SIAM J. MATH. ANAL.
Vol. 44, No. 1, pp. 538–540

ERRATUM: GLOBAL STABILITY OF MONOSTABLE TRAVELING WAVES FOR NONLOCAL TIME-DELAYED REACTION-DIFFUSION EQUATIONS*

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Abstract. This short note is to fix a gap in the proof of Lemma 3.8 in our paper [M. Mei, C. Ou, and X.-Q. Zhao, *SIAM J. Math. Anal.*, 42 (2010), pp. 2762–2790].

Key words. nonlocal reaction-diffusion equations, time delays, traveling waves, global stability

AMS subject classifications. 35K57, 34K20, 92D25

DOI. 10.1137/110850633

In our recent paper [2], in order to get the algebraic stability for critical traveling wavefronts, one of the key steps is to establish the following decay estimate (see [2, Lemma 3.8]):

$$\|\bar{v}(t)\|_{L^\infty_{w_1}(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}}, \tag{0.1}$$

where $w_1(\xi) = e^{-\lambda_*(\xi - x_0)}$ is the weight function for the critical wave case with $c = c_*$, and $\bar{v}(t,\xi)$ is the solution of

$$\begin{cases} \dfrac{\partial \bar{v}}{\partial t} + c_* \dfrac{\partial \bar{v}}{\partial \xi} - D\dfrac{\partial^2 \bar{v}}{\partial \xi^2} + d'(0)\bar{v} - \varepsilon b'(0)\displaystyle\int_{\mathbb{R}} f_\alpha(y)\bar{v}(t-\tau, \xi - y - c_*\tau)dy = 0, \\ \bar{v}(s,\xi) = \bar{v}_0(s,\xi), \quad s \in [-\tau, 0]. \end{cases} \tag{0.2}$$

This was proved with the aid of [2, Lemma 3.7]

$$\|\hat{v}(t)\|_{L^\infty(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}} e^{k_2 t}, \tag{0.3}$$

where $\hat{v}(t,\xi) := e^{k_2 t} w_1(\xi)\bar{v}(t,\xi)$ satisfies (see (3.47)–(3.48) in [2])

$$\begin{cases} \dfrac{\partial \hat{v}}{\partial t} + k_1 \dfrac{\partial \hat{v}}{\partial \xi} - D\dfrac{\partial^2 \hat{v}}{\partial \xi^2} = \varepsilon b'(0) e^{k_2 \tau}\displaystyle\int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_*(y + c_*\tau)}\hat{v}(t-\tau, \xi - y - c_*\tau)dy, \\ \hat{v}(s,\xi) = e^{k_2 s} w_1(\xi)\bar{v}_0(s,\xi) := \hat{v}_0(s,\xi), \quad s \in [-r, 0]. \end{cases} \tag{0.4}$$

Note that $k_1 := c_* - 2D\lambda_*$ and $k_2 := c_*\lambda_* - D\lambda_*^2 + d'(0) > 0$. However, the proof of [2, Lemma 3.7] is incorrect. Indeed, we converted the standing equation (0.4) into an integral form with the regular Green function (the heat kernel without time-delay) $G(t, \xi - \zeta) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(\xi - \zeta + k_1 t)^2}{4Dt}}$, then used the iteration procedure to derive

*Received by the editors October 7, 2011; accepted for publication November 30, 2011; published electronically February 28, 2012.
http://www.siam.org/journals/sima/44-1/85063.html

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the algebraic convergence rate in the case of the critical wave: $C^k(1+t)^{-1/2}$ at the kth iteration for $t\in[(k-1)\tau,k\tau]$. Thus, the constant coefficient C^k is increasing and unbounded as $k\to\infty$. In order to fix such a gap, we derive an equivalent integral equation with the time-delayed Green function and then obtain the decay rates of solutions without using iterations. Inspired by the work of [3], below we provide a new proof for [2, Lemma 3.8].

Proof of Lemma 3.8. Here we need to assume that the initial perturbation around the wavefront $\phi(x+ct)$ satisfies

$$u_0(s,x)-\phi(x+cs)\in C^1([-\tau,0];L^1_w(\mathbb{R})\cap H^1(\mathbb{R})),$$

where $w(x)$ is the weight function given by (2.2) in [2].

Let $\tilde{v}(t,\xi):=w_1(\xi)\bar{v}(t,\xi)$. It then follows from (0.2) that

$$\begin{cases}\dfrac{\partial\tilde{v}}{\partial t}+k_1\dfrac{\partial\tilde{v}}{\partial\xi}-D\dfrac{\partial^2\tilde{v}}{\partial\xi^2}+k_2\tilde{v}=\varepsilon b'(0)\displaystyle\int_{\mathbb{R}}f_\alpha(y)e^{-\lambda_*(y+c_*\tau)}\tilde{v}(t-\tau,\xi-y-c_*\tau)dy,\\ \tilde{v}(s,\xi)=w_1(\xi)\bar{v}_0(s,\xi):=\tilde{v}_0(s,\xi),\quad s\in[-\tau,0].\end{cases}\tag{0.5}$$

