J. Differential Equations 251 (2011) 1305–1331

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Asymptotic convergence to planar stationary waves for multi-dimensional unipolar hydrodynamic model of semiconductors

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ARTICLE INFO

Article history:
Received 4 November 2010
Revised 7 April 2011
Available online 22 April 2011

MSC:
35L50
35L60
35L65
76R50

Keywords:
Euler–Poisson equations
Unipolar hydrodynamic model of semiconductor
Nonlinear damping
Planar stationary waves
Asymptotic convergence
Exponential decay rates

ABSTRACT

In this study, we consider the high dimensional unipolar hydrodynamic model for semiconductors in the form of Euler–Poisson equations. Based on the results that we have obtained in the first part (Huang, et al., 2011 [16]) for the 1-D case, we can further show the stability of planar stationary waves in multi-dimensional case. Utilizing the energy method, we obtain the global existence of the solutions of high dimensional Euler–Poisson equations for the unipolar hydrodynamic model, and prove that the solutions converge to the planar stationary waves time-exponentially.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.04.007

1. Introduction

In this paper, we consider the multi-dimensional isentropic Euler–Poisson equations for the unipolar hydrodynamical model of semiconductor device (for simplicity but without loss of generality, we consider 3-D case throughout the paper)

$$\begin{cases} n_t + \operatorname{div}(n\mathbf{u}) = 0, \\ (n\mathbf{u})_t + \operatorname{div}(n\mathbf{u} \oplus \mathbf{u}) + \nabla p(n) = n\nabla\omega - \dfrac{n\mathbf{u}}{\tau}, \\ \Delta\omega = n - b(x), \end{cases} \tag{1.1}$$

for $x = (x_1, x_2, x_3) \in R^3$, and $t > 0$. Here $n = n(x,t)$, $\mathbf{u} = (u_1, u_2, u_3)(x,t)$, and $\omega = \omega(x,t)$ represent the electron density, the electron velocity and the electric potential, respectively. In what follows, we also denote

$$\mathbf{J} = (J_1, J_2, J_3) := n\mathbf{u} \quad \text{and} \quad \mathbf{E} = (E_1, E_2, E_3) := \nabla\omega \tag{1.2}$$

as the electron current density and the electric field, respectively. The coefficient τ denotes the relaxation time. Since our interest here is the large-time behavior of the solutions rather than the limit of relaxation times, so without loss of generality, we assume throughout this paper $\tau = 1$. The function $b(x)$ stands for the density of fixed, positively charged background ions, the so-called doping profile. $p(n)$ is the pressure–density relation satisfying $p'(n) > 0$ for $n > 0$.

For the system (1.1), the initial conditions are prescribed as

$$\begin{cases} n(x,0) = n_0(x) > 0, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \end{cases} \tag{1.3}$$

with

$$\begin{cases} \lim\limits_{x_1 \to \pm\infty} n_0(x) = n_\pm > 0, \\ \lim\limits_{x_1 \to \pm\infty} \mathbf{u}_0(x) = \mathbf{u}_\pm = (u_\pm, 0, 0), \end{cases} \quad \text{for any fixed } (x_2, x_3) \in R^2, \tag{1.4}$$

and $\nabla\omega(x,t) = \mathbf{E}(x,t)$ satisfies the following boundary condition at $x_1 = -\infty$

$$\lim_{x_1 \to -\infty} \nabla\omega(x,t) = \lim_{x_1 \to -\infty} \mathbf{E}(x,t) = \mathbf{E}_- = (E_-, 0, 0), \tag{1.5}$$

where $n_\pm$, $\mathbf{u}_\pm$ and $\mathbf{E}_-$ are given state constants.

The study on hydrodynamical system of semiconductor devices has been one of hot spots of research in mathematical physics, see [1–16,18–35] and the references therein. Among them, the most studies are related only to the 1-D case, and the study to the n-D case is very limited. For the unipolar isentropic and nonisentropic hydrodynamical equations of semiconductors (one carrier type) in 1-D case, Degond and Markowich [3,4], Fang and Ito [5], Gamba [6], Tsuge [33], and Nishibata and Suzuki [29] investigated the existence and uniqueness of (subsonic) 1-D stationary solutions. Such stationary solutions are also called the (planar) stationary waves to the original equations (1.1). Later on, Luo, Natalini and Xin [23] proved that such stationary solutions for the Cauchy problem are stable time-asymptotically, when the state constants of the current density are zero, i.e., $J_+ = J_- = E_- = 0$ (the switch-off case). Huang, Pan and Yu [17] established a framework for the large time behavior of general uniformly bounded weak entropy solutions to the Cauchy problem of Euler–Poisson system of semiconductor devices. Then they proved the bounded weak entropy solutions converge to the stationary solutions exponentially in time. They had to need such a stiff condition due to a technical difficulty in reformulating the perturbed system in L^2-sense. Recently, we [16] successfully obtained the stability of stationary waves without such a stiff condition by ingeniously constructing a new kind of correct functions to delete the gaps between the original solution and the stationary waves in L^2-space, such a new technique was first introduced by Huang, Mei and Wang [15] for the study of bipolar semiconductor models. For the initial–boundary value problem in 1-D case, Li, Markowich and Mei [19] showed the stability of stationary solutions within a bounded domain $[0,1]$ in the almost flat doping case, which then was improved by Guo and Strauss [7] and Nishibata and Suzuki [29,30] even in the non-flat doping case.

However, for the multi-dimensional case, the stability of the corresponding planar stationary waves is never dealt due to the particular difficulty of the system itself. To solve this problem is our main target in the present paper.

Based on our results in the first part of this series of study [16] for 1-D case, we consider the stability problem of multi-dimensional unipolar hydrodynamic model of semiconductors for multi-dimensional case. By using the basic energy method, we can further prove the stability of the planar stationary waves with exponential decay rates. More precisely, when the initial perturbations around the planar stationary waves are small enough, we prove that the solutions of (1.1) converge time-exponentially to the corresponding planar stationary waves in the form

$$\begin{cases} \left\|(n-\tilde{n})(t)\right\|_{L^\infty} = O(1)e^{-\nu t}, \\ \left\|(\mathbf{J}-\tilde{\mathbf{J}})(t)\right\|_{L^\infty} = O(1)e^{-\nu t}, \qquad \text{for some } \nu > 0. \\ \left\|(\nabla\omega - \tilde{\mathbf{E}})(t)\right\|_{L^\infty} = O(1)e^{-\nu t}, \end{cases} \tag{1.6}$$

Here, $\tilde{\mathbf{J}}(x_1) = (\tilde{J}(x_1), 0, 0)$, $\tilde{\mathbf{E}}(x_1) = (\tilde{E}(x_1), 0, 0)$, and $(\tilde{n}, \tilde{J}, \tilde{E})(x_1)$ is the stationary scalar solution for the system (1.1) in 1-D case, which is called the planar stationary wave for the multi-dimensional solution of (1.1). For details, we refer to Section 2 for the precise definition of planar stationary waves.

The interesting thing of (1.6) is that the current density $\mathbf{J}$ converges to a constant state which is independent of the initial current densities, but is determined by the initial–end state of electron density and the electric field at $x_1 = -\infty$.

Since we consider the stability of planar stationary waves, we need to technically assume in this paper that

$$\begin{cases} b(x) = b(x_1) \in C^4(R) \quad \text{and} \quad \lim\limits_{x_1 \to \pm\infty} b(x_1) = n_\pm, \\ \displaystyle\int_{-\infty}^{0} \left|b(x_1) - n_-\right|^2 dx_1 + \int_{0}^{+\infty} \left|b(x_1) - n_+\right|^2 dx_1 \leqslant C_b \end{cases} \tag{1.7}$$

where $C_b > 0$ is a positive constant.

The rest of this paper is arranged as follows. In Section 2, we state some well-known results on the stationary solutions, and the results in [16] which will be used later in this paper. In Section 3, we reformulate the original system (1.1), then introduce our main results, namely, the stability of the planar stationary wave. In Section 4, the main effort is contributed to prove that the original solutions of the Euler–Poisson equations (1.1) converges to the corresponding 1-D stationary solution, the so-called planar stationary wave.

Notation. Throughout this paper, C_0, C_i, etc. always denote some specific positive constants, and C denotes the generic positive constant. $L^2(R^3)$ is the space of square integrable real valued function defined on R^3 with the norm $\|\cdot\|$, and $H^k(R^3)$ (H^k without any ambiguity) denotes the usual Sobolev space with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$. For the nonnegative multi-indexes $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, we define

$$|\alpha| = |\alpha_1| + |\alpha_2| + |\alpha_3| \tag{1.8}$$

and

$$\alpha \leqslant \beta, \quad \text{if } \alpha_i \leqslant \beta_i, \quad \text{for } i = 1, 2, 3. \tag{1.9}$$

2. Planar stationary waves and some preliminaries

In this section, we are going to introduce the well-known results on the stationary solutions to the corresponding steady-state equation of (1.1), the so-called nonlinear stationary waves. For later need, we will also introduce the stability of stationary waves results for 1-D case showed in [16].

For 1-D case, the system of (1.1), (1.3) and (1.5) can be written as follows

$$\begin{cases} \bar{n}_t + \bar{J}_{x_1} = 0, \\ \bar{J}_t + \left(\dfrac{\bar{J}^2}{\bar{n}} + p(\bar{n})\right)_{x_1} = \bar{n}\bar{E} - \bar{J}, \\ \bar{E}_{x_1} = \bar{n} - b(x_1), \\ (\bar{n}, \bar{J})|_{t=0} = \big(\bar{n}_0(x_1), \bar{J}_0(x_1)\big) \to (n_\pm, J_\pm), \quad \text{as } x_1 \to \pm\infty, \\ \bar{E}|_{x_1=-\infty} = E_-, \end{cases} \tag{2.1}$$

where $J_\pm = n_\pm u_\pm$. The corresponding 1-D steady-state equations are

$$\begin{cases} \tilde{J} = \text{const}, \\ \left(\dfrac{\tilde{J}^2}{\tilde{n}} + p(\tilde{n})\right)_{x_1} = \tilde{n}\tilde{E} - \tilde{J}, \\ \tilde{E}_{x_1} = \tilde{n} - b(x_1), \end{cases} \tag{2.2}$$

with the boundary condition

$$\lim_{x_1 \to -\infty} (\tilde{n}, \tilde{E})(x_1) = (n_-, E_-), \qquad \lim_{x_1 \to -\infty} \tilde{n} = n_+. \tag{2.3}$$

Let

$$b_* = \inf_{x_1 \in R} b(x_1) > 0 \quad \text{and} \quad b^* = \sup_{x_1 \in R} b(x_1) > 0. \tag{2.4}$$

The existence and uniqueness of the stationary wave for the steady-state equations (2.2) and (2.3) are given in [19,23] as follows.

