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STABILITY OF STATIONARY WAVES FOR FULL EULER-POISSON SYSTEM IN MULTI-DIMENSIONAL SPACE

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Abstract. This paper is concerned with the nonisentropic unipolar hydrodynamic model of semiconductors in the form of multi-dimensional full Euler-Poisson system. By heuristically analyzing the exact gaps between the original solutions and the stationary waves at far fields, we ingeniously construct some correction functions to delete these gaps, and then prove the L^∞-stability of stationary waves with an exponential decay rate in 1-D case. Furthermore, based on the 1-D convergence result, we show the stability of planar stationary waves with also some exponential decay rate in m-D case.

1. Introduction.

In this paper, we study the multi-dimensional full Euler-Poisson system

$$
\begin{cases}
n_t + \mathrm{div}(n\mathbf{u}) = 0, \\
(n\mathbf{u})_t + \mathrm{div}(n\mathbf{u}\otimes\mathbf{u}) + \nabla p(n,\theta) = n\nabla\omega - \dfrac{n\mathbf{u}}{\tau_p}, \\
\theta_t + \mathbf{u}\nabla\theta + \dfrac{2}{3}\theta\mathrm{div}\mathbf{u} = \dfrac{2}{3}\dfrac{\kappa}{n}\Delta\theta + \dfrac{2\tau_w-\tau_p}{3\tau_w\tau_p}|\mathbf{u}|^2 - \dfrac{\theta-\theta^0}{\tau_w}, \\
\Delta\omega = n - b(x),
\end{cases}
\tag{1}
$$

for $(x,t)\in\mathbb{R}^m\times\mathbb{R}_+$, with the initial data

$$
\begin{cases}
n(x,0) = n_0(x) \to n_\pm, \\
\mathbf{u}(x,0) = \mathbf{u}_0(x) \to \mathbf{u}_\pm := (u_\pm, 0, \cdots, 0), \quad \text{as } x_1\to\pm\infty, \\
\theta(x,0) = \theta_0(x) \to \theta_\pm,
\end{cases}
\tag{2}
$$

and the boundary condition at far field

$$
\lim_{x_1\to-\infty}\nabla\omega(x,t) = \lim_{x_1\to-\infty}(\partial_{x_1}\omega, \partial_{x_2}\omega, \cdots, \partial_{x_m}\omega) = \mathbf{E}_- := (E_-, 0, \cdots, 0). \tag{3}
$$

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Such a system describes the nonisentropic unipolar hydrodynamical model of semiconductor device, see [17, 28, 39] for details on the applications in semiconductors and in plasma physics. Here $n = n(x,t)$, $\mathbf{u} = (u_1, \cdots, u_m)(x,t)$, $\theta = \theta(x,t)$ and $\omega = \omega(x,t)$ represent the electron density, the electron velocity, the temperature and the electrostatic potential, respectively. The coefficients τ_p, τ_w and κ are positive constants, which represent the momentum relaxation time, the energy relaxation time and the thermal conductivity, respectively. Without loss of generality, we assume $\tau_p = 1$ and $\tau_w = 1$ throughout this paper. The positive constant θ^0 is the lattice temperature of semiconductor device. $n_\pm$, $\mathbf{u}_\pm$, $\theta_\pm$ and $\mathbf{E}_-$ are the initial constant states for the electron density $n(x,0)$, the electron velocity $\mathbf{u}(x,0)$, the temperature $\theta(x,0)$ and the electric field $\nabla\omega(x,0)$ as $x \to \pm\infty$. The function $b(x) > 0$ stands for the density of fixed, positively charged background ions. $p = p(n,\theta)$ is the pressure function, and physically it is given by

$$p(n,\theta) = n\theta. \tag{4}$$

Notice that, we assign the unknown function ω with the Neumann boundary condition (3), rather than with an initial value condition $\omega(x,0)$, because ω satisfies the Poisson equation $(1)_4$. Later on, we will give a more detailed explanation that the boundary condition (3) is necessary and natural. Otherwise, the system will be ill-posed.

The main purpose of this paper is to investigate the large-time behavior of solutions to the system (1) equipped with the initial data (2) and the far-field boundary condition (3). We expect that the solutions of the Euler-Poisson system (1)-(3) converge to their corresponding planar stationary waves, the stationary solutions to the corresponding steady-state equations, even if the system is in the switch-on case, namely, the state constants on the current and the electric field are non-zero.

Let us first investigate the 1-D system of (1) and its corresponding steady-state equations. For sake of simplification, we denote

$$J := nu \quad \text{and} \quad E := \omega_{x_1} \tag{5}$$

as the current for the electrons, and the electric field in 1-D system of (1), respectively. Thus, the 1-D nonisentropic Euler-Poisson system of (1) is written as follows

$$\begin{cases} n_t + J_{x_1} = 0, \\ J_t + \Big(\dfrac{J^2}{n} + p(n,\theta)\Big)_{x_1} = nE - J, \\ \theta_t + \dfrac{J}{n}\theta_{x_1} + \dfrac{2}{3}\Big(\dfrac{J}{n}\Big)_{x_1}\theta = \dfrac{2}{3}\dfrac{\kappa}{n}\theta_{x_1x_1} + \dfrac{1}{3}\dfrac{J^2}{n^2} - (\theta - \theta^0), \\ E_{x_1} = n - b(x_1), \end{cases} \tag{6}$$

equipped with the initial conditions

$$\begin{cases} n(x_1,0) = n_0(x_1) \to n_\pm > 0, \\ J(x_1,0) = J_0(x_1) \to J_\pm, \qquad \text{as } x_1 \to \pm\infty, \\ \theta(x_1,0) = \theta_0(x_1) \to \theta_\pm, \end{cases} \tag{7}$$

and the boundary condition at far field $x_1 = -\infty$

$$\lim_{x_1 \to -\infty} E(x_1,t) = E_-. \tag{8}$$

It must be pointed out that, the boundary condition (8), or replacing it by

$$\lim_{x_1 \to +\infty} E(x_1, t) = E_+, \tag{9}$$

is necessary and natural. Otherwise, as we analyze below in Section 3 for the behavior of the solutions $(n, J, \theta, E)(x_1, t)$ at the far fields $x = \pm\infty$, the functions $J(\pm\infty, t)$ and $E(\pm\infty, t)$ will be underdetermined, which will cause the system to be ill-posed.

Notice also that the initial value of $E(x_1, 0)$ is not necessary to be assigned, but will be automatically determined by the system (6). In fact, integrating $(6)_4$ with respect to x_1 over $(-\infty, x_1]$ and applying (8), we have

$$E(x_1, t) = E_- + \int_{-\infty}^{x_1} [n(y_1, t) - b(y_1)] dy_1.$$

So, the initial data for $E(x_1, t)$ is given by

$$E|_{t=0} = E_- + \int_{-\infty}^{x_1} [n_0(y_1) - b(y_1)] dy_1 =: E_0(x_1). \tag{10}$$

For 1-D nonisentropic Euler-Poisson equations (6), the corresponding steady-state equations are

$$\begin{cases} \tilde{J} = \text{constant}, \\ \Big(\dfrac{\tilde{J}^2}{\tilde{n}} + p(\tilde{n}, \tilde{\theta})\Big)_{x_1} = \tilde{n}\tilde{E} - \tilde{J}, \\ \dfrac{\tilde{J}}{\tilde{n}} \tilde{\theta}_{x_1} + \dfrac{2}{3}\Big(\dfrac{\tilde{J}}{\tilde{n}}\Big)_{x_1} \tilde{\theta} = \dfrac{2}{3}\dfrac{\kappa}{\tilde{n}} \tilde{\theta}_{x_1 x_1} + \dfrac{1}{3}\dfrac{\tilde{J}^2}{\tilde{n}^2} - (\tilde{\theta} - \theta^0), \\ \tilde{E}_{x_1} = \tilde{n} - b(x_1), \\ (\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x_1) \to (n_\pm, \tilde{J}, \tilde{\theta}_\pm, \tilde{E}_\pm), \text{ as } x_1 \to \pm\infty, \end{cases} \tag{11}$$

with

$$\tilde{J} := n_- E_-, \quad \tilde{E}_- := E_-, \quad \tilde{E}_+ := \frac{n_- E_-}{n_+}, \quad \tilde{\theta}_\pm := \theta^0 + \frac{\tilde{J}^2}{3 n_\pm^2}. \tag{12}$$

Such steady-state solutions $(\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x_1)$, the so-called *stationary waves* to (6), can be similarly proved to uniquely exist by the energy method and the fixed point theorem as in [2, 3, 4, 22, 27, 42] in the case of subsonic flow. The steady-state solutions $(\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x \cdot \mathbf{e})$ is called the *planar stationary waves* for the m-D full Euler-Poisson system (1), where $\mathbf{e}$ is the basis of the space $\mathbb{R}^m$. Without loss of generality (by rotating the coordinates), we may take $\mathbf{e} = \mathbf{e}_1 = (1, 0, \cdots, 0)$. So, in this case, the planar stationary waves becomes the 1-D steady-state solutions of (11) and (12): $(\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x \cdot \mathbf{e}_1) = (\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x_1)$.

