DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **36**, Number **3**, March **2016**

doi:10.3934/dcds.2016.36.1331

pp. **1331–1353**

ASYMPTOTIC STABILITY OF NON-MONOTONE TRAVELING WAVES FOR TIME-DELAYED NONLOCAL DISPERSION EQUATIONS

Rui Huang

School of Mathematical Sciences, South China Normal University
Guangzhou, Guangdong, 510631, China

Ming Mei*

Department of Mathematics, Champlain College Saint-Lambert
Quebec, J4P 3P2, Canada
and
Department of Mathematics and Statistics, McGill University
Montreal, Quebec, H3A 2K6, Canada

Kaijun Zhang

School of Mathematics and Statistics, Northeast Normal University
Changchun, Jilin, 130024, China

Qifeng Zhang

School of Science, Zhejiang Sci-Tech University
Hangzhou, Zhejiang, 310018, China

(Communicated by Masaharu Taniguchi)

Abstract. This paper is concerned with the stability of non-monotone traveling waves to a nonlocal dispersion equation with time-delay, a time-delayed integro-differential equation. When the equation is crossing-monostable, the equation and the traveling waves both loss their monotonicity, and the traveling waves are oscillating as the time-delay is big. In this paper, we prove that all non-critical traveling waves (the wave speed is greater than the minimum speed), including those oscillatory waves, are time-exponentially stable, when the initial perturbations around the waves are small. The adopted approach is still the technical weighted-energy method but with a new development. Numerical simulations in different cases are also carried out, which further confirm our theoretical result. Finally, as a corollary of our stability result, we immediately obtain the uniqueness of the traveling waves for the non-monotone integro-differential equation, which was open so far as we know.

1. **Introduction.** Subsequently to our previous study [15] on the stability of monotone traveling waves to the nonlocal dispersion equation, in this paper we further

2010 *Mathematics Subject Classification.* Primary: 35K57; Secondary: 35C07, 92D25.

Key words and phrases. Oscillating wavefronts, asymptotic stability, nonlocal dispersion equation, time-delay, anti-weighted energy method.

∗ Corresponding author.

consider the stability of non-monotone traveling waves to the nonlocal dispersion equation with time-delay

$$\begin{cases} v_t - D(J*v - v) + d(v) = K * b(v(t-r,\cdot)), \\ v(s,x) = v_0(s,x), \quad s \in [-r,0], \ x \in R. \end{cases} \tag{1}$$

This model represents the population dynamics of single species like Australian blowflies [9, 13, 14, 15, 31, 37, 38]. Here, $v(t,x)$ denotes the total population of matured species at time t and location x, $J(x)$ and $K(x)$ are non-negative, unit and symmetric kernels, where, $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, and the convolution

$$J * v = \int_R J(x-y)v(t,y)dy$$

is the rate at which individuals are arriving to position x from all other places, while, the term

$$-v(t,x) = -\int_R J(x-y)v(t,x)dy$$

stands the rate at which they are leaving the location x to travel to all other places. The form $D(J*v-v)$ is called the nonlocal dispersion and represents transportation due to long range dispersion mechanisms, where $D>0$ is the coefficient of spacial diffusion for the species. It can be verified by Fourier transform and Taylor formula [2, 3, 4, 15, 16, 17, 37] that

$$J*v - v \approx Cv_{xx},$$

when $J(x)$ is compactly supported. The parameter $r>0$ is the maturation time for the species, mathematically, we call it the time-delay. $d(v)$ is the death rate function, and $b(v)$ is the birth rate function. Summarizing from the ecological background set-up, we may assume throughout the paper that:

(H$_1$) $d(v)$ is a non-negative, C^2-smooth increasing function, and satisfies $d'(v) \geq d'(0) > 0$ for $v \in [0,\infty)$;

(H$_2$) $b(v)$ is a non-negative, non-monotone, C^2-smooth function, and satisfies $b'(0) \geq |b'(v)|$ for $v \in [0,\infty)$;

(H$_3$) Both kernels $J(x)$ and $K(x)$ are non-negative, symmetric and unit,

$$J(x) \geq 0, \quad J(-x) = J(x), \quad \int_R J(x)dx = 1, \tag{2}$$

$$K(x) \geq 0, \quad K(-x) = K(x), \quad \int_R K(x)dx = 1, \tag{3}$$

and satisfies

$$\int_R |x|J(x)e^{-\eta x}dx < \infty \ \text{ and } \int_R |x|K(x)e^{-\eta x}dx < \infty \text{ for any } \eta > 0; \tag{4}$$

(H$_4$) Two constant equilibria of (1): $v_- = 0$ is unstable and v_+ is stable, namely, $d(0)=0$, $b(0)=0$, $d(v_+) = b(v_+)$, $d'(0) - b'(0) < 0$ and $d'(v_+) - b'(v_+) > 0$.

There are two well-known examples of the equation (1) satisfying (H$_1$)-(H$_4$). One is the so-called nonlocal dispersion Nicholson's blowflies equation [11, 13, 14, 24, 22, 23, 25, 32] with

$$d(v) = \delta v \ \text{ and } \ b(v) = pve^{-av}, \quad \text{for } \delta > 0, \ a > 0, \ p > 0, \tag{5}$$

where $v_- = 0$, $v_+ = \frac{1}{a}\ln\frac{p}{\delta}$. When $\frac{p}{\delta} > e$, the birth rate function $b(v)$ is a uni-modality function with the maximum at $v_* := \frac{1}{a} \in (0, v_+)$, and it can be verified that $|b'(v)| \le b'(0)$ for $v \in [0,\infty)$. The other is the so-called nonlocal dispersion Mackey-Glass equation [20, 12, 18, 22, 23] with

$$d(v) = \delta v \quad \text{and} \quad b(v) = \frac{pv}{1+av^q}, \quad \text{for } \delta > 0,\ a > 0,\ \ p > 0, \tag{6}$$

where $v_- = 0$, $v_+ = \left(\frac{p-\delta}{\delta a}\right)^{\frac{1}{q}}$. When $\frac{p}{\delta} > \frac{q}{q-1}$, the birth rate function $b(v)$ is non-monotone with uni-modality at $v_* := [a(q-1)]^{-1/q} \in (0, v_+)$, and it can be verified that $|b'(v)| \le b'(0)$ for $v \in [0,\infty)$.

The main purpose in this paper is to study the stability for all non-critical traveling waves to (1), particularly, for those oscillating waves. The traveling wavefronts for (1) connecting two steady states $v_\pm$ at far fields are the special solutions to (1) in the form of $v(t,x) = \phi(x+ct)$, namely,

$$\begin{cases} c\phi'(\xi) - D\Big(\displaystyle\int_R J(y)\phi(\xi-y)dy - \phi(\xi)\Big) + d(\phi(\xi)) \\ \qquad\qquad\qquad\qquad = \displaystyle\int_R K(y)b(\phi(\xi-y-cr))dy, \\ \phi(\pm\infty) = v_\pm, \quad \phi(\xi) \ge 0, \end{cases} \tag{7}$$

where $\xi = x+ct$ and $' = \frac{d}{d\xi}$. The existence and uniqueness of the traveling waves for the local/nonlocal dispersion equations with or without time-delay have been intensively studied recently in [5, 6, 7, 8, 15, 31, 36]. When the dynamical system is non-monotone, the effect of time-delay is essential, because, when the time-delay is large, the traveling waves will be oscillating, or even no traveling waves exist. This phenomenon is totally different from the case without time-delay, even if the governing equation is non-monotone. The existence of the monotone/non-monotone traveling waves has been investigated in [37, 38] recently, when the equation is non-monotone. But, the uniqueness of the traveling waves in this case is unknown yet.

Now let us have a brief review on the existence of these waves. Let $\phi(\xi) = \phi(x+ct)$ be a traveling wave of (1), namely, the solution of (7). In order to specify the wave speed c, let us linearize (7) around $\phi = 0$, then

$$c\phi' - D\Big(\int_R J(y)\phi(\xi-y)dy - \phi(\xi)\Big) + d'(0)\phi = b'(0)\int_R K(y)\phi(\xi-y-cr)dy.$$

Heuristically, we expect $\phi(\xi) = O(1)e^{-\lambda|\xi|} = O(1)e^{\lambda\xi} \to 0$ as $\xi \to -\infty$ for some eigenvalue $\lambda > 0$. Substituting this to the above linearized equation, we obtain the following characteristic equation for the pair of (c,λ):

$$c\lambda - D\int_R J(y)e^{-\lambda y}dy + D + d'(0) = b'(0)e^{-\lambda cr}\int_R K(y)e^{-\lambda y}dy. \tag{8}$$

To investigate the admission of (c,λ) to the above characteristic equation, we denote

$$\begin{aligned} G_c(\lambda): &= c\lambda - D\int_R J(y)e^{-\lambda y}dy + D + d'(0), \\ H_c(\lambda): &= b'(0)\int_R K(y)e^{-\lambda(y+cr)}dy. \end{aligned}$$

Since

$$\begin{aligned} &G_c''(\lambda) = -D\int_R y^2 J(y)e^{-\lambda y}dy < 0, \text{ and} \\ &H_c''(\lambda) = b'(0)e^{-\lambda cr}\int_R (y+cr)^2 K(y)e^{-\lambda y}dy > 0, \end{aligned}$$

so $G_c(\lambda)$ is concave downward and $H_c(\lambda)$ is concave upward. Note that $G_c(0) = d'(0) < b'(0) = H_c(0)$, from the graphs of $G_c(\lambda)$ and $H_c(\lambda)$, then $G_c(\lambda)$ and $H_c(\lambda)$ are either touched at a unique tangent point, or intersect with two points, or never cross, based on different value of $c > 0$. For the tangent point (c_*, λ_*), it can be uniquely determined by

$$G_{c_*}(\lambda_*) = H_{c_*}(\lambda_*), \quad G'_{c_*}(\lambda_*) = H'_{c_*}(\lambda_*),$$

namely,

$$c_*\lambda_* - D\int_R J(y)e^{-\lambda_* y}dy + D + d'(0) = b'(0)e^{-\lambda_* c_* r}\int_R K(y)e^{-\lambda_* y}dy, \quad (9)$$

$$c_* + D\int_R yJ(y)e^{-\lambda_* y}dy = -b'(0)e^{-\lambda_* c_* r}\int_R (y + c_* r)K(y)e^{-\lambda_* y}dy. \quad (10)$$

When $c > c_* > 0$, there exist two numbers $0 < \lambda_1 < \lambda_2$, such that the characteristic equation (8) has two solutions (c, λ_1) and (c, λ_2). That is, $G_c(\lambda_i) = H_c(\lambda_i)$ for $i = 1, 2$, and $G_c(\lambda) > H_c(\lambda)$ for $\lambda_1 < \lambda < \lambda_2$, namely,

$$c\lambda_i - D\int_R J(y)e^{-\lambda_i y}dy + D + d'(0) = b'(0)e^{-\lambda_i cr}\int_R K(y)e^{-\lambda_i y}dy, \quad i = 1, 2, \quad (11)$$

and, for $\lambda \in (\lambda_1, \lambda_2)$,

$$c\lambda - D\int_R J(y)e^{-\lambda y}dy + D + d'(0) > b'(0)e^{-\lambda cr}\int_R K(y)e^{-\lambda y}dy. \quad (12)$$

When $0 < c < c_*$, the characteristic equation (8) has no solution. Therefore, as showed in [37, 38], when $c \geq c_* > 0$, the traveling waves of (7) exist, and when $c < c_*$, no traveling waves exist. For the existing traveling waves, they may be non-monotone and oscillatory around v_+ when the time-delay r is suitably large [12, 19, 33, 34]. Furthermore, notice from [9], when $|b'(v_+)| > d'(v_+)$, there exists a number $\bar{r} > 0$, if the time-delay is bigger: $r \geq \bar{r}$, there will be no traveling waves. Such a phenomenon for the local Nicholson's blowflies model has been theoretically studied and numerically reported recently in [12, 19]. The uniqueness of traveling wavefronts in the case of $|b'(v_+)| \geq d'(v_+)$ was proved in [1]. But, to our best knowledge, the uniqueness of the traveling waves for the nonlocal and non-monotone equation (1) in the case $|b'(v_+)| > d'(v_+)$ with $0 < r < \bar{r}$ are still unknown.