Proposition 2.1. *(See [19,23].) Assume that $b'(x_1) \in L^1(R) \cap H^4(R)$ and $b_*\sqrt{p'(b_*)} > |n_- E_-|$, then there exists a unique smooth solution $(\tilde{n}, \tilde{J}, \tilde{E})(x_1)$ of (2.2) and (2.3), which satisfies*

$$\tilde{J} = n_- E_-, \tag{2.5}$$

$$(\tilde{n}, \tilde{E})(+\infty) = \left(n_+, \frac{n_- E_-}{n_+}\right), \tag{2.6}$$

$$b_* \leqslant \tilde{n} \leqslant b^*, \tag{2.7}$$

$$\big|\tilde{n} - b(x_1)\big| = O(1)e^{-n_\pm |x_1|}, \quad \text{as } x_1 \to \pm\infty, \tag{2.8}$$

$$\|\tilde{n} - b\|_{H^3}^2 \leqslant \bar{C}_1(\alpha_1 + \alpha_2 + \alpha_3), \tag{2.9}$$

$$|\tilde{n}_{x_1}| + |\tilde{n}_{x_1x_1}| + |\tilde{n}_{x_1x_1x_1}| + |\tilde{n}_{x_1x_1x_1x_1}| \leqslant \bar{C}_2\alpha_4^{\frac{1}{2}}, \tag{2.10}$$

$$|\tilde{E}| \leqslant \bar{C}_3\big(|E_-| + \alpha_4^{\frac{1}{2}}\big), \tag{2.11}$$

$$|\tilde{E}_{x_1}| + |\tilde{E}_{x_1x_1}| + |\tilde{E}_{x_1x_1x_1}| + |\tilde{E}_{x_1x_1x_1x_1}| \leqslant \bar{C}_4\alpha_4, \tag{2.12}$$

where $\bar{C}_i$ $(i = 1, 2, 3, 4)$ are some positive constants dependent on n_-, E_-, b_* and b^*, α_i $(i = 1, 2, 3, 4)$ are defined as follows

$$\alpha_1 = \|b'\|_{L^2}^2 + \|b'\|_{L^1} + |\log n_+ - \log n_-|, \tag{2.13}$$

$$\alpha_2 = \alpha_1 + \alpha_1^3 + \|b''\|_{L^2}^2 + \|b'\|_{L^4}^4, \tag{2.14}$$

$$\alpha_3 = \alpha_2^3 + \alpha_1^2\alpha_2 + \|b'''\|_{L^2} + \|b''\|_{L^4}^4 + \|b'\|_{L^6}^6, \tag{2.15}$$

$$\begin{aligned}\alpha_4 = {} & \|b'\|_{L^\infty}^2 + \|b'\|_{L^\infty}^6 + \|b''\|_{L^\infty}^2 + \|b'''\|_{L^\infty}^2 + \|b'\|_{L^\infty}^2\|b''\|_{L^\infty}^2 \\ & + \|b''\|_{L^\infty}^2\alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + \|b'\|_{L^\infty}^2\alpha_3^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + \alpha_1\alpha_2 + \alpha_1^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} + \alpha_3^{\frac{1}{2}}\alpha_2^{\frac{1}{2}} \\ & + \alpha_1^{\frac{3}{2}}\alpha_2^{\frac{3}{2}} + \alpha_1^{\frac{1}{2}}\alpha_2\alpha_3^{\frac{1}{2}}.\end{aligned} \tag{2.16}$$

As we know (see [16] for details), since both $\bar{J}(x_1,t) - \tilde{J}(x_1)$, $\bar{E}(x_1,t) - \tilde{E}(x_1) \notin L^2(R)$, such a difficulty comes out from the state constants $J_\pm \neq 0$ and $E_\pm \neq 0$ (the switch-on case), so we need technically to construct the so-called correction function $(\hat{n}, \hat{J}, \hat{E})(x_1,t)$ to delete these gaps, such that

$$\begin{cases} \bar{n}(x_1,t) - \hat{n}(x_1,t) - \tilde{n}(x_1) \in L^2(R), \\ \bar{J}(x_1,t) - \hat{J}(x_1,t) - \tilde{J}(x_1) \in L^2(R), \\ \bar{E}(x_1,t) - \hat{E}(x_1,t) - \tilde{E}(x_1) \in L^2(R), \end{cases}$$

where such a correction function $(\hat{n}, \hat{J}, \hat{E})(x_1,t)$ is constructed in [16] as follows, which depends on the given initial data:

$$\hat{n}(x_1,t) = \begin{cases} \frac{1}{n_+}(A_1(1+\lambda_1)e^{\lambda_1 t} + A_2(1+\lambda_2)e^{\lambda_2 t})m_0(x_1), & \text{for } 1-4n_+ > 0, \\ \frac{1}{n_+}((A_4 + \frac{1}{2}A_3)e^{-\frac{1}{2}t} + \frac{1}{2}A_4te^{-\frac{1}{2}t})m_0(x_1), & \text{for } 1-4n_+ = 0, \\ \frac{1}{2n_+}((A_5 + \sqrt{4n_+-1}\,A_6)\cos(\frac{\sqrt{4n_+-1}}{2}t) & \\ \qquad + (A_6 - \sqrt{4n_+-1}\,A_5)\sin(\frac{\sqrt{4n_+-1}}{2}t))e^{-\frac{1}{2}t}m_0(x_1), & \text{for } 1-4n_+ < 0, \end{cases}$$

and

$$\hat{J}(x_1,t) = \begin{cases} (A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t})\int_{-\infty}^{x_1} m_0(y)\,dy + (J_- - n_-E_-)e^{-t}, \\ \quad \text{for } 1-4n_+ > 0, \\ (A_3e^{-t/2} + A_4te^{-t/2})\int_{-\infty}^{x_1} m_0(y)\,dy + (J_- - n_-E_-)e^{-t}, \\ \quad \text{for } 1-4n_+ = 0, \\ (A_5\cos(\frac{\sqrt{4n_+-1}}{2}t) + A_6\sin(\frac{\sqrt{4n_+-1}}{2}t))e^{-t/2}\int_{-\infty}^{x_1} m_0(y)\,dy + (J_- - n_-E_-)e^{-t}, \\ \quad \text{for } 1-4n_+ < 0, \end{cases}$$

and

$$\hat{E}(x_1,t)=\begin{cases}\frac{1}{n_+}(A_1(1+\lambda_1)e^{\lambda_1 t}+A_2(1+\lambda_2)e^{\lambda_2 t})\int_{-\infty}^{x_1}m_0(y)\,dy, & \text{for } 1-4n_+>0,\\ \frac{1}{n_+}((A_4+\frac{1}{2}A_3)e^{-\frac{1}{2}t}+\frac{1}{2}A_4te^{-\frac{1}{2}t})\int_{-\infty}^{x_1}m_0(y)\,dy, & \text{for } 1-4n_+=0,\\ \frac{1}{2n_+}((A_5+\sqrt{4n_+-1}\,A_6)\cos(\frac{\sqrt{4n_+-1}}{2}t) & \\ \quad+(A_6-\sqrt{4n_+-1}A_5)\sin(\frac{\sqrt{4n_+-1}}{2}t))e^{-\frac{1}{2}t}\int_{-\infty}^{x_1}m_0(y)\,dy, & \text{for } 1-4n_+<0,\end{cases}$$

where $m_0(x_1)\geqslant 0$ satisfies

$$m_0\in C_0^\infty(R),\qquad \operatorname{supp} m_0\subseteq[-L_0,L_0],\qquad \int_R m_0(y)\,dy=1$$

for a large number $L_0>0$, and the other constants are given by

$$\begin{aligned}
\lambda_1&=\frac{-1-\sqrt{|1-4n_+|}}{2},\\
\lambda_2&=\frac{-1+\sqrt{|1-4n_+|}}{2},\\
E_+&=E_-+\int_{-\infty}^{\infty}\big[\bar{n}_0(y)-b(y)\big]\,dy,\\
A_1&=J_+-J_--A_2,\\
A_2&=-\frac{1}{|1-4n_+|}\big[(1+\lambda_1)(J_+-J_-)-n_+E_++n_-E_-\big],\\
A_3&=A_5=J_+-J_-,\\
A_4&=n_+E_+-n_-E_--\frac{1}{2}(J_+-J_-),\\
A_6&=\frac{2}{\sqrt{|4n_+-1|}}\left[n_+E_+-n_-E_--\frac{1}{2}(J_+-J_-)\right].
\end{aligned}$$

Now we introduce the stability results of stationary waves for 1-D case, which were given in our previous work [16].

Proposition 2.2. *(See [16].) Let $(\tilde{n},\tilde{u},\tilde{E})(x_1)$ be stationary wave of (2.2) and (2.3). Denote*

$$\begin{cases} z_0(x_1):=\displaystyle\int_{-\infty}^{x_1}\big[\bar{n}_0(y)-\hat{n}(y,0)-\tilde{n}(y+x_0)\big]\,dy,\\ z_1(x_1):=\bar{J}_0(x_1)-\hat{J}(x_1,0)-\tilde{J}(x_1+x_0),\end{cases}\tag{2.17}$$

and

$$\delta:=|u_+|+|u_-|+|E_-|+|E_+|+\sum_{i=1}^{4}\alpha_i+\|z_0\|_{H^6}+\|z_1\|_{H^5}.\tag{2.18}$$

Then there is a constant $\delta_0>0$ such that when $\delta<\delta_0$, the solution $(\bar{n},\bar{J},\bar{E})$ of the IVP (2.1) is unique and globally exists, and satisfies the following decay estimates

$$\begin{cases} \left\| \partial_t^i \partial_{x_1}^j (\bar{n} - \tilde{n})(t) \right\|_{L^\infty} \leqslant C\delta e^{-\mu t}, \\ \left\| \partial_t^i \partial_{x_1}^j (\bar{u} - \tilde{u})(t) \right\|_{L^\infty} \leqslant C\delta e^{-\mu t}, \quad i = 0, 1;\ j = 0, 1, 2, 3, 4, \\ \left\| \partial_t^i \partial_{x_1}^j (\bar{E} - \tilde{E})(t) \right\|_{L^\infty} \leqslant C\delta e^{-\mu t}, \end{cases} \tag{2.19}$$

where $\tilde{u}(x_1) := \frac{\tilde{J}(x_1)}{\tilde{n}(x_1)}$, $\bar{u}(x_1, t) := \frac{\bar{J}(x_1,t)}{\bar{n}(x_1,t)}$ *and* μ *is a positive constant.*

3. Convergence to planar stationary waves

Let $(n, \mathbf{u}, \omega)(x,t)$ be the solution of the multi-dimensional system (1.1)–(1.5), and let $\bar{\mathbf{u}}(x_1, t) = (\bar{u}(x_1, t), 0, 0)$, $\bar{\mathbf{E}}(x_1, t) = (\bar{E}(x_1, t), 0, 0)$, where $\bar{u}(x_1, t) = \frac{\bar{J}(x_1,t)}{\bar{n}(x_1,t)}$, and $(\bar{n}, \bar{J}, \bar{E})(x_1, t)$ is the scalar solution of (2.1) in 1-D. Now we define

$$\begin{cases} \phi(x,t) := n(x,t) - \bar{n}(x_1, t), \\ \boldsymbol{\Psi}(x,t) := \mathbf{u}(x,t) - \bar{\mathbf{u}}(x_1, t). \end{cases} \tag{3.1}$$

From (1.1) and (2.1), we can reduce the system to

$$\begin{cases} \phi_t + \operatorname{div}(\bar{n}\boldsymbol{\Psi} + \phi\boldsymbol{\Psi} + \phi\bar{\mathbf{u}}) = 0, \\ \boldsymbol{\Psi}_t + \boldsymbol{\Psi} + \dfrac{p'(\bar{n}+\phi)}{\bar{n}+\phi}\nabla\phi - (\nabla\omega - \bar{\mathbf{E}}) = -\mathbf{L}, \\ \operatorname{div}(\nabla\omega - \bar{E}) = \phi, \end{cases} \tag{3.2}$$

where

$$\begin{cases} \mathbf{L} = \theta\nabla\bar{n} + \bar{\mathbf{u}}\nabla\boldsymbol{\Psi} + \boldsymbol{\Psi}\nabla\bar{\mathbf{u}} + \boldsymbol{\Psi}\nabla\boldsymbol{\Psi}, \\ \theta = \dfrac{p'(\bar{n}+\phi)}{\bar{n}+\phi} - \dfrac{p'(\bar{n})}{\bar{n}} \approx \phi. \end{cases} \tag{3.3}$$