When the state constants for the initial electric current are zero, namely,

$$J_- = J_+ = 0, \quad \text{equivalently,} \quad \int_{-\infty}^{\infty} [n_0(x_1) - \tilde{n}(x_1)] dx_1 = 0, \tag{13}$$

it physically stands for the switch-off case (no electric current). In this case, the solution $(n, J, \theta, E)(x, t)$ for the nonisentropic unipolar hydrodynamic model of semiconductor (1) was shown to converge time-asymptotically to the stationary wave $(\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x_1)$ by Ali, Bini and Rionero [2], and Zhu and Hattori [42] in 1-D case, and by Ali [1] in m-D case, respectively.

However, when the state constants of the initial electric current are non-zero, namely,

$$J_+ \neq J_+, \quad \text{or equivalently,} \quad \int_{\mathbb{R}} [n_0(x_1) - \tilde{n}(x_1)] dx_1 \neq 0, \tag{14}$$

it is the physical switch-on case, and this case is more interesting and practical but more difficult and challenging. The stability of (planar) stationary waves in this case remains open so far, because, as we heuristically analyze later in Section 3, there are some L^2-gaps between the original solutions and the corresponding stationary waves. Precisely saying, the original 1-D solutions of (6) at the far fields behave as

$$\begin{aligned}
(n, J, \theta, E)|_{x_1=-\infty} &= (n_-,\ \tilde{J} + O(1)e^{-\nu_0 t},\ \tilde{\theta}_- + O(1)e^{-\nu_0 t},\ \tilde{E}_-),\\
(n, J, \theta, E)|_{x_1=+\infty} &= (n_+,\ \tilde{J} + O(1)e^{-\nu_0 t},\ \tilde{\theta}_+ + O(1)e^{-\nu_0 t},\ \tilde{E}_+ + O(1)e^{-\nu_0 t})
\end{aligned}$$

for some $\nu_0 > 0$, and the steady-state solutions at the far fields are

$$(\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(\pm\infty) = (n_\pm, \tilde{J}, \tilde{\theta}_\pm, \tilde{E}_\pm),$$

where $\tilde{J}$, $\tilde{\theta}_\pm$ and $\tilde{E}_\pm$ are defined in (12). So there are some gaps between the original 1-D solutions and the stationary waves at the far fields

$$(J - \tilde{J}, \theta - \tilde{\theta})|_{x=-\infty} \neq (0, 0), \ \text{ and } \ (J - \tilde{J}, \theta - \tilde{\theta}, E - \tilde{E})|_{x=+\infty} \neq (0, 0, 0).$$

These imply that the perturbations of 1-D solutions around the corresponding steady-state solutions are not in L^2-space:

$$J(x_1, t) - \tilde{J}(x_1) \notin L^2(\mathbb{R}),\ E(x_1, t) - \tilde{E}(x_1) \notin L^2(\mathbb{R}),\ \text{and}\ \theta(x_1, t) - \tilde{\theta}(x_1) \notin L^2(\mathbb{R}).$$

To delete these gaps, the correction functions usually need to be introduced. However, the technique for constructing the correction functions introduced first by Hsiao and Liu [11] for the linear damping case cannot be applied to our case anymore due to the complicated nonlinearity of the system (6). In order to overcome such a difficulty, inspired by our recent works [13, 14, 15, 16] for the unipolar/bipolar semiconductor models (indeed, the unipolar model case considered in [15, 16] can be regarded as a special application of this work), we first make a heuristic analysis on the solutions at far fields and see what the exact gaps in L^2-space are between the 1-D isentropic solutions and the corresponding stationary waves at far filed $x_1 = \pm\infty$, then we try to ingeniously construct some correction functions to delete those L^2-gaps (this is a crucial step), finally we prove the stability of stationary waves by using the technical energy method. More precisely, when the perturbations around the stationary waves with a suitable setting up are small enough, we prove that the solutions of (6) converge exponentially to the corresponding stationary waves in the form

$$\|(n - \tilde{n}, J - \tilde{J}, E - \tilde{E}, \theta - \tilde{\theta})(t)\|_{L^\infty} = O(1)e^{-\mu t}, \tag{15}$$

for some constant $\mu > 0$.

Furthermore, for multi-dimensional (1), we will first prove that, the m-D solutions of (1) converge to the 1-D solutions of (6) when the corresponding initial perturbations are small enough, then based on the crucial convergence result of the 1-D solutions of (6) to the stationary waves, we will further show that the m-D solutions of (6) converge also to the corresponding planar stationary waves time-exponentially. This is our second goal in this paper.

Notice that, the authors [42] claimed that, in the switch-off case (13), the 1-D solutions for the full Euler-Poisson system converge to their corresponding stationary waves; while in the switch-on case (14), the stationary waves are unstable, and the 1-D solutions for full Euler-Poisson system will converge to the other asymptotic profile. However, this is not completely true. Because, although there are some L^2-gaps between the 1-D solutions and the corresponding steady-state solutions, which seems to lead that the 1-D solutions $(n, J, \theta, E)(x_1, t)$ don't converge to the stationary waves $(\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x_1)$ in L^2-sense, but the 1-D solutions of (6) will still converge to the stationary waves in L^∞-sense. For details, see Theorem 3.2 and Corollary 1 below.

Regarding other significant studies on the unipolar or bipolar hydrodynamic system for semiconductor devices, we refer to [3]-[30] and [35]-[41] and the references therein.

This paper related to the nonisentropic full Euler-Poisson system (1) can be regarded as an extension of our recent studies [13, 14, 15, 16] on the stability of stationary/diffusion waves for the unipolar/bipolar hydrodynamic models of semiconductors. The new technique originally introduced in [13] then developed in [15] for constructing the correction functions is applied in this paper, and also plays a key role in the proof of stationary wave stability. The rest of this paper is arranged as follows. In Section 2, we give some well-known results on the stationary solutions. Section 3 is devoted to the convergence to the stationary waves in 1-D case. In Section 4, the main afford is to prove the convergence to the planar stationary waves in m-D case. The crucial step is to prove the convergence of the m-D solutions to the 1-D solutions, when the initial perturbations are small enough. Then based on the convergence result obtained in Section 3, we can immediately gain the stability of planar stationary waves in m-D case.

Notation. Through out this paper, the stationary waves are denoted by $(\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x)$, and the correction functions are denoted by $(\hat{n}, \hat{J}, \hat{\theta}, \hat{E})(x, t)$. C_0, C_i, *et al* always denote some specific positive constants, and C denotes the generic positive constant. $L^2(\mathbb{R}^m)$ is the space of square integrable real valued function defined on $\mathbb{R}^m$ with the norm $\|\cdot\|$, and $H^k(\mathbb{R}^m)$ (H^k without any ambiguity) denotes the usual Sobolev space with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$.

