: w_1(x)\,, \quad x \geq 0\,,\\ w(0,t) = 0\,, \quad t \geq 0\,.\end{cases} \tag{3.15}$$

We reformulate Theorems 3.2–3.3 as follows.

Theorem 3.4. (Algebraic Rate) *Assume that the hypotheses of Theorem* 3.2 *hold. Then, the IBVP* (3.13) *and* (3.15) *has a unique global solution* $w(x,t)$ *satisfying*

$$\sup_{x\in R_+}|(w,w_x,w_t)(x,t)| \leq CN_2^2(1+t)^{-\alpha/2}\,. \tag{3.16}$$

Theorem 3.5. (Exponential Rate) *Assume that the hypotheses of Theorem* 3.3 *hold. Then the IBVP* (3.13) *and* (3.15) *has a unique global solution* $w(x,t)$ *satisfying*

$$w \in C^0(0,\infty; H^2_{Q_2}) \cap L^2(0,\infty; H^2_{Q_2}), \quad w_t \in C^0(0,\infty; H^1_{Q_2}) \cap L^2(0,\infty; H^1_{Q_2}),$$

and

$$|w(\cdot,t)|^2_{2,Q_2} + |w_t(\cdot,t)|^2_{1,Q_2} + \theta \int_0^t [|w(\cdot,\tau)|^2_{2,Q_2} + |w_t(\cdot,\tau)|^2_{1,Q_2}] d\tau \le CN_1^2, \tag{3.17}$$

namely,

$$\sup_{x\in R_+} |(w_x, w_t)(x,t)| + |w(\cdot,t)|^2_{2,Q_2} + |w_t(\cdot,t)|^2_{1,Q_2} \le CN_1^2 e^{-\theta t}, \quad t \ge 0. \tag{3.18}$$

The proofs of Theorems 3.4 and 3.5 will be carried out in Sec. 3.3. First, we will prove the exponential decay rate, then show the algebraic decay rate in Sec. 3.4.

3.3. *Exponential decay rate*

Let $T > 0$. Define the work spaces for (3.13) and (3.15) as

$$X_1(0,T) = \{(w, w_t) \in C^0(0,T; H^2 \times H^1) \cap L^2(0,T; H^2 \times H^1)\},$$

$$X_2(0,T) = \{(w, w_t) \in C^0(0,T; H^2_{Q_2} \times H^1_{Q_2}) \cap L^2(0,T; H^2_{Q_2} \times H^1_{Q_2})\}$$

and denote

$$N_1(T) = \sup_{0\le t\le T} \{|w(\cdot,t)|_2 + |w_t(\cdot,t)|_1\}, \quad t \in [0,T].$$

$$N_2(T) = \sup_{0\le t\le T} \{|w(\cdot,t)|_{2,Q_2} + |w_t(\cdot,t)|_{1,Q_2}\}, \quad t \in [0,T].$$

Now, we prove Theorem 3.5, for which we have to establish the *a priori* estimates in Lemmas 3.5 and 3.6 below.

First, we have more accurate estimates on $d_1(t)$.

Lemma 3.4. *Let* $|d_{10}|^{-1} + x_1^{-1} \ll 1$. *Then* $d_1(t) \in C^3(0,+\infty)$, $d_1'(t) \in L^1(0,+\infty)$, *and* $|d_1(t)| \le C$ *for some constant, and*

$$d_1(t) - s_1 t \le -\frac{1}{2} s_1 t, \quad t \in [0,+\infty) \tag{3.19}$$

and

$$|d_1'(t)| \sim |d_1''(t)| \sim |d_1'''(t)| \sim \{e^{-c_-'(s_1 t - d_{10})} + e^{-c_+(|s|t + x_1)}\}, \quad t \in [0,+\infty) \tag{3.20}$$

and

$$d_1(t) \to d_{1\infty}, \quad as\ t \to +\infty, \tag{3.21}$$

where

$$d_{1\infty} = \frac{1}{u_+ - u_-^1} \left\{ \int_0^\infty [U_s(x + x_1) - u_+] dx + \int_{-\infty}^0 [u_-^1 - U_{s_1}(x + d_{10})] dx \right\}.$$

Proof. By Lemma 2.2 and $d_{10} < 0$, we get

$$u_+ - U_{s_1}(d_{10}) > 0\,, \tag{3.22}$$

which means, in terms of Lemmas 2.1–2.2 and (3.3), that

$$|d_1'(0)| = \frac{1}{u_+ - U_{s_1}(d_{10})}\,|v_- - V_{s_1}(d_{10}) + v_+ - V_s(x_1)| \le \frac{1}{4}s_1\,, \tag{3.23}$$

provided that $|d_{10}|^{-1} + x_1^{-1} \ll 1$.

Therefore, it holds, for some $t_0 > 0$, that

$$|d_1'(t)| \le \frac{1}{2}s_1\,, \quad 0 \le t \le t_0\,, \tag{3.24}$$

which yields, with $d_1(0) = 0$, that

$$|d_1(t)| = \left|\int_0^t d'(\tau)d\tau\right| \le \frac{1}{2}s_1 t\,, \quad 0 \le t \le t_0 \tag{3.25}$$

and

$$d_1(t) - s_1 t \le -\frac{1}{2}s_1 t \le 0\,, \quad 0 \le t \le t_0\,. \tag{3.26}$$

By Lemma 2.2 and (3.26), it follows that

$$u_+ - U_{s_1}(-s_1 t + d_1(t) + d_{10}) \ge u_+ - U_{s_1}(d_{10})\,, \quad 0 \le t \le t_0\,. \tag{3.27}$$

Then, by (3.3), (3.27) and Lemmas 2.1–2.2, we obtain

$$|d_1'(t)| \le \frac{1}{4}s_1\,, \quad 0 \le t \le t_0\,. \tag{3.28}$$

Repeating the above procedure, we can verify that (3.26) and (3.28) hold for all $t \in [0,\infty)$, namely, we have proved (3.19).