The main target of the present paper is to show the stability of the monotone/non-monotone wavefronts to (1) for all $c > c_*$, where the wave speed c under consideration can be allowed sufficiently close to the minimum wave speed c_*. We concentrate ourselves on the non-critical waves with $c > c_*$, and will leave the more challenging case of the critical waves with $c = c_*$ for future. The difficulties we have to face in this paper are the nonlocality and the non-monotonicity for the dispersal reaction-diffusion equation with time-delay (1).

Now let us draw a background picture on the progress of the study in this subject. When the birth rate function $b(v)$ and the death rate function $d(v)$ are monotone for $v \in [0, v_+]$ under consideration, the equation (1) and its traveling waves both are monotone for any time-delay $r > 0$. In this case, when the wave speed is sufficiently large $c \gg 1$, Pan-Li-Lin [31] first showed the local stability of those fast traveling waves by the weighted energy method developed in [24, 25, 22, 23, 28]. Later then, Huang-Mei-Wang [15] showed that, all monotone traveling waves $\phi(x+ct)$ with $c \geq c_*$ are globally stable, and particularly, the non-critical monotone traveling waves $\phi(x+ct)$ with $c > c_*$ are exponentially stable, and the critical monotone waves $\phi(x+$

c_*t) are algebraically stable. The adopted method is the combination of Fourier transform and the weighted energy estimates with the monotone technique. When the birth rate function $b(v)$ is non-monotone for $v \in [0, v_+]$ under consideration, the monotone technique cannot be applied any more, because the equation (1) losses its monotonicity, and the traveling waves will be oscillating or even not exist when the time-delay r is big enough. Following [24, 25, 22, 23, 35], by the regular weighted energy method, Zhang-Ma [38] proved that the waves with sufficiently large $c \gg 1$ are locally stable. The exponential convergence rate is also derived. However, the most interesting cases are for the slower waves with $c > c_*$ (c can be arbitrarily close to c_*), and particularly, the case for the critical oscillating waves with $c = c_*$. Very recently, for the local Nicholson's blowflies equation, Lin-Lin-Lin-Mei [19] succeeded in obtaining the stability of all monotone/non-monotone traveling waves with $c > c_*$, by means of the regular L^2-weighted energy method with a new development by a nonlinear Halanay's inequality. But, from the previous studies [25, 23], we understand that, the regular L^2-weighted energy method cannot be perfectly applied to solve the stability of these slower waves with $c \geq c_*$ for the nonlocal equation (the equation involves integrals). The nonlocal terms usually yield some gaps in the L^2-energy estimates, which cause us to need to take $c \gg 1$ so then we can control these gaps. In order to avoid such a trouble, for the case of nonlocal but still monotone equation, Mei-Ou-Zhao [26], Mei-Wang [27] and Huang-Mei-Wang [15] showed the stability for all (monotone) traveling waves with $c \geq c_*$ by the L^1-energy method but it sufficiently depends on the advantage of the monotonicity of both the equation itself and the traveling waves. Notice that, in this paper the equation (1) is nonlocal and non-monotone, and the traveling waves may be oscillating when r is big, so the above mentioned approaches, including the regular weighted energy method, the monotone method and Fourier transform method, they all seem to fail in obtaining the stability of the wavefronts for (1). Therefore, we need to look for a new strategy to treat this nonlocal and non-monotone case. Inspired by the study on classical Fisher-KPP equation by Moet [30] and the study on hyperbolic p-system by Matsumura-Mei [21], where they introduced a suitable transform function (or say, an anti-weight) to switch the equation to a new equation (we call it the anti-weighted energy method), we realize that some gaps caused by the integral terms in the L^2 weighted energy estimates may not come out with this new transformed equation. On the other hand, we recognize that the oscillations usually occur around the stable node u_+ when $\xi \to +\infty$, and we can come over the difficulty caused by these oscillations by the nonlinear Halanay's inequality [19]. So, with such two observations, we may prove the stability for these oscillatory traveling waves with any $c > c_*$ to the integro-differential equation (1).

The paper is organized as follows. In Section 2, we state our main stability theorems for the non-critical traveling waves to (1), and give the applications to the dispersion Nicholson's blowflies equation and the dispersion Mackey-Glass equation, respectively. Some numerical simulations are also carried out in this section. We test the dispersion Nicholson's blowflies equation with $p/\delta > e^2$ such that the working equation is non-monotone. When the time delay is small, the solution numerically behaves like a stable monotone traveling wave, and when the time-delay is a bit large, the solution then numerically behaves like a stable oscillatory traveling wave. These reported results further confirm our stability theorems. Then, in what follows, we concentrate ourselves on the theoretical proof of the stability theorems. In Section 3, we reformulate the equation (1) to the corresponding perturbed equation around a given non-critical traveling wave, and state the stability of the new perturbed equation. Section 4 is devoted to the proof of stability theorem. The adopted approach is the so-called transformed energy method and Halanay's inequality. In Section 5, as a corollary of our stability theorem, we prove the uniqueness of traveling waves in the non-monotone case, which is still open so far as we know.

In what follows, we always denote a generic constant by $C > 0$, and a specific positive constant by $C_i > 0$ $(i = 0, 1, 2, \cdots)$. Let I be an interval, typically $I = R$. $L^2(I)$ is the space of the square integrable functions defined on I, and $H^k(I)$ $(k \geq 0)$ is the Sobolev space of the L^2-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i}f$ $(i = 1, \cdots, k)$ also belong to $L^2(I)$. Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0, T]; \mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0, T]$ and $L^2([0, T]; \mathcal{B})$ is the space of the $\mathcal{B}$-valued L^2-functions on $[0, T]$.

2. **Main theorems and numerical simulations.** In this section we state the exponential stability of all non-critical monotone/non-monotone traveling waves to (1), and its applications to Nicholson's blowflies equation (5) and Mackey-Glass equation (6), respectively. The proof of this stability theorem will be carried out in the next two sections. To numerically support our theoretical results, we will also present some numerical simulations at the end of this section.

For a given monotone/non-monotone non-critical traveling wave $\phi(x+ct) = \phi(\xi)$ of (7) with $c > c_*$, it is known that

$$\phi(\xi) = O(1)e^{-\lambda_1|\xi|} \to 0 \quad \text{as } \xi \to -\infty,$$

where $\lambda_1 = \lambda_1(c)$ is the eigenvalue specified in (11), and $G_c(\lambda) - H_c(\lambda) > 0$ for $\lambda \in (\lambda_1, \lambda_2)$ (see (12) above). Now we define a weight function as follows

$$w(x) = e^{-2\lambda x}, \quad \text{for } \lambda \in (\lambda_1, \lambda_2). \tag{13}$$

We now state our main stability theorems as follows.

Theorem 2.1 (Stability of monotone/non-monotone traveling waves)**.** *Under the conditions* (H_1)-(H_4), *for any given traveling wave* $\phi(x+ct)$ *with* $c > c_*$ *to Eq.* (1), *whatever it is monotone or non-monotone, suppose that* $V_0(s,x) := v_0(s,x) - \phi(x + cs) \in C([-r, 0]; C(R))$, $\sqrt{w(x)}V_0(s,x) \in C([-r, 0]; H^1(R)) \cap L^2([-r, 0]; H^1(R))$, *and* $\lim_{x\to+\infty} V_0(s,x) =: V_{0,\infty}(s) \in C[-r, 0]$ *exists uniformly with respect to* $s \in [-r, 0]$, *and*

$$\max_{s\in[-r,0]} \|V_0(s)\|_C^2 + \|\sqrt{w}V_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}V_0(s)\|_{H^1}^2 ds \leq \delta_0^2,$$

for some positive number δ_0.

1. *When* $d'(v_+) \geq |b'(v_+)|$, *for any time-delay* $r > 0$, *then the solution* $v(t,x)$ *of* (1) *uniquely and globally exists in time, and satisfies*

$$\begin{aligned} &v(t,x) - \phi(x+ct) \in \mathcal{C}_{unif}[-r, \infty), \\ &\sqrt{w(x)}[v(t,x) - \phi(x+ct)] \in C([-r, \infty); H^1(R)), \\ &\partial_x\big(\sqrt{w(x)}[v(t,x) - \phi(x+ct)]\big) \in L^2([-r, \infty); H^1(R)), \end{aligned} \tag{14}$$

and

$$\sup_{x\in R} |v(t,x) - \phi(x+ct)| \leq Ce^{-\mu t}, \quad t > 0 \tag{15}$$

for some constant $\mu > 0$, where $\mathcal{C}_{unif}[-r,T]$ is defined by, for $0 < T \le \infty$,

$$\mathcal{C}_{unif}[-r,T] := \{ \quad u(t,x) \in C([-r,T]\times R) \text{ such that } \lim_{x\to+\infty} u(t,x) \text{ exists uniformly in } t \in [-r,T]\}.$$

2. *When $d'(v_+) < |b'(v_+)|$, but the time-delay is small: $0 < r < \bar{r}$, where*

$$\bar{r} := \frac{\pi - \arctan(\sqrt{|b'(v_+)|^2 - d'(v_+)^2}/d'(v_+))}{\sqrt{|b'(v_+)|^2 - d'(v_+)^2}}, \tag{16}$$

then the solution $v(t,x)$ of (1) uniquely and globally exists in time, and satisfies (14) and the stability (15).

Since the uniqueness of the traveling waves for the non-monotone nonlocal dispersion equation (1) still remains open, here we address it as a corollary of Theorem 2.1.

