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Stability of non-monotone critical traveling waves for reaction–diffusion equations with time-delay

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Abstract

This paper is concerned with the stability of critical traveling waves for a kind of non-monotone time-delayed reaction–diffusion equations including Nicholson's blowflies equation which models the population dynamics of a single species with maturation delay. Such delayed reaction–diffusion equations possess monotone or oscillatory traveling waves. The latter occurs when the birth rate function is non-monotone and the time-delay is big. It has been shown that such traveling waves $\phi(x+ct)$ exist for all $c \geq c_*$ and are exponentially stable for all wave speed $c > c_*$ [13], where c_* is called the critical wave speed. In this paper, we prove that the critical traveling waves $\phi(x+c_*t)$ (monotone or oscillatory) are also time-asymptotically stable, when the initial perturbations are small in a certain weighted Sobolev norm. The adopted method is the technical weighted-energy method with some new flavors to handle the critical oscillatory waves. Finally, numerical simulations for various cases are carried out to support our theoretical results.

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1. Introduction

Model equations. We are interested in the stability of oscillatory traveling waves for the following time-delayed reaction–diffusion equation

$$\frac{\partial v(t,x)}{\partial t} - D\frac{\partial^2 v(t,x)}{\partial x^2} + dv(t,x) = b(v(t-r,x)), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{1.1}$$

The traveling wave is of the form $\phi(x+ct)$ and the initial data

$$v(s,x) = v_0(s,x), \quad s \in [-r,0], \; x \in \mathbb{R} \tag{1.2}$$

considered is a perturbation of ϕ. This equation models the population dynamics of a single species with nonzero maturation delay, the time required for a newborn to become matured [8,9,16,17,20,28]. Here, $v(t,x)$ represents the mature population at time t and location x, $D > 0$ is the spatial diffusion rate of the mature species, $d > 0$ is the death rate, and $r > 0$ is the maturation delay. The function $b : [0,\infty) \to (0,\infty)$ is called the birth rate function, which is assumed to satisfy the following hypothesis:

(H_1) There are only two equilibria, say $v_\pm$, for the homogeneous part of (1.1). That is, $b(v_\pm) - dv_\pm = 0$. We may take $v_- = 0$ and thus $b(0) = 0$. We further assume that v_- is unstable and v_+ is stable for the homogeneous part of (1.1). That is, $d - b'(0) < 0$ and $d - b'(v_+) > 0$.
(H_2) The uni-modality condition: there is a $v_* \in (0, v_+)$ such that $b(\cdot)$ is increasing on $[0, v_*]$ and decreasing on $[v_*, +\infty)$. In particular, $b'(0) > 0$ and $b'(v_+) < 0$.
(H_3) $b \in C^2[0,\infty)$ and $|b'(v)| \le b'(0)$ for $v \in [0,\infty)$.

Hypothesis (H_1) means that (1.1) is a mono-stable system. A typical example is the classic Fisher–KPP equation

$$v_t - v_{xx} = v(1-v).$$

Hypothesis (H_2) implies that $b(v)$ is not monotone for $v \in [0, v_+]$. As we shall see later this leads to some oscillations for traveling waves when the time-delay r is big.

The hypotheses (H_1)–(H_3) are summarized from the following concrete models.

- *Nicholson's blowflies model* [8,9,16,17,20,28]: The so-called Nicholson's birth rate function (also called the Ricker's type function) is

$$b(v) = pve^{-av}, \quad a > 0, \; p > 0, \tag{1.3}$$

where $p > 0$ is the maximal egg daily production rate per blowfly. Since $b(v)$ reaches its maximum at $v = 1/a$, the biological meaning of $\frac{1}{a} > 0$ is the new-born population at which the population reaches its maximal growth rate. The corresponding constant equilibria are

$$v_- = 0 \text{ and } v_+ = \frac{1}{a} \ln \frac{p}{d}.$$

When $p/d > e$, the birth rate function $b(v)$ is unimodal on $v \in [0, v_+]$, and reaches its unique global maximum at $v_* := \frac{1}{a} \in (0, v_+)$. Furthermore, it can be verified that $|b'(v)| \le b'(0)$ for $v \in [0, \infty)$.

- *Mackey–Glass model* [7,12,16,17]: The birth rate function is the Beverton–Holt function

$$b(v) = \frac{pv}{1 + av^q}, \quad a > 0, \quad p > 0, \quad q > 1, \tag{1.4}$$

where

$$v_- = 0 \text{ and } v_+ = \left(\frac{p-d}{da}\right)^{\frac{1}{q}}.$$

When $\frac{p}{d} > \frac{q}{q-1}$, $b(v)$ is unimodal for $v \in [0, v_+]$, and reaches its unique global maximum at $v_* := [a(q-1)]^{-1/q} \in (0, v_+)$. It can also be verified that $|b'(v)| \le b'(0)$ for $v \in [0, \infty)$.

Traveling waves. A traveling wave for (1.1) is a special solution to (1.1) of the form $\phi(x + ct) \ge 0$ with $\phi(\pm\infty) = v_\pm$. We say such traveling wave connecting v_- to v_+. By plugging it into (1.1), we get that

$$\begin{cases} c\phi'(\xi) - D\phi''(\xi) + d\phi(\xi) = b(\phi(\xi - cr)), \\ \phi(\pm\infty) = v_\pm, \end{cases} \tag{1.5}$$

where $\xi = x + ct$, $' = \frac{d}{d\xi}$, and c is the wave speed. The traveling wave can be viewed as a heteroclinic orbit connecting two equilibria $v_\pm$ from $\xi = -\infty$ to $\xi = \infty$. It can be monotone or oscillatory. The existence and uniqueness of the monotone/oscillatory traveling waves of (1.1) have been studied extensively [1,3–7,14,29,32,33], see the references therein. We briefly describe the results we need below.

(i) Behavior of $\phi(\xi)$ for $\xi \sim -\infty$. Since $\phi(\xi) \to v_-$ as $\xi \to -\infty$, we expect that $\phi(\xi)$ is close to a function $v(\xi)$ which satisfies the linearized equation of (1.5) around v_- for $\xi \sim -\infty$:

$$cv'(\xi) - Dv''(\xi) + dv(\xi) = b'(0)v(\xi - cr), \quad v(-\infty) = 0. \tag{1.6}$$

By plugging $v(\xi) = e^{\lambda\xi}$ into (1.6), we get the following characteristic equation for $\lambda > 0$:

$$c\lambda - D\lambda^2 + d = b'(0)e^{-\lambda cr}. \tag{1.7}$$

Denote

$$F_c(\lambda) := c\lambda - D\lambda^2 + d, \quad G_c(\lambda) := b'(0)e^{-\lambda cr}.$$

As shown in [16] that, for each $r \ge 0$, there exists a unique $c_* = c_*(r) > 0$ at which the two graphs of F_c and G_c are tangent at λ_*. This means that (c_*, λ_*) are determined by

$$F_{c_*}(\lambda_*) = G_{c_*}(\lambda_*), \quad F'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*),$$

namely,

$$c_*\lambda_* - D\lambda_*^2 + d = b'(0)e^{-\lambda_* c_* r} \text{ and } c_* - 2D\lambda_* = -c_* r b'(0)e^{-\lambda_* c_* r}. \tag{1.8}$$

We have

- for $c > c_*$, the characteristic equation (1.7) has two distinct solutions $0 < \lambda_1 < \lambda_2$;
- for $c = c_*$, (1.7) has multiple root $\lambda_1 = \lambda_2 = \lambda_*$;
- for $c < c_*$, (1.7) has no positive root.

When $c < c_*$, there will be no traveling wave for Eq. (1.5). For it would satisfy the linearized equation for $\xi \sim -\infty$, and would have the form $e^{\lambda\xi}$ for $\xi \sim -\infty$, but no such λ can exist. When $c \geq c_* > 0$, on the other hand, the traveling wave $\phi(x + ct)$, if exists, should satisfy

$$\begin{cases} \phi(\xi) = O(1)e^{\lambda_1 \xi} \to 0 & \text{as } \xi \to -\infty, \quad \text{for } c > c_*, \\ \phi(\xi) = O(1)|\xi|e^{\lambda_* \xi} \to 0 & \text{as } \xi \to -\infty, \quad \text{for } c = c_*. \end{cases} \tag{1.9}$$

(ii) Behavior of $\phi(\xi)$ for $\xi \sim \infty$. The asymptotic behavior of the traveling wave ϕ at $\xi = \infty$ is solely determined by the linearized ODE around v_+:

$$c(\phi - v_+)' - D(\phi - v_+)'' + d(\phi - v_+) = b'(v_+)(\phi(\xi - cr) - v_+).$$

Let $v_+ - \phi(\xi) = e^{-\lambda_+ \xi}$ as $\xi \to +\infty$, we get

$$-c\lambda_+ - \lambda_+^2 + d = b'(v_+)e^{\lambda_+ cr},$$

which uniquely solves for $\lambda_+ = \lambda_+(c) > 0$ and $\lambda_+^* = \lambda_+^*(c_*) > 0$. Thus, the asymptotic behavior of the traveling wave as $\xi \to \infty$ is

$$|v_+ - \phi(\xi)| = \begin{cases} O(1)e^{-\lambda_+ \xi}, & \text{for } c > c_*, \\ O(1)e^{-\lambda_+^* \xi}, & \text{for } c = c_*, \end{cases} \quad \text{as } \xi \to \infty.$$

Now we are going to see the possible oscillations for some traveling waves around v_+ when $\xi \gg 1$. Let $x = y/\varepsilon$. Then (1.1) is reduced to

$$v_t(t, y) - \varepsilon^2 D v_{yy}(t, y) + dv(t, y) = b(v(t - r, y)).$$

Let $\eta = t + \frac{y}{c}$ and $v(\eta) = \phi(c\eta) = \phi(c(t + y/c)) = \phi(ct + y)$ be the traveling wave, then $v(\eta)$ satisfies

$$v'(\eta) - \varepsilon^2 D c^{-2} v''(\eta) + dv(\eta) = b(v(\eta - r)). \tag{1.10}$$

For fixed t, when $\xi = x + ct \gg 1$, it is equivalent to $x \gg 1$. Then, for fixed y, $x = \frac{y}{\varepsilon} \gg 1$ implies $\varepsilon \ll 1$. Hence, to discuss the behavior of the traveling waves $\phi(\xi)$ as $\xi \to \infty$ is equivalent to look for the asymptotic behavior of the solution $v(\eta)$ for Eq. (1.10) as $\varepsilon \to 0$. This leads an approximation to (1.10) by

$$v'(\eta) + dv(\eta) = b(v(\eta - r)). \tag{1.11}$$

As shown in [5,6,33], when $b'(v_+) < 0$ and

$$|b'(v_+)| r e^{dr+1} > 1, \tag{1.12}$$

some traveling waves $v(c^{-1}x + t) = \phi(x + ct)$ may slowly oscillate around v_+ as $r \gg 1$. The condition (1.12) for traveling waves with oscillations is equivalent to $r > \underline{r} > 0$, where, $\underline{r}$, given by

$$|b'(v_+)| \underline{r} e^{d\underline{r}+1} = 1, \tag{1.13}$$

is the critical point for the solution to the delayed ODE

$$v'(t) + dv(t) = b'(v_+)v(t - r) \tag{1.14}$$

to possibly occur oscillations [30,33]. That is, for $b'(v_+) < 0$, the solution of the delayed ODE (1.14) is monotone for $0 < r < \underline{r}$ and it may be oscillatory for $r > \underline{r}$ (cf. [30]). However, if $d < |b'(v_+)|$, there exists a Hopf-bifurcation point to (1.14) [2,3]

$$\bar{r} := \frac{\pi - \arctan(\sqrt{|b'(v_+)|^2 - d^2}/d)}{\sqrt{|b'(v_+)|^2 - d^2}}, \tag{1.15}$$

where $\bar{r} > \underline{r}$. When $r \geq \bar{r}$, the solution $v(\eta)$ will not converge to v_+ as $\eta \to \infty$. In this case, the traveling waves do not exist.

(iii) Existence, uniqueness, monotone/oscillatory of the traveling waves.

- When $d \geq |b'(v_+)|$, the traveling wave $\phi(x + ct)$ exists uniquely (up to a shift) for every $c \geq c_* = c_*(r)$, where the time-delay r is allowed to be any number in $[0, \infty)$. If $0 \leq r < \underline{r}$, then these traveling waves are monotone [7]; while, if $r \geq \underline{r}$, then the traveling waves are still monotone for $(c, r) \in [c_*, c^*] \times [\underline{r}, r_0]$, where $c_* = c_*(r)$ is the minimum wave speed as mentioned before, $c^* = c^*(r)$ is given by the characteristic equation for (1.5) around v_+, and r_0 $(> \underline{r})$ is the unique intersection point of two curves $c_*(r)$ and $c^*(r)$; and the traveling waves are oscillating around v_+ for $(c, r) \notin [c_*, c^*] \times [\underline{r}, r_0]$, namely, either $c > c^*$ or $r > r_0$ (cf. [7,13]).
- When $d < |b'(v_+)|$, on the other hand, the traveling wave $\phi(x + ct)$ with $c \geq c_*$ can exist only when $r < \bar{r}$, and no traveling wave can exist for $r \geq \bar{r}$. In the case of $r < \bar{r}$, the waves are monotone for $0 < r < \underline{r}$ and oscillating for $r \in (\underline{r}, \bar{r})$ (cf. [7,13]).

For details, we refer to [3,5,7,14,32,33], see also the summary in [13].

Goal of this paper. The main goal of this paper is to prove the stability of the monotone/oscillatory critical traveling waves $\phi(x + c_*t)$ to the Cauchy problem (1.1) and (1.2). In [13], all non-critical traveling waves $\phi(x + ct)$ with the wave speed $c > c_*$, monotone or oscillatory, are proven to be time-exponentially stable, by the technical weighted-energy method. But the stability of the critical oscillatory wavefronts (traveling waves) $\phi(x + c_*t)$ still remains open. This problem is important because the spreading speeds c of the traveling waves in the biological applications usually are the minimum speed (i.e. the critical speed) [27,36]. This problem is also challenging because the analytical approaches for stability of critical wavefronts by now are very limited, only case by case studies [18,21]. Furthermore, when the critical traveling waves are oscillatory, their stability analysis is even more difficult. In this paper, with some new observations described in the next section, and using the technical weighted-energy method with some new ingredients, fortunately, we can prove the stability of oscillatory critical traveling waves.