2. **Stationary waves.**

In this section, we are going to introduce the well-known results on the stationary solutions to the corresponding steady-state equations of (6), the so-called nonlinear stationary waves:

$$\begin{cases} \tilde{J} = \text{constant}, \\ \Big(\dfrac{\tilde{J}^2}{\tilde{n}} + p(\tilde{n}, \tilde{\theta})\Big)_{x_1} = \tilde{n}\tilde{E} - \tilde{J}, \\ \dfrac{\tilde{J}}{\tilde{n}}\tilde{\theta}_{x_1} + \dfrac{2}{3}\Big(\dfrac{\tilde{J}}{\tilde{n}}\Big)_{x_1}\tilde{\theta} = \dfrac{2}{3}\dfrac{\kappa}{\tilde{n}}\tilde{\theta}_{x_1x_1} + \dfrac{1}{3}\dfrac{\tilde{J}^2}{\tilde{n}^2} - (\tilde{\theta} - \theta^0), \\ \tilde{E}_{x_1} = \tilde{n} - b(x_1), \end{cases} \tag{16}$$

with

$$\lim_{x_1 \to -\infty} (\tilde{n}, \tilde{E})(x_1) = (n_-, E_-), \quad \lim_{x_1 \to +\infty} \tilde{n}(x_1) = n_+. \tag{17}$$

Notice that, $(16)_2$ is equivalent to

$$\Big(\Big(p'_n(\tilde{n}, \tilde{\theta}) - \frac{\tilde{J}^2}{\tilde{n}^2}\Big)\tilde{n}_{x_1}\Big)_{x_1} = \tilde{n}\tilde{E} - \tilde{J} - p'_\theta(\tilde{n}, \tilde{\theta})\tilde{\theta}_{x_1}.$$

In order to keep the uniform ellipticity of the above equation, we need

$$p'_n(\tilde{n}, \tilde{\theta}) - \frac{\tilde{J}^2}{\tilde{n}^2} > 0, \tag{18}$$

that is

$$|\tilde{u}| \text{ (velocity) } = \left|\frac{\tilde{J}}{\tilde{n}}\right| < \sqrt{p_n'(\tilde{n},\tilde{\theta})} =: c(\tilde{n},\tilde{\theta}) \text{ (the sound speed)}.$$

This means that the system under consideration in this paper is a full subsonic flow. Let

$$b_* = \inf_{x_1\in\mathbb{R}} b(x_1) > 0, \text{ and } b^* = \sup_{x_1\in\mathbb{R}} b(x_1) > 0, \tag{19}$$

and

$$\begin{cases} b(x_1) \in C^3(\mathbb{R}^m) \quad \text{and} \quad \lim_{x_1\to\pm\infty} b(x_1) = n_\pm, \\ \displaystyle\int_{-\infty}^0 [b(x_1)-n_-]^2 dx_1 + \int_0^\infty [b(x_1)-n_+]^2 dx_1 \le C_0, \end{cases} \tag{20}$$

for some positive constant C_0. Then we state the existence and uniqueness of the stationary wave for the steady-state equations (16) and (17) as follows.

Proposition 1 ([2, 42])**.** *Assume that $b'(x) \in L^1(\mathbb{R}) \cap H^4(\mathbb{R})$ and $b_*\sqrt{p_n'(b_*,\theta^0)} > |n_-E_-|$, then there exists $\delta_0 > 0$, such that if*

$$|E_-| \le \delta_0, |n_+ - n_-| \le \delta_0, \text{ and } \|b'\|_{L^1} + \|b'\|_{H^4} \le \delta_0, \tag{21}$$

then (16) exists a unique smooth solution $(\tilde{n},\tilde{J},\tilde{\theta},\tilde{E})(x)$ of (16) and (17), which satisfies

$$b_* \le \tilde{n} \le b^*, \tag{22}$$

$$\tilde{J} = n_-E_-, \tag{23}$$

$$\tilde{E}(+\infty) = \frac{n_-E_-}{n_+}, \tag{24}$$

$$\tilde{\theta}(\pm\infty) = \theta^0 + \frac{1}{3}\frac{\tilde{J}^2}{n_\pm^2} \tag{25}$$

$$\|\tilde{n} - b\|_{H^4}^2 \le C_1(\alpha_1 + |n_+ - n_-|), \tag{26}$$

$$|\tilde{E}| + \sum_{i=1}^4 |\partial_x^i \tilde{n}| + |\partial_x^i \tilde{E}| + |\partial_x^i \tilde{\theta}| \le C_2\alpha_2, \tag{27}$$

where $C_i = C_i(n_-,E_-,b_*,b^*) > 0$ and $\alpha_i = \alpha_i(\delta_0) > 0$ are some constants for $i = 1,2$, and α_i satisfies

$$\alpha_i = \alpha_i(\delta_0) \ll 1 \quad \textit{as} \quad \delta_0 \ll 1. \tag{28}$$

Remark 1. To guarantee the system to be subsonic, namely, to hold the condition (18) when $\delta_0 \ll 1$, we need the sufficient condition $b_*\sqrt{p_n'(b_*,\theta^0)} > |n_-E_-|$. Since $p(n,\theta) = n\theta$, then $b_*\sqrt{p_n'(b_*,\theta^0)} > |n_-E_-|$ is equivalent to $b_*\sqrt{\theta^0} > |n_-E_-|$.

3. **Stability of stationary waves in 1-D space.**

For sake of simplification, throughout this section, we still denote the 1-D spatial variable x_1 as $x \in \mathbb{R}$ without confusion.

In order to prove the stability of stationary waves, inspired by [13, 14, 15, 16] for semiconductor models, and initially by [34, 26] for hyperbolic p-system, we need to make a heuristic analysis on the behaviors of the solutions to (6)-(8) at the far fields

$x = \pm\infty$. Then we may understand how big the gaps are between the solutions and the stationary solutions at the far fields. Let

$$\begin{cases} n^{\pm}(t) := n(\pm\infty, t), \\ J^{\pm}(t) := J(\pm\infty, t), \\ \theta^{\pm}(t) := \theta(\pm\infty, t), \\ E^{\pm}(t) := E(\pm\infty, t). \end{cases} \tag{29}$$

From $(6)_1$, since $\partial_x J|_{x=\pm\infty} = 0$, it can be easily seen that

$$n^{\pm}(t) = n(\pm\infty, t) \equiv n_{\pm}. \tag{30}$$

Taking $x \to \pm\infty$ to $(6)_2$ and $(6)_3$, we get four ODEs, i.e.

$$\begin{cases} \dfrac{d}{dt} J^{\pm}(t) = n_{\pm} E^{\pm}(t) - J^{\pm}(t), \\ \dfrac{d}{dt}\theta^{\pm}(t) = \dfrac{1}{3}\dfrac{(J^{\pm}(t))^2}{n_{\pm}^2} - (\theta^{\pm}(t) - \theta^0), \end{cases} \tag{31}$$

and differentiating $(6)_4$ with respect to t and using $(6)_1$, we have

$$E_{xt} = (n - b(x))_t = n_t = -J_x.$$

Integrating it with respect to x over $(-\infty, +\infty)$, we then have

$$\frac{d}{dt} E^{+}(t) - \frac{d}{dt} E^{-}(t) = -J^{+}(t) + J^{-}(t). \tag{32}$$

Notice that, these five equations in (31) and (32) will under-determine six unknown functions $J^{\pm}(t), \theta^{\pm}(t)$ and $E^{\pm}(t)$, which will cause the system (6) to be ill-posed. Thus, naturally we need an extra boundary condition for $E(x, t)$ either at $x = -\infty$ or $x = +\infty$. This indicates the boundary condition (8) (or replacing it with (9)) is necessary and proper.