Corollary 1 (Uniqueness of traveling waves)**.** *Let (H_1)-(H_4) hold, and let either $d'(v_+) \ge |b'(v_+)|$ with any time-delay $r > 0$, or $d'(v_+) < |b'(v_+)|$ but with a small time-delay $0 < r < \bar{r}$, where $\bar{r}$ is defined in (16). Then, for any traveling waves $\phi(x+ct)$ of (1), whatever they are monotone or non-monotone, with the same speed $c > c_*$ and the same exponential decay at $\xi = -\infty$:*

$$\phi(\xi) = O(1)e^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty, \tag{17}$$

they are unique up to shift.

Now, we apply the stability theorem 2.1 to the nonlocal dispersion Nicholson's blowflies equation (1) with $b(v) = pve^{-av}$ and $d(v) = \delta v$. Here, $v_- = 0$, $v_+ = \frac{1}{a}\ln\frac{p}{d}$, and $b(v)$ is non-monotone for $v \in [0,\infty)$ when $\frac{p}{\delta} > e$, and automatically satisfies the conditions (H$_1$)-(H$_4$). As a direct application of Theorem 2.1, we immediately obtain the following stability of monotone/non-monotone traveling waves for the case with the Nicholson's birth rate $b(v) = pve^{-av}$.

Theorem 2.2 (Stability of monotone/non-monotone traveling waves)**.** *Let $b(v) = pve^{-av}$ and $d(v) = \delta > 0$, and the kernels $J(x)$ and $K(x)$ satisfy (H_3). For any given traveling wave $\phi(x+ct)$ with $c > c_*$ connecting with $v_- = 0$ and $v_+ = \frac{1}{a}\ln\frac{p}{\delta}$, whatever it is monotone or non-monotone, suppose that $V_0(s,x) := v_0(s,x) - \phi(x+cs) \in C([-r,0];C(R))$, $\sqrt{w(x)}V_0(s,x) \in C([-r,0];H^1(R)) \cap L^2([-r,0];H^1(R))$, and the limit $\lim_{x\to+\infty} V_0(s,x) =: V_{0,\infty}(s) \in C[-r,0]$ exists uniformly with respect to $s \in [-r,0]$, and*

$$\max_{s\in[-r,0]} \|V_0(s)\|_C^2 + \|\sqrt{w}V_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}V_0(s)\|_{H^1}^2 ds \le \delta_0^2,$$

for some positive number $\delta_0 > 0$.

1. *When $e < \frac{p}{\delta} \le e^2$, for any time-delay $r > 0$, then, the solution $v(t,x)$ of (1) uniquely and globally exists in time in the space (14), and the stability (15) with some constant $\mu > 0$ holds for all $t > 0$.*
2. *When $\frac{p}{\delta} > e^2$ but with a small time-delay $0 < r < \bar{r}$, where*

$$\bar{r} := \frac{\pi - \arctan\sqrt{\ln\frac{p}{\delta}(\ln\frac{p}{\delta} - 2)}}{d\sqrt{\ln\frac{p}{\delta}(\ln\frac{p}{\delta} - 2)}}, \tag{18}$$

then, the solution $v(t,x)$ of (1) uniquely and globally exists in time in the space (14), and the stability (15) with some constant $\mu > 0$ holds for all $t > 0$.

Next, we are going to state the stability result for the nonlocal dispersion Mackey-Glass equation (1) with $b(v) = \frac{pv}{1+av^q}$ for $a > 0$, $p > 0$, $q > 1$, and $d(v) = \delta v$. Here, $v_- = 0$, $v_+ = \left(\frac{p-\delta}{\delta a}\right)^{\frac{1}{q}}$, and $b(v)$ is non-monotone for $v \in [0, \infty)$ when $\frac{p}{\delta} > \frac{q}{q-1}$, and automatically satisfies the conditions (H_1)-(H_4). As a direct application of Theorem 2.1, we immediately obtain the following stability of monotone/non-monotone traveling waves for the case with the Mackey-Glass birth rate $b(v) = \frac{pv}{1+av^q}$.

Theorem 2.3 (Stability of monotone/non-monotone traveling waves)**.** *Let $b(v) = \frac{pv}{1+av^q}$ and $d(v) = \delta > 0$, and the kernels $J(x)$ and $K(x)$ satisfy (H_3). For any given traveling wave $\phi(x + ct)$ with $c > c_*$ connecting with $v_- = 0$ and $v_+ = \left(\frac{p-\delta}{\delta a}\right)^{\frac{1}{q}}$, whatever it is monotone or non-monotone, suppose that $V_0(s, x) := v_0(s, x) - \phi(x + cs) \in C([-r, 0]; C(R))$, $\sqrt{w(x)}V_0(s, x) \in C([-r, 0]; H^1(R)) \cap L^2([-r, 0]; H^1(R))$, and the limit $\lim_{x\to+\infty} V_0(s, x) =: V_{0,\infty}(s) \in C^0[-r, 0]$ exists uniformly with respect to $s \in [-r, 0]$, and*

$$\max_{s\in[-r,0]} \|V_0(s)\|_C^2 + \|\sqrt{w}V_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}V_0(s)\|_{H^1}^2 ds \le \delta_0^2,$$

for some positive number $\delta_0 > 0$.

1. *When $\frac{q}{q-1} < \frac{p}{\delta} \le \frac{q}{q-2}$, for any time-delay $r > 0$, then, the solution $v(t, x)$ of (1) uniquely and globally exists in time in the space (14), and the stability (15) with some constant $\mu > 0$ holds for all $t > 0$.*
2. *When $\frac{p}{\delta} > \frac{q}{q-2}$ but with a small time-delay $0 < r < \bar{r}$, where*

$$\bar{r} := \frac{\pi - \arctan\left(\delta^{-1}\sqrt{[(q-1)\frac{p}{\delta} - q]^2 - \delta^2}\right)}{\sqrt{[(q-1)\frac{p}{\delta} - q]^2 - \delta^2}}, \tag{19}$$

   then, the solution $v(t, x)$ of (1) uniquely and globally exists in time in the space (14), and the stability (15) with some constant $\mu > 0$ holds for all $t > 0$.

Finally, at the end of this section, we are going to report some numerical results, which will further demonstrate that the solution behaves like a monotone traveling wave or an oscillatory traveling wave when the time-delay r is small or big.

Let us consider the nonlocal dispersion Nicholson's blowflies equation (1) by taking the kernels as

$$J(x) = K(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2},$$

and the birth rate function and death rate function selected as Nicholson's type:

$$b(v) = pve^{-av}, \quad d(v) = \delta v.$$

Thus, this equation possesses two constant equilibria: $v_- = 0$ and $v_+ = \frac{1}{a}\ln\frac{p}{\delta}$. When $\frac{p}{\delta} > e$, the birth rate $b(v)$ is non-monotone, where $b(v)$ is increasing for $v \in [0, \frac{1}{a}]$ and decreasing for $v \in [\frac{1}{a}, v_+]$. The condition $d'(v_+) \ge |b'(v_+)|$ is equivalent to $e < \frac{p}{\delta} \le e^2$, and $d'(v_+) < |b'(v_+)|$ is for $\frac{p}{\delta} > e^2$. We choose the initial data

$$v_0(s, x) = \frac{v_+}{1 + e^{-kx}} + 0.1(\cos x)e^{-0.001(x-500)^2}, \quad s \in [-r, 0], \tag{20}$$

which implies

$$|v_0(s, x) - v_\pm| = O(1)e^{-k|x|} \quad \text{as } x \to \pm\infty,$$

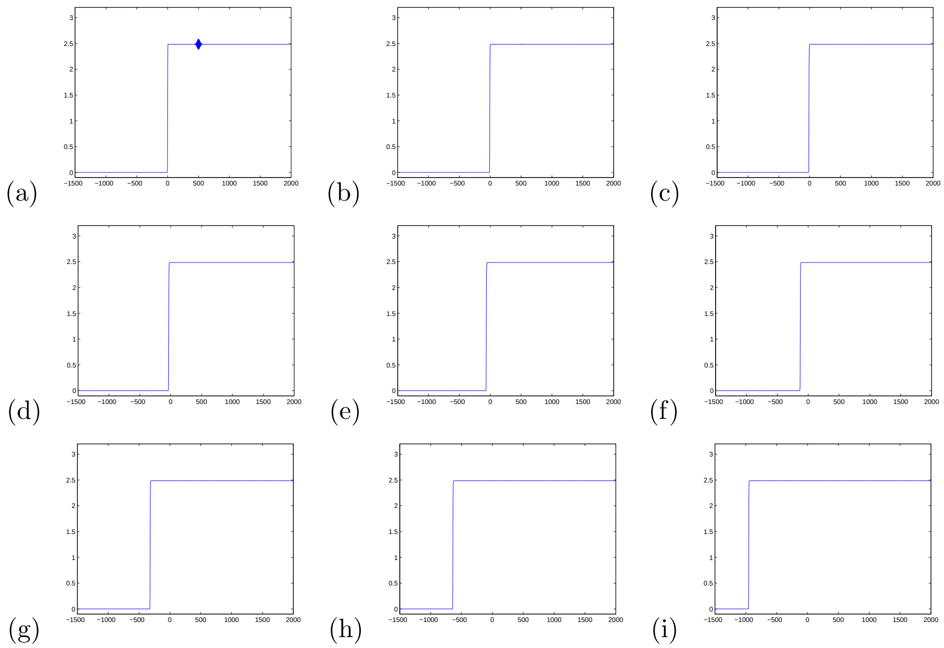

FIGURE 1. Case 1: $\frac{p}{\delta} = 12 > e^2$ with small time-delay $r = 0.1$. From (a) to (i), the solution $v(t, x)$ plots at times $t = 0, 2, 5, 10, 20, 50, 100, 200, 300$, which behaves as a stable monotone traveling wave (no change of the wave's shape after a large time in the sense of stability) and travels from right to left.

where $k > 0$. For simplicity, we take $D = \delta = a = k = 1$, and $p = 12$ so then $\frac{p}{\delta} > e^2$, but leave the time-delay r free. Since $\frac{p}{\delta} > e^2$, from (16), we can calculate $\bar{r} = 2.062047407711962\cdots$. The traveling waves exist for $0 < r < \bar{r}$, and no traveling waves exist for $r \geq \bar{r}$.

Case 1. Convergence to a monotone traveling wave when the time-delay is small. In this case, we take $r = 0.1 < \bar{r}$. The numerical results reported in Figure 1 demonstrate that the solution $v(t, x)$ of (1) behaves like a stable monotone traveling wave after a large time.

Case 2. Convergence to an oscillatory traveling wave when the time-delay is a little big. In this case, we take $r = 1.0 < \bar{r}$. The numerical results reported in Figure 2 demonstrate that the solution $v(t, x)$ of (1) behaves like a stable oscillatory traveling wave after a large time. In order to see how the oscillations of the solution $v(t, x)$ around v_+ behave, let us enlarge $v(t, x)$ at $t = 180$ in Figure 3.