Outline of the paper. The paper is organized as follows. Section 2 contains the main theorems of global existence, uniqueness, uniform boundedness and asymptotic stability. We will work on the perturbed equation. The proof of main theorems will be carried out in Sections 3–5, respectively. In Section 3, we shall prove the global existence and uniqueness of the solution for the perturbed equation, where the initial perturbation can be allowed to be arbitrarily large. In Section 4, when the initial perturbation is suitably small, the solution of the perturbed equation can be proved to be uniformly bounded by the anti-weighted energy method. Based on the uniform boundedness, we shall further prove the asymptotic stability in Section 5. In Section 6, we shall make a remark that our stability theorem for the critical traveling waves is not just valid for the uni-modality birth rate function, it is also valid for multi-modality birth rate functions. Finally, in Section 7, we shall carry out four numerical simulations for Nicholson's blowflies model. The solutions are simulated to behave time-asymptotically stable monotone/non-monotone critical traveling waves in different cases, and the traveling speeds of the solutions $v(t,x)$ are also tested to be close to the corresponding critical wave speeds c_*. These numerical experiments further support our theoretical stability results.

2. Main theorems

The perturbed equation. In order to prove the stability of critical traveling waves, let us reformulate the working equations (1.1) and (1.2) to a perturbed equation around the critical wave.

Let $\phi(x+c_*t)=\phi(\xi)$, $\xi=x+c_*t$, be a given critical traveling wave, and define

$$u(t,\xi):=v(t,x)-\phi(x+c_*t), \qquad u_0(s,\xi):=v_0(s,x)-\phi(x+c_*s).$$

Then, from (1.1)–(1.5), $u(t,\xi)$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t}+c_*\dfrac{\partial u}{\partial \xi}-D\dfrac{\partial^2 u}{\partial \xi^2}+du=P(u(t-r,\xi-c_*r)), & (t,\xi)\in\mathbb{R}_+\times\mathbb{R}, \\ u(s,\xi)=u_0(s,\xi), \quad s\in[-r,0],\ \xi\in\mathbb{R}, \end{cases} \tag{2.1}$$

where

$$P(u):=b(\phi+u)-b(\phi) \tag{2.2}$$

with $u=u(t-r,\xi-c_*r)$ and $\phi=\phi(\xi-c_*r)$. Furthermore, let us linearize the delay term $P(u(t-r,\xi-c_*r))$, we equivalently have

$$\begin{cases} \dfrac{\partial u}{\partial t} + c_* \dfrac{\partial u}{\partial \xi} - D \dfrac{\partial^2 u}{\partial \xi^2} + du - b'(\phi(\xi - c_* r))u(t - r, \xi - c_* r) \\ \qquad\qquad = Q(u(t - r, \xi - c_* r)), \quad (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(s, \xi) = u_0(s, \xi), \quad s \in [-r, 0], \ \xi \in \mathbb{R}, \end{cases} \tag{2.3}$$

where

$$Q(u) := b(\phi + u) - b(\phi) - b'(\phi)u, \tag{2.4}$$

with $\phi = \phi(\xi - c_* r)$ and $u = u(t - r, \xi - c_* r)$, and satisfies, by Taylor's formula,

$$|Q(u)| = O(1)|u|^2. \tag{2.5}$$

Notations. Throughout this paper, we denote a generic constant by $C > 0$, and specific positive constants by $C_i > 0$ ($i = 0, 1, 2, \cdots$). Let $L^2(\mathbb{R})$ denote the space of the square integrable functions, $H^k(\mathbb{R})$ the Sobolev space, $C(\mathbb{R})$ the space of bounded continuous functions equipped with the sup norm. Let $T > 0$ and $\mathcal{B}$ be a Banach space. We denote by $C([0, T]; \mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0, T]$ and $L^2([0, T]; \mathcal{B})$ is the space of the $\mathcal{B}$-valued L^2-functions on $[0, T]$.

Associated with the eigenvalue λ_* (defined in (1.8)), we define a weight function

$$w(\xi) := e^{-2\lambda_* \xi}, \quad \xi \in (-\infty, \infty). \tag{2.6}$$

Notice that $\lim_{\xi \to -\infty} w(\xi) = +\infty$ and $\lim_{\xi \to +\infty} w(\xi) = 0$, because $\lambda_* > 0$. To handle delay equation with delay r, we define $\mathcal{C}_{unif}[-r, T]$, for $0 < T \le \infty$, by

$$\begin{aligned} \mathcal{C}_{unif}[-r, T] := \{ & v(t, x) \in C([-r, T] \times \mathbb{R}) \text{ such that} \\ & \lim_{x \to +\infty} v(t, x) \text{ exists uniformly in } t \in [-r, T], \text{ and} \\ & \lim_{x \to +\infty} v_x(t, x) = \lim_{x \to +\infty} v_{xx}(t, x) = 0 \\ & \text{uniformly with respect to } t \in [-r, T] \}. \end{aligned} \tag{2.7}$$

We also denote

$$\begin{aligned} X_0(-r, 0) = \{ & u | u \in C([-r, 0]; C(\mathbb{R})) \cap \mathcal{C}_{unif}[-r, 0], \\ & \sqrt{w} u \in C([-r, 0]; H^1(\mathbb{R})), \text{ and} \\ & (\sqrt{w} u) \in L^2([-r, 0]; H^2(\mathbb{R})) \}, \end{aligned} \tag{2.8}$$

with

$$M_0^2 := \sup_{t \in [-r, 0]} \left(\|u(t)\|^2_{C(\mathbb{R})} + \|(\sqrt{w} u)(t)\|^2_{H^1(\mathbb{R})} \right) + \int_{-r}^{0} \|(\sqrt{w} u)(s)\|^2_{H^2(\mathbb{R})} ds, \tag{2.9}$$

and

$$
\begin{aligned}
X_{loc}(0,\infty) = \{u | u \in\ & C_{loc}([0,\infty); C(\mathbb{R})) \cap \mathcal{C}_{unif}[0,\infty), \\
& \sqrt{w}u \in C_{loc}([0,\infty); H^1(\mathbb{R})), \text{ and} \\
& \sqrt{w}u \in L^2_{loc}([0,\infty); H^2(\mathbb{R}))\},
\end{aligned} \tag{2.10}
$$

where $L^2_{loc}([0,\infty); H^2(\mathbb{R}))$ is the space whose H^2-valued functions are locally L^2-integrable in $[0,\infty)$, namely, for any $0 < T < \infty$, it holds

$$
\int_0^T \|u(t)\|^2_{H^1} dt \le C_T < \infty.
$$

The local continuous spaces $C_{loc}([0,\infty); C(\mathbb{R}))$ and $C_{loc}([0,\infty); H^1(\mathbb{R}))$ are similarly defined. We further define

$$
\begin{aligned}
X(0,\infty) = \{u | u \in\ & C([0,\infty); C(\mathbb{R})) \cap \mathcal{C}_{unif}[0,\infty), \\
& \sqrt{w}u \in C([0,\infty); H^1(\mathbb{R})), \\
& \sqrt{\phi w}u \in L^2([0,\infty); L^2(\mathbb{R})), \text{ and} \\
& \partial_\xi(\sqrt{w}u) \in L^2([0,\infty); H^1(\mathbb{R}))\},
\end{aligned} \tag{2.11}
$$

with

$$
\begin{aligned}
M^2_\infty := \sup_{t\in(0,\infty)} & \left(\|u(t)\|^2_{C(\mathbb{R})} + \|(\sqrt{w}u)(t)\|^2_{H^1(\mathbb{R})} \right) \\
& + \int_0^\infty \|\sqrt{\phi w}u(s)\|^2_{L^2(\mathbb{R})} ds + \int_0^\infty \|\partial_\xi(\sqrt{w}u)(s)\|^2_{H^1(\mathbb{R})} ds.
\end{aligned} \tag{2.12}
$$

Main results. Now we state the global existence, uniqueness, uniform boundedness and stability for the solution to Eq. (1.1) with a general non-monotone birth rate $b(v)$ as the following three theorems, respectively.

Theorem 2.1 *(Global existence and uniqueness). Assume that* (H_1)–(H_3) *hold. Let* $b'(v_+)$ *and* r *satisfy, either* $d \ge |b'(v_+)|$ *with arbitrarily given* $r > 0$*, or* $d < |b'(v_+)|$ *with* $0 < r < \bar{r}$*, where* $\bar{r}$ *is defined in (1.15). Let* $\phi(x + c_* t) = \phi(\xi)$ *be any given critical traveling wave, and the initial perturbation* $u_0(s,\xi) := v_0(s,\xi) - \phi(\xi) \in X_0(-r,0)$ *be arbitrary, then the solution* $u(t,\xi)$ *of the perturbed equation (2.3) globally and uniquely exists in* $X_{loc}(0,\infty)$.

Theorem 2.2 *(Uniform boundedness). Under the conditions of Theorem 2.1, if the initial perturbation* $u_0 \in X_0(-r,0)$ *is small enough, namely, there exists a constant* $\delta_0 > 0$ *such that* $M_0 \le \delta_0$*, then the solution* $u(t,\xi)$ *of the perturbed equation (2.3) satisfies* $u \in X(0,\infty)$*, and* $u(t,\xi)$ *is uniformly bounded in* $X(0,\infty)$*:*

$$
M^2_\infty \le C M^2_0. \tag{2.13}
$$

Theorem 2.3 *(Stability). Under the conditions in Theorem 2.2, then it holds that*

$$\lim_{t\to\infty}\sup_{\xi\in\mathbb{R}}|u(t,\xi)|=0. \tag{2.14}$$

From Theorems 2.1–2.3, we immediately obtain the stability of the monotone/non-monotone critical traveling waves for Nicholson's blowflies equation and Mackey–Glass equation, respectively.

Corollary 2.1 *(Stability for Nicholson's blowflies equation). Let $b(v)=pve^{-av}$ for $p>0$ and $a>0$. For any given critical traveling wave $\phi(x+c_*t)=\phi(\xi)$, there exists a constant $\delta_0>0$ such that if the initial perturbation $u_0(s,\xi):=v_0(s,\xi)-\phi(\xi)\in X_0(-r,0)$ and satisfies $M_0\le\delta_0$, then:*

1. *when $e<\frac{p}{d}\le e^2$ (equivalently to $d\ge|b'(v_+)|$), for any time-delay $r>0$, then, the solution $u(t,\xi)=v-\phi\in X(0,\infty)$ satisfies (2.14);*
2. *when $\frac{p}{d}>e^2$ (equivalently to $d<|b'(v_+)|$) but with a small time-delay $0<r<\bar{r}$, where*

$$\bar{r}:=\frac{\pi-\arctan\sqrt{\ln\frac{p}{d}(\ln\frac{p}{d}-2)}}{d\sqrt{\ln\frac{p}{d}(\ln\frac{p}{d}-2)}}, \tag{2.15}$$

then, the solution $u(t,\xi)=v-\phi\in X(0,\infty)$ satisfies (2.14).

Corollary 2.2 *(Stability for Mackey–Glass equation). Let $b(v)=\frac{pv}{1+av^q}$ for $p>0$, $q>1$ and $a>0$. For any given critical traveling wave $\phi(x+c_*t)=\phi(\xi)$, there exists a constant $\delta_0>0$ such that if the initial perturbation $u_0(s,\xi):=v_0(s,\xi)-\phi(\xi)\in X_0(-r,0)$ and satisfies $M_0\le\delta_0$, then:*

1. *when $\frac{q}{q-1}<\frac{p}{d}\le\frac{q}{q-2}$ (equivalently to $d\ge|b'(v_+)|$), for any time-delay $r>0$, then, then, the solution $v(t,x)\in X(0,\infty)$ satisfies (2.14);*
2. *when $\frac{p}{d}>\frac{q}{q-2}$ (equivalently to $d<|b'(v_+)|$) but with a small time-delay $0<r<\bar{r}$, where*

$$\bar{r}:=\frac{\pi-\arctan\left(d^{-1}\sqrt{[(q-1)\frac{p}{d}-q]^2-d^2}\right)}{\sqrt{[(q-1)\frac{p}{d}-q]^2-d^2}}, \tag{2.16}$$

then, the solution $v(t,x)\in X(0,\infty)$ satisfies (2.14).

Key observations. The key observations which differ from the proof of the stability of non-critical traveling waves are illustrated briefly below.

Observation 1. The first crucial step for the stability proof in [13] is to get an energy estimate for the perturbed equation in a weighted $L^2_w(\mathbb{R})$-space (see Lemma 3.1 and Lemma 3.2 for the details in [13]):

$$\int_{\mathbb{R}} w(\xi)|u(t,\xi)|^2 d\xi + \int_0^t \int_{\mathbb{R}} \mathcal{A}_{\eta,w}(\xi) w(\xi)|u(s,\xi)|^2 d\xi ds$$
$$\le C \int_{\mathbb{R}} w(\xi)|u_0(0,\xi)|^2 d\xi + O(1) \int_0^t \int_{\mathbb{R}} w(\xi)|u(s,\xi)||u(s-r,\xi-c_*r)|^2 d\xi ds.$$

The function $\mathcal{A}_{\eta,w}(\xi)$ is estimated in (3.11) of [13] as

$$\mathcal{A}_{\eta,w}(\xi) \ge C_1 > 0 \quad \text{with } c > c_*$$

for some positive constant C_1. It is this estimate that allows us to control the nonlinear term on the left-hand-side of the above inequality when the initial perturbation is small enough, and makes us to derive the exponential decay for the perturbed solution. However, when $c = c_*$, we can only have

$$\mathcal{A}_{\eta,w}(\xi) \ge C_1 = 0 \quad \text{for } c = c_*,$$

which seems not enough to control the nonlinear term, and the stability for the critical oscillating waves was open in [13]. But, after a deep observation, here we realize that, when $c = c_*$,

$$\mathcal{A}_{\eta,w}(\xi) \approx O(1)\phi(\xi) = O(1)|\xi|e^{-\lambda_*|\xi|} \to 0 \quad \text{as } \xi \to -\infty.$$

Now we rewrite the nonlinear term by

$$\int_0^t \int_{\mathbb{R}} w(\xi)|u(s,\xi)||u(s-r,\xi-c_*r)|^2 d\xi ds$$
$$= \int_0^t \int_{\mathbb{R}} \phi(\xi) w(\xi) \frac{|u(s,\xi)|}{\phi(\xi)} |u(s-r,\xi-c_*r)|^2 d\xi ds,$$

and recognize that $\frac{|u(s,\xi)|}{\phi(\xi)} = O(1)e^{-\lambda_*|\xi|}$ as $\xi \to -\infty$. Thus, we may control the nonlinear term by the positive term $\int_0^t \int_{\mathbb{R}} \mathcal{A}_{\eta,w}(\xi) w(\xi)|u(s,\xi)|^2 d\xi ds$ when the initial perturbation is small. Such an observation seems to open a door for the proof of the stability of these oscillatory critical traveling waves.