From (8), we have

$$E^{-}(t) = E_{-}. \tag{33}$$

Then, from $(6)_2$, $(6)_3$ and (33), we can easily get

$$J^{-}(t) = (J_{-} - n_{-}E_{-})e^{-t} + n_{-}E_{-}, \tag{34}$$

$$\begin{aligned} \theta^{-}(t) &= (\theta^0 + \frac{\tilde{J}^2}{3n_{-}^2}) + (\theta_{-} - \theta^0 - \frac{\tilde{J}^2}{3n_{-}^2})e^{-t} \\ &\quad - \frac{2\tilde{J}(J_{-} - \tilde{J})}{3n_{-}^2} te^{-t} + \frac{(J_{-} - \tilde{J})^2}{3n_{-}^2}(e^{-t} - e^{-2t}). \end{aligned} \tag{35}$$

From $(6)_2$, we further have

$$\begin{cases} \dfrac{d}{dt} J^{+}(t) = n_{+} E^{+}(t) - J^{+}(t), \\ J^{+}(0) = J_{+}, \end{cases} \tag{36}$$

and from $(6)_4$ and (8), we obtain

$$E^{+}(t) = \lim_{x \to +\infty} E(x, t) = \int_{\mathbb{R}} (n(x, t) - b(x))dx + E_{-}. \tag{37}$$

Differentiating (37) with respect to t, and using $(6)_1$ and (34), we obtain

$$\frac{d}{dt} E^{+}(t) = -\int_{\mathbb{R}} J_x(x, t)dx = -J^{+}(t) + (J_{-} - n_{-}E_{-})e^{-t} + n_{-}E_{-}. \tag{38}$$

From (37), we have

$$E^+(t)|_{t=0} = \int_{\mathbb{R}} (n_0 - b)(x)dx + E_- := E_+. \tag{39}$$

Combining (36), (38) and (39), we obtain

$$\begin{cases} \frac{d}{dt} J^+(t) = n_+ E^+(t) - J^+(t), \\ \frac{d}{dt} E^+(t) = -J^+(t) + (J_- - n_- E_-)e^{-t} + n_- E_-, \\ \frac{d}{dt} \theta^+(t) = \frac{1}{3} \frac{(J^+(t))^2}{n_+^2} - (\theta^+(t) - \theta^0), \\ J^+(0) = J_+, \\ E^+(0) = E_+. \end{cases} \tag{40}$$

We find that $(40)_1, (40)_2$ is a closed system for $J^+(t)$ and $E^+(t)$, so we solve $J^+(t)$ and $E^+(t)$ first, then we can easily obtain $\theta^+(t)$ from $(40)_3$.

Differentiating $(40)_1$ with respect t, we obtain

$$\frac{d^2}{dt^2} J^+(t) = n_+ \frac{d}{dt} E^+(t) - \frac{d}{dt} J^+(t), \tag{41}$$

and substituting $(40)_2$ into (41), then we can reach

$$\begin{cases} \frac{d^2}{dt^2} J^+(t) + \frac{d}{dt} J^+(t) + n_+ J^+(t) = n_+ n_- E_- + n_+ (J_- - n_- E_-)e^{-t}, \\ J^+(0) = J_+, \\ \frac{d}{dt} J^+(0) = n_+ E_+ - J_+. \end{cases} \tag{42}$$

Notice that, the eigenvalues of the second order ODE of (42) are

$$\lambda_1 = \frac{-1 - \sqrt{1 - 4n_+}}{2} \quad \text{and} \quad \lambda_2 = \frac{-1 + \sqrt{1 - 4n_+}}{2}. \tag{43}$$

Thus, according to the signs of $1 - 4n_+$, we can directly but tediously solve the equations (40) and (42) for $J^+(t)$ and $E^+(t)$ as follows.

Case 1: When $1 - 4n_+ > 0$, then

$$J^+(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + (J_- - n_- E_-)e^{-t} + n_- E_-, \tag{44}$$

$$E^+(t) = \frac{1}{n_+} \Big[A_1(1 + \lambda_1)e^{\lambda_1 t} + A_2(1 + \lambda_2)e^{\lambda_2 t} + n_- E_- \Big], \tag{45}$$

where

$$A_1 = J_+ - J_- - A_2, \tag{46}$$

$$A_2 = -\frac{1}{1 - 4n_+} \Big[(1 + \lambda_1)(J_+ - J_-) - n_+ E_+ + n_- E_- \Big]. \tag{47}$$

Case 2: When $1 - 4n_+ = 0$, then

$$J^+(t) = A_3 e^{-\frac{1}{2}t} + A_4 t e^{-\frac{1}{2}t} + (J_- - n_- E_-)e^{-t} + n_- E_-, \tag{48}$$

$$E^+(t) = \frac{1}{n_+} \Big[(A_4 + \frac{1}{2} A_3)e^{-\frac{1}{2}t} + \frac{1}{2} A_4 t e^{-\frac{1}{2}t} + n_- E_- \Big], \tag{49}$$

where

$$A_3 = J_+ - J_-, \tag{50}$$

$$A_4 = n_+ E_+ - n_- E_- - \frac{1}{2}(J_+ - J_-). \tag{51}$$

Case 3: When $1-4n_+<0$, then

$$J^+(t)=\Big(A_5\cos(\frac{\sqrt{4n_+-1}}{2}t)+A_6\sin(\frac{\sqrt{4n_+-1}}{2}t)\Big)e^{-\frac{1}{2}t}$$
$$+(J_- - n_-E_-)e^{-t}+n_-E_-, \tag{52}$$
$$E^+(t)=\frac{n_-E_-}{n_+}+\frac{1}{2n_+}\Big[(A_5+\sqrt{4n_+-1}A_6)\cos(\frac{\sqrt{4n_+-1}}{2}t)$$
$$+(A_6-\sqrt{4n_+-1}A_5)\sin(\frac{\sqrt{4n_+-1}}{2}t)\Big]e^{-\frac{1}{2}t}, \tag{53}$$

where

$$A_5=J_+-J_-, \tag{54}$$
$$A_6=\frac{2}{\sqrt{4n_+-1}}\Big(n_+E_+-n_-E_--\frac{1}{2}(J_+-J_-)\Big). \tag{55}$$

Using $(40)_3$ and (44)-(53), we can obtain

$$\theta^+(t) \;=\; \theta^0+\frac{\tilde{J}^2}{3n_+^2}+(\theta_+-\theta^0-\frac{\tilde{J}^2}{3n_+^2})e^{-t}+O(1)e^{-\mu t} \tag{56}$$

From (8), (30), (34), (44)-(55) and Proposition 1, we have

$$\begin{cases} |n(\pm\infty,t)-\tilde{n}(\pm\infty)|=0,\\ |J(+\infty,t)-\tilde{J}|=O(1)e^{-\mu t},\\ |J(-\infty,t)-\tilde{J}|=O(1)e^{-t},\\ |\theta(\pm\infty,t)-\tilde{\theta}(\pm\infty)|=O(1)e^{-\mu t},\\ E(-\infty,t)=E_-,\\ |E(+\infty,t)-\frac{n_-E_-}{n_+}|=O(1)e^{-\mu t}, \end{cases} \tag{57}$$

for some constant $0<\mu<\frac{1}{2}$.

From the above analysis, we find that there are some gaps between $J(\pm\infty,t)$ and $\tilde{J}=n_-E_-$, $E(+\infty,t)$ and $\tilde{E}(+\infty)=\frac{n_-E_-}{n_+}$, and $\theta(\pm\infty,t)$ and $\tilde{\theta}(\pm\infty)=\tilde{\theta}_\pm$, which lead

$$J(x,t)-\tilde{J},\ E(x,t)-\tilde{E}(x) \text{ and } \theta(x,t)-\tilde{\theta}(x)\notin L^2(\mathbb{R}). \tag{58}$$

To delete these gaps, we need to introduce the correction functions $(\hat{n},\hat{J},\hat{\theta},\hat{E})(x,t)$. After solving $(J^+,E^+)(t)$ from $(40)_1$ and $(40)_2$, we then can solve $(40)_3$ for $\theta^+(t)$. Inspired by this, we are going to construct the correction functions $(\hat{n},\hat{J},\hat{\theta},\hat{E})(x,t)$ in two steps. First of all, we construct $(\hat{n},\hat{J},\hat{E})(x,t)$ from the following system of equations, then we construct $\hat{\theta}(x,t)$ later.