To prove (3.20), we first note that,

$$|v_- - V_{s_1}(-s_1 t + d_1(t) + d_{10})| \le Ce^{-c_-'(\frac{1}{2}s_1 t + |d_{10}|)}$$

and

$$|u_+ - U_{s_1}(-s_1 t + d_1(t) + d_{10})| \sim |u_+ - u_-^1|$$

which means, due to (3.19) and Eq. (3.3), that

$$|d_1'(t)| \le C\{e^{-c_-'(\frac{1}{2}s_1 t + |d_{10}|)} + e^{-c_+(|s|t + x_1)}\}\,.$$

This implies that $d_1'(t) \in L^1(0,\infty)$. Further, we get the boundedness of $|d_1(t)|$

$$|d_1(t)| \le |d_1(0)| + \int_0^t |d_1'(\tau)|d\tau \le C\,.$$

Thus, based on $|d_1(t)| \le C$, Eq. (3.3) and Lemmas 2.1 and 2.2, we obtain (3.20).

Finally, $d_{1\infty}$ can be calculated as shown by Matsumura and Mei in Ref. 26, and we can prove $d_1(t) \to d_{1\infty}$ by the continuity as $t \to \infty$. We omit the details. The proof is complete. □

Now, we are going to prove the basic energy estimate.

Lemma 3.5. *Under the assumptions of Theorem* 3.5, *it holds, for any solution* $w(x,t)$ *of* (3.13) *and* (3.15) *with* $w \in X_2(0,T)$, *that*

$$
\begin{aligned}
&|(w, \sqrt{a}w_x, w_t)(\cdot,t)|^2_{Q_2} + \theta_1 \int_0^t |(w, \sqrt{a}w_x, w_t)(\cdot,\tau)|^2_{Q_2} d\tau \\
&\quad \le C(e^{-\tilde{c}_+ x_1/2} + e^{c'_- d_{10}/2} + |(w_0, w_{0x}, w_1)|^2_{Q_2}), \qquad (3.29)
\end{aligned}
$$

provided that $N_2(T) + |d_{10}|^{-1} + x_1^{-1} \ll 1.$

Proof. Let us denote $L(w) := w_{tt} + w_t - aw_{xx} + f'(U_p)w_x$ and consider the equality

$$
Q_2(w + 2w_t) \cdot L(w) = Q_2(w + 2w_t)(g_1 + g_2)\,. \qquad (3.30)
$$

The left-hand side of (3.30) can be reduced to

$$
\begin{aligned}
&Q_2(w + 2w_t)L(w) \\
&\quad = \partial_t \left\{ Q_2 \left(w_t^2 + ww_t + \frac{1}{2}(1 - Q_{2t}/Q_2)w^2 \right) + aQ_2 w_x^2 \right\} \\
&\qquad + Q_2 \left\{ w_t^2(1 - Q_{2t}/Q_2) + 2(f'(U_p) + aQ_{2x}/Q_2)ww_t + a(1 - Q_{2t}/Q_2)w_x^2 \right\} \\
&\qquad - \partial_x \left\{ aQ_2 ww_x - \frac{1}{2}aQ_{2x}w^2 - \frac{1}{2}Q_2 f'(U_p)w^2 + 2aQ_2 w_t w_x \right\} \\
&\qquad + \frac{1}{2}w^2 \left\{ Q_{2tt} - Q_{2t} - aQ_{2xx} - (Q_2 f'(U_p))_x \right\} \\
&\quad = \partial_t \left\{ Q_2 \left(w_t^2 + ww_t + \frac{1}{2}(1 + (s_1 - d_1'(t))Q_{2x}/Q_2)w^2 \right) + aQ_2 w_x^2 \right\} \\
&\qquad + Q_2 \left\{ w_t^2(1 + (s_1 - d_1'(t))Q_{2x}/Q_2) + 2(f'(U_p) + aQ_{2x}/Q_2)w_x w_t \right. \\
&\qquad \left. + (1 + (s_1 - d_1'(t))Q_{2x}/Q_2)aw_x^2 \right\} \\
&\qquad - \partial_x \left\{ aQ_2 ww_t - \frac{1}{2}aQ_{2x}w^2 - \frac{1}{2}Q_2 f'(U_p)w^2 + 2aQ_2 w_t w_x \right. \\
&\qquad \left. - \frac{1}{2}Q_{2x}d_1'(t)(2s_1 - d_1'(t))w^2 \right\} - \frac{1}{2}w^2(Q_2 h_1)'' U'_{s_1} \\
&\qquad + \left\{ Q_{2x}d_1'(t)(2s_1 - d_1'(t))ww_x - \frac{1}{2}w^2 \partial_x \{Q_2(f'(U_p) - f'(U_{s_1})) + d_1'(t)Q_2\} \right\} \\
&\quad = \partial_t \left\{ G_1(w, w_t) + G_2(w_x) \right\} + G_3(w_t, w_x) - \partial_x G_4(w, w_t, w_x) \\
&\qquad + G_5(w) + G_6(w, w_x)\,. \qquad (3.31)
\end{aligned}
$$