Observation 2. The second crucial step is to get a bound for $\int_0^t \|\partial_\xi u(s)\|^2_{H^1_w} ds$, which, together with the bound of $\|u(t,\cdot)\|_{H^1_w}$, leads to a desired decay of $|u(t,\cdot)|$ as $t \to \infty$. Notice that, when we apply the standard weighted-energy approach like that in [13], we have to fully use the positive term $w(\xi)|u_\xi|^2$ to control the bad term $w(\xi)|uu_\xi|$ by Cauchy–Schwarz inequality, which causes us impossibly to get the desired estimate of $\int_0^t \|\partial_\xi u(s)\|^2_{L^2_w} ds$. So, we need to look for other strategies. Inspired by the classic result [22] for Fisher–KPP equation with compactly supported initial data, and by [15] for p-system of hyperbolic conservation laws, as well as by our recent study [11] for the nonlocal equation (the integro-differential equation), we find that, the so-called anti-weighted-energy method (a special transform to the equation) works out perfectly, and we can get the desired bound for $\int_0^t \|u_\xi(s)\|^2_{H^1_w} ds$ in the first crucial energy estimates. Hence, we are able to get the stability for the critical oscillatory traveling waves.

Observation 3. In the previous studies [13,15–20], the proof approach is the weighted energy method together with the continuity extension method, based on the local existence and the *a priori* estimates. Notice that, for each time-interval $[0, T]$, the uniform convergence $\lim\limits_{\xi\to\infty} u(t, \xi) = 0$ for all $t \in [0, T]$ implies the boundedness $|u(t, \xi)| < \varepsilon$ for $\xi \geq x_0(\varepsilon, T) \gg 1$. Since $x_0(\varepsilon, T)$ can vary in T, this causes us difficultly to get the uniform boundedness $|u(t, \xi)| \leq \varepsilon$ for all $t \in [0, \infty)$ when $\xi \gg 1$. To avoid such a trouble, here we adopt a different approach to get the global existence and the asymptotic stability. That is, we first prove the global existence and uniqueness for $u \in X_{loc}(0, \infty)$ with any initial perturbation $u_0 \in X_0(-r, 0)$; secondly we establish the uniform boundedness of solution (2.13) for all $t \in [0, \infty)$ when the initial perturbation is small; finally we derive the stability (2.14) with the small initial perturbation. In fact, we observe that, the perturbed equation (2.1) is a linear equation for $t \in [0, r]$, because the delay term $P(u(t - r, \xi - c_*r)) = P(u_0(t - r, \xi - c_*r))$ due to $t - r \in [-r, 0]$. Thus, we can easily get the existence and uniqueness of the solution $u \in X_{loc}(0, r)$. Similarly, we may have the existence of the solution $u \in X_{loc}(r, 2r)$, and step by step to get the global existence $u \in X_{loc}(0, \infty)$ including $u \in C_{unif}(0, \infty)$. With this global existence $u \in X_{loc}(0, \infty)$, by the anti-weighted energy method, we can further establish the uniform boundedness of the solution $M_\infty \leq CM_0$ for $u(t, \xi) \in X(0, \infty)$ with $t \in [0, \infty)$ as well as the convergence (2.14).

Remarks. 1. In above main theorems, for the global existence of the solution, the initial data $u_0 \in X_0(-r, 0)$ can be arbitrarily large. However, to have the uniform boundedness (2.13) and the asymptotic stability (2.14), the initial data u_0 must be small enough.

2. In Theorem 2.3, the critical traveling waves $\phi(x + c_*t)$, no matter they are monotone or oscillatory, are proven to be time-asymptotically stable when the initial perturbations are small enough. However, the expecting optimal convergence rate $O(t^{-\frac{1}{2}})$ is unable to get at this moment, due to some technical restrictions. To obtain such convergence rate result, to our best knowledge, the usual approaches are: either the monotone technology plus the optimal decay estimates on the corresponding linearized equations [16,18], or the Fourier's transform plus energy estimates [10,21], or the approximate Green function method [23,35], or the multiplier method [24,31]. Unfortunately, we may not be able to adopt the monotone method, because our equation is lack of monotonicity and the traveling waves may be oscillatory; nor the Fourier's transform method, because the correspondingly linearized equation is with variable coefficients that depend on the wavefronts $\phi(x + c_*t)$, rather than some constants, which makes us impossible to carry out Fourier transform; nor the approximate Green function method, because it is really complicated and difficult to construct a suitable approximate-Green-function due to the effect of the time-delay; nor the multiplier method because of the bad effect by the time-delay. So, the optimal convergent rate result to critical oscillatory traveling waves $\phi(x + c_*t)$ remains open, which will be our future target.

3. Global existence and uniqueness

In this section, we are going to prove Theorem 2.1, namely, the global existence and uniqueness of the solution for the Cauchy problem (2.1).

When $t \in [0, r]$, Eq. (2.1) is linear, because $t - r \in [-r, 0]$ such that $P(u(t - r, \xi - c_* r)) = P(u_0(t - r, \xi - c_* r))$. Thus, the solution of (2.1) can be explicitly and uniquely solved by, for $t \in [0, r]$,

$$u(t,\xi) = e^{-dt} \int_{\mathbb{R}} G(\eta, t) u_0(0, \xi - \eta) d\eta + \int_0^t e^{-d(t-s)} \int_{\mathbb{R}} G(\eta, t-s) P(u_0(s - r, \xi - \eta - c_* r)) d\eta ds, \tag{3.1}$$

where $G(\eta, t)$ is the heat kernel

$$G(\eta, t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(\eta + c_* t)^2}{4Dt}}.$$

When $u_0 \in X_0(-r, 0)$, we are going to prove $u \in X_{loc}(0, r)$.

Multiplying (2.1) by $w(\xi)u$, and using Cauchy–Schwarz inequality

$$|D w_\xi u u_\xi| \le D w u_\xi^2 + \frac{D}{4} (\frac{w_\xi}{w})^2 w u^2,$$

and integrating it with respect to ξ over $\mathbb{R}$, we then have

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{w} u(t)\|_{L^2}^2 + m_1 \|\sqrt{w} u(t)\|_{L^2}^2 \le \int_{\mathbb{R}} w(\xi) u(t, \xi) P(u_0(t - r, \xi - c_* r)) d\xi, \tag{3.2}$$

where

$$m_1 := c_* \lambda_* - D\lambda_*^2 + d = b'(0) e^{-c_* \lambda_* r} > 0.$$

Again, by using Cauchy–Schwarz inequality, the right-hand-side of (3.2) can be estimated by

$$\begin{aligned} &\int_{\mathbb{R}} w(\xi) u(t, \xi) P(u_0(t - r, \xi - c_* r)) d\xi \\ &\le C \int_{\mathbb{R}} w(\xi) |u(t, \xi)| |u_0(t - r, \xi - c_* r)| d\xi \\ &\le \varepsilon \|\sqrt{w} u(t)\|_{L^2}^2 + \frac{C}{4\varepsilon} \|\sqrt{w} u_0(t - r)\|_{L^2}^2 \end{aligned}$$

for some small constant $\varepsilon > 0$. Substituting this into (3.2), we have

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{w} u(t)\|_{L^2}^2 + (m_1 - \varepsilon) \|\sqrt{w} u(t)\|_{L^2}^2 \le C \|\sqrt{w} u_0(t - r)\|_{L^2}^2. \tag{3.3}$$

Integrating (3.3) over $[0, t]$ for $t \in [0, r]$, and taking $\varepsilon < m_1$, we get

$$
\begin{aligned}
&\|\sqrt{w}u(t)\|_{L^2}^2 + \int_0^t \|\sqrt{w}u(s)\|_{L^2}^2 ds \\
&\le C\|\sqrt{w}u_0(0)\|_{L^2}^2 + C\int_0^t \|\sqrt{w}u_0(s-r)\|_{L^2}^2 ds \\
&= C\|\sqrt{w}u_0(0)\|_{L^2}^2 + C\int_{-r}^{t-r} \|\sqrt{w}u_0(s)\|_{L^2}^2 ds \\
&\le C\|\sqrt{w}u_0(0)\|_{L^2}^2 + C\int_{-r}^{0} \|\sqrt{w}u_0(s)\|_{L^2}^2 ds \\
&< \infty, \quad \text{for } t \in [0, r].
\end{aligned} \tag{3.4}
$$

On the other hand, we multiply (2.1) by $w(\xi)u$, and integrate it with respect to ξ over $\mathbb{R}$, but use Cauchy–Schwarz inequality in a different form

$$
|Dw_\xi u u_\xi| \le \frac{D}{2} w u_\xi^2 + \frac{D}{2}(\frac{w_\xi}{w})^2 w u^2,
$$

we then get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{w}u(t)\|_{L^2}^2 + \frac{D}{2}\|\sqrt{w}u_\xi(t)\|_{L^2}^2 \\
&\quad \le m_2\|\sqrt{w}u(t)\|_{L^2}^2 + \int_{\mathbb{R}} w(\xi)u(t,\xi)P(u_0(t-r,\xi-c_*r))d\xi,
\end{aligned} \tag{3.5}
$$

where $m_2 := |2D\lambda_*^2 - c_*\lambda_* - d|$. Integrating (3.5) over $[0, t]$, and using Cauchy–Schwarz inequality to the nonlinear term, and the bounded estimate for $\int_0^t \|\sqrt{w}u(s)\|_{L^2}^2 ds$ in (3.4), we get the estimate for $\int_0^t \|\sqrt{w}u_\xi(s)\|_{L^2}^2 ds$:

$$
\begin{aligned}
&\|\sqrt{w}u(t)\|_{L^2}^2 + \int_0^t \|\sqrt{w}u_\xi(s)\|_{L^2}^2 ds \\
&\le C\|\sqrt{w}u_0(0)\|_{L^2}^2 + C\int_{-r}^{0} \|\sqrt{w}u_0(s)\|_{L^2}^2 ds \\
&< \infty, \quad \text{for } t \in [0, r].
\end{aligned} \tag{3.6}
$$

Similarly, differentiating (2.1) with respect to ξ and multiplying it by $w(\xi)u_\xi(t,\xi)$, and integrating the resultant equation over $[0, t] \times \mathbb{R}$ for $t \in [0, r]$, we can prove

$$\begin{aligned}
&\|\sqrt{w}u_\xi(t)\|_{L^2}^2+\int_0^t \|\sqrt{w}u_{\xi\xi}(s)\|_{L^2}^2 ds\\
&\quad\le C\|\sqrt{w}u_{0,\xi}(0)\|_{L^2}^2+C\int_{-r}^0 \|\sqrt{w}u_0(s)\|_{H^1}^2 ds\\
&\quad<\infty,\quad \text{for } t\in[0,r].
\end{aligned}\tag{3.7}$$

From (3.1) and the property of heat kernel: $\int_{\mathbb{R}} G(\xi,t)d\xi=1$, we have

$$\begin{aligned}
\|u(t)\|_C&\le e^{-dt}\|u_0(0)\|_C+C\sup_{s-r\in[-r,0]}\|u_0(s-r)\|_C\int_0^t e^{-d(t-s)}ds\\
&\le e^{-dt}\|u_0(0)\|_C+C\sup_{t-r\in[-r,0]}\|u_0(t-r)\|_C\\
&<\infty,\quad \text{for } t\in[0,r].
\end{aligned}\tag{3.8}$$

On the other hand, since $u_0\in C_{unif}(-r,0)$, namely, $\lim_{\xi\to\infty}u_0(t,\xi)=:u_{0,\infty}(t)\in C[-r,0]$ and $\lim_{\xi\to\infty}\partial_\xi^k u_0(t,\xi)=0$ all exist uniformly in t for $k=1,2$, we can prove $u\in\mathcal{C}_{unif}[0,r]$. In fact,

$$\begin{aligned}
\lim_{\xi\to\infty}u(t,\xi)&=e^{-dt}\int_{\mathbb{R}}G(\eta,t)\lim_{\xi\to\infty}u_0(0,\xi-\eta)d\eta\\
&\quad+\int_0^t e^{-d(t-s)}\int_{\mathbb{R}}G(\eta,t-s)\lim_{\xi\to\infty}P(u_0(s-r,\xi-\eta-c_*r))d\eta ds\\
&=u_{0,\infty}(0)e^{-dt}\int_{\mathbb{R}}G(\eta,t)d\eta\\
&\quad+\int_0^t e^{-d(t-s)}P(u_{0,\infty}(s-r))\int_{\mathbb{R}}G(\eta,t-s)d\eta ds\\
&=u_{0,\infty}(0)e^{-dt}+\int_0^t e^{-d(t-s)}P(u_{0,\infty}(s-r))ds\\
&=:g_1(t),\quad \text{uniformly with respect to } t\in[0,r].
\end{aligned}\tag{3.9}$$

Similarly, noting the facts

$$G(\eta,t)|_{\eta=\pm\infty}=0 \ \text{ and } \ (\partial_\eta G(\eta,t))|_{\eta=\pm\infty}=0,$$

we can prove that, for $k=1,2$,

$$
\begin{aligned}
\lim_{\xi\to\infty} \partial_\xi^k u(t,\xi) &= e^{-dt} \int_{\mathbb{R}} \partial_\eta^k G(\eta,t) \lim_{\xi\to\infty} u_0(0,\xi-\eta)d\eta \\
&\quad + \int_0^t e^{-d(t-s)} \int_{\mathbb{R}} \partial_\eta^k G(\eta,t-s) \lim_{\xi\to\infty} P(u_0(s-r,\xi-\eta-c_*r))d\eta ds \\
&= u_{0,\infty}(0)e^{-dt} \int_{\mathbb{R}} \partial_\eta^k G(\eta,t)d\eta \\
&\quad + \int_0^t e^{-d(t-s)} P(u_{0,\infty}(s-r)) \int_{\mathbb{R}} \partial_\eta^k G(\eta,t-s)d\eta ds \\
&= 0, \quad \text{uniformly with respect to } t\in[0,r]. \qquad (3.10)
\end{aligned}
$$

Thus, (3.4)–(3.10) imply $u \in X_{loc}(0,r)$ and

$$
\begin{aligned}
&\|u(t)\|_C^2 + \|\sqrt{w}u(t)\|_{H^1}^2 + \int_0^t \|\sqrt{w}u(s)\|_{H^2}^2 ds \\
&\le C\Big(\|u_0(0)\|_C^2 + \|\sqrt{w}u_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}u_0(s)\|_{H^2}^2 ds\Big), \quad t\in[0,r], \qquad (3.11)
\end{aligned}
$$

for some $C > 1$.