Notice that

$$\operatorname{curl}(\nabla\omega) = \mathbf{0} \quad \text{and} \quad \operatorname{curl}(\bar{\mathbf{E}}) = \mathbf{0} \quad \text{which imply} \quad \operatorname{curl}(\nabla\omega - \bar{\mathbf{E}}) = \mathbf{0}, \tag{3.4}$$

so there exists a function $H(x,t)$ such that

$$\nabla H = \nabla\omega - \bar{\mathbf{E}}. \tag{3.5}$$

Thus, we can reduce (3.2) into

$$\begin{cases} \phi_t + \operatorname{div}(\bar{n}\boldsymbol{\Psi} + \phi\boldsymbol{\Psi} + \phi\bar{\mathbf{u}}) = 0, \\ \boldsymbol{\Psi}_t + \boldsymbol{\Psi} + \dfrac{p'(\bar{n}+\phi)}{\bar{n}+\phi}\nabla\phi - \nabla H = -\mathbf{L}, \\ \Delta H = \phi, \end{cases} \tag{3.6}$$

with the initial data

$$\begin{cases} \phi(x,0) = n_0(x) - \bar{n}_0(x_1) =: \phi_0(x) \in H^3(R^3), \\ \boldsymbol{\Psi}(x,0) = \mathbf{u}_0(x) - \bar{\mathbf{u}}_0(x_1) =: \boldsymbol{\Psi}_0(x) \in H^3(R^3), \end{cases} \tag{3.7}$$

and the boundary condition

$$\left|\nabla H(x,t)\right| \to 0 \quad \text{as } |x| \to +\infty. \tag{3.8}$$

We also define

$$\begin{cases} \Delta H_0(x) := n_0(x) - \bar{n}_0(x_1), \\ H_0(x) \to 0, \quad \text{as } |x| \to +\infty, \end{cases} \tag{3.9}$$

and

$$\eta := \left\|(\phi_0, \boldsymbol{\Psi}_0, \nabla H_0)\right\|_{H^3(R^3)}. \tag{3.10}$$

Now we are ready to state the stability results for the planar stationary waves in R^3 as follows.

Theorem 3.1. *Let $b(x) = b(x_1)$ satisfy (1.7) and let $\delta + \eta \ll 1$, where δ is defined in (2.18). Then there exists a unique global smooth solution $(n, \mathbf{u}, \nabla\omega)$ for 3-D unipolar hydrodynamic model for semiconductor system (1.1)–(1.5) and satisfies*

$$n - \bar{n}, \mathbf{u} - \bar{\mathbf{u}}, \nabla\omega - \bar{\mathbf{E}} \in C\big([0,\infty), H^3\big(R^3\big)\big) \tag{3.11}$$

and

$$\left\|(n - \bar{n}, \mathbf{u} - \bar{\mathbf{u}}, \nabla\omega - \bar{\mathbf{E}})(t)\right\|_{H^3} \leqslant C\eta e^{-\nu t}, \quad \textit{for some constant } \nu > 0, \tag{3.12}$$

which implies, by Sobolev's inequalities in this 3-D case, that

$$\left\|(n - \bar{n}, \mathbf{u} - \bar{\mathbf{u}}, \nabla\omega - \bar{\mathbf{E}})(t)\right\|_{C^1} \leqslant C\eta e^{-\nu t} \tag{3.13}$$

and

$$\left\|(n - \bar{n}, \mathbf{u} - \bar{\mathbf{u}}, \nabla\omega - \bar{\mathbf{E}})(t)\right\|_{L^p} \leqslant C\eta e^{-\nu t}, \quad \textit{for } 2 \leqslant p \leqslant \infty. \tag{3.14}$$

As we mentioned before (originally, see the first part of [16]), there are some L^p-gaps ($1 \leqslant p < \infty$) between the 1-D solution $(\bar{n}, \bar{\mathbf{u}}, \bar{\mathbf{E}})(x,t)$ and the 1-D stationary wave $(\tilde{n}, \tilde{\mathbf{u}}, \tilde{\mathbf{E}})(x,t)$, so one has only the L^∞-convergence (2.19) of 1-D solution $(\bar{n}, \bar{\mathbf{u}}, \bar{\mathbf{E}})(x,t)$ and the 1-D stationary wave $(\tilde{n}, \tilde{\mathbf{u}}, \tilde{\mathbf{E}})(x,t)$. Therefore, from Theorem 3.1 and Proposition 2.2, one can immediately obtain the following L^∞-stability of the planar stationary wave.

Corollary 3.2 *(Convergence to planar stationary waves). Under the conditions of Theorem 3.1 and Proposition 2.2, the solution $(n, \mathbf{u}, \nabla\omega)$ for 3-D system (1.1)–(1.5) converges to its planar stationary wave $(\tilde{n}, \tilde{u}, \tilde{E})(x_1)$ (the steady-state solution of (2.2) and (2.3)) as follows*

$$\begin{cases} \left\|(n - \tilde{n})(t)\right\|_{L^\infty} \leqslant O(1)e^{-\nu t}, \\ \left\|(\mathbf{u} - \tilde{\mathbf{u}})(t)\right\|_{L^\infty} \leqslant O(1)e^{-\nu t}, \\ \left\|(\nabla\omega - \tilde{\mathbf{E}})(t)\right\|_{L^\infty} \leqslant O(1)e^{-\nu t}, \end{cases} \tag{3.15}$$

where $\tilde{\mathbf{u}}(x_1) = (\tilde{u}(x_1), 0, 0)$, $\tilde{\mathbf{E}}(x_1) = (\tilde{E}(x_1), 0, 0)$.

4. A priori estimates

In order to prove Theorem 3.1 by the energy method with the continuous extension argument (cf. [27]), as we know, the crucial step is to establish the *a priori* estimates for the solution. This will be our main target in this section.

Letting $T \in (0, +\infty]$, we define the solution space for

$$X(T) = \left\{(\phi, \boldsymbol{\Psi}, \nabla H)(x,t) \;\middle|\; (\phi, \boldsymbol{\Psi}, \nabla H)(x,t) \in C\left(0, T; H^3\left(R^3\right)\right),\ 0 \leqslant t \leqslant T\right\} \tag{4.1}$$

with the norm

$$N(T)^2 =: \sup_{0 \leqslant t \leqslant T} \left\|(\phi, \boldsymbol{\Psi}, \nabla H)(t)\right\|_{H^3}^2. \tag{4.2}$$

Let $N(T)^2 \leqslant \varepsilon^2$, where ε is sufficiently small which will be determined later. It is noted that, (4.2) with the Sobolev inequality $\|f\|_{L^\infty(R^3)} \leqslant C\|f\|^{1/4}\|\nabla^2 f\|^{3/4}$ gives

$$\sum_{k=0}^{1} \left\|\nabla^k(\phi, \boldsymbol{\Psi}, \nabla H)(t)\right\|_{L^\infty(R^3)} \leqslant C\varepsilon. \tag{4.3}$$

It is easy to verify from (3.1) and (4.3) that, there exists a positive constant c such that

$$0 < \frac{1}{c} \leqslant n = \phi + \bar{n} \leqslant c. \tag{4.4}$$

Remark 4.1. Before we deal with the *a priori* estimates, we can get an estimate about $H(x,t)$, which will be used later. Noticing that $\Delta H(x,t), \nabla H(x,t) \in L^2(R^3)$, we can easily obtain

$$\begin{aligned} \left|H(x,t)\right| &= \left|\int_{R^3} \frac{1}{|x-y|}\phi(y,t)\,dy + C\right| \\ &= \left|\int_{R^3} \frac{1}{|x-y|}\Delta H(y,t)\,dy + C\right| \\ &= \left|\int_{R^3} \nabla\left(\frac{1}{|x-y|}\right)\nabla H(y,t)\,dy + C\right| \leqslant \tilde{C} < +\infty. \end{aligned} \tag{4.5}$$

Now we are going to establish the *a priori* estimates.

Proposition 4.2 *(A priori estimate). It holds that*

$$\left\|(\nabla H, \phi, \boldsymbol{\Psi})(t)\right\|_{H^3}^2 \leqslant C\left\|(\nabla H_0, \phi_0, \boldsymbol{\Psi}_0)\right\|_{H^3}^2 e^{-\mu t}, \tag{4.6}$$

provided $\varepsilon + \delta \ll 1$.

In order to prove Proposition 4.2, we are going to establish the L^2-energy estimate for the solution first, then to establish for the first, the second, and the third derivatives of the solution.

Now we prove our first L^2-energy estimate. Multiplying $(3.6)_2$ by $-(\bar{n} + \phi)\nabla H$ and integrating the resultant equation over R^3 with respect to x, we obtain

$$
\begin{aligned}
&-\int_{R^3}(\bar{n}+\phi)\boldsymbol{\Psi}_t\nabla H\,dx-\int_{R^3}(\bar{n}+\phi)\boldsymbol{\Psi}\nabla H\,dx\\
&\quad-\int_{R^3}p'(\bar{n}+\phi)\nabla\phi\nabla H\,dx+\int_{R^3}(\bar{n}+\phi)|\nabla H|^2\,dx\\
&=\int_{R^3}(\bar{n}+\phi)\mathbf{L}\nabla H\,dx. \qquad (4.7)
\end{aligned}
$$

Differentiating $(3.6)_3$ with respect to t twice, and utilizing $(3.6)_1$, we have

$$
\Delta H_{tt}=-\operatorname{div}\big((\bar{n}+\phi)\boldsymbol{\Psi}_t+\phi_t\boldsymbol{\Psi}+\bar{n}_t\boldsymbol{\Psi}+\phi_t\bar{\mathbf{u}}+\phi\bar{\mathbf{u}}_t\big). \qquad (4.8)
$$