As $a > 0$ is big enough, it holds $|Q_{2x}/Q_2| = O(1)\frac{u_+ - u_-^1}{a - s_1^2} \ll 1$. With a similar argument as in Refs. 17, 28 and 29, we have

$$\Delta_1 =: 1 - 2\left[1 + (s_1 - d_1'(t))\frac{Q_{2x}}{Q_2}\right] = -\left[1 + 2(s_1 - d_1'(t))\frac{Q_{2x}}{Q_2}\right] \le -C < 0\,, \quad (3.32)$$

$$\Delta_3 =: 4\left(f'(U_p) + a\frac{Q_{2x}}{Q_2}\right)^2 - 4a\left(1 + (s_1 - d_1'(t))\frac{Q_{2x}}{Q_2}\right)^2 \le -C < 0\,, \quad (3.33)$$

provided that $|d_{10}|^{-1} + x_1^{-1} \ll 1$, where Δ_1 and Δ_3 are the discriminates of G_1 and G_3 respectively. Thus, we have

$$G_1(w, w_t) + G_2(w_x) \ge C^{-1}(w^2 + w_t^2 + aw_x^2)\,, \quad (3.34)$$

$$G_3(w, w_t) \ge C^{-1}(w_x^2 + w_t^2)\,. \quad (3.35)$$

By (2.14) and Lemma 2.2, we have

$$G_5(w) = O(1)Q_2(U_{s_1})w^2\,. \quad (3.36)$$

By Lemmas 2.1–2.2 and Lemma 3.4, the term $G_6(w, w_x)$ can be estimated as

$$|G_6(w, w_x)| \le \frac{1}{2}C^{-1}Q_2(U_{s_1})\left(w_x^2 + w^2\right)\,, \quad (3.37)$$

provided that $|d_{10}|^{-1} + x_1^{-1} \ll 1$.

Interagting (3.30) over $[0, +\infty) \times [0, t]$, using (3.34)–(3.37), and the zero-boundary conditions $w|_{x=0} = w|_{x=\infty} = w_t|_{x=0} = w_t|_{x=\infty} = 0$ which implies $G_4(w, w_t, w_x)|_{x=0} = G_4(w, w_t, w_x)|_{x=\infty} = 0$, we obtain

$$\begin{aligned} &|(w, \sqrt{a}w_x, w_t)(\cdot, t)|_{Q_2}^2 + 2\theta_1\int_0^t |(w, \sqrt{a}w_x, w_t)(\cdot, \tau)|_{Q_2}^2 d\tau \\ &\quad \le C_1\left(\left|\int_0^t\int_0^{+\infty} Q_2(w + 2w_t)(g_1 + g_2)dxd\tau\right| + |(w_0, w_{0x}, w_1)|_{Q_2}^2\right)\,, \end{aligned} \quad (3.38)$$

for some positive constants θ_1 and C_1.

Due to the Taylor's formula and the Cauchy inequality, the second equality of (3.14) can be reduced to

$$\begin{aligned} |g_2(x,t)| &= |f'(U_{s_1})(U_s - u_+ + w_x) + O(1)(U_s - u_+ + w_x)^2 - f'(U_p)w_x \\ &\quad - f'(u_+)(U_s - u_+) - O(1)(U_s - u_+)^2| \\ &= |[f'(U_{s_1}) - f'(U_p)]w_x + [f'(U_{s_1}) - f'(u_+)](U_s - u_+) \\ &\quad + O(1)(U_s - u_+ + w_x)^2 - O(1)(U_s - u_+)^2| \\ &= |O(1)(U_s - u_+)w_x + O(1)(U_s - u_+)| \\ &\quad + |O(1)(U_s - u_+ + w_x)^2 - O(1)(U_s - u_+)^2| \\ &\le O(1)w_x^2 + O(1)|U_s - u_+| + O(1)|U_s - u_+|^2\,. \end{aligned}$$

Thus, using the above estimate, the Cauchy inequality and the Sobolev inequality, we can control the first term in the right-hand side of (3.38) on $g_2(x,t)$ as follows:

$$
\begin{aligned}
&C_1\left|\int_0^t\int_0^{+\infty} Q_2(w+2w_t)g_2dxd\tau\right| \\
&\quad\le CN_2(T)\int_0^t\int_0^{+\infty} Q_2w_x^2dxd\tau+\frac{1}{2}\theta_1\int_0^t\int_0^{+\infty} Q_2(w^2+w_t^2)dxd\tau \\
&\qquad + C_{\theta_1}\int_0^t\int_0^{+\infty} Q_2(U_{s_1}(x+\eta_1))(U_s(x+\eta_2)-u_+)^2dxd\tau\,, \qquad (3.39)
\end{aligned}
$$

where C_{θ_1} is a positive constant depending on θ_1. Using Lemmas 2.1–2.3, it can be shown that

$$
\begin{aligned}
&Q_2(U_{s_1}(x+\eta_1))(U_s(x+\eta_2)-U_+)^2 \\
&\sim\begin{cases} C\exp\left\{\frac{1}{2}c'_+(x+\eta_1)-2c_+(x+\eta_2)\right\}\,, & \text{for } x>-\eta_1\,, \\ C\exp\left\{-\frac{1}{2}c'_-(x+\eta_1)-2c_+(x+\eta_2)\right\}\,, & \text{for } x<-\eta_1\,, \end{cases} \\
&\sim\begin{cases} C\exp\left\{-\frac{1}{2}(4c_+-c'_+)x+2c_+st-2c_+x_1+c'_-d_{10}/2\right\}\,, \quad \text{for } x>-\eta_1\,, \\ C\exp\left\{-\frac{1}{2}(4c_++c'_-)x+\frac{5}{8}\left(c'_-s_1+\frac{16}{5}c_+s\right)t-\frac{1}{2}\left(4c_++\frac{d_{10}}{x_1}c'_-\right)x_1\right\}\,, \\ \qquad\qquad \text{for } x<-\eta_1\,. \end{cases}
\end{aligned}
$$

This yields

$$
\begin{aligned}
&C\int_0^t\int_0^{+\infty} Q_2(U_{s_1}(x+\eta_1))(U_s(x+\eta_2)-u_+)^2dxd\tau \\
&\quad\le C\int_0^t\left(\int_0^{-\eta_1}\exp\left\{-\frac{1}{2}(4c_+-c'_+)x+2c_+s\tau-2c_+x_1+c'_-d_{10}/2\right\}\right. \\
&\qquad+\int_{-\eta_1}^{+\infty}\exp\left\{-\frac{1}{2}(4c_++c'_-)x+\frac{5}{8}\left(c'_-s_1+\frac{16}{5}c_+s\right)\tau\right. \\
&\qquad-\left.\left.\frac{1}{2}\left(4c_++\frac{d_{10}}{x_1}c'_-\right)x_1\right\}\right)dxd\tau \\
&\quad\le C(e^{-\tilde{c}_+x_1/2}+e^{c'_-d_{10}/2})\,. \qquad (3.40)
\end{aligned}
$$