Let $(\hat{n},\hat{J},\hat{E})(x,t)$ be the solutions to the following linear equations

$$\begin{cases} \hat{n}_t+\hat{J}_x=0,\\ \hat{J}_t=\breve{n}\hat{E}-\hat{J},\\ \hat{E}_x=\hat{n},\\ \hat{J}(x,t)\to J^\pm(t)-n_-E_- & \text{as } x\to\pm\infty,\\ \hat{E}(x,t)\to 0 & \text{as } x\to-\infty,\\ \hat{E}(x,t)\to E^+(t)-\frac{n_-E_-}{n_+} & \text{as } x\to+\infty. \end{cases} \tag{59}$$

In order to get $(\hat{n}, \hat{J}, \hat{E})(x,t)$ to (59), we consider the following linear system with some tricky selection on $\breve{n} = \breve{n}(x)$, $\hat{J}(x,t)$ and $\hat{E}(x,0)$

$$\begin{cases} \hat{J}_t(x,t) = \breve{n}(x)\hat{E}(x,t) - \hat{J}(x,t), \\ \hat{E}_t(x,t) = -\hat{J}(x,t) + (J_- - n_-E_-)e^{-t}, \\ \hat{J}(x,0) = (J^-(0) - n_-E_-) + (J^+ - J^-)(0)\displaystyle\int_{-\infty}^x m_0(y)dy, \\ \hat{E}(x,0) = (E^+(0) - \frac{n_-E_-}{n_+})\displaystyle\int_{-\infty}^x m_0(y)dy, \end{cases} \tag{60}$$

where $m_0(x)$ and $\breve{n}(x)$ are also ingeniously selected as

$$\begin{cases} m_0(x) \geq 0, \ m_0 \in C_0^\infty(\mathbb{R}), \ \text{supp } m_0 \subseteq [-L_0, L_0], \ \displaystyle\int_{\mathbb{R}} m_0(y)dy = 1 \\ \breve{n}(x) = n_- + (n_+ - n_-)\displaystyle\int_{-\infty}^{x+2L_0} m_0(y)dy \end{cases} \tag{61}$$

with some constant $L_0 > 0$.

When $x < -L_0$, we have $\hat{E}(x,0) \equiv 0$. So, it can be easily seen that (60) possesses the particular solutions

$$\hat{J}(x,t) = (J_- - n_-E_-)e^{-t}, \quad \hat{E}(x,t) = 0, \quad \text{for } -\infty < x < -L_0. \tag{62}$$

When $x \geq -L_0$, we have $\breve{n}(x) \equiv n_+$. Similarly to the previous but complicated calculation, we can solve (60) as the following. However, we can verify that these solutions imply also the solutions given in (62) for $x < -L_0$. Therefore, we summarize them as follows.

Case 1: When $1 - 4n_+ > 0$, then, for $x \in \mathbb{R}$

$$\hat{J}(x,t) = \left(A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}\right)\int_{-\infty}^x m_0(y)dy + (J_- - n_-E_-)e^{-t}, \tag{63}$$

$$\hat{E}(x,t) = \frac{1}{n_+}\left(A_1(1+\lambda_1)e^{\lambda_1 t} + A_2(1+\lambda_2)e^{\lambda_2 t}\right)\int_{-\infty}^x m_0(y)dy, \tag{64}$$

thus, we define

$$\hat{n}(x,t) = \frac{1}{n_+}\left(A_1(1+\lambda_1)e^{\lambda_1 t} + A_2(1+\lambda_2)e^{\lambda_2 t}\right)m_0(x). \tag{65}$$

Then we can verify that $(\hat{n}, \hat{J}, \hat{E})(x,t)$ satisfy (59) for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

Case 2: When $1 - 4n_+ = 0$, then, for $x \in \mathbb{R}$

$$\hat{J}(x,t) = \left(A_3e^{-\frac{1}{2}t} + A_4te^{-\frac{1}{2}t}\right)\int_{-\infty}^x m_0(y)dy + (J_- - n_-E_-)e^{-t}, \tag{66}$$

$$\hat{E}(x,t) = \frac{1}{n_+}\left((A_4 + \frac{1}{2}A_3)e^{-\frac{1}{2}t} + \frac{1}{2}A_4te^{-\frac{1}{2}t}\right)\int_{-\infty}^x m_0(y)dy, \tag{67}$$

thus, we define

$$\hat{n}(x,t) = \frac{1}{n_+}\left((A_4 + \frac{1}{2}A_3)e^{-\frac{1}{2}t} + \frac{1}{2}A_4te^{-\frac{1}{2}t}\right)m_0(x). \tag{68}$$

Then we can verify that $(\hat{n}, \hat{J}, \hat{E})(x,t)$ satisfy (59) for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

Case 3: When $1-4n_+<0$, then, for $x\in\mathbb{R}$

$$\begin{aligned}\hat{J}(x,t) &= \Big(A_5\cos(\frac{\sqrt{4n_+-1}}{2}t)+A_6\sin(\frac{\sqrt{4n_+-1}}{2}t)\Big)e^{-\frac{1}{2}t}\int_{-\infty}^{x}m_0(y)dy\\ &\quad+(J_- -n_-E_-)e^{-t}, \qquad (69)\\ \hat{E}(x,t) &= \frac{1}{2n_+}\Big((A_5+\sqrt{4n_+-1}A_6)\cos(\frac{\sqrt{4n_+-1}}{2}t)\\ &\quad+(A_6-\sqrt{4n_+-1}A_5)\sin(\frac{\sqrt{4n_+-1}}{2}t)\Big)e^{-\frac{1}{2}t}\int_{-\infty}^{x}m_0(y)dy, \qquad (70)\end{aligned}$$

thus, we define

$$\begin{aligned}\hat{n}(x,t) &= \frac{1}{2n_+}\Big((A_5+\sqrt{4n_+-1}A_6)\cos(\frac{\sqrt{4n_+-1}}{2}t)\\ &\quad+(A_6-\sqrt{4n_+-1}A_5)\sin(\frac{\sqrt{4n_+-1}}{2}t)\Big)e^{-\frac{1}{2}t}m_0(x). \qquad (71)\end{aligned}$$

Then we can verify that $(\hat{n},\hat{J},\hat{E})(x,t)$ satisfy (59) for $(x,t)\in\mathbb{R}\times\mathbb{R}^+$.

Now we are going to construct the correction function $\hat{\theta}(x,t)$. In order to delete the gap $\theta(x,t)-\tilde{\theta}(x)$, we denote

$$\hat{\theta}^{\pm}(t)=\theta^{\pm}(t)-\tilde{\theta}(\pm\infty), \tag{72}$$

and from $(6)_3$ and $(11)_3$, it satisfy

$$\begin{cases}\dfrac{d\hat{\theta}^{\pm}(t)}{dt}=\Big(\dfrac{(J^{\pm}(t))^2}{3n_{\pm}^2}-\dfrac{\tilde{J}^2}{3n_{\pm}^2}\Big)-\hat{\theta}^{\pm}(t),\\ \hat{\theta}^{\pm}(t)|_{t=0}=\theta_{\pm}-\tilde{\theta}(\pm\infty)=\theta_{\pm}-\theta^0-\frac{\tilde{J}^2}{3n_{\pm}^2}.\end{cases} \tag{73}$$

Let

$$\hat{\theta}(x,t)=\hat{\theta}_-(t)\Big(1-\int_{-\infty}^{x}m_0(y)dy\Big)+\hat{\theta}_+(t)\int_{-\infty}^{x}m_0(y)dy, \tag{74}$$

then $\hat{\theta}(x,t)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\theta}^{\pm}(x,t) &= -\hat{\theta}^{\pm}(x,t)+\Big(1-\int_{-\infty}^{x}m_0(y)dy\Big)\Big(\frac{(J^-(t))^2}{3n_-^2}-\frac{\tilde{J}^2}{3n_-^2}\Big)\\ &\quad+\Big(\frac{(J^+(t))^2}{3n_+^2}-\frac{\tilde{J}^2}{3n_+^2}\Big)\int_{-\infty}^{x}m_0(y)dy\\ &=: -\hat{\theta}^{\pm}(x,t)+h(x,t). \qquad (75)\end{aligned}$$

From (68)-(71) and (74), we realize that these correction functions decay time-exponentially as follow.

Lemma 3.1. *There hold*

$$\|(\hat{n},\hat{J},\hat{\theta},\hat{E})(t)\|_{L^\infty(\mathbb{R})}\le C\sigma e^{-\nu_0 t} \tag{76}$$

and

$$\text{supp }\hat{n}=\text{supp }m_0\subseteq[-L_0,L_0] \tag{77}$$

for $\sigma:=|J_+|+|J_-|+|E_-|+|E_+|+|\theta_--\theta^0|+|\theta_+-\theta^0|$ *and* $0<\nu_0<\frac{1}{2}$.