Next multiplying (4.8) by H and integrating it by parts with respect to x over R^3 and using (4.5), we obtain

$$
\begin{aligned}
&-\int_{R^3}(\bar{n}+\phi)\boldsymbol{\Psi}_t\nabla H\,dx\\
&=\int_{R^3}\operatorname{div}\big((\bar{n}+\phi)\boldsymbol{\Psi}_t\big)H\,dx\\
&=-\int_{R^3}\Delta H_{tt}H\,dx-\int_{R^3}\operatorname{div}(\phi_t\boldsymbol{\Psi}+\bar{n}_t\boldsymbol{\Psi}+\phi_t\bar{\mathbf{u}}+\phi\bar{\mathbf{u}}_t)H\,dx\\
&=\int_{R^3}\nabla H_{tt}\nabla H\,dx+\int_{R^3}(\phi_t\boldsymbol{\Psi}+\bar{n}_t\boldsymbol{\Psi}+\phi_t\bar{\mathbf{u}}+\phi\bar{\mathbf{u}}_t)\nabla H\,dx\\
&=\frac{d}{dt}\bigg(\int_{R^3}\nabla H\nabla H_t\,dx\bigg)-\big\|\nabla H_t(t)\big\|^2+\int_{R^3}(\phi_t\boldsymbol{\Psi}+\bar{n}_t\boldsymbol{\Psi}+\phi_t\bar{\mathbf{u}}+\phi\bar{\mathbf{u}}_t)\nabla H\,dx\\
&\geqslant\frac{d}{dt}\bigg(\int_{R^3}\nabla H\nabla H_t\,dx\bigg)-\big\|\nabla H_t(t)\big\|^2-C(\delta+\varepsilon)\big\|(\nabla\phi,\nabla\boldsymbol{\Psi},\nabla H,\phi,\boldsymbol{\Psi})(t)\big\|^2, \qquad (4.9)
\end{aligned}
$$

where, in order to prove the last estimate in (4.9), namely,

$$
\begin{aligned}
&\int_{R^3}(\phi_t\boldsymbol{\Psi}+\bar{n}_t\boldsymbol{\Psi}+\phi_t\bar{\mathbf{u}}+\phi\bar{\mathbf{u}}_t)\nabla H\,dx\\
&\qquad\geqslant-C(\delta+\varepsilon)\big\|(\nabla\phi,\nabla\boldsymbol{\Psi},\nabla H,\phi,\boldsymbol{\Psi})(t)\big\|^2, \qquad (4.10)
\end{aligned}
$$

we have used the Cauchy–Schwarz inequality $|ab|\leqslant\frac{1}{2}a^2+\frac{1}{2}b^2$, the *a priori* assumption (4.3), and the following smallness

$$
\big|\partial_t^i\partial_{x_1}^j(\bar{n},\bar{\mathbf{u}})\big|\leqslant C\delta,\quad \text{for } i=0,1;\ j=0,1,2,3,4, \qquad (4.11)
$$

which will be frequently used later. Such a smallness (4.11) can be easily obtained from Propositions 2.1 and 2.2. Here, in the estimate of (4.10), the following estimate is needed too:

$$\|\phi_t\|^2 \leqslant C\|\nabla\boldsymbol{\Psi}\|^2 + C(\delta+\varepsilon)\big\|(\nabla\phi, \phi, \boldsymbol{\Psi})\big\|^2, \tag{4.12}$$

which can be easily obtained from $(3.6)_1$ and (4.11).

Differentiating $(3.6)_3$ with respect to t, and utilizing $(3.6)_1$, we have

$$\Delta H_t = -\operatorname{div}\big((\bar n+\phi)\Psi + \phi\bar{\mathbf{u}}\big). \tag{4.13}$$

Integrating $(4.13)\cdot H$ by parts with respect to x over R^3 and using (4.5), we obtain

$$\begin{aligned}
-\int_{R^3}(\bar n+\phi)\boldsymbol{\Psi}\nabla H\,dx &= \int_{R^3}\operatorname{div}\big((\bar n+\phi)\boldsymbol{\Psi}\big)H\,dx \\
&= -\int_{R^3}\Delta H_t H\,dx - \int_{R^3}\operatorname{div}(\phi\bar{\mathbf{u}})H\,dx \\
&= \frac{d}{dt}\bigg(\int_{R^3}\frac{|\nabla H|^2}{2}\,dx\bigg) + \int_{R^3}\phi\bar{\mathbf{u}}\nabla H\,dx \\
&\geqslant \frac{d}{dt}\bigg(\int_{R^3}\frac{|\nabla H|^2}{2}\,dx\bigg) - C(\delta+\varepsilon)\big\|(\phi,\nabla H)(t)\big\|^2.
\end{aligned} \tag{4.14}$$

Noticing $(3.6)_3$, we obtain

$$\begin{aligned}
-\int_{R^3}p'(\bar n+\phi)\nabla\phi\nabla H\,dx &= -\int_{R^3}p'(\bar n+\phi)\nabla(\Delta H)\nabla H\,dx \\
&\geqslant 2C_0\big\|\nabla^2 H(t)\big\|^2 - C(\delta+\varepsilon)\big\|\nabla H(t)\big\|^2
\end{aligned} \tag{4.15}$$

and

$$\int_{R^3}(\bar n+\phi)|\nabla H|^2\,dx \geqslant 2C_0\big\|\nabla H(t)\big\|^2. \tag{4.16}$$

Using the Cauchy inequality and $(3.3)_2$, we obtain

$$\begin{aligned}
\int_{R^3}(\bar n+\phi)\mathbf{L}\nabla H\,dx &= \int_{R^3}(\bar n+\phi)(\theta\nabla\bar n + \bar u\nabla\boldsymbol{\Psi} + \boldsymbol{\Psi}\nabla\bar u + \boldsymbol{\Psi}\nabla\boldsymbol{\Psi})\nabla H\,dx \\
&\leqslant C(\delta+\varepsilon)\big\|(\nabla\boldsymbol{\Psi}, \nabla H, \phi, \boldsymbol{\Psi})(t)\big\|^2,
\end{aligned} \tag{4.17}$$

then, substituting (4.9), (4.14)–(4.17) into (4.7) and noticing the smallness of δ and ε, we obtain

$$\begin{aligned}
&\frac{d}{dt}\bigg(\int_{R^3}\nabla H\nabla H_t + \frac{|\nabla H|^2}{2}\,dx\bigg) + C_0\big\|\big(\nabla H, \nabla^2 H\big)(t)\big\|^2 \\
&\quad\leqslant \big\|\nabla H_t(t)\big\|^2 + C(\delta+\varepsilon)\big\|(\nabla\boldsymbol{\Psi}, \nabla\phi, \phi, \boldsymbol{\Psi})(t)\big\|^2.
\end{aligned} \tag{4.18}$$

From $(3.6)_3$, we can easily obtain

$$\left\|\nabla^2 H(t)\right\|^2 = \left\|\phi(t)\right\|^2, \tag{4.19}$$

and multiplying (4.13) by H_t and integrating it by parts, we obtain

$$\left\|\nabla H_t(t)\right\|^2 \leqslant C\left\|\boldsymbol{\Psi}(t)\right\|^2 + C\delta\left\|\phi(t)\right\|^2. \tag{4.20}$$

Combining (4.19) and (4.20), we reduce (4.18) into

$$\frac{d}{dt}F_{11}(t) + C_0\left\|(\nabla H, \phi)(t)\right\|^2 \leqslant C\left\|\boldsymbol{\Psi}(t)\right\|^2 + C(\delta+\varepsilon)\left\|(\nabla\boldsymbol{\Psi}, \nabla\phi)(t)\right\|^2, \tag{4.21}$$

where

$$F_{11}(t) = \int_{R^3}\left[\nabla H\nabla H_t + \frac{|\nabla H|^2}{2}\right]dx. \tag{4.22}$$

On the other hand, by taking $(\bar{n}+\phi)(3.6)_2\cdot\boldsymbol{\Psi}$ and integrating the resultant equation, we get

$$\begin{aligned}
&\int_{R^3}(\bar{n}+\phi)\boldsymbol{\Psi}_t\boldsymbol{\Psi}\,dx + \int_{R^3}(\bar{n}+\phi)|\boldsymbol{\Psi}|^2\,dx \\
&\quad + \int_{R^3}p'(\bar{n}+\phi)\nabla\phi\boldsymbol{\Psi}\,dx - \int_{R^3}(\bar{n}+\phi)\nabla H\boldsymbol{\Psi}\,dx \\
&= -\int_{R^3}(\bar{n}+\phi)\mathbf{L}\boldsymbol{\Psi}\,dx.
\end{aligned} \tag{4.23}$$

Furthermore, we can prove

$$\int_{R^3}(\bar{n}+\phi)\boldsymbol{\Psi}_t\boldsymbol{\Psi}\,dx \geqslant \frac{d}{dt}\left(\int_{R^3}\frac{1}{2}(\bar{n}+\phi)|\boldsymbol{\Psi}|^2\,dx\right) - C(\delta+\varepsilon)\left\|\boldsymbol{\Psi}(t)\right\|^2 \tag{4.24}$$

and

$$\int_{R^3}(\bar{n}+\phi)|\boldsymbol{\Psi}|^2\,dx \geqslant C_0\left\|\boldsymbol{\Psi}(t)\right\|^2, \tag{4.25}$$

and using $(3.6)_1$, we obtain

$$\begin{aligned}
&\int_{R^3}p'(\bar{n}+\phi)\nabla\phi\boldsymbol{\Psi}\,dx \\
&= -\int_{R^3}\frac{p'(\bar{n}+\phi)}{\bar{n}+\phi}\phi\,\mathrm{div}\big((\bar{n}+\phi)\boldsymbol{\Psi}\big)\,dx - \int_{R^3}(\bar{n}+\phi)\nabla\left(\frac{p'(\bar{n}+\phi)}{\bar{n}+\phi}\right)\phi\boldsymbol{\Psi}\,dx
\end{aligned}$$

$$\geqslant \int_{R^3} \frac{p'(\bar{n}+\phi)}{\bar{n}+\phi}\phi\phi_t\,dx + \int_{R^3} \frac{p'(\bar{n}+\phi)}{\bar{n}+\phi}\phi\,\mathrm{div}(\bar{\mathbf{u}}\phi)\,dx - C(\delta+\varepsilon)\big\|(\boldsymbol{\Psi},\phi)(t)\big\|^2$$

$$\geqslant \frac{d}{dt}\left(\int_{R^3} \frac{p'(\bar{n}+\phi)}{2(\bar{n}+\phi)}\phi^2\,dx\right) - C(\delta+\varepsilon)\big\|(\boldsymbol{\Psi},\nabla\phi,\phi)(t)\big\|^2. \tag{4.26}$$

Similarly to (4.14), we have

$$-\int_{R^3} (\bar{n}+\phi)\boldsymbol{\Psi}\nabla H\,dx \geqslant \frac{d}{dt}\left(\int_{R^3} \frac{|\nabla H|^2}{2}\,dx\right) - C(\delta+\varepsilon)\big\|(\phi,\nabla H)(t)\big\|^2 \tag{4.27}$$

and

$$-\int_{R^3} (\bar{n}+\phi)\mathbf{L}\boldsymbol{\Psi}\,dx = \int_{R^3} (\bar{n}+\phi)(\theta\nabla\bar{n} + \bar{\mathbf{u}}\nabla\boldsymbol{\Psi} + \boldsymbol{\Psi}\nabla\bar{\mathbf{u}} + \boldsymbol{\Psi}\nabla\boldsymbol{\Psi})\boldsymbol{\Psi}\,dx$$

$$\leqslant C(\delta+\varepsilon)\big\|(\phi,\boldsymbol{\Psi})(t)\big\|^2. \tag{4.28}$$

Substituting (4.24)–(4.28) into (4.23) and noticing the smallness of δ and ε, we obtain

$$\frac{d}{dt}F_{12}(t) + C_0\big\|\boldsymbol{\Psi}(t)\big\|^2 \leqslant C(\delta+\varepsilon)\big\|(\nabla H,\nabla\phi,\phi)(t)\big\|^2, \tag{4.29}$$

where

$$F_{12}(t) = \int_{R^3} \left[\frac{|\nabla H|^2}{2} + \frac{p'(\bar{n}+\phi)}{2(\bar{n}+\phi)}\phi^2 + \frac{1}{2}(\bar{n}+\phi)|\boldsymbol{\Psi}|^2\right]dx. \tag{4.30}$$

Then, from (4.21) and (4.29), we have established the first energy estimate as follows.

Lemma 4.3. *Let N_1 be a positive and large number, and $F_1(t) := F_{11}(t) + N_1F_{12}(t)$. Then,*

$$\frac{d}{dt}F_1(t) + C_1\big\|(\nabla H,\phi,\boldsymbol{\Psi})\big\|^2 \leqslant C(\delta+\varepsilon)\big\|(\nabla\phi,\nabla\boldsymbol{\Psi})(t)\big\|^2 \tag{4.31}$$

and

$$C_{11}\big\|(\nabla H,\phi,\boldsymbol{\Psi})(t)\big\|^2 \leqslant F_1(t) \leqslant C_{12}\big\|(\nabla H,\phi,\boldsymbol{\Psi})(t)\big\|^2, \tag{4.32}$$

where C_1, C_{11} and C_{12} are some positive constants.