Therefore, we have, in terms of (3.39)–(3.40), that

$$
\begin{aligned}
&C_1\left|\int_0^t\int_0^{+\infty} Q_2(w+2w_t)g_2dxd\tau\right| \\
&\quad\le CN_2(T)\int_0^t\int_0^{+\infty} Q_2w_x^2dxd\tau+\frac{1}{2}\theta_1\int_0^t\int_0^{+\infty} Q_2(w^2+w_t^2)dxd\tau \\
&\qquad + C(e^{-\tilde{c}_+x_1/2}+e^{c'_-d_{10}/2})\,. \qquad (3.41)
\end{aligned}
$$

Similarly, we can prove

$$\begin{aligned}
&\left|\int_0^t\int_0^{+\infty} Q_2(w+2w_t)g_1 dxd\tau\right| \\
&\quad\le \frac{1}{4}\theta_1\int_0^t\int_0^{+\infty} Q_2(w^2+w_t^2)dxd\tau \\
&\qquad + C\int_0^t\int_0^{+\infty} Q_2(U_{s_1}(x+\eta_1))(|d_1'(\tau)U_{s_1}'(x+\eta_1)| + |d_1'(\tau)(U_{s_1}(x+\eta_1)-u_+)| \\
&\qquad + |U_{s_1}'(x+\eta_1)(U_{s_1}(x+\eta_1)-u_+)|)^2 dxd\tau \\
&\quad\le \frac{1}{4}\theta_1\int_0^t\int_0^{+\infty} Q_2(w^2+w_t^2)dxd\tau + C(e^{-c_+x_1/2}+e^{c_-'d_{10}/2})\,. \qquad (3.42)
\end{aligned}$$

With the help of (3.41) and (3.42), we obtain Lemma 3.5 from (3.38), provided that $N_2(T)+|d_{10}|^{-1}+x_1^{-1}\ll 1$. □

Similarly, consider the equality

$$Q_2(w_t+2w_{tt})\partial_t L(w) = Q_2(w_t+2w_{tt})(g_{1t}+g_{2t})\,. \qquad (3.43)$$

Integrating (3.43) over $[0,+\infty)\times[0,t]$, using the similar argument as used in Lemma 3.5 and the zero-boundary condition $w|_{x=0}=w|_{x=\infty}=w_t|_{x=0}=w_t|_{x=\infty}=0$, we get the higher order energy estimate:

Lemma 3.6. *Under the assumptions of Theorem* 3.5, *it holds, for any solution* $w(x,t)$ *of the IBVP* (3.13) *and* (3.15) *with* $w\in X_2(0,T)$, *that*

$$\begin{aligned}
&|(w_t,\sqrt{a}w_{xx},w_{xt})(\cdot,t)|_{Q_2}^2 + \theta_3\int_0^t |(w_t,\sqrt{a}w_{xx},w_{xt})(\cdot,\tau)|_{Q_2}^2 d\tau \\
&\quad\le C(e^{-\tilde{c}_+x_1/2}+e^{c_-'d_{10}/2}+|(w_0,w_{0x},w_1)|_{1,Q_2}^2)
\end{aligned}$$

provided that $N_2(T)+|d_{10}|^{-1}+x_1^{-1}\ll 1$.

3.4. *Algebraic decay rate*

Here, we prove the algebraic decay rates. Let us define $\bar{u}:=(u_++u_-^1)/2$. Since U_{s_1} is strictly increasing in R, there exists a unique number $\eta^*\in R$ such that $U_{s_1}(\eta^*)=\bar{u}$. Denote $K(x,t)=(1+t)^\gamma\langle(\eta-\eta^*)/a\rangle^\beta Q_1(U_{s_1}(\eta))$, $\bar{K}(x,t)=(1+t)^\gamma\langle(\eta-\eta^*)/a\rangle^\beta$, i.e. $K(x,t)=\bar{K}(x,t)Q_1(U_{s_1}(x+\eta_1))$, where $\eta=x+\eta_1$. Multiplying Eq. (3.10) by $2K(x,t)w$ and $2K(x,t)w_t$, respectively, we have

$$2K(x,t)w\cdot L(w) = 2K(x,t)w(g_1+g_2)\,, \qquad (3.44)$$

$$2K(x,t)w_t\cdot L(w) = 2K(x,t)w_t(g_1+g_2)\,. \qquad (3.45)$$

Combining (3.44)$\times\frac{1}{2}$ + (3.45), we obtain, by a straightforward but tedious calculation as was made in Refs. 17 and 28, that

$$\begin{aligned}
&\left\{Kw_t^2 + Kw_t w + \frac{1}{2}(K + (s_1 - d_1'(t))K_x)w^2 + aKw_x^2\right\}_t \\
&\quad - \frac{\gamma}{1+t}\left[Kw_t^2 + Kww_t + \frac{1}{2}(K + (s_1 - d_1'(t))K_x)w^2 + aKw_x^2\right] \\
&\quad + \{(K + (s_1 - d_1'(t))K_x)w_t^2 + 2(f'(U_p)K + aK_x)w_x w_t \\
&\quad + a(K + (s_1 - d_1'(t))K_x)w_x^2\} \\
&\quad + \{(a - (s_1 - d_1'(t))^2)\bar{K}_x Q_1(U_{s_1}) + \bar{K}Q_1(U_{s_1})_x(2s_1 - d_1'(t))d_1'(t)\}ww_x \\
&\quad + \frac{1}{2}P_\beta w^2 - \partial_x B(x,t) = K(w + 2w_t)(g_1 + g_2)\,, \qquad (3.46)
\end{aligned}$$