When $t \in [r, 2r]$, Eq. (2.1) with the initial data $u(s,\xi)$ for $s \in [0,r]$ is still linear because the source term $P(u(t-r,\xi-c_*r))$ is known due to $t-r \in [0,r]$ and $u(t-r,\xi-c_*r)$ is solved in (3.1). So the solution $u(t,\xi)$ for $t \in [r,2r]$ is uniquely and explicitly given by,

$$
\begin{aligned}
u(t,\xi) &= e^{-dt} \int_{\mathbb{R}} G(\eta,t)u(r,\xi-\eta)d\eta \\
&\quad + \int_r^t e^{-d(t-s)} \int_{\mathbb{R}} G(\eta,t-s)P(u(s-r,\xi-\eta-c_*r))d\eta ds. \qquad (3.12)
\end{aligned}
$$

By taking the same estimates as in (3.4)–(3.10), we can prove $u \in X_{loc}(r,2r)$, that is

$$
\begin{aligned}
&\|u(t)\|_C^2 + \|\sqrt{w}u(t)\|_{H^1}^2 + \int_r^t \|\sqrt{w}u(s)\|_{H^2}^2 ds \\
&\le C\Big(\|u(r)\|_C^2 + \|\sqrt{w}u(r)\|_{H^1}^2 + \int_0^r \|\sqrt{w}u(s)\|_{H^2}^2 ds\Big) \\
&\le C^2\Big(\|u_0(0)\|_C^2 + \|\sqrt{w}u_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}u_0(s)\|_{H^2}^2 ds\Big), \quad t\in[r,2r], \qquad (3.13)
\end{aligned}
$$

where, in the last step we used (3.11); and as similarly shown in (3.9) and (3.10) that

$$\begin{aligned}
\lim_{\xi\to\infty} u(t,\xi) &= e^{-dt}\int_{\mathbb{R}} G(\eta,t)\lim_{\xi\to\infty} u(r,\xi-\eta)d\eta \\
&\quad + \int_r^t e^{-d(t-s)}\int_{\mathbb{R}} G(\eta,t-s)\lim_{\xi\to\infty} P(u(s-r,\xi-\eta-c_*r))d\eta ds \\
&= g_1(r)e^{-dt}\int_{\mathbb{R}} G(\eta,t)d\eta \\
&\quad + \int_r^t e^{-d(t-s)}P(g_1(s-r))\int_{\mathbb{R}} G(\eta,t-s)d\eta ds \\
&= g_1(r)e^{-dt} + \int_r^t e^{-d(t-s)}P(g_1(s-r))ds \\
&=: g_2(t), \quad \text{uniformly with respect to } t\in[r,2r],
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\xi\to\infty} \partial_\xi^k u(t,\xi) &= e^{-dt}\int_{\mathbb{R}} \partial_\eta^k G(\eta,t)\lim_{\xi\to\infty} u(r,\xi-\eta)d\eta \\
&\quad + \int_r^t e^{-d(t-s)}\int_{\mathbb{R}} \partial_\eta^k G(\eta,t-s)\lim_{\xi\to\infty} P(u(s-r,\xi-\eta-c_*r))d\eta ds \\
&= g_1(r)e^{-dt}\int_{\mathbb{R}} \partial_\eta^k G(\eta,t)d\eta \\
&\quad + \int_r^t e^{-d(t-s)}P(g_1(s-r))\int_{\mathbb{R}} \partial_\eta^k G(\eta,t-s)d\eta ds \\
&= 0, \quad \text{uniformly with respect to } t\in[r,2r].
\end{aligned}$$

Repeating the above procedure, step by step, we can prove that $u\in X_{loc}((n-1)r,nr)$ uniquely exists, and satisfies

$$\begin{aligned}
&\|u(t)\|_C^2 + \|\sqrt{w}u(t)\|_{H^1}^2 + \int_{(n-1)r}^t \|\sqrt{w}u(s)\|_{H^2}^2 ds \\
&\le C^n\Big(\|u_0(0)\|_C^2 + \|\sqrt{w}u_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}u_0(s)\|_{H^2}^2 ds\Big)
\end{aligned} \tag{3.14}$$

for $t \in [(n-1)r, nr]$, and finally we prove that u is unique, and $u \in X_{loc}(0,\infty)$ with, for any $T > 0$, that

$$\|u(t)\|_C^2 + \|\sqrt{w}u(t)\|_{H^1}^2 + \int_0^t \|\sqrt{w}u(s)\|_{H^2}^2 ds$$

$$\le C_T\Big(\|u_0(0)\|_C^2 + \|\sqrt{w}u_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\sqrt{w}u_0(s)\|_{H^2}^2 ds\Big), \quad t \in [0,T]. \tag{3.15}$$

The proof is complete. □

4. Uniform boundedness

This section is devoted to the proof of Theorem 2.2. For the global solution of (2.3), $u \in X_{loc}(0,\infty)$, when the initial perturbation $u_0 \in X_0(-r,0)$ is small enough, we are going to prove $u \in X(0,\infty)$ by deriving the uniform boundedness (2.13).

As we have stated before, since the weighted-energy estimates in [13,16] to (2.3) cannot yield the boundedness of $\int_0^t \|u_\xi(s)\|_{L_w^2}^2 ds$, due to a full use of the positive term u_ξ^2 is needed to control the term uu_ξ by applying Cauchy–Schwarz inequality, we have to look for a different approach. Here we adopt the so-called anti-weighted method [22,15,11]. That is, we take the following transformation (or say, anti-weight)

$$u(t,\xi) = [w(\xi)]^{-\frac{1}{2}}\tilde{u}(t,\xi), \quad \text{i.e.,} \quad \tilde{u}(t,\xi) = \sqrt{w(\xi)}u(t,\xi) = e^{-\lambda_*\xi}u(t,\xi), \tag{4.1}$$

we get the following equations for the new unknown $\tilde{u}(t,\xi)$

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} - D\dfrac{\partial^2 \tilde{u}}{\partial \xi^2} + k_1\dfrac{\partial \tilde{u}}{\partial \xi} + k_2\tilde{u} - b'(\phi(\xi - c_*r))e^{-\lambda_* c_* r}\tilde{u}(t-r,\xi - c_*r) \\ \qquad\qquad = \tilde{Q}(\tilde{u}(t-r,\xi - c_*r)), \quad (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}, \\ \tilde{u}(s,\xi) = \sqrt{w(\xi)}u(s,\xi) =: \tilde{u}_0(s,\xi), \quad s \in [-r,0], \ \xi \in \mathbb{R}, \end{cases} \tag{4.2}$$

where

$$k_1 := c_* - 2D\lambda_*, \quad k_2 := c_*\lambda_* + d - D\lambda_*^2 \tag{4.3}$$

satisfying (by (1.8))

$$k_2 = c_*\lambda_* + d - D\lambda_*^2 = b'(0)e^{-\lambda_* c_* r},$$

and

$$\tilde{Q}(\tilde{u}) = e^{-\lambda_*\xi}Q(u) \tag{4.4}$$

satisfying (by Taylor's expansion formula)

$$|\tilde{Q}(\tilde{u})| \le Ce^{-\lambda_* \xi}|u|^2 = \frac{C}{\sqrt{w(\xi)}}|\tilde{u}|^2. \tag{4.5}$$

Now we are going to establish the uniform boundedness of the solution $u \in X(0,\infty)$ by several lemmas.

Lemma 4.1. *It holds that*

$$\begin{aligned}
&\|\tilde{u}(t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}} A(\xi)|\tilde{u}(s,\xi)|^2 d\xi ds + 2D\int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds \\
&\le \|\tilde{u}_0(0)\|_{L^2}^2 + b'(0)e^{-\lambda_* c_* r}\int_{-r}^0 \int_{\mathbb{R}} |\tilde{u}_0(s,\xi)|^2 d\xi ds \\
&\quad + C\int_0^t \int_{\mathbb{R}} [w(\xi)]^{-\frac{1}{2}}|\tilde{u}(s,\xi)||\tilde{u}(s-r,\xi-c_*r)|^2 d\xi ds,
\end{aligned} \tag{4.6}$$

where

$$A(\xi) := e^{-\lambda_* c_* r}\Big(2b'(0) - |b'(\phi(\xi - c_* r))| - |b'(\phi(\xi))|\Big). \tag{4.7}$$

Proof. Multiplying Eq. (4.2) by $\tilde{u}$ and integrating it with respect to ξ and t over $\mathbb{R} \times [0,t]$, we have

$$\begin{aligned}
&\|\tilde{u}(t)\|_{L^2}^2 + 2D\int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 + 2k_2 \int_0^t \int_{\mathbb{R}} |\tilde{u}(s,\xi)|^2 d\xi ds \\
&- 2e^{-\lambda_* c_* r}\int_0^t \int_{\mathbb{R}} b'(\phi(\xi - c_* r))\tilde{u}(s,\xi)\tilde{u}(s-r,\xi-c_*r) d\xi ds \\
&= \|\tilde{u}_0(0)\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}} \tilde{u}(s,\xi)\tilde{Q}(\tilde{u}(t-r,\xi-c_*r)) d\xi ds.
\end{aligned} \tag{4.8}$$

By using Cauchy–Schwarz inequality, we can estimate

$$\begin{aligned}
&\left|2e^{-\lambda_* c_* r}\int_0^t \int_{\mathbb{R}} b'(\phi(\xi - c_* r))\tilde{u}(s,\xi)\tilde{u}(s-r,\xi-c_*r) d\xi ds\right| \\
&\le e^{-\lambda_* c_* r}\int_0^t \int_{\mathbb{R}} |b'(\phi(\xi - c_* r))||\tilde{u}(s,\xi)|^2 d\xi ds
\end{aligned}$$

$$+ e^{-\lambda_* c_* r} \int_0^t \int_{\mathbb{R}} |b'(\phi(\xi - c_* r))||\tilde{u}(s - r, \xi - c_* r)|^2 d\xi ds$$

$$\text{(change of variables: } \xi - c_* r \to \xi,\ s - r \to s)$$

$$= e^{-\lambda_* c_* r} \int_0^t \int_{\mathbb{R}} |b'(\phi(\xi - c_* r))||\tilde{u}(s, \xi)|^2 d\xi ds$$

$$+ e^{-\lambda_* c_* r} \int_{-r}^{t-r} \int_{\mathbb{R}} |b'(\phi(\xi))||\tilde{u}(s, \xi)|^2 d\xi ds$$

$$\le e^{-\lambda_* c_* r} \int_0^t \int_{\mathbb{R}} \Big(|b'(\phi(\xi - c_* r))| + |b'(\phi(\xi))|\Big)|\tilde{u}(s, \xi)|^2 d\xi ds$$

$$+ b'(0) e^{-\lambda_* c_* r} \int_{-r}^0 \int_{\mathbb{R}} |\tilde{u}_0(s, \xi)|^2 d\xi ds, \tag{4.9}$$

where we used the condition (H_3) for $|b'(\phi)| \le b'(0)$ in the last step. On the other hand, noting (4.5), we can estimate the nonlinear term as follows

$$\Big| \int_0^t \int_{\mathbb{R}} \tilde{u}(s, \xi) \tilde{Q}(\tilde{u}(t - r, \xi - c_* r)) d\xi ds \Big|$$

$$\le C \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{w(\xi)}} |\tilde{u}(s, \xi)||\tilde{u}(s - r, \xi - c_* r)|^2 d\xi ds. \tag{4.10}$$

Substituting (4.9) and (4.10) to (4.8), we have

$$\|\tilde{u}(t)\|_{L^2}^2 + 2D \int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}} A(\xi)|\tilde{u}(s, \xi)|^2 d\xi ds$$

$$\le \|\tilde{u}_0(0)\|_{L^2}^2 + b'(0) e^{-\lambda_* c_* r} \int_{-r}^0 \int_{\mathbb{R}} |\tilde{u}_0(s, \xi)|^2 d\xi ds$$

$$+ C \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{w(\xi)}} |\tilde{u}(s, \xi)||\tilde{u}(s - r, \xi - c_* r)|^2 d\xi ds, \tag{4.11}$$

where

$$
\begin{aligned}
A(\xi) &:= 2k_2 - e^{-\lambda_* c_* r}\Big(|b'(\phi(\xi - c_* r))| + |b'(\phi(\xi))|\Big) \\
&= 2[c_*\lambda_* - D\lambda_*^2 + d] - e^{-\lambda_* c_* r}\Big(|b'(\phi(\xi - c_* r))| + |b'(\phi(\xi))|\Big) \\
&= e^{-\lambda_* c_* r}\Big(2b'(0) - |b'(\phi(\xi - c_* r))| - |b'(\phi(\xi))|\Big).
\end{aligned}
$$

Here, we used (4.3) and (1.8), namely, $k_2 = c_*\lambda_* - D\lambda_*^2 + d = b'(0)e^{-\lambda_* c_* r}$. □

Lemma 4.2. *It holds that*

$$A(\xi) \geq C_2\phi(\xi) \geq 0 \tag{4.12}$$

for some positive constant C_2.