Now we are going to establish the second energy estimate for the solution with the first derivatives.

Let β_1 be nonnegative multi-index, and $|\beta_1| = 1$. Taking $\partial_x^{\beta_1}((\bar{n}+\phi)(3.6)_2)\cdot(-\partial_x^{\beta_1}\nabla H)$ and integrating the resultant equation, we have

$$
\begin{aligned}
&- \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}\big((\bar{n}+\phi)\boldsymbol{\Psi}_t\big)\partial_x^{\beta_1}\nabla H\,dx - \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}\big((\bar{n}+\phi)\boldsymbol{\Psi}\big)\partial_x^{\beta_1}\nabla H\,dx \\
&\quad - \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}\big(p'(\bar{n}+\phi)\nabla\phi\big)\partial_x^{\beta_1}\nabla H\,dx + \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}\big((\bar{n}+\phi)\nabla H\big)\partial_x^{\beta_1}\nabla H\,dx \\
&= \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}\big((\bar{n}+\phi)\mathbf{L}\big)\partial_x^{\beta_1}\nabla H\,dx. \qquad (4.33)
\end{aligned}
$$

Integrating $\partial_x^{\beta_1}(4.8)\cdot\partial_x^{\beta_1}H$ by parts with respect to x over R^3, we obtain

$$
\begin{aligned}
&- \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}\big((\bar{n}+\phi)\boldsymbol{\Psi}_t\big)\partial_x^{\beta_1}\nabla H\,dx \\
&= \frac{d}{dt}\bigg(\int_{R^3} \nabla^2 H\nabla^2 H_t\,dx\bigg) - \big\|\nabla^2 H_t(t)\big\|^2 + \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}(\phi_t\Psi + \bar{n}_t\boldsymbol{\Psi} + \phi_t\bar{\mathbf{u}} + \phi\bar{\mathbf{u}}_t)\partial_x^{\beta_1}\nabla H\,dx \\
&= \frac{d}{dt}\bigg(\int_{R^3} \nabla^2 H\nabla^2 H_t\,dx\bigg) - \big\|\nabla^2 H_t(t)\big\|^2 - \sum_{|\beta_1|=1} \int_{R^3} (\phi_t\boldsymbol{\Psi} + \bar{n}_t\boldsymbol{\Psi} + \phi_t\bar{\mathbf{u}} + \phi\bar{\mathbf{u}}_t)\partial_x^{\beta_1+\beta_1}\nabla H\,dx \\
&\geqslant \frac{d}{dt}\bigg(\int_{R^3} \nabla^2 H\nabla^2 H_t\,dx\bigg) - \big\|\nabla^2 H_t(t)\big\|^2 - C(\delta+\varepsilon)\big\|(\nabla\phi, \nabla\Psi, \phi, \boldsymbol{\Psi})\big\|^2. \qquad (4.34)
\end{aligned}
$$

Similarly, integrating $\partial_x^{\beta_1}(4.13)\cdot\partial_x^{\beta_1}H$ by parts, we get

$$
\begin{aligned}
&- \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}\big((\bar{n}+\phi)\boldsymbol{\Psi}\big)\partial_x^{\beta_1}\nabla H\,dx \\
&= \frac{d}{dt}\bigg(\int_{R^3} \frac{|\nabla^2 H|^2}{2}\,dx\bigg) - \sum_{|\beta_1|=1} \int_{R^3} (\phi\bar{\mathbf{u}})\partial_x^{\beta_1+\beta_1}\nabla H\,dx \\
&\geqslant \frac{d}{dt}\bigg(\int_{R^3} \frac{|\nabla^2 H|^2}{2}\,dx\bigg) - C(\delta+\varepsilon)\big\|(\nabla\phi, \phi)(t)\big\|^2, \qquad (4.35)
\end{aligned}
$$

where we used

$$
\big\|\nabla^3 H(t)\big\| = \big\|\nabla\phi(t)\big\|. \qquad (4.36)
$$

Noticing (4.36), we obtain

$$
\begin{aligned}
&- \sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1}\big(p'(\bar{n}+\phi)\nabla\phi\big)\partial_x^{\beta_1}\nabla H\,dx \\
&\geqslant - \sum_{|\beta_1|=1} \int_{R^3} p'(\bar{n}+\phi)\nabla\partial_x^{\beta_1}\Delta H\partial_x^{\beta_1}\nabla H\,dx - C(\delta+\varepsilon)\big\|\big(\nabla^3 H, \nabla^2 H\big)(t)\big\|^2
\end{aligned}
$$

$$\geqslant 3C_0\left\|\nabla^3 H(t)\right\|^2 - C(\delta+\varepsilon)\left\|\left(\nabla^3 H, \nabla^2 H\right)(t)\right\|^2$$
$$\geqslant 2C_0\left\|\nabla\phi(t)\right\|^2 - C(\delta+\varepsilon)\left\|\phi(t)\right\|^2, \tag{4.37}$$

and

$$\sum_{|\beta_1|=1}\int_{R^3}\partial_x^{\beta_1}\left((\bar n+\phi)\nabla H\right)\partial_x^{\beta_1}\nabla H\,dx \geqslant C_0\left\|\nabla^2 H(t)\right\|^2 - C(\delta+\varepsilon)\left\|\nabla H(t)\right\|^2$$
$$= C_0\left\|\phi(t)\right\|^2 - C(\delta+\varepsilon)\left\|\nabla H(t)\right\|^2, \tag{4.38}$$

and

$$\sum_{|\beta_1|=1}\int_{R^3}\partial_x^{\beta_1}\left((\bar n+\phi)\mathbf{L}\right)\partial_x^{\beta_1}\nabla H\,dx \leqslant C(\delta+\varepsilon)\left\|(\nabla\phi, \nabla\boldsymbol{\Psi}, \phi, \boldsymbol{\Psi})\right\|^2, \tag{4.39}$$

and by integrating $\partial_x^{\beta_1}(4.13)\cdot\partial_x^{\beta_1}H_t$ by parts to have

$$\left\|\nabla^2 H_t(t)\right\|^2 \leqslant C\left\|\nabla\boldsymbol{\Psi}(t)\right\|^2 + C(\delta+\varepsilon)\left\|(\nabla\phi, \phi, \boldsymbol{\Psi})(t)\right\|^2, \tag{4.40}$$

then, by substituting (4.34)–(4.39) into (4.33), and noticing the estimates of (4.40) and the smallness of δ, ε, we obtain

$$\frac{d}{dt}F_{21}(t) + C_0\left\|(\nabla\phi, \phi)(t)\right\|^2 \leqslant C\left\|\nabla\boldsymbol{\Psi}(t)\right\|^2 + C(\delta+\varepsilon)\left\|(\nabla H, \boldsymbol{\Psi})(t)\right\|^2, \tag{4.41}$$

where

$$F_{21}(t) = \int_{R^3}\left[\nabla^2 H\nabla^2 H_t + \frac{|\nabla^2 H|^2}{2}\right]dx. \tag{4.42}$$

On the other hand, by taking $\partial_x^{\beta_1}(3.6)_2\cdot\partial_x^{\beta_1}\boldsymbol{\Psi}$ and integrating the resultant equation, we get

$$\frac{d}{dt}\left(\int_{R^3}\frac{|\nabla\boldsymbol{\Psi}|^2}{2}\,dx\right) + \left\|\nabla\boldsymbol{\Psi}(t)\right\|^2$$
$$+ \sum_{|\beta_1|=1}\int_{R^3}\partial_x^{\beta_1}\left(\frac{p'(\bar n+\phi)}{\bar n+\phi}\nabla\phi\right)\partial_x^{\beta_1}\boldsymbol{\Psi}\,dx - \sum_{|\beta_1|=1}\int_{R^3}\nabla\partial_x^{\beta_1}H\partial_x^{\beta_1}\boldsymbol{\Psi}\,dx$$
$$= -\sum_{|\beta_1|=1}\int_{R^3}\partial_x^{\beta_1}\mathbf{L}\partial_x^{\beta_1}\boldsymbol{\Psi}\,dx. \tag{4.43}$$

Notice also that

$$\sum_{|\beta_1|=1} \int_{R^3} \partial_x^{\beta_1} \left(\frac{p'(\bar{n}+\phi)}{\bar{n}+\phi} \nabla\phi \right) \partial_x^{\beta_1} \boldsymbol{\Psi} \, dx$$

$$\geqslant -C(\delta+\varepsilon) \left\| (\nabla\phi, \nabla\boldsymbol{\Psi})(t) \right\|^2 - \sum_{|\beta_1|=1} \int_{R^3} \frac{p'(\bar{n}+\phi)}{\bar{n}+\phi} \partial_x^{\beta_1} \phi \partial_x^{\beta_1} \operatorname{div} \boldsymbol{\Psi} \, dx. \tag{4.44}$$

In order to control the last integral of (4.44), we note that for index α, by using $(3.6)_1$,

$$\begin{aligned} \partial_x^\alpha \operatorname{div} \boldsymbol{\Psi} = -\frac{1}{\bar{n}+\phi} \Bigg\{ & \partial_x^\alpha \phi_t + \sum_{|\gamma|=1,\, \gamma \leqslant \alpha}^{|\alpha|} C_\gamma \partial_x^\gamma \bar{n} \partial_x^{\alpha-\gamma} \operatorname{div} \boldsymbol{\Psi} \\ & + \sum_{|\gamma|=1,\, \gamma \leqslant \alpha}^{|\alpha|} C_\gamma \partial_x^\gamma \phi \partial_x^{\alpha-\gamma} \operatorname{div} \boldsymbol{\Psi} \\ & + \partial_x^\alpha [\phi \operatorname{div} \bar{\mathbf{u}} + \bar{\mathbf{u}} \nabla\phi + \boldsymbol{\Psi} \nabla\bar{n} + \boldsymbol{\Psi} \nabla\phi] \Bigg\}. \end{aligned} \tag{4.45}$$