where

$$\begin{aligned}
B(x,t) = \Big\{&a\bar{K}Q_1 ww_x - \frac{1}{2}a(\bar{K}_x Q_1 + \bar{K}Q_{1x})w^2 - \frac{1}{2}\bar{K}Q_1 f'(U_p)w^2 \\
&+ \frac{1}{2}w^2\bar{K}Q_{1x}(s_1 - d_1'(t))d_1'(t) + \frac{1}{2}w^2\bar{K}_x Q_1(a - (s_1 - d_1'(t))^2) \\
&+ 2a\bar{K}Q_1 w_t w_x\Big\} \qquad (3.47)
\end{aligned}$$

and

$$\begin{aligned}
P_\beta(z) := &-\bar{K}_x(Q_1h_1)'(U_{s_1}) - \bar{K}(Q_1h_1)''(U_{s_1})U_{s_1}' + (d_1''(t) - d_1'(t))\bar{K}_x Q_1 \\
&- d_1'(t)\bar{K}Q_{1x} - ((f'(U_p) - f'(U_{s_1}))K)_x\,. \qquad (3.48)
\end{aligned}$$

It is clear that

$$\left|\frac{K_x}{K}\right| = \left|\frac{\bar{K}Q_{1x} + \bar{K}_x Q_1}{\bar{K}Q_1}\right| \le \left|\frac{Q_{1x}}{Q_1}\right| + \left|\frac{\beta}{a}\frac{(x + \eta_1 - \eta^*)/a}{\langle(x + \eta_1 - \eta^*)/a\rangle^2}\right| \le \frac{C}{a} \ll 1 \qquad (3.49)$$

for $a \gg 1$. Denoting by D_7 and D_8 the discriminates of G_7 and G_8 respectively, we have, due to (3.49), that

$$D_7 = -1 - 2(s_1 - d_1'(t))K_x/K < 0\,, \qquad (3.50)$$

$$D_8 = 4[(f' + aK_x/K)^2 - a(1 + (s_1 - d_1'(t))K_x/K)^2] < 0\,. \qquad (3.51)$$

Thus, we get

$$G_7 := Kw_t^2 + Kw_t w + \frac{1}{2}(K + (s_1 - d_1'(t))K_x)w^2 \ge C^{-1}K(w^2 + w_t^2)\,, \qquad (3.52)$$

$$\begin{aligned}
G_8 := &(K + (s_1 - d_1'(t))K_x)w_t^2 + 2(f'(U)K + aK_x)w_x w_t \\
&+ a(K + (s_1 - d_1'(t))K_x)w_x^2 \ge C^{-1}K(w_x^2 + w_t^2)\,. \qquad (3.53)
\end{aligned}$$

By Lemma 2.1, (2.12) and (3.20), (3.49) we have

$$
\begin{aligned}
&(d_1''(t) - d_1'(t))\bar{K}_x Q_1 - d_1'(t)\bar{K} Q_{1x} - (f'(U_p) - f'(U_{s_1})K)_x \\
&\quad \le C(|d_1'(t)| + |U_s - u_+| + |U_s'|)\bar{K} \\
&\quad \le C(e^{-c_+ x_1/2} + e^{c_-' d_{10}/2})(1+t)^\gamma \langle (\eta - \eta^*)/a \rangle^{\beta - 1} .
\end{aligned}
$$

Therefore, similarly to that in Ref. 17 we can prove that $P_\beta(z)$ satisfies the following lemma.

Lemma 3.7. *Let α be a given positive number. For $\beta \in [0, \alpha]$, there exists a constant $c_1 > 0$ independent of β such that*

$$
P_\beta(\eta) \ge c_1 \beta (1+t)^\gamma \langle (\eta - \eta^*)/a \rangle^{\beta - 1} \quad \text{for any} \quad \eta \in R , \tag{3.54}
$$

provided that $x_1^{-1} + |d_{10}|^{-1} \ll 1$.

Integrating (3.46) over $[0, +\infty) \times [0, t]$, and using (3.53) and (3.54), we obtain, via similar argument in Refs. 17 and 28, the following estimates:

Lemma 3.8. *For any $t \in [0, T]$, it holds, for any $\gamma \ge 0$ and $\beta \in [0, \alpha]$, that*

$$
\begin{aligned}
&(1+t)^\gamma |(w, w_x, w_t)(t)|_\beta^2 + \int_0^t (1+\tau)^\gamma (\beta |w(\cdot, \tau)|_{\beta-1}^2 + |(w_x, w_t)(\cdot, \tau)|_\beta^2) d\tau \\
&\quad \le C \left\{ |(w_0, w_{0x}, w_1)|_\beta^2 + e^{-c_+ x_1/2} + e^{c_-' d_{10}/2} \right\} \\
&\qquad + C\beta \int_0^t (1+\tau)^\gamma \|w_x(\cdot, \tau)\|^2 d\tau + C\gamma \int_0^t (1+\tau)^{\gamma-1} |(w, w_x, w_t)(\cdot, \tau)|_\beta^2 d\tau .
\end{aligned} \tag{3.55}
$$

Moreover, it holds

$$
\begin{aligned}
&(1+t)^\gamma |(w, w_x, w_t)(t)|_{\alpha-\gamma}^2 + (\alpha - \gamma) \int_0^t (1+\tau)^\gamma |w(\cdot, \tau)|_{\alpha-\gamma-1}^2 d\tau \\
&\qquad + \int_0^t (1+\tau)^\gamma |(w_x, w_t)(\tau)|_{\alpha-\gamma}^2 d\tau \\
&\quad \le C \left(|(w_0, w_{0x}, w_1)|_\alpha^2 + e^{-c_+ x_1/2} + e^{c_-' d_{10}/2} \right)
\end{aligned} \tag{3.56}
$$

for γ integer in $[0, \alpha]$, provided that $N_1(T) + |d_{10}|^{-1} + x_1^{-1} \ll 1$.