Proof. From the condition (H_2), we note that $b'(\phi) > 0$ for $0 \leq \phi < v_*$, $b'(\phi) < 0$ for $v_* < \phi < \infty$, and $b'(\phi) = 0$ for $\phi = v_*$. We note also from the second equation of (1.9) that $\phi(\xi) = C_3|\xi|e^{-\lambda_*|\xi|} \to 0$ and $\phi(\xi - c_* r) = C_3|\xi - c_* r|e^{-\lambda_*|\xi - c_* r|} \to 0$ as $\xi \to -\infty$ for some positive constant $C_3 > 0$, which gives

$$\lim_{\xi \to -\infty} \frac{\phi(\xi - c_* r)}{\phi(\xi)} = e^{-\lambda_* c_* r}.$$

On the other hand, the condition $|b'(v)| \leq b'(0)$ for $v \geq 0$ (see (H_3)) and $b'(v) > 0$ in $[0, v_*]$ (see (H_2)) implies

$$b''(v) < 0 \quad \text{for } v \text{ near } 0.$$

Thus, by Taylor's expansion formula, there exist some positive numbers $\tilde{\phi}_1 \in (0, \phi(\xi))$ and $\tilde{\phi}_2 \in (0, \phi(\xi - c_* r))$, such that

$$
\begin{aligned}
\lim_{\xi \to -\infty} \frac{A(\xi)}{\phi(\xi)} &= \lim_{\xi \to -\infty} e^{-\lambda_* c_* r} \frac{[b'(0) - b'(\phi(\xi)] + [b'(0) - b'(\phi(\xi - c_* r)]}{\phi(\xi)} \\
&= \lim_{\xi \to -\infty} e^{-\lambda_* c_* r} \frac{-b''(\tilde{\phi}_1)\phi(\xi) - b''(\tilde{\phi}_2)\phi(\xi - c_* r)}{\phi(\xi)} \\
&= e^{-\lambda_* c_* r}[|b''(0)| + |b''(0)|e^{-\lambda_* c_* r}] \\
&=: C_4 > 0.
\end{aligned}
$$

Namely, there exists a negative number $\xi_* < 0$ with $|\xi_*| \gg 1$ such that

$$\frac{A(\xi)}{\phi(\xi)} \geq C_5 > 0 \quad \text{for } \xi \in (-\infty, \xi_*], \tag{4.13}$$

where C_5 is a positive constant.

On the other hand, when $\xi \in [\xi_*, \infty)$, the monotone/non-monotone critical waves $\phi(\xi)$ and $\phi(\xi - c_* r)$ are bounded, i.e.,

$$0 < m \le \phi(\xi) \le M, \text{ and } 0 < m \le \phi(\xi - c_* r) \le M$$

for some positive constants m and M. Actually, here $m = \phi(\xi_* - c_* r) > 0$. In fact, from the geometric analysis of the monotone/non-monotone traveling waves [7,32,33], see also our numerical simulations presented in Figs. 1(b), 3(b), 5(b), and 7(b) in the last section, we know that, the possible oscillations of the non-monotone traveling waves $\phi(\xi)$ occur near $+\infty$, and the waves are still monotone when ξ near $-\infty$. Thus, from the condition (H$_3$), i.e., $b'(0) > |b'(v)|$ for $v \in [0, \infty)$, we have

$$\begin{aligned}\frac{A(\xi)}{\phi(\xi)} &= e^{-\lambda_* c_* r} \frac{2b'(0) - |b'(\phi(\xi)| - |b'(\phi(\xi - c_* r)|}{\phi(\xi)} \\ &\ge 2e^{-\lambda_* c_* r} \frac{b'(0) - \max_{v \in [m,M]} |b'(v)|}{M} \\ &=: C_6 > 0. \end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14), we have proved (4.12) for some positive constant C_2. □

Based on Lemmas 4.1 and 4.2, now we can establish the first key energy estimate.

Lemma 4.3. *There exists $\delta_1 > 0$, when $M_\infty \le \delta_1$, then*

$$\begin{aligned} &\|\tilde{u}(t)\|_{L^2}^2 + \int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds + \int_0^t \int_{\mathbb{R}} \phi(\xi) w(\xi) |u(s,\xi)|^2 d\xi ds \\ &\le C_7 \Big(\|\tilde{u}_0(0)\|_{L^2}^2 + \int_{-r}^0 \|\tilde{u}_0(s)\|_{L^2}^2 ds \Big) \le C_7 M_0^2, \quad t \in [0,\infty) \end{aligned} \tag{4.15}$$

where C_7 is a positive constant.

Proof. Since $\tilde{u}(t,\xi) = \sqrt{w(\xi)} u(t,\xi)$, by Lemma 4.2, then the second term of the left-hand-side of (4.6) can be written as

$$\int_0^t \int_{\mathbb{R}} A(\xi) |\tilde{u}(s,\xi)|^2 d\xi ds \ge C_2 \int_0^t \int_{\mathbb{R}} \phi(\xi) w(\xi) |u(s,\xi)|^2 d\xi ds. \tag{4.16}$$

This can be used to control the nonlinear term in (4.6).

Now we are going to estimate the nonlinear term in (4.6). It can be reformed as

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{w(\xi)}} |\tilde{u}(s,\xi)| |\tilde{u}(s-r, \xi - c_* r)|^2 d\xi ds \\ &= \int_0^t \int_{\mathbb{R}} w(\xi - c_* r) |u(s,\xi)| |u(s-r, \xi - c_* r)|^2 d\xi ds \end{aligned}$$

$$= \int_0^t \int_{\mathbb{R}} \phi(\xi - c_* r) w(\xi - c_* r) \frac{w(\xi)}{w(\xi - c_* r)} \frac{|u(s,\xi)|}{\phi(\xi - c_* r)} |u(s-r, \xi - c_* r)|^2 d\xi ds. \quad (4.17)$$

So we need further to estimate $\frac{|u(s,\xi)|}{\phi(\xi - c_* r)}$.

Notice from (1.9) that the critical wave $\phi(x + c_* t)$ is positive and bounded, and $\lim_{\xi \to +\infty} \phi(\xi) = v_+$ and $\phi(\xi) = O(1)|\xi| e^{\lambda_* \xi} \to 0$ as $\xi \to -\infty$, so, there exists a number ξ_1 near $-\infty$, i.e., $\xi_1 < 0$ and $|\xi_1| \gg 1$, such that

$$\phi(\xi) = O(1)|\xi| e^{\lambda_* \xi} \text{ for } \xi \in (-\infty, \xi_1), \quad \text{and } \phi(\xi) = O(1) \text{ for } \xi \in [\xi_1, \infty).$$

Thus, by the definition of $w(\xi) = e^{-2\lambda_* \xi}$, we can verify that

$$\frac{1}{\phi(\xi - c_* r)} \le \begin{cases} C\sqrt{w(\xi)}, & \text{for } \xi \in (-\infty, \xi_1) \\ C, & \text{for } \xi \in [\xi_1, \infty), \end{cases}$$

for some positive constant C. This with the definition of solution space $X(-r, \infty)$ and the definition of M_∞ (see (2.13)) as well as Sobolev inequality guarantees

$$\begin{aligned} \sup_{\xi \in \mathbb{R}} \frac{|u(t,\xi)|}{\phi(\xi - c_* r)} &\le \sup_{\xi \in (-\infty, \xi_1)} C\sqrt{w}|u(t,\xi)| + \sup_{\xi \in [\xi_1, \infty)} C|u(t,\xi)| \\ &\le C \sup_{\xi \in \mathbb{R}} \sqrt{w}|u(t,\xi)| + C \sup_{\xi \in \mathbb{R}} |u(t,\xi)| \\ &\le C\|\sqrt{w} u(t)\|_{H^1} + C\|u(t)\|_C \\ &\le C M_\infty. \end{aligned} \quad (4.18)$$

Thus, applying the above estimates (4.18) and the fact $\frac{w(\xi)}{w(\xi - c_* r)} = e^{-2\lambda_* c_* r}$, from (4.17) we can estimate the nonlinear term in (4.6) as follows

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{w(\xi)}} |\tilde{u}(s,\xi)||\tilde{u}(s-r, \xi - c_* r)|^2 d\xi ds \\ &= \int_0^t \int_{\mathbb{R}} \phi(\xi - c_* r) w(\xi - c_* r) \frac{w(\xi)}{w(\xi - c_* r)} \frac{|u(s,\xi)|}{\phi(\xi - c_* r)} |u(s-r, \xi - c_* r)|^2 d\xi ds \\ &\le C M_\infty) \int_0^t \int_{\mathbb{R}} \phi(\xi - c_* r) w(\xi - c_* r) |u(s-r, \xi - c_* r)|^2 d\xi ds \\ &\qquad \text{(change of variables } : \xi - c_* r \to \xi, s - r \to s) \\ &= C M_\infty \int_{-r}^{t-r} \int_{\mathbb{R}} \phi(\xi) w(\xi) |u(s,\xi)|^2 d\xi ds \end{aligned}$$

$$\leq CM_\infty \int_0^t \int_{\mathbb{R}} \phi(\xi)w(\xi)|u(s,\xi)|^2 d\xi ds + CM_\infty \int_{-r}^0 \int_{\mathbb{R}} \phi(\xi)w(\xi)|u_0(s,\xi)|^2 d\xi ds$$

$$\leq CM_\infty \int_0^t \int_{\mathbb{R}} \phi(\xi)w(\xi)|u(s,\xi)|^2 d\xi ds + C \int_{-r}^0 \|\tilde{u}_0(s)\|_{L^2}^2 ds. \tag{4.19}$$

Finally, substituting (4.16) and (4.19) to (4.6), we prove

$$\|\tilde{u}(t)\|_{L^2}^2 + \int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds + [C_2 - C_8 M_\infty] \int_0^t \int_{\mathbb{R}} \phi(\xi)w(\xi)|u(s,\xi)|^2 d\xi ds$$

$$\leq C_9 \Big(\|\tilde{u}_0(0)\|_{L^2}^2 + \int_{-r}^0 \|\tilde{u}_0(s)\|_{L^2}^2 ds \Big),$$

for some positive constants C_8 and C_9, which immediately implies (4.15) by taking M_∞ to be small, for example, let

$$0 < M_\infty \leq \delta_1 := \frac{C_2}{2C_8}, \tag{4.20}$$

then the corresponding constant C_7 in (4.15) is

$$C_7 := \frac{C_9}{\min\{1, \frac{C_2}{2}\}}.$$

The proof is complete. □

Similarly, the estimate for $\tilde{u}_\xi$ can be established as follows.

Lemma 4.4. *When $M_\infty \leq \delta_1$, then*

$$\|\tilde{u}_\xi(t)\|_{L^2}^2 + \int_0^t \|\tilde{u}_{\xi\xi}(s)\|_{L^2}^2 ds \leq C_{10}(M_\infty + 1)M_0^2, \quad t \in [0, \infty) \tag{4.21}$$

and

$$\int_0^t \Big| \frac{d}{ds} \|\tilde{u}_\xi(s)\|_{L^2}^2 \Big| ds \leq C_{11}(M_\infty + 1)M_0^2, \quad t \in [0, \infty) \tag{4.22}$$

where C_{10} and C_{11} are some positive constants.

Proof. Differentiating (4.2) with respect to ξ and multiplying the resultant equation by $\tilde{u}_\xi$ and integrating it with respect to ξ over $\mathbb{R}$, we have

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{u}_\xi(t)\|_{L^2}^2 + 2D\|\tilde{u}_{\xi\xi}(t)\|_{L^2}^2 + 2k_2\|\tilde{u}_\xi(t)\|_{L^2}^2 \\
&\quad = 2e^{-\lambda_* c_* r}\int_{\mathbb{R}} b'(\phi(\xi - c_* r))\tilde{u}_\xi(t,\xi)\tilde{u}_\xi(t-r,\xi-c_*r)d\xi \\
&\qquad + 2e^{-\lambda_* c_* r}\int_{\mathbb{R}} b''(\phi(\xi - c_* r))\phi'(\xi - c_* r)\tilde{u}_\xi(t,\xi)\tilde{u}(t-r,\xi-c_*r)d\xi \\
&\qquad + 2\int_{\mathbb{R}} \tilde{u}_\xi(t,\xi)\partial_\xi \tilde{Q}(\tilde{u}(t-r,\xi-c_*r))d\xi \\
&\quad =: I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{4.23}$$

Integrating it over $[0,t]$, we get

$$\begin{aligned}
&\|\tilde{u}_\xi(t)\|_{L^2}^2 + 2D\int_0^t \|\tilde{u}_{\xi\xi}(s)\|_{L^2}^2 ds + 2k_2\int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds \\
&\quad = \|\tilde{u}_{0,\xi}(0)\|_{L^2}^2 + \int_0^t [I_1(s)+I_2(s)+I_3(s)]ds.
\end{aligned} \tag{4.24}$$

By using the estimate (4.15), Cauchy–Schwarz inequality, the change of variables in (4.9), and the facts: $|\phi'(\xi)| \le C\phi(\xi)$, and $|\tilde{u}(t,\xi)| = e^{-\lambda_*\xi}|u(t,\xi)| = \sqrt{w(\xi)}|u(t,\xi)| \le CM_\infty$ for $(\xi,t) \in \mathbb{R}\times\mathbb{R}_+$, we can similarly estimate the nonlinear terms as

$$\begin{aligned}
&\int_0^t |I_1(s)|ds \\
&\quad \le C\int_0^t\int_{\mathbb{R}} [|\tilde{u}_\xi(s,\xi)|^2 + |\tilde{u}_\xi(s-r,\xi-c_*r)|^2]d\xi ds \\
&\quad \le C\int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds + C\int_{-r}^0 \|\tilde{u}_{0,\xi}(s)\|_{L^2}^2 ds \\
&\quad \le C\Big(\|\tilde{u}_0(0)\|_{L^2}^2 + \int_{-r}^0 \|\tilde{u}_{0,\xi}(s)\|_{L^2}^2 ds\Big) \\
&\quad \le CM_0^2,
\end{aligned} \tag{4.25}$$

and

$$
\begin{aligned}
&\int_0^t |I_2(s)|ds \\
&\le C\int_0^t \int_{\mathbb{R}} [|\tilde{u}_\xi(s,\xi)|^2 + |\phi'(\xi - c_* r)|^2 |\tilde{u}(s-r, \xi - c_* r)|^2] d\xi ds \\
&= C\int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds + C\int_0^t \int_{\mathbb{R}} |\phi'(\xi - c_* r)|^2 w(\xi - c_* r)|u(s-r, \xi - c_* r)|^2 d\xi ds \\
&\le C\int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds + C\int_0^t \int_{\mathbb{R}} \phi(\xi - c_* r) w(\xi - c_* r)|u(s-r, \xi - c_* r)|^2 d\xi ds \\
&= C\int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds + C\int_{-r}^{t-r} \int_{\mathbb{R}} \phi(\xi) w(\xi)|u(s, \xi)|^2 d\xi ds \\
&\le C\int_0^t \|\tilde{u}_\xi(s)\|_{L^2}^2 ds + C\int_0^t \int_{\mathbb{R}} \phi(\xi) w(\xi)|u(s, \xi)|^2 d\xi ds + C\int_{-r}^0 \|\tilde{u}_{0,\xi}(s)\|_{L^2}^2 ds \\
&\le C\Big(\|\tilde{u}_0(0)\|_{L^2}^2 + \int_{-r}^0 \|\tilde{u}_{0,\xi}(s)\|_{L^2}^2 ds\Big) \\
&\le CM_0^2,
\end{aligned}
\tag{4.26}
$$

and finally

$$
\begin{aligned}
&\int_0^t |I_3(s)|ds \\
&\le C\int_0^t \int_{\mathbb{R}} |\tilde{u}_\xi(s,\xi)||\tilde{u}_\xi(s-r, \xi - c_* r)||\tilde{u}(s-r, \xi - c_* r)| d\xi ds \\
&\le CM_\infty \int_0^t \int_{\mathbb{R}} |\tilde{u}_\xi(s,\xi)||\tilde{u}_\xi(s-r, \xi - c_* r)| d\xi ds \\
&\le CM_\infty \int_0^t \int_{\mathbb{R}} [|\tilde{u}_\xi(s,\xi)|^2 + |\tilde{u}_\xi(s-r, \xi - c_* r)|^2] d\xi ds
\end{aligned}
$$

$$
\begin{aligned}
&\le CM_\infty\Big(\int_0^t \|\tilde{u}_\xi(s)|_{L^2}^2 ds + \int_{-r}^0 \|\tilde{u}_{0,\xi}(s)\|_{L^2}^2 ds\Big) \\
&\le CM_\infty\Big(\|\tilde{u}_0(0)\|_{L^2}^2 + \int_{-r}^0 \|\tilde{u}_{0,\xi}(s)\|_{L^2}^2 ds\Big) \\
&\le CM_\infty M_0^2
\end{aligned}
\tag{4.27}
$$

provided $M_\infty \le \delta_1$.