3. **Reformulation of the problem.**

This section is devoted to the proof of Theorem 2.1 for the stability of those monotone or non-monotone traveling waves of (1) .

Let $\phi(x + ct) = \phi(\xi)$ be a given traveling wave of (1) with speed $c > c_*$ (no matter it is monotone or non-monotone), and $v(t, x)$ be the solution of (1) with a small initial perturbation around the wave $\phi(x + cs)$ for $s \in [-r, 0]$. Denote

$$V(t, \xi) := v(t, x) - \phi(x + ct) = v(t, \xi - ct) - \phi(\xi),$$

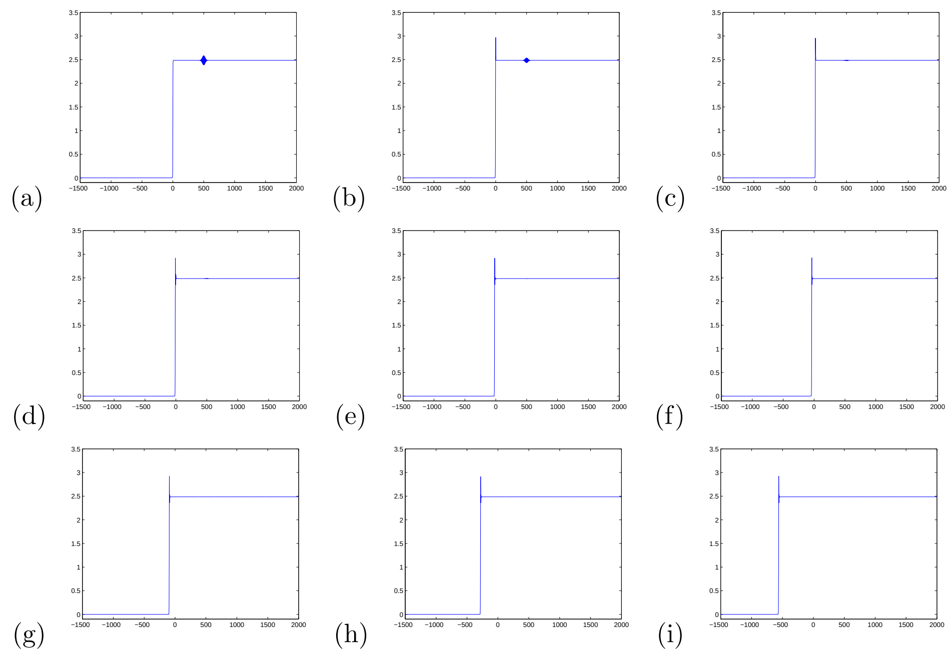

FIGURE 2. Case 2: $\frac{p}{\delta} = 12 > e^2$ with small time-delay $r = 1.0$. From (a) to (i), the solution $v(t,x)$ plots at times $t = 0,\ 2,\ 5,\ 10,\ 20,\ 50,\ 150,\ 200,\ 300$, which behaves as a stable oscillatory traveling wave (no change of the wave's shape after a large time in the sense of stability) and travels from right to left.

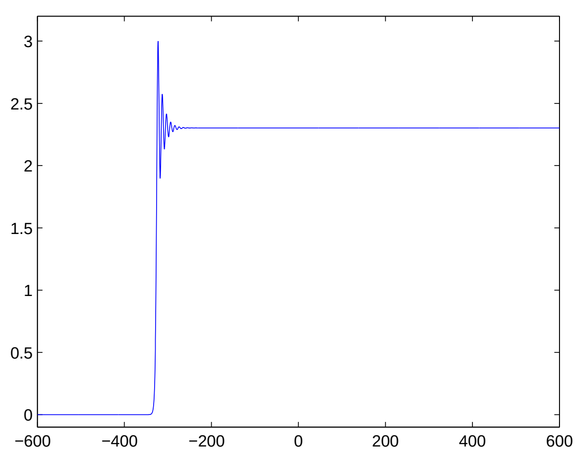

FIGURE 3. Case 2: $\frac{p}{\delta} = 12 > e^2$ with small time-delay $r = 1.0$. In order to see how the oscillations of $v(t,x)$ behave, we present a large scale plot for $v(t,x)$ at $t = 180$.

$$V_0(s,\xi) := v_0(s,x) - \phi(x+cs).$$

Then $V(t,\xi)$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t} + c\dfrac{\partial V}{\partial \xi} - D(J*V-V) + d'(\phi)V \\ \qquad - \displaystyle\int_R K(y)b'(\phi(\xi-cr-y))V(t-r,\xi-y-cr)dy \\ \qquad = -P(V(t,\xi)) + \displaystyle\int_R K(y)Q(V(t-r,\xi-y-cr))dy, \quad t>0,\ \xi\in R, \\ V(s,\xi) = V_0(s,\xi),\ s\in[-r,0], \quad \xi\in R, \end{cases} \tag{21}$$

where

$$P(V) := d(\phi+V) - d(\phi) - d'(\phi)V \tag{22}$$

with $\phi=\phi(\xi)$ and $V=V(t,\xi)$, and

$$Q(V) := b(\phi+V) - b(\phi) - b'(\phi)V \tag{23}$$

with $\phi=\phi(\xi-cr)$ and $V=V(t-r,\xi-cr)$. By Taylor's expansion formula, we know

$$|P(V)| \le C|V|^2 \ \text{ and } \ |Q(V)| \le C|V|^2 \tag{24}$$

for some positive constant C.

Let $0\le T\le\infty$, we define the solution space for (21) as follows

$$\begin{aligned} X(-r,T) = \{V \quad | \quad & V\in C([-r,T];C(R))\cap\mathcal{C}_{unif}[-r,T], \\ & \sqrt{w}V\in C([-r,T];H^1(R)), \text{ and} \\ & \sqrt{w}V\in L^2([-r,T];H^1(R))\} \end{aligned} \tag{25}$$

equipped with the norm

$$N(T)^2 = \sup_{t\in[-r,T]}\Big(\|V(t)\|_C^2 + \|(\sqrt{w}V)(t)\|_{H^1}^2 + \int_{-r}^t \|(\sqrt{w}V)(t)\|_{H^1}^2 dt\Big). \tag{26}$$

Now we state the corresponding stability theorem for the initial value problem (21) as follows.

Theorem 3.1 (Stability). *Under the conditions (H_1)-(H_4), suppose that $V_0(s,\xi)\in X(-r,0)$ and $N(0)\le\delta_0$.*

1. *When $d'(v_+)\ge|b'(v_+)|$, for any time-delay $r>0$, then the solution $V(t,\xi)$ of (21) uniquely and globally exists in $X(-r,\infty)$, and satisfies*
$$\sup_{\xi\in R}|V(t,\xi)| \le Ce^{-\mu t}, \quad t>0 \tag{27}$$
for some constant $\mu>0$.
2. *When $d'(v_+)<|b'(v_+)|$, but the time-delay is small: $0<r<\overline{r}$ for a specified $\overline{r}$ in (16), then the solution $V(t,\xi)$ of (21) uniquely and globally exists in $X(-r,\infty)$, and satisfies the stability (27).*

Notice that, Theorem 2.1 is equivalent to Theorem 3.1, and Theorem 3.1 can be proved by the continuity extension method [24, 25] based on the following local existence and the *a priori* energy estimates.

Proposition 1 (Local existence). *Let $(H_1)-(H_4)$ hold, and let $d'(v_+)\ge|b'(v_+)|$ for any $r>0$ or $d'(v_+)<|b'(v_+)|$ for $0<r<\overline{r}$ hold. Suppose $V_0(s,\xi)\in X(-r,0)$, and $N(0)\le\delta_1$ for a given positive constant $\delta_1>0$, then there exists a small $t_0=t_0(\delta_1)>0$ such that the local solution $V(t,\xi)$ of (21) uniquely exists for*

$t \in [-r, t_0]$, and satisfies $V \in X(-r, t_0)$ and $N(t_0) \le C_1\ N(0)$ for some constant $C_1 > 1$.

Proposition 2 (A priori estimates)**.** *Assume that $(H_1) - (H_4)$ hold, and $d'(v_+) \ge |b'(v_+)|$ for any $r > 0$ or $d'(v_+) < |b'(v_+)|$ for $0 < r < \overline{r}$ hold. Let $V \in X(-r, T)$ be a local solution of (21) for a given constant $T > 0$, then there exist positive constants $\delta_2 > 0$, $C_2 > 1$ and $\mu > 0$ independent of T and $V(t,\xi)$ such that, when $N(T) \le \delta_2$, then*

$$\|V(t)\|_C^2 + \|(\sqrt{w}V)(t)\|_{H^1}^2 + \int_0^t e^{-2\mu(t-s)}\|(\sqrt{w}V)(s)\|_{H^1}^2 ds \le C_2 e^{-2\mu t} N(0)^2 \quad (28)$$

holds for $t \in [0, T]$.

The local existence (Proposition 1) can be similarly proved by the standard iteration technique (c.f. [19] and the references therein). The detail of the proof is omitted. The *a priori* estimates (Proposition 2) will be the main effort in the paper, and will be proved in next section.

4. A priori estimates.

In this section, we are going to establish the *a priori* estimates. The adopted approach is the so-called transformed energy method combining with Fourier transform and nonlinear Halanay's inequality.

Let $V(t,\xi) \in X(-r, T)$ be the local solution of the Cauchy problem (21). So, $V \in C_{uinf}[-r, T]$. Based on this, we first derive the boundedness of V as well as its exponential decay in time, when ξ is sufficiently close to ∞.

Lemma 4.1. *There exist a large number $x_0 \gg 1$ (independent of t) and a number $\mu_1 > 0$, such that:*

1. *When $d'(v_+) \ge |b'(v_+)|$, for all $r > 0$, then*
$$\|V(t)\|_{L^\infty[x_0,\infty)} \le Ce^{-\mu_1 t}\|V_0\|_{L^\infty([-r,0]\times R)} \quad (29)$$
provided $N(T) \ll 1$;
2. *When $d'(v_+) < |b'(v_+)|$, but for $0 < r < \overline{r}$, where $\overline{r}$ is defined in (16), namely,*
$$\overline{r} = \frac{\pi - \arctan(\sqrt{|b'(v_+)|^2 - d'(v_+)^2}/d'(v_+))}{\sqrt{|b'(v_+)|^2 - d'(v_+)^2}},$$
then the exponential decay (29) holds for $N(T) \ll 1$.

Proof. Since $V(t,\xi) \in X(-r, T)$, by the definition of $\mathcal{C}_{unif}[0, T]$, we have that $\lim_{\xi\to+\infty} V(t,\xi)$ exists uniformly with respect to $t \in [0, T]$.