Now we are going to make a perturbation of (6) to the steady equations (16). Noticing (6), (16), (59) and (75), we have

$$\begin{cases} (n-\hat{n}-\tilde{n})_t + (J-\hat{J}-\tilde{J})_x = 0, \\ (J-\hat{J}-\tilde{J})_t + (p(n,\theta)-p(\tilde{n},\tilde{\theta}))_x \\ \qquad\qquad = nE - \tilde{n}\tilde{E} - \breve{n}\hat{E} - (J-\hat{J}-\tilde{J}) - \Big(\dfrac{J^2}{n} - \dfrac{\tilde{J}^2}{\tilde{n}}\Big)_x, \\ (\theta-\tilde{\theta}-\hat{\theta})_t + (\dfrac{J}{n}\theta_x - \dfrac{\tilde{J}}{\tilde{n}}\tilde{\theta}_x) + \dfrac{2}{3}[(\dfrac{J}{n})_x\theta - (\dfrac{\tilde{J}}{\tilde{n}})_x\tilde{\theta}] \\ \qquad\qquad = \dfrac{2\kappa}{3}(\dfrac{\theta_{xx}}{n} - \dfrac{\tilde{\theta}_{xx}}{\tilde{n}}) + \Big(\dfrac{J^2}{3n^2} - \dfrac{\tilde{J}^2}{3\tilde{n}^2} - h(x,t)\Big) - (\theta-\tilde{\theta}-\hat{\theta}), \\ (E-\hat{E}-\tilde{E})_x = n - \hat{n} - \tilde{n}, \end{cases} \tag{78}$$

with zero-perturbations (the gaps are removed by the constructed correction functions)

$$\begin{cases} n(\pm\infty,t) - \hat{n}(\pm\infty,t) - \tilde{n}(\pm\infty) = 0, \\ J(\pm\infty,t) - \hat{J}(\pm\infty,t) - \tilde{J}(\pm\infty) = 0, \\ \theta(\pm\infty,t) - \hat{\theta}(\pm\infty,t) - \tilde{\theta}(\pm\infty) = 0, \\ E(\pm\infty,t) - \hat{E}(\pm\infty,t) - \tilde{E}(\pm\infty) = 0, \\ \displaystyle\int_{\mathbb{R}} [n(x,t) - \hat{n}(x,t) - \tilde{n}(x)]dx = \int_{\mathbb{R}} [n_0(x) - \hat{n}(x,0) - \tilde{n}(x)]dx = 0. \end{cases} \tag{79}$$

Thus, let

$$\begin{cases} \phi := n - \hat{n} - \tilde{n}, \\ \psi := J - \hat{J} - \tilde{J}, \\ \zeta := \theta - \hat{\theta} - \tilde{\theta}, \\ e := E - \hat{E} - \tilde{E}, \end{cases} \tag{80}$$

namely,

$$\begin{cases} e_x = n - \hat{n} - \tilde{n} = \phi, \\ -e_t = J - \hat{J} - \tilde{J} = \psi, \end{cases} \tag{81}$$

we deduce (78) into

$$\begin{cases} e_{tt} + e_t - (\tilde{\theta}e_x + \tilde{n}\zeta)_x + \tilde{n}e = R_{1x} - R_2 + R_{3x}, \\ \zeta_t + \zeta = \dfrac{2\kappa}{3\tilde{n}}\zeta_{xx} + R_4 + R_5 + R_6 + R_7, \end{cases} \tag{82}$$

where

$$\begin{cases} R_1 = e_x\zeta + e_x\hat{\theta} + \hat{n}\theta + \tilde{n}\hat{\theta}, \\ R_2 = (e_x + \hat{n})(e + \tilde{E} + \hat{E}) + (\tilde{n} - \breve{n})\hat{E}, \\ R_3 = \dfrac{J^2}{n} - \dfrac{\tilde{J}^2}{\tilde{n}}, \\ R_4 = \dfrac{2\kappa}{3}[\frac{\hat{\theta}_{xx}}{\tilde{n}} - \dfrac{e_x + \hat{n}}{n\tilde{n}}(\zeta_{xx} + \tilde{\theta}_{xx} + \hat{\theta}_{xx})], \\ R_5 = -(\dfrac{J}{n}\theta_x - \dfrac{\tilde{J}}{\tilde{n}}\tilde{\theta}_x), \\ R_6 = -\dfrac{2}{3}[(\dfrac{J}{n})_x\theta - (\frac{\tilde{J}}{\tilde{n}})_x\tilde{\theta}], \\ R_7 = \dfrac{J^2}{3n^2} - \dfrac{\tilde{J}^2}{\tilde{3}n^2} - h(x,t), \end{cases} \tag{83}$$

with initial data

$$\begin{cases} e(x,0) = E_0(x) - \hat{E}(x,0) - \tilde{E}(x), \\ e_x(x,0) = n_0(x) - \hat{n}(x,0) - \tilde{n}(x), \\ e_t(x,0) = -J_0(x) + \hat{J}(x,0) + \tilde{J}(x), \\ \zeta(x,0) = \theta_0(x) - \tilde{\theta}(x,0) - \hat{\theta}(x,0), \end{cases} \tag{84}$$

where $E_0(x)$ is defined by

$$E_0(x) = \int_{-\infty}^{x} (n_0(y) - b(y))dy + E_-. \tag{85}$$

Theorem 3.2. *Let $\delta := |n_+ - n_-| + |J_+| + |J_-| + |E_-| + |E_+| + |\theta_+ - \theta^0| + |\theta_- - \theta^0| + \alpha_1 + \alpha_2$, $\Phi_0 := \|e(0)\|_{H^3(\mathbb{R})} + \|(e_t,\zeta)(0)\|_{H^2(\mathbb{R})}$, then, there is a $\tilde{\delta}_0 > 0$ such that when $\delta + \Phi_0 < \tilde{\delta}_0$, the solutions (n, J, θ, E) of IVP (6) and (7) are unique and globally exist, and satisfy*

$$\|e(t)\|_{H^3(\mathbb{R})} + \|(e_t,\zeta)(t)\|_{H^2(\mathbb{R})} \le C(\delta + \Phi_0)^{\frac{1}{2}} e^{-\nu t} \tag{86}$$

where ν is a positive constant.

From (80) and (81), applying Sobolev inequality $H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$ to (86) in Theorem 3.2, we have

$$\|(n - \hat{n} - \tilde{n}, J - \hat{J} - \tilde{J}, \theta - \hat{\theta} - \tilde{\theta}, E - \hat{E} - \tilde{E})(t)\|_{L^\infty(\mathbb{R})} \le C(\delta + \Phi_0)^{\frac{1}{2}} e^{-\nu t}.$$

Thanks also to (76) in Lemma 3.1, namely, $\|(\hat{n}, \hat{J}, \hat{\theta}, \hat{E})(t)\|_{L^\infty(\mathbb{R})} \le C\sigma e^{-\nu_0 t}$, we then immediately obtain the convergence of 1-D solution $(n, J, \theta, E)(x, t)$ to its corresponding steady-state solution $(\tilde{n}, \tilde{J}, \tilde{\theta}, \tilde{E})(x)$ as follows.

Corollary 1 (Stability of Stationary Waves)**.** *Under the conditions of Theorem 3.2, we have*

$$\begin{cases} \|(n - \tilde{n})(t)\|_{L^\infty(\mathbb{R})} \le C(\delta + \Phi_0)^{\frac{1}{2}} e^{-\hat{\mu} t}, \\ \|(J - \tilde{J})(t)\|_{L^\infty(\mathbb{R})} \le C(\delta + \Phi_0)^{\frac{1}{2}} e^{-\hat{\mu} t}, \\ \|(\theta - \tilde{\theta})(t)\|_{L^\infty(\mathbb{R})} \le C(\delta + \Phi_0)^{\frac{1}{2}} e^{-\hat{\mu} t}, \\ \|(E - \tilde{E})(t)\|_{L^\infty(\mathbb{R})} \le C(\delta + \Phi_0)^{\frac{1}{2}} e^{-\hat{\mu} t}, \end{cases} \tag{87}$$

where $\hat{\mu} = \min\{\nu, \nu_0\} > 0$.