Thus, substituting (4.25)–(4.27) to (4.24) and integrating the resultant equation with respect to t over $[0, t]$, we get

$$
\begin{aligned}
&\|\tilde{u}_\xi(t)\|_{L^2}^2 + \int_0^t \|\tilde{u}_{\xi\xi}(s)\|_{L^2}^2 ds \\
&\qquad \le C_{10}(M_\infty + 1)\Big(\|\tilde{u}_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\tilde{u}_0(s)\|_{H^1}^2 ds\Big) \\
&\qquad \le C_{10}(M_\infty + 1)M_0^2
\end{aligned}
$$

for some constant $C_{10} > 0$, provided $M_\infty \le \delta_1$. This proves (4.21).

Next, we prove (4.22). From (4.23), we have

$$
\Big|\frac{d}{dt}\|\tilde{u}_\xi(t)\|_{L^2}^2\Big| \le 2D\|\tilde{u}_{\xi\xi}(t)\|_{L^2}^2 + 2k_2\|\tilde{u}_\xi(t)\|_{L^2}^2 + |I_1(t)| + |I_2(t)| + |I_3(t)|.
$$

Integrating it over $[0, t]$ and using (4.15), (4.22) and (4.25)–(4.27), we have

$$
\begin{aligned}
&\int_0^t \Big|\frac{d}{ds}\|\tilde{u}_\xi(s)\|_{L^2}^2\Big| ds \\
&\qquad \le C_{11}(M_\infty + 1)\Big(\|\tilde{u}_0(0)\|_{H^1}^2 + \int_{-r}^0 \|\tilde{u}_0(s)\|_{H^1}^2 ds\Big) \\
&\qquad \le C_{11}(M_\infty + 1)M_0^2
\end{aligned}
$$

for some constant $C_{11} > 0$, provided $M_\infty \le \delta_1$. This proves (4.22). $\square$

We now prove the boundedness for $\|u(t)\|_C = \|w^{-\frac{1}{2}}\tilde{u}(t)\|_C$ uniformly in $t \in [0, \infty)$. Since $u \in X(0, \infty)$, so $u \in C_{unif}(0, \infty)$, namely, $\lim_{\xi\to+\infty} u(t, \xi) = u(t, \infty) =: z(t)$ exists uniformly for $t \in [-r, \infty]$, and $\lim_{\xi\to+\infty} u_\xi(t, \xi) = 0$ and $\lim_{\xi\to+\infty} u_{\xi\xi}(t, \xi) = 0$ are uniformly for $t \in [-r, \infty)$. Let us take the limits to (2.3) as $\xi \to +\infty$, then

$$\begin{cases} z'(t) + dz(t) - b'(v_+)z(t-r) = Q(z(t-r)), \\ z(s) = z_0(s), \ \ s \in [-r, 0]. \end{cases} \tag{4.28}$$

As shown in [13], we have the following exponential decay for $z(t)$.

Lemma 4.5. *(See [13].) When $d \geq |b'(v_+)|$ with arbitrary time-delay $r > 0$, or $d < |b'(v_+)|$ but with a small time-delay $0 < r < \bar{r}$, where $\bar{r}$ is defined in (1.15), then*

$$|u(t, \infty)| = |z(t)| \leq CM_0 e^{-\mu t}, \ \ t > 0, \tag{4.29}$$

for some $0 < \mu = \mu(p, d, r, b'(v_+)) < d$, provided with $|z_0| \ll 1$.

Now we can prove the boundedness of u in $C(\mathbb{R})$.

Lemma 4.6. *It holds that*

$$\|u(t)\|_C \leq C_{12}\sqrt{M_\infty + 1}\ M_0, \ \ t \in [0, \infty) \tag{4.30}$$

provided $M_\infty \leq \delta_1$.

Proof. Notice that, $\lim_{\xi \to +\infty} u(t, \xi) = u(t, \infty) =: z(t)$ uniformly with respect to $t \in [0, \infty)$, that is, for any given $\varepsilon_0 > 0$ (we may choose it less than or equal to M_0), there exists a large number $x_0 = x_0(\varepsilon_0) \gg 1$ (independent of $t \in [0, \infty)$, because of the uniform convergence) such that, when $\xi \geq x_0$, then

$$|u(t, \xi) - z(t)| < \varepsilon_0 \ \text{ uniformly in } t \in [0, \infty).$$

This implies, with the help of (4.29) for $|u(t, \infty)| = |z(t)| \leq CM_0 e^{-\mu t} \leq CM_0$, that

$$\sup_{x \in [x_0, \infty)} |u(t, \xi)| < CM_0 + \varepsilon_0 < CM_0, \ \ \text{uniformly in } t \in [0, \infty). \tag{4.31}$$

Applying the fact $\sqrt{w(\xi)} = e^{-\lambda_* \xi} \geq e^{-\lambda_* x_0}$ for $\xi \in (-\infty, x_0]$, Sobolev inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, and the energy estimates (4.15) and (4.21), we have

$$\begin{aligned} \sup_{\xi \in (-\infty, x_0]} |u(t, \xi)| &\leq \sup_{\xi \in [-\infty, x_0]} \left| \frac{\sqrt{w(\xi)}}{e^{-\lambda_* x_0}} u(t, \xi) \right| \\ &= e^{\lambda_* x_0} \sup_{\xi \in [-\infty, x_0]} |\sqrt{w(\xi)} u(t, \xi)| \\ &\leq C\|\sqrt{w} u(t)\|_{H^1} \leq C\sqrt{M_\infty + 1}\ M_0, \quad t \in [0, \infty). \end{aligned} \tag{4.32}$$

Thus, (4.31) and (4.32) imply (4.30) for some positive constant C_{12}. $\square$

Proof of Theorem 2.2. Add (4.15), (4.21) and (4.30) together, we have

$$\|\tilde{u}(t)\|_{H^1}^2 + \|u(t)\|_C^2 + \int_0^t \|(\sqrt{\phi w}u)(s)\|_{L^2}^2 ds + \int_0^t \|\tilde{u}_\xi(s)\|_{H^1}^2 ds$$
$$\le C(M_\infty + 1)M_0^2, \quad t \in [0, \infty). \tag{4.33}$$

Notice that $\tilde{u} = \sqrt{w}u$, then (4.33) is reduced to

$$M_\infty^2 := \sup_{t\in[0,\infty)} \left\{ \|\sqrt{w}u(t)\|_{H^1}^2 + \|u(t)\|_C^2 \right\}$$
$$+ \int_0^\infty \|(\sqrt{\phi w}u)(s)\|_{L^2}^2 ds + \int_0^\infty \|(\sqrt{w}u)_\xi(s)\|_{H^1}^2 ds$$
$$\le C_{13}(M_\infty + 1)M_0^2. \tag{4.34}$$

In order to guarantee the *a priori* estimates $M_\infty \le \delta_1$ (see (4.20)) and $M_\infty \le \sqrt{C_{13}(M_\infty + 1)}M_0$ (see (4.34)), we take $\delta_0 > 0$ in Theorem 2.3 as

$$\delta_0 := \frac{\delta_1}{\sqrt{C_{13}(\delta_1 + 1)}}, \tag{4.35}$$

when $M_0 \le \delta_0$, then we can guarantee

$$M_\infty \le \sqrt{C_{13}(M_\infty + 1)}M_0 \le \sqrt{C_{13}(\delta_1 + 1)}\delta_0 = \delta_1,$$

and

$$M_\infty^2 \le C_{13}(M_\infty + 1)M_0^2 \le C_{13}(\delta_1 + 1)M_0^2 =: C_{14}M_0^2.$$

This proves the uniform boundedness (2.13). □

5. Asymptotic stability

This section is to devoted to the proof of the asymptotic stability (2.14). From (2.13) and (4.22), when $M_0 \le \delta_0$, we have

$$\|u(t)\|_C^2 + \|(\sqrt{w}u)(t)\|_{H^1}^2 + \int_0^t \|(\sqrt{\phi w}u)(s)\|_{L^2}^2 ds$$
$$+ \int_0^t \|\partial_\xi(\sqrt{w}u)(s)\|_{H^1}^2 ds + \int_0^t \left| \frac{d}{ds} \|\partial_\xi(\sqrt{w}u)(s)\|_{L^2}^2 \right| ds$$
$$\le CM_0^2, \quad \text{for } t \in [0, \infty). \tag{5.1}$$

Set

$$g(t) := \|\tilde{u}_\xi(t)\|_{L^2}^2 = \|\partial_\xi(\sqrt{w}u)(t)\|_{L^2}^2.$$

From (5.1), we know that

$$0 \le g(t) \le CM_0^2, \quad \int_0^\infty g(t)dt \le CM_0^2, \quad \text{and} \quad \int_0^\infty |g'(t)|dt \le CM_0^2.$$

This implies

$$\lim_{t\to\infty} g(t) = 0, \quad \text{i.e.,} \quad \lim_{t\to\infty} \|\tilde{u}_\xi(t)\|_{L^2}^2 = 0. \tag{5.2}$$

By using Sobolev inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$

$$\|\tilde{u}(t)\|_C \le \sqrt{2}\|\tilde{u}(t)\|^{\frac{1}{2}}\|\tilde{u}_\xi(t)\|^{\frac{1}{2}},$$

and the boundedness of $\|\tilde{u}(t)\| = \|(\sqrt{w}u)(t)\| \le CM_0$ and the convergence of (5.2), we then prove

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} |\sqrt{w(\xi)}u(t,\xi)| = \lim_{t\to\infty} \|\tilde{u}(t)\|_C = 0. \tag{5.3}$$

Now, we are going to prove the convergence

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} |u(t,\xi)| = 0.$$

To prove such a stability relation, let us start from the far field $\xi \gg 1$. By the same fashion as shown in Lemma 4.6, the solution $z(t) = u(t,\infty)$ to the delayed ODE (4.28) decays exponentially

$$|u(t,\infty)| = |z(t)| \le CM_0 e^{-\mu t}, \quad \text{for all } t \in [0,\infty), \tag{5.4}$$

when $d \ge |b'(v_+)|$ with arbitrary time-delay $r > 0$, or $d < |b'(v_+)|$ but with a small time-delay $0 < r < \bar{r}$. Back to (2.1) and (2.2), we can write the solution in the integral form represented by the heat kernel $G(t-s, \xi-\eta)$:

$$\begin{aligned} u(t,\xi) = {} & e^{-dt}\int_{\mathbb{R}} G(\eta,t)u(0,\xi-\eta)d\eta \\ & + \int_0^t e^{-d(t-s)}\int_{\mathbb{R}} G(\eta,t-s)P(u(s-r,\xi-\eta-c_*r))d\eta ds. \end{aligned} \tag{5.5}$$

Multiplying (5.5) by $e^{\mu t}$ where $0 < \mu < d$ is specified in (4.29), and noting $|P(u)| \le C|u|$, then we get

$$
\begin{aligned}
|e^{\mu t} u(t,\xi)| \le{}& e^{-(d-\mu)t} \int_{\mathbb{R}} G(\eta,t)|u_0(0,\xi-\eta)|d\eta \\
&+ e^{\mu t} \int_0^t e^{-d(t-s)} \int_{\mathbb{R}} G(\eta,t-s)|P(u(s-r,\xi-\eta-c_* r))|d\eta ds \\
\le{}& e^{-(d-\mu)t} \int_{\mathbb{R}} G(\eta,t)|u_0(0,\xi-\eta)|d\eta \\
&+ Ce^{\mu t} \int_0^t e^{-d(t-s)} \int_{\mathbb{R}} G(\eta,t-s)|u(s-r,\xi-\eta-c_* r)|d\eta ds. \qquad (5.6)
\end{aligned}
$$

Since $u \in X(0,\infty)$ is the global solution of (2.3), namely $u \in C_{unif}(0,\infty)$, then $u(t,\xi) \to u(t,\infty) = z(t)$ as $\xi \to \infty$ uniformly in $t \in [0,\infty)$. By applying the property of the heat kernel and the exponential decay (5.4), then from (5.6) we obtain

$$
\begin{aligned}
\lim_{\xi\to\infty} |e^{\mu t} u(t,\xi)| \le{}& e^{-(d-\mu)t} \int_{\mathbb{R}} G(\eta,t) \lim_{\xi\to\infty} |u_0(0,\xi-\eta)|d\eta \\
&+ Ce^{\mu t} \int_0^t e^{-d(t-s)} \int_{\mathbb{R}} G(\eta,t-s) \lim_{\xi\to\infty} |u(s-r,\xi-\eta-c_* r)|d\eta ds \\
\le{}& |u_{0,\infty}(0)| e^{-(d-\mu)t} \int_{\mathbb{R}} G(\eta,t)d\eta \\
&+ Ce^{\mu t} \int_0^t e^{-d(t-s)} |z(s-r)| \int_{\mathbb{R}} G(\eta,t-s)d\eta ds \\
\le{}& |u_{0,\infty}(0)| e^{-(d-\mu)t} + Ce^{\mu t} \int_0^t e^{-d(t-s)} e^{-\mu(s-r)} ds \\
={}& |u_{0,\infty}(0)| e^{-(d-\mu)t} + Ce^{\mu_1 r} \int_0^t e^{-d(t-s)} e^{\mu(t-s)} ds \\
={}& |u_{0,\infty}(0)| e^{-(d-\mu)t} + \frac{Ce^{\mu_1 r}}{d-\mu}[1 - e^{-(d-\mu)t}] \\
\le{}& C, \qquad \text{uniformly in all } t > 0. \qquad (5.7)
\end{aligned}
$$