Let us go back to the original equations (1) and (10), and denote

$$\mathcal{V}(t,x) = v(t,x) - \phi(x + ct).$$

Namely, $\mathcal{V}(t,x) = V(t,\xi)$ and satisfies

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t} - D(J * \mathcal{V} - \mathcal{V}) + d'(\phi)\mathcal{V} \\ \qquad - \displaystyle\int_R K(y) b'(\phi(x + c(t-r) - y))\mathcal{V}(t-r, x-y)dy \\ \qquad = -P(\mathcal{V}(t,x)) + \displaystyle\int_R K(y) Q(\mathcal{V}(t-r, x-y))dy, \quad t > 0,\ \xi \in R, \\ \mathcal{V}(s,x) = \mathcal{V}_0(s,x),\ s \in [-r, 0], \quad x \in R, \end{cases} \quad (30)$$

Denote $z(t) := V(t,\infty) = \mathcal{V}(t,\infty)$ and $z_0(s) := \mathcal{V}_0(s,\infty)$ for $s \in [-r,0]$. Since $V \in C_{unif}[0,T]$, namely, $\lim_{\xi\to\infty} V(t,\xi) = \lim_{x\to\infty} \mathcal{V}(t,x) = z(t)$ is uniformly in t, we have

$$\lim_{x\to\infty} \mathcal{V}_t(t,x) = z_t(t), \quad \lim_{x\to\infty} d'(\phi(x+ct))\mathcal{V}_t(t,x) = d'(v_+))z_t(t),$$

and

$$\lim_{x\to+\infty} J*\mathcal{V} = \lim_{\xi\to+\infty} \int_R J(y)\mathcal{V}(t,x-y)dy = \int_R J(y)z(t)dy = z(t)\int_R J(y)dy = z(t),$$

and

$$\begin{aligned}
&\lim_{x\to+\infty} \int_R K(y)b'(\phi(x+c(t-r)-y))\mathcal{V}(t-r,x-y)dy\\
&= \int_R K(y)b'(v_+)z(t-r)dy = b'(v_+)z(t-r)\int_R K(y)dy\\
&= b'(v_+)z(t-r),
\end{aligned}$$

and

$$\lim_{x\to\infty} P(\mathcal{V}(t,x)) = P(z(t)),$$

and

$$\lim_{x\to+\infty} \int_R K(y)Q(\mathcal{V}(t-r,x-y)dy = \lim_{x\to+\infty} \int_R K(y)Q(z(t-r))dy = Q(z(t-r)),$$

all of these limits are uniformly with respect to $t \in [0,T]$. Thus, by taking $x \to +\infty$ to equation (30), we have

$$\begin{cases} z'(t) + d'(v_+)z(t) - b'(v_+)z(t-r) = -P(z(t)) + Q(z(t-r)), \\ z(s) = z_0(s), \ s \in [-r,0]. \end{cases} \tag{31}$$

Applying the nonlinear Halanay's inequality given in [19], we have that: when $d'(v_+) \geq |b'(v_+)|$, for all $r > 0$, then

$$|z(t)| \leq C\|z_0\|_{L^\infty(-r,0)}e^{-\mu_1 t} \tag{32}$$

for some $0 < \mu_1 < d'(0)$, provided $N(T) \ll 1$; while, when $d'(v_+) < |b'(v_+)|$, but for $0 < r < \bar{r}$, where $\bar{r}$ is defined in (16), then the above decay estimate (32) holds for $N(T) \ll 1$.

From (30), it is equivalent to

$$\begin{aligned}
&(e^{d'(0)t}\mathcal{V})_t - D(J*\mathcal{V}-\mathcal{V}) + [d'(\phi) - d'(0)]\mathcal{V}\\
&\qquad - \int_R K(y)b'(\phi(x+c(t-r)-y))\mathcal{V}(t-r,x-y)dy\\
&\quad = -P(\mathcal{V}(t,x)) + \int_R K(y)Q(\mathcal{V}(t-r,x-y))dy.
\end{aligned}$$

Integrating the above equation with respect to t over $[0,t]$, we get

$$\begin{aligned}
&\mathcal{V}(t,x)\\
= \ & e^{-d'(0)t}\Big[\mathcal{V}_0(0,x) + D\int_0^t (J*\mathcal{V}-\mathcal{V})(s,x)ds - \int_0^t [d'(\phi)-d'(0)]\mathcal{V}(s,x)ds\\
&+ \int_0^t\int_R K(y)b'(\phi(x+c(s-r)-y))\mathcal{V}(s-r,x-y)dyds\\
&- \int_0^t P(\mathcal{V}(s,x))ds + \int_0^t\int_R K(y)Q(\mathcal{V}(s-r,x-y))dyds\Big].
\end{aligned}$$

So, we then have, for $0 < \mu_1 < d'(0)$,

$$\begin{aligned} e^{\mu_1 t}\mathcal{V}(t,x) \quad = \quad & e^{-[d'(0)-\mu_1]t}\Big[\mathcal{V}_0(0,x) + D\int_0^t (J*\mathcal{V}-\mathcal{V})(s,x)ds \\ & -\int_0^t [d'(\phi)-d'(0)]\mathcal{V}(s,x)ds \\ & +\int_0^t\int_R K(y)b'(\phi(x+c(s-r)-y))\mathcal{V}(s-r,x-y)dyds \\ & -\int_0^t P(\mathcal{V}(s,x))ds + \int_0^t\int_R K(y)Q(\mathcal{V}(s-r,x-y))dyds\Big]. \end{aligned} \tag{33}$$

Taking the limits to (33) as $x \to \infty$, and noting all these limits are uniformly in t, and applying the facts $|P(z)| \le Cz^2$, $|Q(z)| \le Cz^2$ and the decay estimate (32) for $z(t)$, then we have

$$\begin{aligned} & \lim_{x\to\infty} e^{\mu_1 t}\mathcal{V}(t,x) \\ = \quad & e^{-[d'(0)-\mu_1]t}\Big[\lim_{x\to\infty}\mathcal{V}_0(0,x) + D\int_0^t \lim_{x\to\infty}(J*\mathcal{V}-\mathcal{V})(s,x)ds \\ & -\int_0^t \lim_{x\to\infty}[d'(\phi)-d'(0)]\mathcal{V}(s,x)ds \\ & +\int_0^t \lim_{x\to\infty}\int_R K(y)b'(\phi(x+c(s-r)-y))\mathcal{V}(s-r,x-y)dyds \\ & -\int_0^t \lim_{x\to\infty} P(\mathcal{V}(s,x))ds + \int_0^t \lim_{x\to\infty}\int_R K(y)Q(\mathcal{V}(s-r,x-y))dyds\Big] \\ = \quad & e^{-[d'(0)-\mu_1]t}\Big[z_0(0) + D\int_0^t (J*z(s)-z(s))ds \\ & -[d'(v_+)-d'(0)]\int_0^t z(s)ds + \int_0^t\int_R K(y)b'(v_+)z(s-r)dyds \\ & -\int_0^t P(z(s))ds + \int_0^t\int_R K(y)Q(z(s-r))dyds\Big] \\ = \quad & e^{-[d'(0)-\mu_1]t}\Big[z_0(0) - [d'(v_+)-d'(0)]\int_0^t z(s)ds \\ & +b'(v_+)\int_0^t z(s-r)dyds - \int_0^t P(z(s))ds + \int_0^t Q(z(s-r))ds\Big] \\ \le \quad & Ce^{-[d'(0)-\mu_1]t}\Big[|z_0(0)| + \int_0^t |z(s)|ds + \int_0^t |z(s-r)|ds \\ & + \int_0^t |z(s)|^2 ds + \int_0^t |z(s-r))|^2 ds\Big] \\ \le \quad & Ce^{-[d'(0)-\mu_1]t}\Big[|z_0(0)| + \int_0^t e^{-\mu_1 s}ds + \int_0^t e^{-\mu_1(s-r)}ds \\ & + \int_0^t e^{-2\mu_1 s}ds + \int_0^t e^{-2\mu_1(s-r)}ds\Big] \\ \le \quad & C, \quad \text{uniformly in } t. \end{aligned} \tag{34}$$

Thus, there exists a number $x_0 \gg 1$ independent of t, such that when $x \geq x_0$, we have:

$$\sup_{x\in[x_0,\infty)} |\mathcal{V}(t,x)| \leq Ce^{-\mu_1 t}\|\mathcal{V}_0\|_{L^\infty([-r,0]\times R)}, \quad t\geq 0. \tag{35}$$

either for $d'(v_+) \geq |b'(v_+)|$ with any $r>0$, or for $d'(v_+) < |b'(v_+)|$ but with $0<r<\bar{r}$. Again, notice that $V(t,\xi)=\mathcal{V}(t,x)$ and $\xi = x+ct \geq x \geq x_0$ for $x\geq x_0$ and $t\geq 0$, then (35) immediately implies,

$$\sup_{\xi\in[x_0,\infty)} |V(t,\xi)| \leq Ce^{-\mu_1 t}\|V_0\|_{L^\infty([-r,0]\times R)}, \quad t\geq 0 \tag{36}$$

either for $d'(v_+) \geq |b'(v_+)|$ with any $r>0$, or for $d'(v_+) < |b'(v_+)|$ but with $0<r<\bar{r}$. Thus, (29) is proved. □

Now we are going to establish the *a priori* estimates (28). Different from the standard weighted energy method by multiplying (21) by $w(\xi)V(t,\xi)$, we adopt the transformed energy method. First of all, we shift $V(t,\xi)$ to $V(t,\xi+x_0)$ by the constant x_0 given in Lemma 4.1, then we introduce the following transformation

$$U(t,\xi) = \sqrt{w(\xi)}V(t,\xi+x_0) = e^{-\lambda\xi}V(t,\xi+x_0), \tag{37}$$

where $e^{-\lambda\xi}\to\infty$ as $\xi\to-\infty$, and $e^{-\lambda\xi}\to 0$ as $\to+\infty$. Substituting $V=w^{-1/2}U$ to (21), then we derive the following equation for $U(t,\xi)$

$$\begin{cases} \dfrac{\partial U}{\partial t} + c\dfrac{\partial U}{\partial \xi} - D\displaystyle\int_R J(y)e^{-\lambda y}U(t,\xi-y)dy + [c\lambda + D + d'(\phi(\xi+x_0))]U \\ \quad -\displaystyle\int_R K(y)b'(\phi(\xi-y-cr+x_0))e^{-\lambda(y+cr)}U(t-r,\xi-y-cr)dy \\ = -\sqrt{w(\xi)}P(V(t,\xi+x_0)) + \displaystyle\int_R K(y)\sqrt{w(\xi)}Q(V(t-r,\xi-y-cr+x_0))dy, \\ U|_{t=s} = \sqrt{w(\xi)}V_0(s,\xi+x_0) =: U_0(s,\xi), \quad (s,\xi)\in[-r,0]\times R. \end{cases} \tag{38}$$

Next, we prove the *a priori* estimates (28) by serval lemmas.