Remark 2. Our stability of stationary waves holds for the more general case (switch-on case), which significantly improves and develops the existing stability results [1, 2, 42] in the switch-off case.

3.1. **A priori estimates.** It is known that Theorem 3.2 can be proved by the classical energy method with the continuation argument based on the local existence and the *a priori* estimates (c.f. [31, 32, 33]). Since the local existence of the solutions of (82), (84) can be proved in the standard iteration method together with the energy estimates, so the main effort in this subsection is to establish the *a priori* estimates for the solutions, which is usually technical and crucial in the proof of stability.

Let $T \in (0, +\infty]$, we define the solution space, for $0 \le t \le T$,

$$X(T) = \left\{ (e,\zeta)(x,t) \middle| \partial_t^j e \in C(0,T; H^{3-j}(\mathbb{R})), \zeta \in C(0,T; H^2(\mathbb{R})), j = 0, 1, \right\}$$

with the norm

$$N(T)^2 := \sup_{0 \le t \le T} \left\{ \|e(t)\|^2_{H^3(\mathbb{R})} + \|(e_t,\zeta)(t)\|^2_{H^2(\mathbb{R})} \right\}. \tag{88}$$

Let $N(T)^2 \leq \varepsilon^2$, where ε is sufficiently small which will be determined later. It is noted that, (88) with Sobolev inequality $\|\partial_x^k f\|_{L^\infty(\mathbb{R})} \leq C\|\partial_x^k f\|^{1/2}\|\partial_x^{k+1} f\|^{1/2}$ gives

$$\sum_{k=0}^{2} \|\partial_x^k e(t)\|_{L^\infty(\mathbb{R})} + \sum_{k=0}^{1} \|\partial_x^k (e_t, \zeta)(t)\|_{L^\infty(\mathbb{R})} \leq C\varepsilon. \tag{89}$$

It is easy to verify from (81) and the conditions of Theorem 3.2 that, there exists a positive constant c such that

$$0 < \frac{1}{c} \leq n = e_x + \hat{n} + \tilde{n} \leq c. \tag{90}$$

Now we are going to establish the *a priori* estimates.

Lemma 3.3. *Let μ_1 be constant and suitably large, then it holds that*

$$\frac{d}{dt}F_1(t) + C_1 F_1(t) + C_1\|\zeta_x(t)\|^2 \leq C(\delta+\varepsilon)\|(e_{xx}, e_{xt})(t)\|^2 + C\delta e^{-\nu_0 t}, \tag{91}$$

where

$$\begin{cases} F_1(t) = \dfrac{d}{dt}\displaystyle\int_{\mathbb{R}} \left(e e_t + \frac{1}{2}e^2 + \frac{\mu_1}{2}(e_t^2 + \tilde{n}e^2 + \tilde{\theta}e_x^2) + \frac{3\mu_1 \tilde{n}^2}{4\tilde{\theta}}\zeta^2 \right) dx \\ C_{11}\|(\zeta, e, e_x, e_t)(t)\|^2 \leq F_1(t) \leq C_{12}\|(\zeta, e, e_x, e_t)(t)\|^2 \end{cases} \tag{92}$$

provided $\varepsilon + \delta \ll 1$.

Proof. Multiplying (82) by e and integrate it over $(-\infty, +\infty)$, we obtains

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \left(\frac{1}{2}e^2 + e_t e\right) dx + \int_{\mathbb{R}} \tilde{n}e^2 dx + \int_{\mathbb{R}} \tilde{\theta}e_x^2 dx \\ &= \|e_t\|^2 - \int_{\mathbb{R}} \tilde{n}e_x\zeta dx + \int_{\mathbb{R}} (R_{1x} - R_2 + R_{3x}) e dx. \end{aligned} \tag{93}$$

It is easily to obtain

$$-\int_{\mathbb{R}} \tilde{n}e_x\zeta dx \leq \frac{1}{4}\int_{\mathbb{R}} \tilde{\theta}e_x^2 dx + C\|\zeta(t)\|^2. \tag{94}$$

Noticing Proposition 1 and Lemma 3.1, then by Cauchy's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}} R_{1x} e dx &= \int_{\mathbb{R}} (e_x\zeta + e_x\hat{\theta} + \hat{n}\theta + \tilde{n}\hat{\theta})_x e dx \\ &= -\int_{\mathbb{R}} (e_x\zeta + e_x\hat{\theta} + \hat{n}\theta)e_x dx + \int_{\mathbb{R}} (\tilde{n}_x\hat{\theta} + \tilde{n}\hat{\theta}_x) e dx \\ &\leq C(\delta+\varepsilon)\|(e, e_x)(t)\|^2 + C\delta e^{-\nu_0 t}. \end{aligned} \tag{95}$$

From (20) and Proposition 1, one can prove

$$\int_{\mathbb{R}} (\tilde{n} - \breve{n})^2 dx \leq C. \tag{96}$$

Thus using (96), one then has

$$\begin{aligned} -\int_{\mathbb{R}} R_2 e dx &= -\int_{\mathbb{R}} \left((e_x + \hat{n})(e + \tilde{E} + \hat{E}) + (\tilde{n} - \breve{n})\hat{E} \right) e dx \\ &\leq C(\delta+\varepsilon)\|(e, e_x)(t)\|^2 + C\delta e^{-\nu_0 t}, \end{aligned} \tag{97}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} R_{3x} e dx &= -\int_{\mathbb{R}} \left(\frac{J^2}{n} - \frac{\tilde{J}^2}{\tilde{n}} \right) e_x dx \\
&= -\int_{\mathbb{R}} \left(\frac{e_t^2 - 2e_t(\tilde{J} + \hat{J})}{n} + \frac{2\tilde{J}\hat{J} + \hat{J}^2}{n} - \tilde{J}^2 (\frac{1}{n} - \frac{1}{\tilde{n}}) \right) e_x dx \\
&\le C(\delta + \varepsilon) \|(e_t, e_x)(t)\|^2 - \int_{\mathbb{R}} \left(\frac{2\tilde{J}\hat{J} + \hat{J}^2}{n} \right)_x e dx + C\delta e^{-\nu_0 t} \\
&\le C(\delta + \varepsilon) \|(e, e_t, e_x)(t)\|^2 + C\delta e^{-\nu_0 t}. \qquad (98)
\end{aligned}$$

Substituting (95), (97), (94) and (98) into (161), one finally obtains

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} \left(e e_t + \frac{1}{2} e^2 \right) dx + 2C_0 \|(e, e_x)\|^2 \\
&\le \|e_t(t)\|^2 + C\|\zeta(t)\|^2 + C(\delta + \varepsilon)\|(e, e_t, e_x)(t)\|^2 + C\delta e^{-\nu_0 t}. \qquad (99)
\end{aligned}$$

Multiplying (82) by e_t and integrating it over $(-\infty, +\infty)$, one obtains

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left(e_t^2 + \tilde{n} e^2 \right) dx + \|e_t(t)\|^2 - \int_{\mathbb{R}} (\tilde{\theta} e_x + \tilde{n}\zeta)_x e_t dx \\
&= \int_{\mathbb{R}} (R_{1x} - R_2 + R_{3x}) e_t dx. \qquad (100)
\end{aligned}$$

The third term of left hand side can be estimated as

$$\begin{aligned}
&-\int_{\mathbb{R}} (\tilde{\theta} e_x + \tilde{n}\zeta)_x e_t dx \\
&= \int_{\mathbb{R}} \tilde{\theta} e_x e_{xt} dx - \int_{\mathbb{R}} (\tilde{n}_x \zeta + \tilde{n}\zeta_x) e_t dx \\
&\ge \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \tilde{\theta} e_x^2 dx - \int_{\mathbb{R}} \tilde{n}\zeta_x e_t dx - C(\delta + \varepsilon)\|(e_t, \zeta)(t)\|^2 - C\delta e^{-\nu_0 t}. \quad (101)
\end{aligned}$$