This quickly implies that, there exists a number $x_1 \gg 1$ (independent of t), such that when $\xi \geq x_1$, then

$$\sup_{\xi\in[x_1,\infty)} |u(t,\xi)| \leq Ce^{-\mu t}, \quad t > 0. \tag{5.8}$$

Notice that, $\sqrt{w(\xi)} = e^{-\lambda_* \xi} \geq e^{-\lambda_* x_1}$ for $\xi \in (-\infty, x_1]$, then (5.3) implies

$$\begin{aligned}
\lim_{t\to\infty} \sup_{\xi\in(-\infty,x_1]} |u(t,\xi)| &\leq \lim_{t\to\infty} \sup_{\xi\in(-\infty,x_1]} \left| \frac{\sqrt{w(\xi)}}{e^{-\lambda_* x_1}} u(t,\xi) \right| \\
&\leq e^{\lambda_* x_1} \lim_{t\to\infty} \sup_{\xi\in R} |\sqrt{w(\xi)} u(t,\xi)| \\
&= 0.
\end{aligned}$$

This with (5.8) together proves

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}} |u(t,\xi)| = 0.$$

The proof is complete. $\square$

6. A remark on the multi-modality birth rate function

We consider a more general case of (1.1):

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} - D\dfrac{\partial^2 v(t,x)}{\partial x^2} + d(v(t,x)) = b(v(t-r,x)), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ v(s,x) = v_0(s,x), \quad s \in [-r,0], \ x \in \mathbb{R}. \end{cases} \tag{6.1}$$

Here the death rate function $d(v)$ and birth rate function $b(v)$ satisfy

($\mathcal{H}_1$) there are only two constant equilibria $v_\pm$ of (6.1), with $v_- = 0$ being unstable and v_+ being stable, that is, $b(v_\pm) - d(v_\pm) = 0$, $d'(0) - b'(0) < 0$ and $d'(v_+) - b'(v_+) > 0$;
($\mathcal{H}_2$) the multi-modality condition: $b \in C^2[0,\infty)$ and $b(v) \geq 0$ on $[0,\infty)$ can be finitely multi-modal;
($\mathcal{H}_3$) $d \in C^2[0,\infty)$, $d(v) \geq 0$, $d'(v) \geq d'(0) > 0$ and $|b'(v)| \leq b'(0)$ for $v \in [0,\infty)$.

Although the uni-modality condition (H_2) for the birth rate is practical and summarized from the biological models of populations like Nicholson's blowflies equation and Mackey–Glass equation, we can generalize it to include multi-modal functions. In fact in Section 3 when we prove the *a priori* energy estimates, we didn't use the uni-modality condition, but the condition (H_3) with $d'(v) \geq d'(0) > 0$ and $|b'(v)| \leq b'(0)$ for $v \in [0,\infty)$ is essential.

Without any difficulty, as shown in Sections 3–5 we can similarly prove the following stability of the critical monotone/non-monotone traveling waves.

Theorem 6.1 *(Stability in more general case). Let the birth rate function $d(v)$ and $b(v)$ be general and satisfy $(\mathcal{H}_1)$–$(\mathcal{H}_3)$, and assume either $d'(v_+) \ge |b'(v_+)|$ with any time-delay $r > 0$, or $d'(v_+) < |b'(v_+)|$ but with a small time-delay $0 < r < \bar{r}$, where $\bar{r}$ is defined by*

$$\bar{r} := \frac{\pi - \arctan(\sqrt{|b'(v_+)|^2 - |d'(v_+)|^2}/d'(v_+))}{\sqrt{|b'(v_+)|^2 - |d'(v_+)|^2}}.$$

For any given critical traveling wave $\phi(x + c_* t)$ to Eq. (1.1), whatever it is monotone or non-monotone, suppose that the initial perturbation $u_0(s,x) := v_0(s,x) - \phi(x + c_* s) \in C([-r,0]; C(\mathbb{R}))$, $\sqrt{w(x)}u_0(s,x) \in C([-r,0]; H^1(\mathbb{R})) \cap L^2([-r,0]; H^1(\mathbb{R}))$, and $\lim_{x\to+\infty}[v_0(s,x) - \phi(x + c_* s)] =: u_{0,\infty}(s) \in C[-r,0]$ exists uniformly with respect to $s \in [-r,0]$. Then, there exists a constant $\delta_0 > 0$ independent of x and t, when the initial perturbation is small

$$\max_{s\in[-r,0]} \|u_0(s)\|_C^2 + \|\sqrt{w}u_0(s)\|_{H^1}^2 + \int_{-r}^{0} \|\sqrt{w}u_0(s)\|_{H^1}^2 ds \le \delta_0^2,$$

the solution $v(t,x)$ of (1.1) and (1.2) is unique and globally exists in time, and satisfies

$$\begin{aligned} &v(t,x) - \phi(x + c_* t) \in C([-r,\infty); C(\mathbb{R})) \cap \mathcal{C}_{unif}[-r,\infty), \\ &\sqrt{w(x)}[v(t,x) - \phi(x + c_* t)] \in C([-r,\infty); H^1(\mathbb{R})), \\ &\partial_x\Big(\sqrt{w(x)}[v(t,x) - \phi(x + c_* t)]\Big) \in L^2([-r,\infty); H^1(\mathbb{R})), \end{aligned} \tag{6.2}$$

and

$$\lim_{t\to\infty} \sup_{x\in R} |v(t,x) - \phi(x + c_* t)| = 0, \tag{6.3}$$

where $\mathcal{C}_{unif}[-r,T]$ for $0 < T \le \infty$ is defined in (2.7).

7. Numerical simulations

In this section, we are going to carry out some numerical simulations, which will also perfectly support our theoretical stability results for the critical traveling waves.

We consider Nicholson's blowflies equation

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} - D\dfrac{\partial^2 v(t,x)}{\partial x^2} + dv(t,x) = pv(t-r,x)e^{-av(t-r,x)}, & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ v|_{t=s} = v_0(s,x), \quad (s,x) \in [-r,0] \times \mathbb{R}. \end{cases} \tag{7.1}$$

It possesses two constant equilibria $v_- = 0$ and $v_+ = \frac{1}{a}\ln\frac{p}{d}$. When $\frac{p}{d} > e$, the birth rate function $b(v) = pve^{-av}$ is unimodal, and satisfies $b'(0) > |b'(v)|$ for $v \in (0,\infty)$. The condition $d \ge |b'(v_+)|$ is equivalent to $e < \frac{p}{d} \le e^2$, and $d > |b'(v_+)|$ is equivalent to $\frac{p}{d} > e^2$.

For simplicity, throughout this section we fix $D = d = a = 1$, and leave p, r and the initial data $v_0(s,x)$ free. For the critical traveling waves $\phi(x + c_* t)$, the critical wave speed c_* is uniquely determined by (1.8), that is,

$$c_*\lambda_* - D\lambda_*^2 + d = b'(0)e^{-\lambda_* c_* r}, \quad c_* - 2D\lambda_* = -c_* r b'(0)e^{-\lambda_* c_* r},$$

and the critical waves $\phi(x + c_*t) = \phi(\xi)$ spatial-exponentially decays as

$$\phi(\xi) = O(1)|\xi|e^{-\lambda_*|\xi|} \quad \text{as} \quad \xi \to -\infty.$$

From our stability theorem, the initial data of Eq. (7.1) is expected to be

$$\lim_{x\to-\infty} v_0(s,x) = 0, \quad \lim_{x\to\infty} v_0(s,x) = v_+ \quad \text{uniformly in } s \in [-r,0],$$

and particularly,

$$e^{-\lambda_* x}|v_0(s,x) - \phi(x + c_*s)| \to 0 \quad \text{as} \quad x \to -\infty, \text{ uniformly in } s \in [-r,0].$$

So, let us select the initial data in the form of

$$v_0(s,x) = \begin{cases} |x|e^{-\lambda_*|x|}, & \text{as } x \le 0, \ s \in [-r,0], \\ \frac{v_+ x}{x + e^{-\lambda_* x}}, & \text{as } x \ge 0, \ s \in [-r,0]. \end{cases} \tag{7.2}$$

Clearly, such a $v_0(s,x)$ is continuous in $[-r,0] \times \mathbb{R}$. The expected critical wave speed is $c_* = c_*(\lambda_*)$ determined by (1.8), and the targeted wave is $\phi(x + c_*t)$ such that

$$\lim_{x\to-\infty} \sup_{s\in[-r,0]} \frac{v_0(s,x)}{\phi(x + c_*s)} = 1 \tag{7.3}$$

and

$$\lim_{x\to-\infty} \sup_{s\in[-r,0]} e^{\lambda_*|x|}|v_0(s,x) - \phi(x + c_*s)| = 0.$$

This is similar to the case without time-delay for the classic Fisher–KPP equation [25,26,34].

Based on the structure of the traveling waves [7] and our theoretical stability results, we realize that when $e < \frac{p}{d} \le e^2$, if the time-delay is small such that $0 < r < \underline{r}$, where $\underline{r}$ defined in (1.13) is the critical number for occurring oscillations of the corresponding delayed ODE (1.14), then, we obviously expect that the solution $v(t,x)$ of (7.1) large-time behaves like a monotone critical traveling wave $\phi(x + c_*t)$; and if the time-delay is large such that $r \ge \underline{r}$, then we expect that the solution $v(t,x)$ of (7.1) large-time behaves like an oscillatory critical traveling wave $\phi(x + c_*t)$; while, when $\frac{p}{d} > e^2$, if the time-delay is small such that $0 < r < \underline{r}$, we still expect that the solution $v(t,x)$ of (7.1) large-time behaves like a monotone critical traveling wave $\phi(x + c_*t)$; and if the time-delay is such that $\underline{r} \le r \le \bar{r}$, then we expect that the solution $v(t,x)$ of (7.1) large-time behaves like an oscillatory critical traveling wave $\phi(x + c_*t)$, where $\bar{r}$ is given in (2.15). Therefore, we take 4 cases to carry out our numerical simulations, see Table 1 for the details.

The computational scheme with a constrained mesh reported in this section is based on the following Crank–Nicholson scheme for the time derivative and a central scheme for the spatial derivative:

Table 1
Different cases for selection of p, r and the initial data $v_0(s,x)$.

Case	p	r	Zone of $\frac{p}{d}$	Zone of r	λ_* for v_0	Behavior of $v(t,x)$
1	6	0.2	$\frac{p}{d} \in (e, e^2)$	$r < \underline{r}$	$1.871\cdots$	monotone critical wave with $c_* = 2.564\cdots$
2	6	10	$\frac{p}{d} \in (e, e^2)$	$r > \underline{r}$	$0.864\cdots$	oscillatory critical wave with $c_* = 0.287\cdots$
3	10	0.1	$\frac{p}{d} > e^2$	$r < \underline{r}$	$2.561\cdots$	monotone critical wave with $c_* = 3.688\cdots$
4	10	2	$\frac{p}{d} > e^2$	$\underline{r} < r < \bar{r}$	$1.266\cdots$	oscillatory critical wave with $c_* = 1.041\cdots$

$$\begin{aligned}
&\frac{v_i^{n+1}-v_i^n}{\Delta t} - \frac{D}{2(\Delta x)^2}\Big[(v_{i+1}^{n+1}-2v_i^{n+1}+v_{i-1}^{n+1}) + (v_{i+1}^n - 2v_i^n + v_{i-1}^n)\Big] + \frac{d}{2}(v_i^{n+1}+v_i^n) \\
&\quad = \frac{p}{2}(v_i^{n-m}+v_i^{n+1-m})\exp\Big[-\frac{a}{2}(v_i^{n-m}+v_i^{n+1-m})\Big], \qquad (7.4)
\end{aligned}$$

where $m = r/\Delta t$. Although (7.1) is nonlinear and the Crank–Nicholson scheme is implicit, the scheme (7.4) is explicit since the nonlinear term is delayed. The advantage of such a scheme is unconditionally stable and the solutions can be easily computed. Thus, there is no restriction on the step size in the scheme (7.4), and this numerical scheme (7.4) is second-order accurate in both spatial and temporal directions. The original initial value problem (7.1) is for x in the whole space $(-\infty, \infty)$, but numerically we have to impose a finite computational domain (L_a, L_b) for x with some selected large numbers L_a and L_b. Although the step sizes Δx and Δt can be large due to the unconditional stability of the scheme, we still choose them as small as possible so that the numerical results can much precisely and clearly illustrate our theoretical results.

Next, we report the numerical simulations in four test cases.