Lemma 4.2. *It holds that*

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{L^2} + \mu_2\|U(t)\|^2_{L^2} + C_3\big[\|U(t)\|^2_{L^2} - \|U(t-r)\|^2_{L^2}\big] \leq I_1(t)+I_2(t), \tag{39}$$

where

$$\begin{aligned} \mu_2 : &= c\lambda + D + d'(0) - D\int_R J(y)e^{-\lambda y}dy - b'(0)e^{-\lambda cr}\int_R K(y)e^{-\lambda y}dy \\ &> 0, \quad (\textit{ see } (12)), \end{aligned} \tag{40}$$

$$C_3 := \frac{1}{2}b'(0)e^{-\lambda cr}\int_R K(y)e^{-\lambda y}dy > 0, \tag{41}$$

$$I_1(t) := -\int_R \sqrt{w(\xi)}U(t,\xi)P(V(t,\xi+x_0))d\xi, \tag{42}$$

$$I_2(t) := \int_R U(t,\xi)\left(\int_R K(y)\sqrt{w(\xi)}Q(V(t-r,\xi-y-cr+x_0))dy\right)d\xi. \tag{43}$$

Proof. Multiplying (38) by U and integrating the resultant equation over R with respect to ξ, we have

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|^2_{L^2} + \int_R \big[c\lambda + D + d'(\phi(\xi+x_0))\big]U^2(t,\xi)d\xi$$

$$
\begin{aligned}
&- D\int_R U(t,\xi)\left(\int_R J(y)e^{-\lambda y}U(t,\xi-y)dy\right)d\xi\\
&- \int_R U(t,\xi)\left(\int_R K(y)b'(\phi(\xi-y-cr+x_0))e^{-\lambda(y+cr)}U(t-r,\xi-y-cr)dy\right)d\xi\\
&= I_1(t)+I_2(t).
\end{aligned}
\tag{44}
$$

We denote the terms in equation (44) by

$$
\begin{aligned}
I_3 :=& \int_R \left[c\lambda + D + d'(\phi(\xi+x_0))\right]U^2(t,\xi)d\xi,\\
I_4 :=& D\int_R U(t,\xi)\left(\int_R J(y)e^{-\lambda y}U(t,\xi-y)dy\right)d\xi,\\
I_5 :=& \int_R U(t,\xi)\left(\int_R K(y)b'(\phi(\xi-y-cr+x_0))e^{-\lambda(y+cr)}U(t-r,\xi-y-cr)dy\right)d\xi.
\end{aligned}
$$

Now we give the estimates of $I_3 - I_5$ respectively. First of all, from (H_1), namely, $d'(\phi) > d'(0)$ for $\phi > 0$, we immediately have

$$
I_3 = \int_R \left[c\lambda + D + d'(\phi(\xi+x_0))\right]U^2(t,\xi)d\xi \geq \left[c\lambda + D + d'(0)\right]\|U(t)\|^2_{L^2}. \tag{45}
$$

Next, by the Hölder inequality and the properties of Fourier transform (Parseval's equality), we can get the following optimal estimate

$$
\begin{aligned}
|I_4| &\leq D\int_R |U(t,\xi)|\left|\int_R J(y)e^{-\lambda y}U(t,\xi-y)dy\right|d\xi\\
&\leq D\|U(t)\|_{L^2}\left\|(Je^{-\lambda y})*U\right\|_{L^2}\\
&= D\|U(t)\|_{L^2}\left\|\mathcal{F}\left[(Je^{-\lambda y})*U\right]\right\|_{L^2}\\
&= D\|U(t)\|_{L^2}\left\|\mathcal{F}[Je^{-\lambda y}]\cdot\mathcal{F}[U]\right\|_{L^2}\\
&= D\|U(t)\|_{L^2}\left(\int_R \left|\mathcal{F}[J(y)e^{-\lambda y}](\eta)\right|^2\cdot\left|\mathcal{F}[U](t,\eta)\right|^2 d\eta\right)^{1/2}\\
&= D\|U(t)\|_{L^2}\left(\int_R \left|\int_R e^{-\mathrm{i}y\eta}J(y)e^{-\lambda y}dy\right|^2\cdot\left|\mathcal{F}[U](t,\eta)\right|^2 d\eta\right)^{1/2}\\
&\leq D\|U(t)\|_{L^2}\left(\int_R \left(\int_R \left|e^{-\mathrm{i}y\eta}J(y)e^{-\lambda y}\right|dy\right)^2\cdot\left|\mathcal{F}[U](t,\eta)\right|^2 d\eta\right)^{1/2}\\
&= D\|U(t)\|_{L^2}\left(\int_R \left(\int_R J(y)e^{-\lambda y}dy\right)^2\cdot\left|\mathcal{F}[U](t,\eta)\right|^2 d\eta\right)^{1/2}\\
&= D\left(\int_R J(y)e^{-\lambda y}dy\right)\|U(t)\|_{L^2}\left(\int_R \left|\mathcal{F}[U](t,\eta)\right|^2 d\eta\right)^{1/2}\\
&= D\left(\int_R J(y)e^{-\lambda y}dy\right)\|U(t)\|^2_{L^2}.
\end{aligned}
\tag{46}
$$

Similarly, noticing that $|b'(\phi)| \leq b'(0)$, by the properties of Fourier transform, we can derive the following optimal L^2-energy estimate

$|I_5|$

$$\le \int_R |U(t,\xi)| \cdot \left| \int_R K(y) b'(\phi(\xi - y - cr + x_0)) e^{-\lambda(y+cr)} U(t-r, \xi - y - cr) dy \right| d\xi$$
$$\le b'(0) \int_R |U(t,\xi)| \cdot \left| \int_R K(y) e^{-\lambda(y+cr)} U(t-r, \xi - y - cr) dy \right| d\xi$$
$$= b'(0) \int_R |U(t,\xi)| \cdot \left| \left(K(y) e^{-\lambda(y+cr)} \right) * U(t-r, \xi - cr) \right| d\xi$$
$$\le b'(0) \|U(t)\|_{L^2} \left\| \left(\left(K(y) e^{-\lambda(y+cr)} \right) * U \right) (t-r) \right\|_{L^2}$$
$$= b'(0) \|U(t)\|_{L^2} \left\| \mathcal{F} \left[K(y) e^{-\lambda(y+cr)} * U \right] (t-r) \right\|_{L^2}$$
$$= b'(0) \|U(t)\|_{L^2} \left\| \mathcal{F} \left[K(y) e^{-\lambda(y+cr)} \right] \cdot \mathcal{F}[U](t-r) \right\|_{L^2}$$
$$= b'(0) \|U(t)\|_{L^2} \left(\int_R \left| \int_R K(y) e^{-\lambda(y+cr)} \cdot e^{-\mathrm{i} y\eta} dy \right|^2 \cdot |\mathcal{F}[U](t-r,\eta)|^2 d\eta \right)^{1/2}$$
$$\le b'(0) \|U(t)\|_{L^2} \left(\int_R \left| \int_R K(y) e^{-\lambda(y+cr)} dy \right|^2 \cdot |\mathcal{F}[U](t-r,\eta)|^2 d\eta \right)^{1/2}$$
$$= b'(0) \left(\int_R K(y) e^{-\lambda(y+cr)} dy \right) \|U(t)\|_{L^2} \|\mathcal{F}[U](t-r)\|_{L^2}$$
$$= b'(0) \left(\int_R K(y) e^{-\lambda(y+cr)} dy \right) \|U(t)\|_{L^2} \|U(t-r)\|_{L^2}$$
$$\le b'(0) \left(\int_R K(y) e^{-\lambda(y+cr)} dy \right) \left(\frac{1}{2} \|U(t)\|_{L^2}^2 + \frac{1}{2} \|U(t-r)\|_{L^2}^2 \right). \tag{47}$$

Substituting (45), (46) and (47) to (44), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 + \left(c\lambda + D + d'(0) - D\int_R J(y)e^{-\lambda y}dy \right) \|U(t)\|_{L^2}^2 \\
&\quad -\frac{1}{2} b'(0) \Big(\int_R K(y) e^{-\lambda(y+cr)} dy \Big) \left[\|U(t)\|_{L^2}^2 + \|U(t-r)\|_{L^2}^2 \right] \\
&\le I_1(t) + I_2(t),
\end{aligned}$$

namely,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 + \frac{1}{2} b'(0) \Big(\int_R K(y) e^{-\lambda(y+cr)} dy \Big) \left[\|U(t)\|_{L^2}^2 - \|U(t-r)\|_{L^2}^2 \right] \\
&\quad + \left(c\lambda + D + d'(0) - D\int_R J(y)e^{-\lambda y}dy - b'(0) \int_R K(y) e^{-\lambda(y+cr)} dy \right) \|U(t)\|_{L^2}^2 \\
&\le I_1(t) + I_2(t),
\end{aligned}$$

which immediately implies (39). The proof is complete. □

Lemma 4.3. *There exists $0 < \mu < \mu_2$ such that*

$$\begin{aligned}
&\|U(t)\|_{L^2}^2 + \int_0^t e^{-2\mu(t-s)} \|U(s)\|_{L^2}^2 ds \\
&\le C e^{-2\mu t} \Big(\|U_0(0)\|_{L^2}^2 + \int_{-r}^0 e^{2\mu s} \|U_0(s)\|_{L^2}^2 ds \Big)
\end{aligned} \tag{48}$$

provided $N(T) \ll 1$.