So, utilizing (4.45), we have

$$\begin{aligned} & -\sum_{|\beta_1|=1} \int_{R^3} \frac{p'(\bar{n}+\phi)}{\bar{n}+\phi} \partial_x^{\beta_1} \phi \partial_x^{\beta_1} \operatorname{div} \boldsymbol{\Psi} \, dx \\ & = \sum_{|\beta_1|=1} \int_{R^3} \frac{p'(\bar{n}+\phi)}{(\bar{n}+\phi)^2} \partial_x^{\beta_1} \phi \Bigg\{ \partial_x^{\beta_1} \phi_t + \sum_{|\gamma|=1,\, \gamma \leqslant \beta_1}^{|\beta_1|} C_\gamma \partial_x^\gamma \bar{n} \partial_x^{\beta_1-\gamma} \operatorname{div} \boldsymbol{\Psi} \\ & \quad + \sum_{|\gamma|=1,\, \gamma \leqslant \beta_1}^{|\beta_1|} C_\gamma \partial_x^\gamma \phi \partial_x^{\beta_1-\gamma} \operatorname{div} \boldsymbol{\Psi} + \partial_x^{\beta_1} [\phi \operatorname{div} \bar{\mathbf{u}} + \bar{\mathbf{u}} \nabla\phi + \boldsymbol{\Psi} \nabla\bar{n} + \boldsymbol{\Psi} \nabla\phi] \Bigg\} dx \\ & \geqslant \frac{d}{dt} \left(\int_{R^3} \frac{p'(\bar{n}+\phi)}{2(\bar{n}+\phi)^2} |\nabla\phi|^2 \, dx \right) - C(\delta+\varepsilon) \left\| (\nabla\boldsymbol{\Psi}, \nabla\phi, \Psi, \phi)(t) \right\|^2, \end{aligned} \tag{4.46}$$

where we have used the following estimates to complete (4.46):

$$\sum_{|\beta_1|=1} \int_{R^3} \frac{p'(\bar{n}+\phi)}{(\bar{n}+\phi)^2} \partial_x^{\beta_1} \phi \partial_x^{\beta_1} \phi_t \, dx \geqslant \frac{d}{dt} \left(\int_{R^3} \frac{p'(\bar{n}+\phi)}{2(\bar{n}+\phi)^2} |\nabla\phi|^2 \, dx \right) - C(\delta+\varepsilon) \left\| \nabla\phi(t) \right\|^2, \tag{4.47}$$

and

$$\sum_{|\beta_1|=1} \int_{R^3} \frac{p'(\bar{n}+\phi)}{(\bar{n}+\phi)^2} \partial_x^{\beta_1} \phi \sum_{|\gamma|=1,\, \gamma \leqslant \beta_1}^{|\beta_1|} C_\gamma \partial_x^\gamma \bar{n} \partial_x^{\beta_1-\gamma} \operatorname{div} \boldsymbol{\Psi} \, dx \geqslant -C(\delta+\varepsilon) \left\| (\nabla\phi, \nabla\boldsymbol{\Psi})(t) \right\|^2, \tag{4.48}$$

and

$$\sum_{|\beta_1|=1}\int_{R^3}\frac{p'(\bar n+\phi)}{(\bar n+\phi)^2}\partial_x^{\beta_1}\phi\sum_{|\gamma|=1,\,\gamma\leqslant\beta_1}^{|\beta_1|}C_\gamma\,\partial_x^{\gamma}\phi\partial_x^{\beta_1-\gamma}\operatorname{div}\boldsymbol{\Psi}\,dx\geqslant -C(\delta+\varepsilon)\big\|(\nabla\phi,\nabla\boldsymbol{\Psi})(t)\big\|^2, \tag{4.49}$$

and

$$\begin{aligned}&\sum_{|\beta_1|=1}\int_{R^3}\frac{p'(\bar n+\phi)}{(\bar n+\phi)^2}\partial_x^{\beta_1}\phi\partial_x^{\beta_1}(\phi\operatorname{div}\bar{\mathbf u}+\bar{\mathbf u}\nabla\phi+\boldsymbol{\Psi}\nabla\bar n+\boldsymbol{\Psi}\nabla\phi)\,dx\\&\quad\geqslant -C(\delta+\varepsilon)\big\|(\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi})(t)\big\|^2.\end{aligned} \tag{4.50}$$

Applying (4.46) to (4.44), we obtain

$$\begin{aligned}&\sum_{|\beta_1|=1}\int_{R^3}\partial_x^{\beta_1}\left(\frac{p'(\bar n+\phi)}{\bar n+\phi}\nabla\phi\right)\partial_x^{\beta_1}\boldsymbol{\Psi}\,dx\\&\quad\geqslant\frac{d}{dt}\left(\int_{R^3}\frac{p'(\bar n+\phi)}{2(\bar n+\phi)^2}|\nabla\phi|^2\,dx\right)-C(\delta+\varepsilon)\big\|(\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi})(t)\big\|^2.\end{aligned} \tag{4.51}$$

Notice that

$$\begin{aligned}-\sum_{|\beta_1|=1}\int_{R^3}\nabla\partial_x^{\beta_1}H\partial_x^{\beta_1}\boldsymbol{\Psi}\,dx&\leqslant\frac18\big\|\nabla\boldsymbol{\Psi}(t)\big\|^2+C\big\|\nabla^2H(t)\big\|^2\\&=\frac18\big\|\nabla\boldsymbol{\Psi}(t)\big\|^2+C\big\|\phi(t)\big\|^2\end{aligned} \tag{4.52}$$

and

$$-\sum_{|\beta_1|=1}\int_{R^3}\partial_x^{\beta_1}\mathbf{L}\partial_x^{\beta_1}\boldsymbol{\Psi}\,dx\leqslant C(\delta+\varepsilon)\big\|(\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi})(t)\big\|^2. \tag{4.53}$$

Substituting (4.51)–(4.53) into (4.43), and noticing the smallness of $\delta+\varepsilon$, we obtain

$$\frac{d}{dt}F_{22}(t)+\frac34\big\|\nabla\boldsymbol{\Psi}(t)\big\|^2\leqslant C\big\|\phi(t)\big\|^2+C(\delta+\varepsilon)\big\|(\nabla\phi,\boldsymbol{\Psi})(t)\big\|^2, \tag{4.54}$$

where

$$F_{22}(t)=\int_{R^3}\left(\frac{|\nabla\boldsymbol{\Psi}|^2}{2}+\frac{p'(\bar n+\phi)}{2(\bar n+\phi)^2}|\nabla\phi|^2\right)dx. \tag{4.55}$$

Then, from (4.41) and (4.54), we have established our second energy estimate as follows.

Lemma 4.4. *Let N_2 be a positive and suitably large number, and $F_2(t):=F_{21}(t)+N_2F_{22}(t)$. Then it holds*

$$\frac{d}{dt}F_2(t)+C_2\big\|(\nabla\phi,\nabla\boldsymbol{\Psi})(t)\big\|^2\leqslant C\big\|(\nabla H,\phi,\boldsymbol{\Psi})(t)\big\|^2 \tag{4.56}$$

and

$$C_{21}\big\|\big(\nabla^2 H,\nabla\phi,\nabla\boldsymbol{\Psi}\big)(t)\big\|^2-C(\delta+\varepsilon)\big\|(\phi,\boldsymbol{\Psi})(t)\big\|^2$$
$$\leqslant F_2(t)\leqslant C_{22}\big\|\big(\nabla^2 H,\nabla\phi,\nabla\boldsymbol{\Psi}\big)(t)\big\|^2+C(\delta+\varepsilon)\big\|(\phi,\boldsymbol{\Psi})(t)\big\|^2, \tag{4.57}$$

where C_2, C_{21} and C_{22} are some positive constants.

Next, we are going to establish the third energy estimate for the solution with the second derivatives.

Let β_2 be nonnegative multi-index, and $|\beta_2|=2$. Taking $\partial_x^{\beta_2}((\bar n+\phi)(3.6)_2)\cdot(-\partial_x^{\beta_2}\nabla H)$ and integrating the resultant equation, one obtains

$$\begin{aligned}
&-\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big((\bar n+\phi)\boldsymbol{\Psi}_t\big)\partial_x^{\beta_2}\nabla H\,dx-\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big((\bar n+\phi)\boldsymbol{\Psi}\big)\partial_x^{\beta_2}\nabla H\,dx\\
&\quad-\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big(p'(\bar n+\phi)\nabla\phi\big)\partial_x^{\beta_2}\nabla H\,dx\\
&\quad+\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big((\bar n+\phi)\nabla H\big)\partial_x^{\beta_2}\nabla H\,dx\\
&=\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big((\bar n+\phi)\mathbf{L}\big)\partial_x^{\beta_2}\nabla H\,dx.
\end{aligned}\tag{4.58}$$

Using the similar technique as before, we can estimate

$$\begin{aligned}
&-\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big((\bar n+\phi)\boldsymbol{\Psi}_t\big)\partial_x^{\beta_2}\nabla H\,dx\\
&\quad\geqslant\frac{d}{dt}\bigg(\int_{R^3}\nabla^3 H\nabla^3 H_t\,dx\bigg)-\big\|\nabla^3 H_t(t)\big\|^2\\
&\qquad-C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2,
\end{aligned}\tag{4.59}$$

and

$$\begin{aligned}
&-\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big((\bar n+\phi)\boldsymbol{\Psi}\big)\partial_x^{\beta_2}\nabla H\,dx\\
&\quad\geqslant\frac{d}{dt}\bigg(\int_{R^3}\frac{|\nabla^3 H|^2}{2}\,dx\bigg)-C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla\phi,\phi\big)(t)\big\|^2,
\end{aligned}\tag{4.60}$$

and

$$
\begin{aligned}
&-\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big(p'(\bar{n}+\phi)\nabla\phi\big)\partial_x^{\beta_2}\nabla H\,dx\\
&\geqslant -\sum_{|\beta_2|=2}\int_{R^3}p'(\bar{n}+\phi)\nabla\partial_x^{\beta_2}\Delta H\partial_x^{\beta_2}\nabla H\,dx-C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla\phi\big)(t)\big\|^2\\
&\geqslant 3C_0\big\|\nabla^4 H(t)\big\|^2-C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla\phi\big)(t)\big\|^2\\
&\geqslant 2C_0\big\|\nabla^2\phi(t)\big\|^2-C(\delta+\varepsilon)\big\|\nabla\phi(t)\big\|^2,
\end{aligned}\tag{4.61}
$$

and

$$
\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big((\bar{n}+\phi)\nabla H\big)\partial_x^{\beta_2}\nabla H\,dx\geqslant-\frac{C_0}{2}\big\|\nabla^2\phi(t)\big\|^2-C\big\|(\phi,\nabla H)(t)\big\|^2,\tag{4.62}
$$

and

$$
\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\big((\bar{n}+\phi)\mathbf{L}\big)\partial_x^{\beta_2}\nabla H\,dx\leqslant C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2,\tag{4.63}
$$

and

$$
\big\|\nabla^3 H_t(t)\big\|^2\leqslant C\big\|\nabla^2\boldsymbol{\Psi}(t)\big\|^2+C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2.\tag{4.64}
$$

Substituting (4.59)–(4.63) into (4.58), and utilizing (4.64), we obtain

$$
\frac{d}{dt}F_{31}(t)+C_0\big\|\nabla^2\phi(t)\big\|^2\leqslant C\big\|\nabla^2\boldsymbol{\Psi}(t)\big\|^2+C\big\|(\nabla\boldsymbol{\Psi},\nabla\phi,\nabla H,\boldsymbol{\Psi},\phi)(t)\big\|^2,\tag{4.65}
$$

where

$$
F_{31}(t)=\int_{R^3}\bigg[\nabla^3 H\nabla^3 H_t+\frac{|\nabla^3 H|^2}{2}\bigg]dx.\tag{4.66}
$$

Taking $\partial_x^{\beta_2}(3.6)_2\cdot\partial_x^{\beta_2}\boldsymbol{\Psi}$ and integrating the resultant equation, we have

$$
\begin{aligned}
&\frac{d}{dt}\bigg(\int_{R^3}\frac{|\nabla^2\boldsymbol{\Psi}|^2}{2}\,dx\bigg)+\big\|\nabla^2\boldsymbol{\Psi}(t)\big\|^2\\
&\quad+\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\bigg(\frac{p'(\bar{n}+\phi)}{\bar{n}+\phi}\nabla\phi\bigg)\partial_x^{\beta_2}\boldsymbol{\Psi}\,dx-\sum_{|\beta_2|=2}\int_{R^3}\nabla\partial_x^{\beta_2}H\partial_x^{\beta_2}\boldsymbol{\Psi}\,dx\\
&=-\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\mathbf{L}\partial_x^{\beta_2}\boldsymbol{\Psi}\,dx.
\end{aligned}\tag{4.67}
$$