The estimate (3.56) can be derived from (3.55), with a similar argument as used in the Cauchy problem in Ref. 17 (the original idea can be found in the work by Kawashima and Matsumura[12]). Based on this lemma, as in Ref. 29 (see Lemma 5.2 therein) or in Ref. 32 for the Burger's equation, we may immediately get the following optimal decay rate.

Lemma 3.9. *It holds for any* $\varepsilon > 0$

$$(1+t)^{\alpha}\|(w,w_x,w_t)(t)\|^2 + (1+t)^{-\varepsilon}\int_0^t (1+\tau)^{\alpha+\varepsilon}\|(w_x,w_t)(\cdot,\tau)\|^2 d\tau$$
$$\leq C\left(|(w_0,w_{0x},w_1)|_{\alpha}^2 + e^{-c_+x_1/2} + e^{c'_-d_{10}/2}\right). \tag{3.57}$$

For the higher derivatives of the solution, by a similar procedure as used in Lemmas 3.6 and 3.8, we can have the following estimates.

Lemma 3.10. *It holds for any* $\varepsilon > 0$

$$(1+t)^{\alpha}\|\partial_x(w,w_x,w_t)(t)\|^2 + (1+t)^{-\varepsilon}\int_0^t (1+\tau)^{\alpha+\varepsilon}\|\partial_x(w_x,w_t)(\cdot,\tau)\|^2 d\tau$$
$$\leq C(\|(w_0,w_{0x},w_1)\|_2^2 + |(w_0,w_{0x},w_1)|_{\alpha}^2 + e^{-c_+x_1/2} + e'_-d_{10}/2). \tag{3.58}$$

Combining Lemmas 3.8 and 3.10, we complete the proof of Theorem 3.4.

4. IBVP for Degenerate Case

Due to Oleinik's entropy condition, it always holds, from (2.5) and (2.9), that $f'(u_+) \leq s < 0 < s_1$. Therefore, we only consider the case when the forward travelling wave is degenerate, i.e.

$$f'(u_+) = s < f'(u_-) \tag{DE}$$

and assume that for an integer $n > 0$ it holds

$$f(u) = su + (u-u_+)^{n+1}, \quad \text{as } u \text{ approaches } u_+. \tag{4.1}$$

As mentioned in Secs. 1 and 2, the front wave $(U_{s_1}, V_{s_1})(x - s_1t)$ is chosen to be nondegenerate, namely, which satisfies the Laxian entropy condition

$$f'(u_+) < s_1 < f'(u_-^1).$$

The boundary perturbations cannot be well-controlled like the nondegenerate case $f'(u_+) < s < f'(u_-)$ and $f'(u_+) < s_1 < f'(u_-^1)$. In fact, if we still consider that the initial data are a perturbation of (U_p, V_p) denoted by (3.2), we will find from (3.3) that

$$|d_1(t)| \sim O(1)\int_0^t \{e^{-c'_-|s_1t+d_1(t)+d_{10}|} + |-s\tau + x_1|^{-1/n}\}d\tau \to +\infty,$$
$$\text{as } t \to +\infty. \tag{4.2}$$

Thus, a shift function $d_s(x,t)$ should be used for the backward travelling wave to control the bounds of $|d_1(t)|$. We consider the following two simple cases.

$$\begin{aligned} d_s(x,t) &= (x - st + \alpha_0)^k, \quad k > n, \quad \alpha_0 > 0, \\ d_s(x,t) &= e^{c_0(x-st+\alpha_0)}, \quad c_0 > 0, \quad \alpha_0 > 0. \end{aligned} \tag{4.3}$$

4.1. *Main results*

We first state our essential assumption of this section. It holds

$$\int_0^{\infty} [u_0(x) - U_{s_1}(x + d_{20}) - U_s(x + d_s(x, 0)) + u_+]dx = 0 \tag{4.4}$$

for either shift

$$d_s(x, t) = (x - st + \alpha_0)^k\,, \quad k > n\,, \quad \alpha_0 > 0$$

or

$$d_s(x, t) = e^{c_0(x - st + \alpha_0)}\,, \quad c_0 > 0\,, \quad \alpha_0 > 0$$

under consideration, where d_{20} and α_0 are any given constants satisfying $0 < -d_{20} \leq \alpha_0$.

Denote

$$\begin{aligned}(U_p, V_p) = (U_{s_1}, V_{s_1})(x - s_1 t + d_2(t) + d_{20})\\ + (U_s, V_s)(x - st + d_s(x, t)) - (u_+, v_+)\,,\end{aligned} \tag{4.5}$$

with $0 < -d_{20} \leq \alpha_0$, and $d_2(t)$ to be chosen as a solution of the following ODE

$$\begin{cases} d_2'(t)[u_+ - U_{s_1}(d_{20} - s_1 t + d_2(t))] \\ \qquad = v_- - V_{s_1}(d_{20} - s_1 t + d_2(t)) + v_+ - V_s(x_1 - st), \\ \qquad - \displaystyle\int_0^{+\infty} (\partial_t d_s(x, t) U_s'(x - st + d_s(x, t)) \\ \qquad + \partial_x d_s(x, t) V_s'(x - st + d_s(x, t)))dx\,, \\ d_2(0) = 0\,. \end{cases} \tag{4.6}$$

As in the last section, it can be proved that $d_2(t) \in C^1(0, +\infty)$, $d_2'(t) \in L^1(0, +\infty)$, and $d_2(t) \to d_{2\infty} < +\infty$ as $t \to +\infty$, where the value $d_{2\infty}$ can be determined by

$$\begin{aligned} d_{2\infty} = \frac{1}{u_+ - u_-^1} \Bigg\{ \int_0^{+\infty} \left(U_s \left(x + d_s \left(0, -\frac{y}{s} \right) \right) - u_+ \right) dx \\ + \int_{-\infty}^{0} (u_-^1 - U_{s_1}(x + d_{20}))dx \Bigg\}\,. \end{aligned}$$

Set

$$w_0(x) = -\int_x^{+\infty} (u_0(y) - U_p(y, 0))dy\,, \quad z_0(x) = v_0(x) - V_p(x, 0)\,.$$

Corresponding to Sec. 3.1, we have the following theorem on the existence of global smooth solutions for the IBVP (1.1) and (1.2).