Proof. Multiplying the inequality (39) by $e^{2\mu t}$ and integrating the resultant inequality with respect to t over $[0,t]$, where $\mu>0$ will be selected later, we have

$$\begin{aligned} & e^{2\mu t}\|U(t)\|_{L^2}^2+2(\mu_2-\mu)\int_0^t e^{2\mu s}\|U(s)\|_{L^2}^2 ds \\ & +2C_3\int_0^t e^{2\mu s}[\|U(s)\|_{L^2}^2-\|U(s-r)\|_{L^2}^2]ds \\ & \le \|U_0(0)\|_{L^2}^2+2\int_0^t e^{2\mu s}\big[I_1(s)+I_2(s)\big]ds. \end{aligned} \tag{49}$$

Notice that, by the change of variable $s-r\to s$,

$$\begin{aligned} & \int_0^t e^{2\mu s}\|U(s-r)\|_{L^2}^2 ds \\ & = \int_{-r}^{t-r} e^{2\mu(s+r)}\|U(s)\|_{L^2}^2 ds \\ & = \int_{-r}^{0} e^{2\mu(s+r)}\|U(s)\|_{L^2}^2 ds+\int_0^{t-r} e^{2\mu(s+r)}\|U(s)\|_{L^2}^2 ds \\ & \le \int_{-r}^{0} e^{2\mu(s+r)}\|U_0(s)\|_{L^2}^2 ds+\int_0^{t} e^{2\mu(s+r)}\|U(s)\|_{L^2}^2 ds. \end{aligned} \tag{50}$$

Substituting (50) to (49), we have

$$\begin{aligned} & e^{2\mu t}\|U(t)\|_{L^2}^2+2\left[(\mu_2-\mu)+C_3(1-e^{2\mu r})\right]\int_0^t e^{2\mu s}\|U(s)\|_{L^2}^2 ds \\ & \le C\Big(\|U_0(0)\|_{L^2}^2+\int_{-r}^0 e^{2\mu s}\|U_0(s)\|_{L^2}^2 ds\Big) \\ & \quad +2\int_0^t e^{2\mu s}\big[I_1(s)+I_2(s)\big]ds. \end{aligned} \tag{51}$$

Now we can choose $0<\mu<\mu_2$ to be small such that

$$C_4:=(\mu_2-\mu)+C_3(1-e^{2\mu r})>0.$$

Then

$$\begin{aligned} & \|U(t)\|_{L^2}^2+2C_4\int_0^t e^{-2\mu(t-s)}\|U(s)\|_{L^2}^2 ds \\ & \le Ce^{-2\mu t}\left(\|U_0(0)\|_{L^2}^2+\int_{-r}^0 e^{2\mu s}\|U_0(s)\|_{L^2}^2 ds\right) \\ & \quad +2\int_0^t e^{-2\mu(t-s)}\big[I_1(s)+I_2(s)\big]ds. \end{aligned} \tag{52}$$

Next, we estimate the nonlinear terms involving I_1 and I_2. Since $V(t,\xi)\in X(0,T)$, namely, $V\in C^0(R)$, we have

$$|V(t,\xi+x_0)|\le CN(T).$$

Thus, from (42) and (43), by Taylor's expansion (see (24)), namely,

$$\begin{aligned} & |P(V(s,\xi+x_0))|\le CV^2(s,\xi+x_0), \\ & |Q(u(s-r,\xi-y-cr+x_0))|\le CV^2(s-r,\xi-y-cr+x_0), \end{aligned}$$

and noting $U(t,\xi)=\sqrt{w(\xi)}V(t,\xi+x_0)=e^{-\lambda\xi}V(t,\xi+x_0)$ and $U(t,\xi-y-cr)=\sqrt{w(\xi-y-cr)}V(t,\xi-y-cr+x_0)=e^{-\lambda(\xi-y-cr)}V(t,\xi-y-cr+x_0)$, we can estimate

$$\begin{aligned}
&2\int_0^t e^{-2\mu(t-s)}I_1(s)ds\\
&=-2\int_0^t e^{-2\mu(t-s)}\int_R \sqrt{w(\xi)}U(s,\xi)P(V(s,\xi+x_0))d\xi ds\\
&\le C\int_0^t e^{-2\mu(t-s)}\int_R \sqrt{w(\xi)}|U(s,\xi)||V(s,\xi+x_0)|^2 d\xi ds\\
&=C\int_0^t e^{-2\mu(t-s)}\int_R |U(s,\xi)|^2|V(s,\xi+x_0)|d\xi ds\\
&\le CN(T)\int_0^t e^{-2\mu(t-s)}\int_R |U(s,\xi)|^2 d\xi ds\\
&=CN(T)\int_0^t e^{-2\mu(t-s)}\|U(s)\|_{L^2}^2 ds, \qquad (53)
\end{aligned}$$

and

$$\begin{aligned}
&2\int_0^t e^{-2\mu(t-s)}I_2(s)ds\\
&=2\int_0^t e^{-2\mu(t-s)}\int_R \sqrt{w(\xi)}U(s,\xi)\left(\int_R K(y)Q(V(s-r,\xi-y-cr+x_0))dy\right)d\xi ds\\
&\le C\int_0^t e^{-2\mu(t-s)}\int_R \sqrt{w(\xi)}|U(s,\xi)|\cdot\left(\int_R K(y)V^2(s-r,\xi-y-cr+x_0)dy\right)d\xi ds\\
&=C\int_0^t e^{-2\mu(t-s)}\int_R |U(s,\xi)|\cdot\left(\int_R K(y)\sqrt{w(\xi)}V^2(s-r,\xi-y-cr+x_0)dy\right)d\xi ds\\
&=C\int_0^t e^{-2\mu(t-s)}\int_R |U(s,\xi)|\cdot\left(\int_R K(y)e^{-\lambda\xi}V^2(s-r,\xi-y-cr+x_0)dy\right)d\xi ds\\
&=C\int_0^t e^{-2\mu(t-s)}\int_R |U(s,\xi)|\\
&\quad\times\left(\int_R K(y)e^{-\lambda(\xi-y-cr)}e^{-\lambda(y+cr)}V^2(s-r,\xi-y-cr+x_0)dy\right)d\xi ds\\
&=C\int_0^t e^{-2\mu(t-s)}\int_R |U(s,\xi)|\\
&\quad\times\left(\int_R K(y)e^{-\lambda(y+cr)}|U(s-r,\xi-y-cr)||V(s-r,\xi-y-cr+x_0)|dy\right)d\xi ds\\
&\le CN(T)\int_0^t e^{-2\mu(t-s)}\\
&\quad\times\int_R |U(s,\xi)|\cdot\left(\int_R K(y)e^{-\lambda(y+cr)}|U(s-r,\xi-y-cr)|dy\right)d\xi ds\\
&\le CN(T)\int_0^t e^{-2\mu(t-s)}\left(\int_R K(y)e^{-\lambda(y+cr)}dy\right)\left[\frac{1}{2}\|U(s)\|_{L^2}^2+\frac{1}{2}\|U(s-r)\|_{L^2}^2\right]ds\\
&\qquad\qquad\text{(obtained by the same fashion in (47))}
\end{aligned}$$

$$
\begin{aligned}
&= CN(T)\left(\int_R K(y)e^{-\lambda(y+cr)}dy\right)\int_0^t e^{-2\mu(t-s)}\left[\frac{1}{2}\|U(s)\|_{L^2}^2+\frac{1}{2}\|U(s-r)\|_{L^2}^2\right]ds\\
&\le CN(T)\left(\int_R K(y)e^{-\lambda(y+cr)}dy\right)\int_0^t e^{-2\mu(t-s)}\|U(s)\|_{L^2}^2ds\\
&\quad + CN(T)\left(\int_R K(y)e^{-\lambda(y+cr)}dy\right)e^{-2\mu t}\int_{-r}^0 e^{2\mu s}\|U_0(s)\|_{L^2}^2ds\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(obtained by the same fashion in (50))}\\
&\le CN(T)\int_0^t e^{-2\mu(t-s)}\|U(s)\|_{L^2}^2ds + Ce^{-2\mu t}\int_{-r}^0 e^{2\mu s}\|U_0(s)\|_{L^2}^2ds. \qquad (54)
\end{aligned}
$$

Substituting (53) and (54) to (52), we have

$$
\begin{aligned}
&\|U(t)\|_{L^2}^2 + [2C_4 - CN(T)]\int_0^t e^{-2\mu(t-s)}\|U(s)\|_{L^2}^2ds\\
&\le Ce^{-2\mu t}\left(\|U_0(0)\|_{L^2}^2 + \int_{-r}^0 e^{2\mu s}\|U_0(s)\|_{L^2}^2ds\right). \qquad (55)
\end{aligned}
$$

Let $N(T) \ll 1$, we immediately obtain (48). The proof is complete. $\square$

Next we derive the estimates for the higher order derivatives of the solution.

Lemma 4.4. *It holds that*

$$
\begin{aligned}
&\|U_\xi(t)\|_{L^2}^2 + \int_0^t e^{-2\mu(t-s)}\|U_\xi(s)\|_{L^2}^2ds\\
&\le Ce^{-2\mu t}\Big(\|U_0(0)\|_{H^1}^2 + \int_{-r}^0 e^{2\mu s}\|U_0(s)\|_{H^1}^2ds\Big) \qquad (56)
\end{aligned}
$$

provided $N(T) \ll 1$.

Proof. Differentiating (38) with respect to ξ and multiplying it by $\frac{\partial U}{\partial \xi}$, then integrating the resultant equation with respect to ξ and t over $R\times[0,t]$, we can similarly prove (56) provided $N(T) \ll 1$. The detail is omitted. $\square$

Thus, combining (48) and (56), we have established the following energy estimates.

Lemma 4.5. *It holds that*

$$
\|U(t)\|_{H^1}^2 + \int_0^t e^{-2\mu(t-s)}\|U(s)\|_{H^1}^2ds \le C\delta^2 e^{-2\mu t}, \qquad (57)
$$

namely,

$$
\|\sqrt{w}V(t)\|_{H^1}^2 + \int_0^t e^{-2\mu(t-s)}\|\sqrt{w}V(s)\|_{H^1}^2ds \le C\delta^2 e^{-2\mu t}, \qquad (58)
$$

provided $N(T) \ll 1$, *where*

$$
\delta^2 := \|\sqrt{w}V_0(0)\|_{H^1}^2 + \int_{-r}^0 e^{2\mu s}\|\sqrt{w}V_0(s)\|_{H^1}^2ds. \qquad (59)
$$

From (57), by Sobolev's inequality $H^1(R) \hookrightarrow C^0(R)$, we get

$$
|U(t,\xi)| \le C\|U(t)\|_{H^1} \le C\delta e^{-\mu t}.
$$

Notice that $U(t,\xi) = \sqrt{w(\xi)}V(t,\xi+x_0) = e^{-\lambda\xi}V(t,\xi+x_0)$, and $\sqrt{w(\xi)} = e^{-\lambda\xi} \geq 1$ for $\xi \in (-\infty, 0]$, we then get

$$\sup_{\xi\in(-\infty,0]} |V(t,\xi+x_0)| \leq C\delta e^{-\mu t}.$$

This proves the following estimate for the unshifted $V(t,\xi)$.

Lemma 4.6. *It holds that*

$$\|V(t)\|_{L^\infty(-\infty,x_0]} \leq C\delta e^{-\mu t}, \tag{60}$$

provided $N(T) \ll 1$.

Finally, combining Lemma 4.6 and Lemma 4.1, we prove

Lemma 4.7. *For* $0 < \mu < \min\{\mu_1, \mu_2\}$ *and* $N(T) \ll 1$*, it holds that:*

1. *When* $d'(v_+) \geq |b'(v_+)|$*, for all* $r > 0$*, then*
$$\|V(t)\|_{L^\infty(R)} \leq C\delta e^{-\mu t}; \tag{61}$$
2. *When* $d'(v_+) < |b'(v_+)|$*, but for* $0 < r < \overline{r}$*, where* $\overline{r}$ *is defined in* (16)*, then*
$$\|V(t)\|_{L^\infty(R)} \leq C\delta e^{-\mu t}. \tag{62}$$

5. **Uniqueness of traveling waves.** This section is devoted to the proof of Corollary 1, the uniqueness of the traveling waves in the non-monotone case for the nonlocal dispersion equation (1), which was not solved in [37, 38].