Using similar technique as before, we can estimate

$$\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\left(\frac{p'(\bar{n}+\phi)}{\bar{n}+\phi}\nabla\phi\right)\partial_x^{\beta_2}\boldsymbol{\Psi}\,dx$$
$$\geqslant\frac{d}{dt}\left(\int_{R^3}\frac{p'(\bar{n}+\phi)}{2(\bar{n}+\phi)^2}\left|\nabla^2\phi\right|^2dx\right)-C(\delta+\varepsilon)\left\|\left(\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\right)(t)\right\|^2,\tag{4.68}$$

and

$$-\sum_{|\beta_2|=2}\int_{R^3}\nabla\partial_x^{\beta_2}H\partial_x^{\beta_2}\boldsymbol{\Psi}\,dx\leqslant\frac{1}{8}\left\|\nabla^2\Psi(t)\right\|^2+C\left\|\nabla^3H(t)\right\|^2$$
$$=\frac{1}{8}\left\|\nabla^2\boldsymbol{\Psi}(t)\right\|^2+C\left\|\nabla\phi(t)\right\|^2,\tag{4.69}$$

and

$$-\sum_{|\beta_2|=2}\int_{R^3}\partial_x^{\beta_2}\mathbf{L}\partial_x^{\beta_2}\boldsymbol{\Psi}\,dx\leqslant C(\delta+\varepsilon)\left\|\left(\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\right)(t)\right\|^2.\tag{4.70}$$

Substituting (4.51)–(4.53) into (4.43), and noticing the smallness of $\delta+\varepsilon$, we obtain

$$\frac{d}{dt}F_{32}(t)+\frac{3}{4}\left\|\nabla^2\boldsymbol{\Psi}(t)\right\|^2\leqslant C\left\|\nabla\phi(t)\right\|^2+C(\delta+\varepsilon)\left\|\left(\nabla^2\phi,\nabla\boldsymbol{\Psi},\boldsymbol{\Psi},\phi\right)(t)\right\|^2,\tag{4.71}$$

where

$$F_{32}(t)=\int_{R^3}\left(\frac{|\nabla^2\boldsymbol{\Psi}|^2}{2}+\frac{p'(\bar{n}+\phi)}{2(\bar{n}+\phi)^2}\left|\nabla^2\phi\right|^2\right)dx.\tag{4.72}$$

Then, from (4.71) and (4.65), we immediately obtain the third energy estimate for the solution as follows.

Lemma 4.5. *Let N_3 be a positive number which is suitably large, and $F_3(t):=F_{31}(t)+N_3F_{32}(t)$. Then it holds that*

$$\frac{d}{dt}F_3(t)+C_3\left\|\left(\nabla^2\phi,\nabla^2\boldsymbol{\Psi}\right)(t)\right\|^2\leqslant C\left\|(\nabla\phi,\nabla\boldsymbol{\Psi},\nabla H,\phi,\boldsymbol{\Psi})(t)\right\|^2\tag{4.73}$$

and

$$C_{31}\left\|\left(\nabla^3H,\nabla^2\phi,\nabla^2\boldsymbol{\Psi}\right)(t)\right\|^2-C(\delta+\varepsilon)\left\|(\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi})(t)\right\|^2$$
$$\leqslant F_3(t)\leqslant C_{32}\left\|\left(\nabla^3H,\nabla^2\phi,\nabla^2\boldsymbol{\Psi}\right)(t)\right\|^2+C(\delta+\varepsilon)\left\|(\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi})(t)\right\|^2,\tag{4.74}$$

where C_3, C_{31} and C_{32} are some positive constants.

Finally, we are going to establish the fourth energy estimate for the solution with the third derivatives.

Let β_3 be nonnegative multi-index, and $|\beta_3|=3$. Taking $\partial_x^{\beta_3}((\bar{n}+\phi)(3.6)_2)\cdot(-\partial_x^{\beta_3}\nabla H)$ and integrating the resultant equation, we have

$$
\begin{aligned}
&-\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big((\bar{n}+\phi)\boldsymbol{\Psi}_t\big)\partial_x^{\beta_3}\nabla H\,dx-\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big((\bar{n}+\phi)\boldsymbol{\Psi}\big)\partial_x^{\beta_3}\nabla H\,dx\\
&\quad-\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big(p'(\bar{n}+\phi)\nabla\phi\big)\partial_x^{\beta_3}\nabla H\,dx+\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big((\bar{n}+\phi)\nabla H\big)\partial_x^{\beta_3}\nabla H\,dx\\
&=\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big((\bar{n}+\phi)\mathbf{L}\big)\partial_x^{\beta_3}\nabla H\,dx.
\end{aligned}
\tag{4.75}
$$

Using similar technique as before, we can estimate

$$
\begin{aligned}
&-\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big((\bar{n}+\phi)\boldsymbol{\Psi}_t\big)\partial_x^{\beta_3}\nabla H\,dx\\
&\quad\geqslant\frac{d}{dt}\bigg(\int_{R^3}\nabla^4 H\nabla^4 H_t\,dx\bigg)-\big\|\nabla^4 H_t(t)\big\|^2\\
&\qquad-C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2,
\end{aligned}
\tag{4.76}
$$

and

$$
\begin{aligned}
&-\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big((\bar{n}+\phi)\boldsymbol{\Psi}\big)\partial_x^{\beta_3}\nabla H\,dx\\
&\quad\geqslant\frac{d}{dt}\bigg(\int_{R^3}\frac{|\nabla^4 H|^2}{2}\,dx\bigg)-C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^2\phi,\nabla\phi,\phi\big)(t)\big\|^2,
\end{aligned}
\tag{4.77}
$$

and

$$
\begin{aligned}
&\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big((\bar{n}+\phi)\nabla H\big)\partial_x^{\beta_3}\nabla H\,dx\\
&\quad\geqslant-\frac{C_0}{2}\big\|\nabla^3\phi(t)\big\|^2-C\big\|\big(\nabla^2\phi,\nabla\phi,\phi,\nabla H\big)(t)\big\|^2,
\end{aligned}
\tag{4.78}
$$

and

$$
\begin{aligned}
&\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big((\bar{n}+\phi)\mathbf{L}\big)\partial_x^{\beta_3}\nabla H\,dx\\
&\quad\leqslant C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2,
\end{aligned}
\tag{4.79}
$$

and

$$
\big\|\nabla^4 H_t(t)\big\|^2\leqslant C\big\|\nabla^3\boldsymbol{\Psi}(t)\big\|^2+C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2.
\tag{4.80}
$$

Here, the third term of the right-hand side of (4.75) can be estimated as follows

$$
\begin{aligned}
&-\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\big(p'(\bar n+\phi)\nabla\phi\big)\partial_x^{\beta_3}\nabla H\,dx\\
&\quad\geqslant-\sum_{|\beta_3|=3}\int_{R^3}p'(\bar n+\phi)\nabla\partial_x^{\beta_3}\Delta H\partial_x^{\beta_3}\nabla H\,dx-C\sum_{|\beta_3|=3}\int_{R^3}\big|\nabla(\bar n+\phi)\nabla^3\phi\partial_x^{\beta_3}\nabla H\big|\,dx\\
&\qquad-C\sum_{|\beta_3|=3}\int_{R^3}\big(\big|\nabla(\bar n+\phi)\big|^2+\big|\nabla^2(\bar n+\phi)\big|\big)\big|\nabla^2\phi\partial_x^{\beta_3}\nabla H\big|\,dx\\
&\qquad-C\sum_{|\beta_3|=3}\int_{R^3}\big(\big|\nabla(\bar n+\phi)\big|^3+\big|\nabla(\bar n+\phi)\nabla^2(\bar n+\phi)\big|+\big|\nabla^3(\bar n+\phi)\big|\big)\big|\nabla\phi\partial_x^{\beta_3}\nabla H\big|\,dx\\
&\quad\geqslant 3C_0\big\|\nabla^3\phi(t)\big\|^2-C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla\phi\big)(t)\big\|^2,
\end{aligned}
\tag{4.81}
$$

where we have used the Gagliardo–Nirenberg inequality to obtain the following estimate to complete (4.81):

$$
\begin{aligned}
\int_{R^3}\big|\nabla^2\phi\big|^2\big|\nabla^4H\big|\,dx&\leqslant\big\|\nabla^2\phi(t)\big\|_{L^4}^2\big\|\nabla^4H(t)\big\|\\
&\leqslant C\big\|\nabla^2\phi(t)\big\|^{\frac12}\big\|\nabla^3\phi(t)\big\|^{\frac32}\big\|\nabla^4H(t)\big\|\\
&\leqslant C\varepsilon\big\|\big(\nabla^3\phi,\nabla^2\phi\big)(t)\big\|^2.
\end{aligned}
\tag{4.82}
$$

Substituting (4.76)–(4.79) and (4.81) into (4.75), and utilizing (4.80), we obtain

$$
\frac{d}{dt}F_{41}(t)+C_0\big\|\nabla^3\phi(t)\big\|^2\leqslant C\big\|\nabla^3\boldsymbol{\Psi}(t)\big\|^2+C\big\|\big(\nabla^2\boldsymbol{\Psi},\nabla^2\phi,\nabla\boldsymbol{\Psi},\nabla\phi,\nabla H,\boldsymbol{\Psi},\phi\big)(t)\big\|^2,\tag{4.83}
$$

where

$$
F_{41}(t)=\int_{R^3}\bigg[\nabla^4H\nabla^4H_t+\frac{|\nabla^4H|^2}{2}\bigg]dx.\tag{4.84}
$$

Similarly, by taking $\partial_x^{\beta_3}(3.6)_2\cdot\partial_x^{\beta_3}\boldsymbol{\Psi}$ and integrating the resultant equation, we have

$$
\begin{aligned}
&\frac{d}{dt}\bigg(\int_{R^3}\frac{|\nabla^3\boldsymbol{\Psi}|^2}{2}\,dx\bigg)+\big\|\nabla^3\boldsymbol{\Psi}(t)\big\|^2\\
&\qquad+\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\bigg(\frac{p'(\bar n+\phi)}{\bar n+\phi}\nabla\phi\bigg)\partial_x^{\beta_3}\boldsymbol{\Psi}\,dx-\sum_{|\beta_3|=3}\int_{R^3}\nabla\partial_x^{\beta_3}H\partial_x^{\beta_3}\boldsymbol{\Psi}\,dx\\
&\quad=-\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\mathbf{L}\partial_x^{\beta_3}\boldsymbol{\Psi}\,dx.
\end{aligned}
\tag{4.85}
$$

On the other hand, the Cauchy inequality and the Gagliardo–Nirenberg inequality imply that

$$
\begin{aligned}
&\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\left(\frac{p'(\bar n+\phi)}{\bar n+\phi}\nabla\phi\right)\partial_x^{\beta_3}\boldsymbol{\Psi}\,dx\\
&\quad\geqslant-\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{\bar n+\phi}\partial_x^{\beta_3}\phi\partial_x^{\beta_3}\operatorname{div}\boldsymbol{\Psi}\,dx-C\int_{R^3}\left|\nabla(\bar n+\phi)\nabla^3\phi\nabla^3\boldsymbol{\Psi}\right|dx\\
&\qquad-C\int_{R^3}\left(\left|\nabla(\bar n+\phi)\right|^2+\left|\nabla^2(\bar n+\phi)\right|\right)\left|\nabla^2\phi\nabla^3\boldsymbol{\Psi}\right|dx\\
&\qquad-C\int_{R^3}\left(\left|\nabla(\bar n+\phi)\right|^3+\left|\nabla^2(\bar n+\phi)\nabla(\bar n+\phi)\right|+\left|\nabla^3(\bar n+\phi)\right|\right)\left|\nabla\phi\nabla^3\boldsymbol{\Psi}\right|dx\\
&\quad\geqslant-\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{\bar n+\phi}\partial_x^{\beta_3}\phi\partial_x^{\beta_3}\operatorname{div}\boldsymbol{\Psi}\,dx-C\int_{R^3}\left|\nabla^2\phi\right|^2\left|\nabla^3\boldsymbol{\Psi}\right|dx\\
&\qquad-C(\delta+\varepsilon)\left\|\left(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\right)(t)\right\|^2\\
&\quad\geqslant-\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{\bar n+\phi}\partial_x^{\beta_3}\phi\partial_x^{\beta_3}\operatorname{div}\boldsymbol{\Psi}\,dx\\
&\qquad-C(\delta+\varepsilon)\left\|\left(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\right)(t)\right\|^2.
\end{aligned}
\tag{4.86}
$$