Theorem 4.1. (Convergence) *Let $a > 0$ be a large constant. Suppose that $f \in C^3$, conditions (DE), (4.4), (2.3), (2.5), (2.8) and (2.9) hold, $w_0 \in H^2$, and $z_0 \in H^1$. Assume that it holds*

(i) $k > 2n$, $\alpha_0 > 0$, *for* $d_s(x,t) = (x - st + \alpha_0)^k$,
(ii) $c_0 > 0$, $\alpha_0 \geq |d_{20}|$, *for* $d_s(x,t) = e^{c_0(x-st+\alpha_0)}$.

Then there exists a $\varepsilon_1 > 0$, *such that if* $\|w_0\|_2 + \|z_0\|_1 + |d_{20}|^{-1} + \alpha_0^{-1} < \varepsilon_1$, *a global smooth solution* $(u,v)(x,t)$ *to* (1.1) *and* (1.2) *exists and satisfies*

$$\sup_{x\in R_+} |(u,v)(x,t) - (U_p, V_p)(x,t)| \to 0\,, \quad \textit{as } t \to +\infty\,.$$

To obtain the following theorems on the exponential decay rate, we only consider the case when $d_s(x,t) = e^{c_0(x-st+\alpha_0)}$ with

$$c_0 > \frac{n}{4}\max\left\{c'_-,\ c'_+,\ -\frac{2c'_-s_1}{s}\right\}. \tag{4.7}$$

Theorem 4.2. (Exponential Rate) *Assume that the hypotheses of Theorem* 4.1 *and* (4.7) *hold. Suppose* $w_0 \in H^2_{Q_{0,2}}$, $z_0 \in H^1_{Q_{0,2}}$. *There exist constants* $\varepsilon_2 > 0$ *and* $\theta = \theta(|u_+ - u^1_-|, |u_+ - u_-|, a) > 0$ *such that if* $a(|w_0|_{2,Q_{0,2}} + |z_0|_{1,Q_{0,2}} + |d_{20}|^{-1} + \alpha_0^{-1}) \leq \varepsilon_2$, *then the IBVP* (1.1) *and* (1.2) *has a unique global solution* $(u,v)(x,t)$ *satisfying*

$$u - U_p \in C^0(0,\infty; H^1_{Q_2}) \cap L^2(0,\infty; H^1_{Q_2})\,,$$
$$v - V_p \in C^0(0,\infty; H^1_{Q_2}) \cap L^2(0,\infty; H^1_{Q_2})$$

and

$$\sup_{x\in R_+} |(u,v)(x,t) - (U_p,V_p)(x,t)| \leq CN_1 e^{-\theta t/2}\,, \tag{4.8}$$

where $N_1 = |w_0|_{2,Q_{0,2}} + |z_0|_{1,Q_{0,2}} + e^{c'_- d_{10}/4} + e^{-\tilde{c}_0\alpha_0/4}$ *with* $\tilde{c}_0 = \min\{c_0/n,\ \frac{4}{n}c_0 - c'_-\}$.

For algebraic decay rates, we are able to deal with the two cases given by (4.6). We have

Theorem 4.3. (Algebraic Rate) *Assume that the hypotheses of Theorem* 4.1 *hold. Suppose* $w_0 \in L^2_\alpha \cap H^2$, $z_0 \in L^2_\alpha \cap H^1$ *for some* $\alpha > 0$ *satisfying that*

(i) $\alpha > 0$, *for* $d_s(x,t) = e^{c_0(x-st+\alpha_0)}$,
(ii) $0 < \alpha < -2 + 2k/n$, *for* $d_s(x,t) = (x - st + \alpha_0)^k$.

Then, if (w_0, z_0) *is small enough in* $(L^2_\alpha \cap H^2) \times (L^2_\alpha \cap H^1)$, *the IBVP* (1.1) *and* (1.2) *has a unique global solution* $(u,v)(x,t)$ *satisfying*

$$\sup_{x\in R_+} |(u,v)(x,t) - (U_p,V_p)(x,t)| \leq CN_2(1+t)^{-\alpha/2}\,, \tag{4.9}$$

where $N_2 = |(w_0,z_0)|_\alpha + \|w_0\|_2 + \|z_0\|_1 + e^{c'_- d_{10}/4} + \tilde{c}$ *with*

$$\tilde{c} = \begin{cases} e^{-c_0\alpha_0/4n}\,, & \textit{for } d_s(x,t) = e^{c_0(x-st+\alpha_0)}\,, \\ \alpha_0^{(2+\alpha-2k/n)/4}\,, & \textit{for } d_s(x,t) = (x-st+\alpha_0)^k\,. \end{cases}$$

4.2. *Reformulation of original problems and proofs*

The reformulation of the problems is similar to that in Sec. 3.2. Setting

$$\partial_x w(x,t) = u(x,t) - U_p(x,t),\ z(y,t) = v(x,t) - V_p(x,t)\,, \quad t \geq 0\,,$$

where $(U_p, V_p)(x,t)$ is given by (4.5), we have

$$\begin{cases} \partial_t w + d_2'(t)[U_{s_1}(x+\eta_1) - u_+] + z \\ \qquad - \displaystyle\int_x^{+\infty} (\partial_t d_s(y,t) U_s'(y - st + d_s(y,t)) \\ \qquad + \partial_x d_s(y,t) V_s'(y - st + d_s(y,t)))dy = 0\,, \\ \partial_t z + d_2'(t)\partial_x V_{s_1}(x+\eta_1) + a\partial_x^2 w + z\partial_t d_s(x,t) V_s' + a\partial_x d_s(x,t) U_s' \\ \qquad = f(U_p + w_x) - f(U_{s_1}(x+\eta_1)) - f(U_s(x+\eta_2)) + f(u_+)\,, \end{cases}$$