Assume that $\phi_1(x+ct)$ and $\phi_2(x+ct)$ are two different traveling waves with the same speed $c > c_*$ and the same exponential decay at $-\infty$:

$$\phi_1(\xi) = \mathcal{C}_1 e^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty,$$

and

$$\phi_2(\xi) = \mathcal{C}_2 e^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty,$$

for some positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\lambda_1 = \lambda_1(c) > 0$ is defined in (11). Let us shift $\phi_2(x+ct)$ to $\phi_2(x+ct+x_*)$ with some constant shift x_*. By taking $\xi \to -\infty$, obviously $\xi + x_* < 0$, then

$$\phi_2(\xi+x_*) = \mathcal{C}_2 e^{-\lambda_1|\xi+x_*|} = \mathcal{C}_2 e^{\lambda_1(\xi+x_*)} = \mathcal{C}_2 e^{\lambda_1 x_*} e^{-\lambda_1|\xi|} = \mathcal{C}_1 e^{-\lambda_1|\xi|} \quad \text{as } \xi \to -\infty$$

by selecting x_* as

$$x_* = \frac{1}{\lambda_1} \ln \frac{\mathcal{C}_1}{\mathcal{C}_2}.$$

Thus, we have

$$|\phi_2(\xi+x_*) - \phi_1(\xi)| = O(1)e^{-\alpha|\xi|} \quad \text{for } \alpha > \lambda_1 \quad \text{as } \xi \to -\infty.$$

This implies

$$\sqrt{w(\xi)}[\phi_2(\xi+x_*) - \phi_1(\xi)] = e^{-\lambda_1(\xi-x_0)}[\phi_2(\xi+x_*) - \phi_1(\xi)] \in C(R) \cap H^1(R).$$

Now we take the initial data for equation (1) by

$$v_0(s,x) = \phi_2(x+cs+x_*), \ x \in R, \ s \in [-r, 0].$$

Obviously, with such a selected initial data, the corresponding solution to (1) is

$$v(t,x) = \phi_2(x+ct+x_*).$$

Applying the stability theorem 2.1, when $d'(v_+) \geq |b'(v_+)|$ with any time-delay $r > 0$, or when $d'(v_+) < |b'(v_+)|$ but with $0 < r < \overline{r}$, then

$$\lim_{t\to\infty} \sup_{x\in R} |\phi_2(x+ct+x_*) - \phi_1(x+ct)| = 0,$$

namely, $\phi_2(x + ct + x_*) = \phi_1(x + ct)$ for all $x \in R$ as $t \gg 1$. This proves the uniqueness of the traveling waves up to a constant shift.

Acknowledgments. The research of RH was supported in part by NNSFC No.11071099 and CPSF No. 2015M572301. The research of MM was supported in part by NSERC grant RGPIN 354724-11, FQRNT grant 192571 and Fédération de cégeps. The research of KJZ was supported in part by NSFC No.11371082. The research of QFZ was supported in part by the Joint Training Ph.D Program of China Scholarship Council (201306160037).

REFERENCES

[1] M. Aguerra, C. Gomez and S. Trofimchuk, On uniqueness of semi-wavefronts, *Math. Ann.*, **354** (2012), 73–109.

[2] F. Andreu-Vaillo, J. M. Mazon, J. D. Rossi and J. J. Toledo-Melero, *Nonlocal Diffusion Problems*, Math. Surveys and Monographs, Vol. 165, Amer. Math. Soc., 2010.

[3] E. Chasseigne, M. Chaves and J. Rossi, Asymptotic behavior for nonlocal diffusion equations, *J. Math. Pure Appl.*, **86** (2006), 271–291.

[4] C. Cortazar, M. Elgueta, J. D. Rossi and N. Wolanski, How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems, *Arch. Rational Mech. Anal.*, **187** (2008), 137–156.

[5] J. Coville, On uniqueness and monotonicity of solutions of non-local reaction-diffusion equation, *Annali. di Matematica Pura Appl.*, **185** (2006), 461–485.

[6] J. Coville, J. Dávila and S. Martínez, Nonlocal anisotropic dispersal with monostable nonlinearity, *J. Differential Equations*, **244** (2008), 3080–3118.

[7] J. Coville and L. Dupaigne, On a non-local equation arising in population dynamics, *Proc. Roy. Soc. Edinburgh Sect. A*, **137** (2007), 727–755.

[8] J. Coville and L. Dupaigne, Propagation speed of travelling fronts in non local reaction-diffusion equations, Nonlinear Anal., **60** (2005), 797–819.

[9] J. Fang and X.-Q. Zhao, Existence and uniqueness of traveling waves for non-monotone integral equations with applications, *J. Differential Equations*, **248** (2010), 2199–2226.

[10] P. C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomathematics, **28**, Springer-Verlag, Berlin-New York, 1979.

[11] W. S. C. Gurney, S. P. Blythe and R. M. Nisbet, Nicholson's blowflies revisited, *Nature*, **287** (1980), 17–21.

[12] A. Gomez and S. Trofimchuk, Global continuation of monotone wavefronts, *J. London Math. Soc.*, **89** (2014), 47–68.

[13] S. A. Gourley, J. W.-H. So and J. Wu, Nonlocalily of reaction-diffusion equations induced by delay: Biological modeling and nonlinear dynamics, (Russian) *Sovrem. Mat. Fundam. Napravl.*, **1** (2003), 84–120; translation in *J. Math. Sci.,* **124** (2004), 5119–5153.

[14] S. A. Gourley and J. Wu, Delayed non-local diffusive systems in biological invasion and disease spread, in *Nonlinear Dynamics and Evolution Equations* (eds. H. Brunner, X.-Q. Zhao and X. Zou), Fields Institute Communications, 48, Amer. Math. Soc., Providence, RI, 2006, 137–200.

[15] R. Huang, M. Mei and Y. Wang, Planar traveling waves for nonlocal dispersion equation with monostable nonlinearity, *Discret. Contin. Dyn. Syst. A*, **32** (2012), 3621–3649.

[16] L. Ignat and J. D. Rossi, Decay estimates for nonlocal problems via energy methods, *J. Math. Pure Appl.(9),* **92** (2009), 163–187.

[17] L. Ignat and J. D. Rossi, A nonlocal convolution-diffusion equation, *J. Func. Anal.*, **251** (2007), 399–437.

[18] D. Liang and J. Wu, Travelling waves and numerical approximations in a reaction advection diffusion equation with nonlocal delayed effects, *J. Nonlinear Sci.*, **13** (2003), 289–310.

[19] C.-K. Lin, C.-T. Lin, Y. P. Lin and M. Mei, Exponential stability of nonmonotone traveling waves for Nicholson's blowflies equation, *SIAM J. Math. Anal.*, **46** (2014), 1053–1084.

[20] M. C. Mackey and L. Glass, Oscillation and chaos in physiological control systems, *Science*, **197** (1977), 287–289.

[21] A. Matsumura and M. Mei, Nonlinear stability of viscous shock profile for a non-convex system of viscoelasticity, *Osaka J. Math.*, **34** (1997), 589–603.

[22] M. Mei, C.-K. Lin, C.-T. Lin and J. W.-H. So, Traveling wavefronts for time-delayed reaction-diffusion equation. I. Local nonlinearity, *J. Differential Equations*, **247** (2009), 495–510.

[23] M. Mei, C.-K. Lin, C.-T. Lin and J. W.-H. So, Traveling wavefronts for time-delayed reaction-diffusion equation. II. Nonlocal nonlinearity, *J. Differential Equations*, **247** (2009), 511–529.

[24] M. Mei, J. W.-H. So, M. Li and S. Shen, Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion, *Proc. Roy. Soc. Edinburgh Sec. A*, **134** (2004), 579–594.

[25] M. Mei and J. W.-H. So, Stability of strong travelling waves for a non-local time-delayed reaction-diffusion equation, *Proc. Roy. Soc. Edinburgh Sec. A*, **138** (2008), 551–568.

[26] M. Mei, C. Ou and X.-Q. Zhao, Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations, *SIAM J. Math. Anal.*, **42** (2010), 2762–2790; Erratum, *SIAM J. Math. Anal.,* **44** (2012), 538–540.

[27] M. Mei and Y. Wang, Remark on stability of traveling waves for nonlocal Fisher-KPP equations, *Int. J. Numer. Anal. Model. Seris B*, **2** (2011), 379–401.

[28] M. Mei and Y. S. Wong, Novel stability results for traveling wavefronts in an age-structured reaction-diffusion equations, *Math. Biosci. Engin.,* **6** (2009), 743–752.

[29] J. A. J. Metz and O. Diekmann, *The dynamics of Physiologically Structured Populations*, Springer, New York, 1986.

[30] H. J. K. Moet, A note on the asymptotic behavior of solutions of the KPP equation, *SIAM J. Math. Anal.*, **10** (1979), 728–732.

[31] S. Pan, W.-T. Li and G. Lin, Existence and stability of traveling wavefronts in a nonlocal diffusion equation with delay, *Nonlinear Anal.*, **72** (2010), 3150–3158.

[32] J. W.-H. So, J. Wu and X. Zou, A reaction-diffusion model for a single species with age structure: I. Traveling wavefronts on unbounded domains, *Roy. Soc. London Proc. Series A Math. Phys. Eng. Sci.*, **457** (2001), 1841–1853.

[33] E. Trofimchuk and S. Trofimchunk, Admissible wavefront speeds for a single species reaction-diffusion equation with delay, *Discrete Contin. Dyn. Syst. A*, **20** (2008), 407–423.

[34] E. Trofimchuk, V. Tkachenko and S. Trofimchuk, Slowly oscillating wave solutions of a single species reaction-diffusion equation with delay, *J. Differential Equations*, **245** (2008), 2307–2332.

[35] S.-L. Wu, H.-Q. Zhao and S.-Y. Liu, Asymptotic stability of traveling waves for delayed reaction-diffusion equations with crossing-monostability, *Z. Angew. Math. Phys.*, **62** (2011), 377–397.

[36] H. Yagisita, Existence and nonexistence of traveling waves for a nonlocal monostable equation, *Publ. Res. Inst. Math. Sci.*, **45** (2009), 925–953.

[37] G.-B. Zhang, Traveling waves in a nonlocal dispersal population model with age-structure, *Nonlinear Anal.*, **74** (2011), 5030–5047.

[38] G.-B. Zhang and R. Ma, Spreading speeds and traveling waves for a nonlocal dispersal equation with convolution-type crossing-monostable nonlinearity, *Z. Ang. Math. Phys.*, **64** (2013), 1643–1659.

Received December 2014; revised March 2015.

E-mail address: huang@scnu.edu.cn
E-mail address: ming.mei@mcgill.ca, mmei@champlaincollege.qc.ca
E-mail address: zhangkj201@nenu.edu.cn
E-mail address: zhangqifeng0504@gmail.com