Case 1. $e < \frac{p}{d} \le e^2$ and $r < \underline{r}$, the solution $v(t,x)$ converges to a monotone critical traveling wave $\phi(x + c_* t)$. We take $p = 6$ and $r = 0.2$. Clearly, when $e < \frac{p}{d} < e^2$, then $v_+ = \frac{1}{a}\ln\frac{p}{d} > v_* = \frac{1}{a}$. This implies that the birth rate function $b(v)$ for $v \in [0, v_+]$ is non-monotone but concave downward only. The time-delay $r = 0.2$ is small, and satisfies $r < \underline{r} = 0.333027\cdots$, where $\underline{r}$ is given by (1.13), which is the critical number of the time-delay for the delayed ODE (1.14) possessing oscillatory solutions. By calculating (1.8), we get the critical wave speed $c_* = 2.564\cdots$ and the corresponding eigenvalue $\lambda_* = 1.871\cdots$. Since $r < \underline{r}$, these critical waves $\phi(x + c_* t)$ may not be oscillating. In fact, according to Gomez and Trofimchuk's analysis [7], they are monotone. Now we set the initial data as in (7.2) with $\lambda_* = 1.871\cdots$. According to our stability Theorem 2.1, we expect that the original solution $v(t,x)$ of (7.1) time-asymptotically converges to a certain critical traveling wave $\phi(x + c_* t)$ with $c_* = 2.564\cdots$. In fact, as numerically demonstrated in Fig. 1, we can see that the solution behaves exactly like a monotone traveling wave. In order to get the traveling speed for the solution $v(t,x)$, let us exam it from the contour graph (Fig. 2). The slope of the contour line is just the wave speed. Since the contour line passes through the points $(0,0)$ and $(-300, 116.9709\cdots)$, so the speed can be estimated as

$$c = \frac{|x_2 - x_1|}{|t_2 - t_1|} = \frac{|-300-0|}{|116.9709\cdots - 0|} = 2.564\cdots,$$

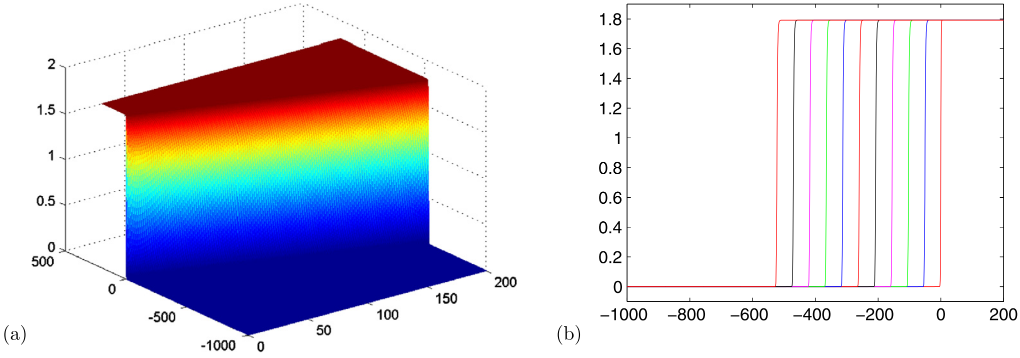

Fig. 1. Case 1: $e < \frac{p}{d} = 6 < e^2$ with small time-delay $r < \underline{r}$. (a) 3D-graphs of $v(t,x)$. (b) 2D-graphs of $v(t,x)$ at $t = 0,\ 20,\ 40,\ \cdots,\ 180,\ 200$. The solution behaves like a stable monotone wavefront traveling from right to left.

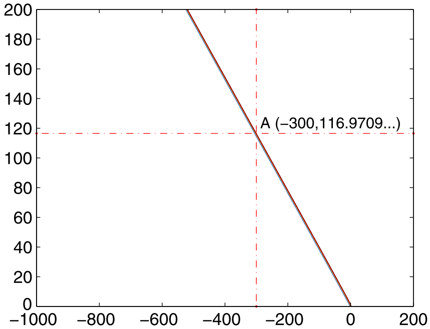

Fig. 2. Case 1: $e < \frac{p}{d} = 6 < e^2$ with small time-delay $r > \underline{r}$. The contour line showed in above indicates that the solution $v(t,x)$ travels with a speed of $c = 2.564\cdots$, which is just the critical wave speed $c_* = 2.564\cdots$.

which is exactly equal to the predicated critical wave speed c_*. Thus, we can verify that the solution $v(t,x)$ behaves like the (monotone) critical traveling wave $\phi(x + c_*t)$ with $c_* = 2.564\cdots$.

Case 2. $e < \frac{p}{d} \leq e^2$ *and* $r > \underline{r}$*, the solution* $v(t,x)$ *converges to an oscillatory critical traveling wave* $\phi(x + c_*t)$. In this case, we take $p = 6$ and $r = 10$. From (1.13) and (1.8), we have $\underline{r} = 0.33302758\cdots$, $c_* = 0.287\cdots$ and $\lambda_* = 0.864\cdots$. Since $r = 10 > \underline{r} = 0.33302758\cdots$, these critical traveling waves $\phi(x + c_*t)$ are oscillatory. Now we take the initial data in (7.2) with $\lambda_* = 0.864\cdots$. Fig. 3 shows that the solution $v(t,x)$ behaves like an oscillatory traveling wave, and the contour line given in Fig. 4 implies that the solution $v(t,x)$ travels with a speed

$$c = \frac{|x_2 - x_1|}{|t_2 - t_1|} = \frac{|-90 - 0|}{|313.348\cdots - 0|} = 0.287\cdots,$$

which is exactly equal to the predicated critical wave speed c_*. This numerically indicates that, after a large time, the solution $v(t,x)$ behaves like an oscillating critical wave $\phi(x + c_*t)$ with $c_* = 0.287\cdots$. Namely, the solution $v(t,x)$ converges to the oscillatory critical traveling wave $\phi(x + c_*t)$.

Case 3. $\frac{p}{d} \geq e^2$ *and* $r < \underline{r}$*, the solution* $v(t,x)$ *converges to a monotone critical traveling wave* $\phi(x + c_*t)$. Now let us take $p = 10$ and $r = 0.1$. Similarly, (1.13) and (1.8) determine

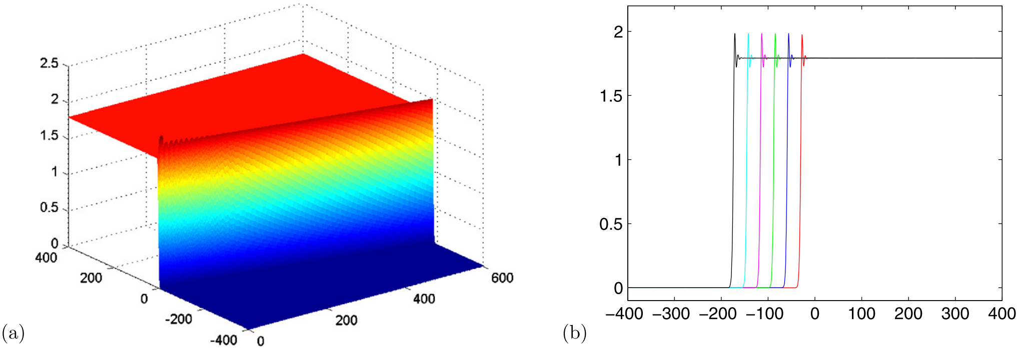

Fig. 3. Case 2: $e < \frac{p}{d} = 6 < e^2$ with big time-delay $r = 10 > \underline{r}$. (a) 3D-graphs of $v(t, x)$. (b) 2D-graphs of $v(t, x)$ at $t = 120,\ 240,\ 360,\ 480,\ 500,\ 620$. The solution behaves like a stable oscillatory wavefront traveling from right to left.

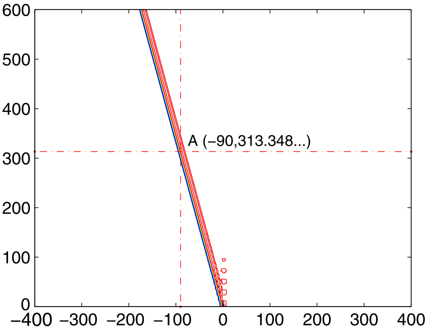

Fig. 4. Case 2: $e < \frac{p}{d} = 6 < e^2$ with big time-delay $r = 10 > \underline{r}$. The contour line showed in above indicates that the solution $v(t, x)$ travels with a speed of $c = 0.287\cdots$, which is just the critical wave speed $c_* = 0.287\cdots$.

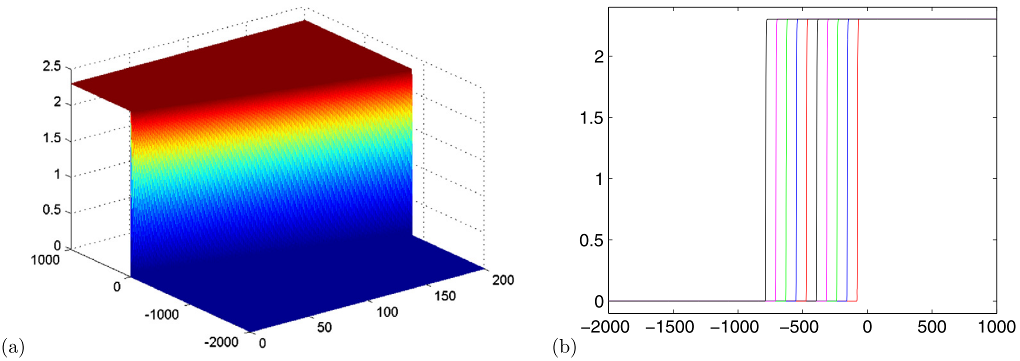

Fig. 5. Case 3: $\frac{p}{d} = 10 > e^2$ with small time-delay $r = 0.1 < \underline{r}$. (a) 3D-graphs of $v(t, x)$. (b) 2D-graphs of $v(t, x)$ at $t = 20,\ 40,\ 60,\ \cdots,\ 200$. The solution behaves like a stable oscillatory wavefront traveling from right to left.

$\underline{r} = 3.034694\cdots$, $c_* = 3.688\cdots$ and $\lambda_* = 2.561\cdots$. Since $r < \underline{r}$, the critical traveling waves $\phi(x + c_*t)$ are monotone. When we take the initial data $v_0(s, x)$ in (7.2) with $\lambda_* = 2.561\cdots$, as numerically demonstrated in Fig. 5, the solution $v(t, x)$ behaves like a monotone traveling wave, and Fig. 6 further confirms that the solution $v(t, x)$ travels with a speed

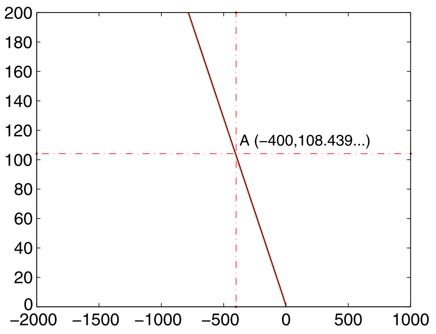

Fig. 6. Case 3: $\frac{p}{d} = 10 > e^2$ with small time-delay $r = 0.1 < \underline{r}$. The contour line showed in above indicates that the solution $v(t, x)$ travels with a speed of $c = 3.688\cdots$, which is just the critical wave speed $c_* = 3.688\cdots$.

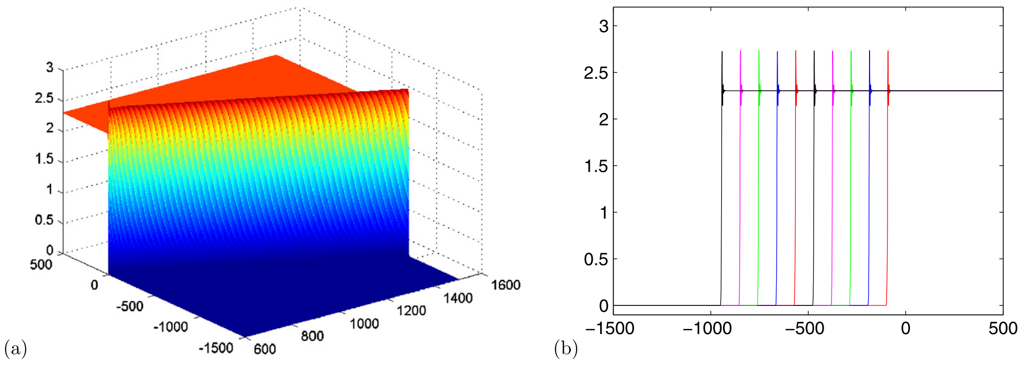

Fig. 7. Case 4: $\frac{p}{d} = 10 > e^2$ with time-delay $\underline{r} < r = 2 < \bar{r}$. (a) 3D-graphs of $v(t, x)$. (b) 2D-graphs of $v(t, x)$ at $t = 690,\ 780,\ 870,\ 960,\ 1050,\ \cdots,\ 1410,\ 1500$. The solution behaves like a stable oscillatory wavefront traveling from right to left.

$$c = \frac{|x_2 - x_1|}{|t_2 - t_1|} = \frac{|-400 - 0|}{|108.439\cdots - 0|} = 3.688\cdots,$$

which is exactly equal to the predicated critical wave speed c_*. This implies that the solution $v(t, x)$ behaves like the corresponding monotone critical traveling wave $\phi(x + c_* t)$ with $c_* = 3.688\cdots$.

Case 4. $\frac{p}{d} \geq e^2$ and $\underline{r} < r < \bar{r}$, the solution $v(t, x)$ converges to an oscillatory critical traveling wave $\phi(x + c_* t)$. In the last case, we choose $p = 10$ and $r = 2$. A simple calculation from (1.13), (2.16) and (1.8) gives $\underline{r} = 0.2254\cdots$, $\bar{r} = 3.034694\cdots$, $c_* = 1.041\cdots$ and $\lambda_* = 1.266\cdots$. In this case, $r = 2$ satisfies $\underline{r} = 0.2254\cdots < r < \bar{r} = 3.034694\cdots$, the critical waves are non-monotone. Now we take the initial data $v_0(s, x)$ in (7.2) with $\lambda_* = 1.266\cdots$. The numerical results showed in Fig. 7 demonstrates that the solution $v(t, x)$ behaves like a non-monotone traveling wave with oscillations around v_+, and the contour line presented in Fig. 8 confirms that the solution $v(t, x)$ travels with a speed

$$c = \frac{|x_2 - x_1|}{|t_2 - t_1|} = \frac{|-500 - 0|}{|480.089\cdots - 0|} = 1.041\cdots,$$

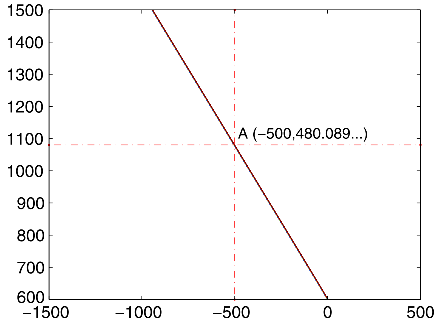

Fig. 8. Case 4: $\frac{p}{d} = 10 > e^2$ with time-delay $\underline{r} < r = 2 < \bar{r}$. The contour line showed in above indicates that the solution $v(t, x)$ travels with a speed of $c = 1.041\cdots$, which is just the critical wave speed $c_* = 1.041\cdots$.

which is exactly equal to the predicated critical wave speed c_*. So, $v(t, x)$ behaves like an oscillatory critical traveling wave $\phi(x + c_*t)$ with $c_* = 1.041\cdots$.

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