with $\eta_1 =: -s_1 t + d_2(t) + d_{20}$ and $\eta_2 =: -st + d_s(0,t)$. Where $d_2(t)$ satisfies the ordinary differential equation given in (4.6). Then it holds

$$\int_0^{+\infty} (u - U_p)dx = \int_0^{\infty} [u_0(x) - U_p(x,0)]dx = 0 \tag{4.10}$$

due to the essential assumption (4.4). Thus, w satisfies the following equation

$$L_1(w) =: w_{tt} + w_t - aw_{xx} + f'(U_p)w_x = g_3(x,t) + g_4(x,t)\,, \tag{4.11}$$

where

$$\begin{aligned} g_3(x,t) &= d_2'(t)V_{s_1}'(x+\eta_1) + (s_1 - d_2'(t))d_1'(t)U_{s_1}'(x+\eta_1) \\ &\quad + (d_1'(t) + d_2''(t))[u_+ - U_{s_1}(x+\eta_1)]\,, \\ &\quad + \partial_t d_s(x,t)V_s'(x+\eta_2) + a\partial_x d_s(x,t)U_s'(x+\eta_2) \\ &\quad + \int_x^{+\infty} \partial_t\{\partial_t d_s(y,t)U_s'(y+\eta_2) + \partial_x d_s(y,t)V_s'(y+\eta_2)\}dy \\ &\quad + \int_x^{+\infty} \{\partial_t d_s(y,t)U_s'(y+\eta_2) + \partial_x d_s(y,t)V_s'(y+\eta_2)\}dy\,, \\ g_4(x,t) &= -\{f(U_p + w_x) - f(U_{s_1}(x+\eta_1)) - f(U_s(x+\eta_2)) \\ &\quad - f'(U_p)w_x + f(u_+)\}\,. \end{aligned} \tag{4.12}$$

The corresponding initial and boundary values are

$$\begin{cases} w(x,0) = w_0(x)\,, \quad w_t(x,0) = -z_0(x) + z_0(0)\dfrac{u_+ - U_{s_1}(x+d_{20})}{u_+ - U_{s_1}(d_{20})} =: w_1(x),\ x \geq 0\,, \\ w(0,t) = 0\,, \quad t \geq 0\,. \end{cases} \tag{4.13}$$

We have the reformulation of Theorems 4.1–4.3 as follows.

Theorem 4.4. (Convergence) *Assume that the hypotheses of Theorem* 4.1 *hold. Then, the IBVP* (4.11) *and* (4.2) *has a unique global solution* $w(x,t)$ *satisfying*

$$w \in C^0(0,\infty; H^3) \cap L^2(0,\infty; H^3_{Q_2})\,, \quad w_t \in C^0(0,\infty; H^2_{Q_2}) \cap L^2(0,\infty; H^2_{Q_2})$$

and

$$\sup_{x\in R_+} |(w, w_t)(x,t)| \to 0\,, \quad \text{as } t \to +\infty\,.$$

Theorem 4.5. (Exponential Rate) *Assume that the hypotheses of Theorem* 3.3 *hold. Then, the IBVP* (3.13) *and* (3.15) *has a unique global solution* $w(x,t)$ *satisfying*

$$w \in C^0(0,\infty; H^2_{Q_2}) \cap L^2(0,\infty; H^2_{Q_2})\,, \quad w_t \in C^0(0,\infty; H^1_{Q_2}) \cap L^2(0,\infty; H^1_{Q_2})$$

and

$$|w(\cdot,t)|^2_{2,Q_2} + |w_t(\cdot,t)|^2_{1,Q_2} + \theta \int_0^t [|w(\cdot,\tau)|^2_{2,Q_2} + |w_t(\cdot,\tau)|^2_{1,Q_2}]d\tau \le CN_1^2\,, \tag{4.14}$$

namely,

$$\sup_{x\in R_+} |(w_x, w_t)(x,t)| + |w(\cdot,t)|^2_{2,Q_2} + |w_t(\cdot,t)|^2_{1,Q_2} \le CN_1^2 e^{-\theta t}\,, \quad t \ge 0\,. \tag{4.15}$$

Theorem 4.6. (Algebraic Rate) *Assume that the hypotheses of Theorem* 4.3 *hold. Then, the IBVP* (3.13) *and* (3.15) *has a unique global solution* $w(x,t)$ *satisfying*

$$\sup_{x\in R_+} |(w, w_x, w_t)(x,t)| \le CN_2^2 (1+t)^{-\alpha/2}\,. \tag{4.16}$$

The procedure to prove the above theorems on the existence of solution and its exponential and algebraic decay rates is similar to those in Refs. 6, 17 and 28 and in Sec. 3 with the help of Lemma 4.11 on $d_2(t)$. We omit the details.

Lemma 4.11. *Under the assumptions of Theorem* 4.1, *it holds*

$$d_2(t) - s_1 t \le 0\,, \quad t \in [0,T] \tag{4.17}$$

and

(i) *for* $d_s(x,t) = (x - st + \alpha_0)^k$,

$$|\partial_t^i d_2'(t)| \sim \{e^{c'_-(-s_1 t + d_2(t) + d_{20})} + (x - st + \alpha_0)^{-i-k/n}\}\,, \quad i = 1,2\,,\ t \in [0,T]\,, \tag{4.18}$$

(ii) *for* $d_s(x,t) = e^{c_0(x-st+\alpha_0)}$,

$$|\partial_t^i d_2'(t)| \sim \{e^{c'_-(-s_1 t + d_2(t) + d_{20})} + e^{\frac{1}{n}c_0(st-\alpha_0)}\}, \quad i = 1, 2,\ t \in [0,T]\,. \tag{4.19}$$

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