Taking the Fourier transform to (0.5), we have

$$\frac{d\check{v}}{dt}+A(\eta)\check{v}=B(\eta)\check{v}(t-\tau,\eta)\quad\text{and}\quad\check{v}(s,\eta)=\check{v}_0(s,\eta),\ s\in[-\tau,0],\tag{0.6}$$

where $\check{v}(t,\eta)=\mathcal{F}[\tilde{v}]$ is the Fourier transform of $\tilde{v}$, and

$$A(\eta)=D\eta^2+k_2+\mathrm{i}k_1\eta,\quad B(\eta)=\varepsilon b'(0)\int_{\mathbb{R}}f_\alpha(y)e^{-\lambda_*(y+c_*\tau)}e^{-\mathrm{i}(y+c_*\tau)\eta}dy.$$

It is easy to see that [1, Theorem 1] is still valid provided that the initial function $\varphi(t)$ is continuous on $[-\tau,0]$, and $\varphi'(t)$ is continuous for all $t\in[-\tau,0]$ except for finite many points where both left and right limits of $\varphi'(t)$ exist. Accordingly, we solve the above time-delayed equation (0.6) as

$$\begin{aligned}\check{v}(t,\eta)&=e^{-A(\eta)(t+\tau)}e_\tau^{\mathcal{B}(\eta)t}\check{v}_0(-\tau,\eta)\\&\quad+\int_{-\tau}^0e^{-A(\eta)(t-s)}e_\tau^{\mathcal{B}(\eta)(t-\tau-s)}\left[\frac{d}{ds}\check{v}_0(s,\eta)+A(\eta)\check{v}_0(s,\eta)\right]ds,\end{aligned}\tag{0.7}$$

where

$$\mathcal{B}(\eta):=B(\eta)e^{A(\eta)\tau},\tag{0.8}$$

and $e_\tau^{\mathcal{B}(\eta)t}$ is the delayed exponential function defined by

$$e_\tau^{\mathcal{B}(\eta)t}=\begin{cases}0, & -\infty<t<-\tau,\\ 1, & -\tau\le t<0,\\ 1+\frac{\mathcal{B}(\eta)t}{1!}, & 0\le t<\tau,\\ 1+\frac{\mathcal{B}(\eta)t}{1!}+\frac{\mathcal{B}(\eta)^2(t-\tau)^2}{2!}, & \tau\le t<2\tau,\\ \vdots & \vdots\\ 1+\frac{\mathcal{B}(\eta)t}{1!}+\frac{\mathcal{B}(\eta)^2(t-\tau)^2}{2!}+\cdots+\frac{\mathcal{B}(\eta)^m[t-(m-1)\tau]^m}{m!}, & (m-1)\tau\le t<m\tau,\\ \vdots & \vdots\end{cases}$$

Taking the inverse Fourier transform to (0.7), we then get

$$\begin{aligned}\tilde{v}(t,\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}x\cdot\eta} e^{-A(\eta)(t+\tau)} e_{\tau}^{\mathcal{B}(\eta)t} \check{v}_0(-\tau,\eta)d\eta \\ &\quad + \int_{-\tau}^{0} \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}x\cdot\eta} e^{-A(\eta)(t-s)} e_{\tau}^{\mathcal{B}(\eta)(t-\tau-s)} \\ &\quad \times \left[\frac{d}{ds}\check{v}_0(s,\eta) + A(\eta)\check{v}_0(s,\eta)\right] d\eta ds. \end{aligned} \tag{0.9}$$

By [3, Theorem 2.3], as applied to (0.9), it follows that

$$\|\tilde{v}(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-\frac{1}{2}} e^{-\varepsilon_1(c_1-c_3)t}, \tag{0.10}$$

where $0 < \varepsilon_1 < 1$ is a specified constant, c_1 and c_3 are positive constants given by

$$c_1 := k_2 = c_*\lambda_* - D\lambda_*^2 + d'(0) > 0, \tag{0.11}$$

and

$$\begin{aligned} c_3 :&= \varepsilon b'(0)\int_{\mathbb{R}} f_\alpha(y) e^{-\lambda_*(y_1+c_*\tau)}dy \\ &= \varepsilon b'(0)\int_{\mathbb{R}} f_{\alpha 1}(y_1) e^{-\lambda_*(y_1+c_*\tau)}dy_1 \\ &= \varepsilon b'(0) e^{\alpha\lambda_*^2 - \lambda_* c_*\tau} > 0. \end{aligned} \tag{0.12}$$

In the case where $c = c_*$, we see from [2, Lemma 2.1] that

$$\varepsilon b'(0) e^{\alpha\lambda_*^2 - \lambda_* c_*\tau} = c_*\lambda_* - D\lambda_*^2 + d'(0),$$

and hence $c_1 = c_3$. It then follows from (0.10) that

$$\|\tilde{v}(t)\|_{L^\infty(\mathbb{R})} \le Ct^{-\frac{1}{2}},$$

which is equivalent to

$$\|\bar{v}(t)\|_{L^\infty_{w_1}(\mathbb{R})} \le Ct^{-\frac{1}{2}}.$$

This completes the proof of Lemma 3.8. ∎

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