Using (4.45) and the Gagliardo–Nirenberg inequality again, we can control the first term of the right-hand side of (4.86) as follows:

$$
\begin{aligned}
&-\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{\bar n+\phi}\partial_x^{\beta_3}\phi\partial_x^{\beta_3}\operatorname{div}\boldsymbol{\Psi}\,dx\\
&\quad=\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{(\bar n+\phi)^2}\partial_x^{\beta_3}\phi\Bigg\{\partial_x^{\beta_3}\phi_t+\sum_{|\gamma|=1,\,\gamma\leqslant\beta_3}^{|\beta_3|}C_\gamma\partial_x^{\gamma}\bar n\partial_x^{\beta_3-\gamma}\operatorname{div}\boldsymbol{\Psi}\\
&\qquad+\sum_{|\gamma|=1,\,\gamma\leqslant\beta_3}^{|\beta_3|}C_\gamma\partial_x^{\gamma}\phi\partial_x^{\beta_3-\gamma}\operatorname{div}\boldsymbol{\Psi}+\partial_x^{\beta_3}[\phi\operatorname{div}\bar u+\bar u\nabla\phi+\boldsymbol{\Psi}\nabla\bar n+\boldsymbol{\Psi}\nabla\phi]\Bigg\}dx\\
&\quad\geqslant\frac{d}{dt}\left(\int_{R^3}\frac{p'(\bar n+\phi)}{2(\bar n+\phi)^2}\left|\nabla^3\phi\right|^2dx\right)\\
&\qquad-C(\delta+\varepsilon)\left\|\left(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\right)(t)\right\|^2,
\end{aligned}
\tag{4.87}
$$

where we have used the following estimates to complete (4.87):

$$
\begin{aligned}
&\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{(\bar n+\phi)^2}\partial_x^{\beta_3}\phi\partial_x^{\beta_3}\phi_t\,dx\\
&\quad\geqslant\frac{d}{dt}\left(\int_{R^3}\frac{p'(\bar n+\phi)}{2(\bar n+\phi)^2}\left|\nabla^3\phi\right|^2dx\right)-C(\delta+\varepsilon)\left\|\nabla^3\phi(t)\right\|^2,
\end{aligned}
\tag{4.88}
$$

and

$$\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{(\bar n+\phi)^2}\partial_x^{\beta_3}\phi\cdot\sum_{|\gamma|=1,\,\gamma\leqslant\beta_3}^{|\beta_3|}C_\gamma\,\partial_x^\gamma\bar n\partial_x^{\beta_3-\gamma}\operatorname{div}\boldsymbol{\Psi}\,dx$$
$$\leqslant -C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\boldsymbol{\Psi},\nabla\boldsymbol{\Psi}\big)(t)\big\|^2, \tag{4.89}$$

and

$$\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{(\bar n+\phi)^2}\partial_x^{\beta_3}\phi\sum_{|\gamma|=1,\,\gamma\leqslant\beta_3}^{|\beta_3|}C_\gamma\,\partial_x^\gamma\phi\partial_x^{\beta_3-\gamma}\operatorname{div}\boldsymbol{\Psi}\,dx$$
$$\geqslant -C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi}\big)(t)\big\|^2-C\int_{R^3}\big|\nabla^3\phi\nabla^2\phi\nabla^2\boldsymbol{\Psi}\big|\,dx$$
$$\geqslant -C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi}\big)(t)\big\|^2-C\big\|\nabla^3\phi(t)\big\|\big\|\nabla^2\phi(t)\big\|_{L^4}\big\|\nabla^2\boldsymbol{\Psi}(t)\big\|_{L^4}$$
$$\geqslant -C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi}\big)(t)\big\|^2-C\big\|\nabla^3\phi(t)\big\|^{\frac{7}{4}}\big\|\nabla^2\phi(t)\big\|^{\frac{1}{4}}\big\|\nabla^3\boldsymbol{\Psi}(t)\big\|^{\frac{3}{4}}\big\|\nabla^2\boldsymbol{\Psi}(t)\big\|^{\frac{1}{4}}$$
$$\geqslant -C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi}\big)(t)\big\|^2, \tag{4.90}$$

and

$$\sum_{|\beta_3|=3}\int_{R^3}\frac{p'(\bar n+\phi)}{(\bar n+\phi)^2}\partial_x^{\beta_3}\phi\partial_x^{\beta_3}(\phi\operatorname{div}\bar{\mathbf u}+\bar{\mathbf u}\nabla\phi+\boldsymbol{\Psi}\nabla\bar n+\boldsymbol{\Psi}\nabla\phi)\,dx$$
$$\geqslant -C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2-C\int_{R^3}\big|\nabla^3\phi\nabla^2\phi\nabla^2\boldsymbol{\Psi}\big|\,dx$$
$$\geqslant -C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2. \tag{4.91}$$

Applying (4.87) into (4.86), we obtain

$$\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\left(\frac{p'(\bar n+\phi)}{\bar n+\phi}\nabla\phi\right)\partial_x^{\beta_3}\Psi\,dx$$
$$\geqslant \frac{d}{dt}\left(\int_{R^3}\frac{p'(\bar n+\phi)}{2(\bar n+\phi)^2}\big|\nabla^3\phi\big|^2\,dx\right)$$
$$-C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2. \tag{4.92}$$

Since the Cauchy inequality implies

$$-\sum_{|\beta_3|=3}\int_{R^3}\nabla\partial_x^{\beta_3}H\partial_x^{\beta_3}\boldsymbol{\Psi}\,dx\leqslant\frac{1}{8}\big\|\nabla^3\boldsymbol{\Psi}(t)\big\|^2+C\big\|\nabla^4H(t)\big\|^2$$
$$=\frac{1}{8}\big\|\nabla^3\boldsymbol{\Psi}(t)\big\|^2+C\big\|\nabla^2\phi(t)\big\|^2, \tag{4.93}$$

and the Cauchy inequality and the Gagliardo–Nirenberg inequality imply that

$$-\sum_{|\beta_3|=3}\int_{R^3}\partial_x^{\beta_3}\mathbf{L}\partial_x^{\beta_3}\boldsymbol{\Psi}\,dx\leqslant C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2, \tag{4.94}$$

by substituting (4.92)–(4.94) into (4.85), and noticing the smallness of $\delta+\varepsilon$, we obtain

$$\frac{d}{dt}F_{42}(t)+\frac{3}{4}\big\|\nabla^3\boldsymbol{\Psi}(t)\big\|^2\leqslant C\big\|\nabla^2\phi(t)\big\|^2+C(\delta+\varepsilon)\big\|\big(\nabla^3\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\boldsymbol{\Psi},\phi\big)(t)\big\|^2, \tag{4.95}$$

where

$$F_{42}(t)=\int_{R^3}\left(\frac{|\nabla^3\boldsymbol{\Psi}|^2}{2}+\frac{p'(\bar{n}+\phi)}{2(\bar{n}+\phi)^2}\big|\nabla^3\phi\big|^2\right)dx. \tag{4.96}$$

Thus, from (4.95) and (4.83), we have established the fourth energy estimate for the solution with the third derivatives as follows.

Lemma 4.6. *Let N_4 be a positive constant which is suitably large, and $F_4(t):=F_{41}(t)+N_4F_{42}(t)$. Then it holds*

$$\frac{d}{dt}F_4(t)+C_4\big\|\big(\nabla^3\phi,\nabla^3\boldsymbol{\Psi}\big)(t)\big\|^2\leqslant C\big\|\big(\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\nabla H,\phi,\boldsymbol{\Psi}\big)(t)\big\|^2 \tag{4.97}$$

and

$$\begin{aligned}&C_{41}\big\|\big(\nabla^4H,\nabla^3\phi,\nabla^3\boldsymbol{\Psi}\big)(t)\big\|^2-C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2\\&\quad\leqslant F_4(t)\leqslant C_{42}\big\|\big(\nabla^3H,\nabla^3\phi,\nabla^3\boldsymbol{\Psi}\big)(t)\big\|^2\\&\qquad+C(\delta+\varepsilon)\big\|\big(\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla\phi,\nabla\boldsymbol{\Psi},\phi,\boldsymbol{\Psi}\big)(t)\big\|^2,\end{aligned} \tag{4.98}$$

where C_4, C_{41} and C_{42} are some positive constants.

Proof of Proposition 4.2. Let λ_1, λ_2 and λ_3 be suitably large numbers, and applying Lemmas 4.3–4.6, we then obtain

$$\begin{aligned}&\frac{d}{dt}\big(\lambda_3\big[\lambda_2\big(\lambda_1F_1(t)+F_2(t)\big)+F_3(t)\big]+F_4(t)\big)\\&\quad+C_5\big\|\big(\nabla^4H,\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^3H,\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla^2H,\nabla\phi,\nabla\boldsymbol{\Psi},\nabla H,\phi,\boldsymbol{\Psi}\big)(t)\big\|^2\leqslant 0\end{aligned} \tag{4.99}$$

and

$$\begin{aligned}&C_{51}\big\|\big(\nabla^4H,\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^3H,\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla^2H,\nabla\phi,\nabla\boldsymbol{\Psi},\nabla H,\phi,\boldsymbol{\Psi}\big)(t)\big\|^2\\&\quad\leqslant\lambda_3\big[\lambda_2\big(\lambda_1F_1(t)+F_2(t)\big)+F_3(t)\big]+F_4(t)\\&\quad\leqslant C_{52}\big\|\big(\nabla^4H,\nabla^3\phi,\nabla^3\boldsymbol{\Psi},\nabla^3H,\nabla^2\phi,\nabla^2\boldsymbol{\Psi},\nabla^2H,\nabla\phi,\nabla\boldsymbol{\Psi},\nabla H,\phi,\boldsymbol{\Psi}\big)(t)\big\|^2\end{aligned} \tag{4.100}$$

provided with $\delta+\varepsilon\ll 1$, where C_5, C_{51} and C_{52} are some positive constants.

Thus, applying the Gronwall inequality to (4.99), we obtain

$$\left\|(\nabla H, \phi, \boldsymbol{\Psi})(t)\right\|_{H^3}^2 \leqslant C\left\|(\nabla H_0, \phi_0, \boldsymbol{\Psi}_0)\right\|_{H^3}^2 e^{-\nu t} \tag{4.101}$$

for some positive constant $\nu > 0$, provided with $\delta + \varepsilon \ll 1$. The proof is complete. □

Acknowledgments

The research of FMH was supported in part by NSFC Grant No. 10825102 for distinguished youth scholar, NSFC–NSAF Grant No. 10676037 and 973 project of China under Grant No. 2006CB805902, the research of MM was supported in part by Natural Sciences and Engineering Research Council of Canada under the NSERC grant RGPIN 354724-08, the research of HMY was supported in part by NSFC Grant No. 10901095 and the Promotive research fund for excellent young and middle-aged scientists of Shandong Province Grant No. BS2010SF025.

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