By using the fact $\int_{\mathbb{R}} \tilde{n}_x^2 dx \le C$ and (96), we estimate the right hand side of (100) as follows

$$\begin{aligned}
\int_{\mathbb{R}} R_{1x} e_t dx &= \int_{\mathbb{R}} (e_x \zeta + e_x \hat{\theta} + \hat{n}\theta + \tilde{n}\hat{\theta})_x e_t dx \\
&= -\int_{\mathbb{R}} (e_x \zeta + e_x \hat{\theta} + \hat{n}\theta) e_{xt} dx + \int_{\mathbb{R}} (\tilde{n}_x \hat{\theta} + \tilde{n}\hat{\theta}_x) e_t dx \\
&\le C(\delta + \varepsilon)\|(e, e_t, e_x)(t)\|^2 + C\delta e^{-\nu_0 t}, \qquad (102)
\end{aligned}$$

and

$$\begin{aligned}
-\int_{\mathbb{R}} R_2 e_t dx &= -\int_{\mathbb{R}} \left((e_x + \hat{n})(e + \tilde{E} + \hat{E}) + (\tilde{n} - \breve{n})\hat{E} \right) e_t dx \\
&\le C(\delta + \varepsilon)\|(e, e_x, e_t)(t)\|^2 + C\delta e^{-\nu_0 t}. \qquad (103)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} R_{3x} e_t dx &= \int_{\mathbb{R}} \Big(\frac{J^2}{n} - \frac{\tilde{J}^2}{\tilde{n}}\Big)_x e_t dx \\
&= \int_{\mathbb{R}} \Big(\frac{2J}{n}(-e_{xt} + \hat{J}_x) - \frac{J^2}{n}(e_x + \hat{n}_x) \\
&\qquad + O(1)\tilde{n}_x(e_t + \hat{J} + e_x + \hat{n})\Big) e_t dx \\
&\leq C(\delta+\varepsilon)\|(e_t, e_x, e_{xt}, e_{xx})(t)\|^2 + C\delta e^{-\nu_0 t}. \qquad (104)
\end{aligned}$$

Substituting (101)-(104) into (100), one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big(e_t^2 + \tilde{n}e^2 + \tilde{\theta}e_x^2\Big)dx + \frac{7}{8}\|e_t(t)\|^2 - \int_{\mathbb{R}} \tilde{n}\zeta_x e_t dx \\
&\leq C(\delta+\varepsilon)\|(e, e_t, e_x, e_{xt}, e_{xx}, \zeta)(t)\|^2 + C\|\zeta_x(t)\|^2 + C\delta e^{-\nu_0 t}. \qquad (105)
\end{aligned}$$

Multiplying $(82)_2$ by $\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta$ and integrating the resultant equation with respect to x, over $(-\infty, +\infty)$, we obtain

$$\frac{d}{dt}\Big(\int_{\mathbb{R}} \frac{3\tilde{n}^2}{4\tilde{\theta}}\zeta^2 dx\Big) + C_0\|\zeta\|^2 \leq \int_{\mathbb{R}} \frac{\kappa\tilde{n}}{\tilde{\theta}}\zeta_{xx}\zeta dx + \int_{\mathbb{R}} (R_4 + R_5 + R_6 + R_7)\zeta dx. \quad (106)$$

Integrating by parts and using Cauchy's inequality and noticing Proposition 1 and Lemma 3.1, then the right hand side terms of (106) can be estimates as follows

$$\int_{\mathbb{R}} \frac{\kappa\tilde{n}}{\tilde{\theta}}\zeta_{xx}\zeta dx \leq -\int_{\mathbb{R}} \frac{\kappa\tilde{n}}{\tilde{\theta}}\zeta_x^2 dx + C(\delta+\varepsilon)\|(\zeta, \zeta_x)(t)\|^2, \qquad (107)$$

and

$$\begin{aligned}
\int_{\mathbb{R}} R_4 \frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta dx &= -\frac{2\kappa}{3}\int_{\mathbb{R}} (\zeta_{xx} + \tilde{\theta}_{xx} + \hat{\theta}_{xx})\zeta \frac{e_x + \hat{n}}{n\tilde{n}} \frac{3\tilde{n}^2}{2\tilde{\theta}} dx \\
&\leq C(\delta+\varepsilon)\|(\zeta, \zeta_x, e_x)(t)\|^2 + C\delta e^{-\nu_0 t}, \qquad (108)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} R_5 \frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta dx &= -\int_{\mathbb{R}} \Big(\frac{J}{n}\theta_x - \frac{\tilde{J}}{\tilde{n}}\tilde{\theta}_x\Big)\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta dx \\
&= -\int_{\mathbb{R}} \frac{J}{n}(\zeta_x + \hat{\theta}_x)\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta dx - \int_{\mathbb{R}} \Big(\frac{J}{n} - \frac{\tilde{J}}{\tilde{n}}\Big)\zeta\tilde{\theta}_x \frac{3\tilde{n}^2}{2\tilde{\theta}} dx \\
&\leq C(\delta+\varepsilon)\|(\zeta, \zeta_x, e_x, e_t)(t)\|^2 + C\delta e^{-\nu_0 t}, \qquad (109)
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}} R_6 \frac{3\tilde{n}^2}{2\tilde{\theta}} \zeta dx \\
&= -\frac{2}{3} \int_{\mathbb{R}} \left[\Big(\frac{J}{n}\Big)_x \theta - \Big(\frac{\tilde{J}}{\tilde{n}}\Big)_x \tilde{\theta} \right] \zeta \frac{3\tilde{n}^2}{2\tilde{\theta}} dx \\
&= -\frac{2}{3} \int_{\mathbb{R}} \left[\Big(\frac{J}{n}\Big)_x (\zeta + \hat{\theta})\zeta - \Big(\frac{J}{n} - \frac{\tilde{J}}{\tilde{n}}\Big)_x \zeta \tilde{\theta} \right] \frac{3\tilde{n}^2}{2\tilde{\theta}} dx \\
&\le C \int_{\mathbb{R}} (|e_{xt}| + |\hat{J}_x| + |e_{xx}| + |\tilde{n}_x| + |\hat{n}_x|) |(\zeta + \hat{\theta})\zeta| dx + \int_{\mathbb{R}} \frac{\tilde{n}^2}{n} e_{xt} \zeta dx \\
&\quad - C \int_{\mathbb{R}} \Big(|\hat{J}_x \zeta| + |J(e_{xx} + \hat{n}_x)\zeta| + |\tilde{n}_x \zeta (e_t + \hat{J} + e_x + \hat{n})| \Big) dx \\
&\le - \int_{\mathbb{R}} \tilde{n} e_t \zeta_x dx + C(\delta + \varepsilon) \|(\zeta, \zeta_x, e_x, e_t, e_{xx}, e_{xt})(t)\|^2 + C\delta e^{-\nu_0 t}. \qquad (110)
\end{aligned}
$$

In order to estimate the last term of the right hand side of (106), we need the following estimates, noticing that for $|x| \ge L_0$, we have

$$
\hat{J}^2 + 2\hat{J}\tilde{J} = (1 - g(x))((J^-(t))^2 - \tilde{J}^2) + g(x)((J^+(t))^2 - \tilde{J}^2), \qquad (111)
$$

with $g(x) := \int_{-\infty}^{x} m_0(y) dy$, and further have

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{3\tilde{n}^2}{2\tilde{\theta}} \Big(\frac{\hat{J}^2}{n^2} + \frac{2\hat{J}\tilde{J}}{n^2} - h(x,t) \Big) \zeta dx \\
&= \Big\{ \int_{-\infty}^{-L_0} + \int_{L_0}^{-\infty} + \int_{-L_0}^{L_0} \Big\} \frac{3\tilde{n}^2}{2\tilde{\theta}} \Big(\frac{\hat{J}^2}{n^2} + \frac{2\hat{J}\tilde{J}}{n^2} - h(x,t) \Big) \zeta dx \\
&= \int_{-\infty}^{-L_0} \frac{3\tilde{n}^2}{2\tilde{\theta}} \zeta (\hat{J}^2 + 2\hat{J}\tilde{J}) \Big(\frac{1}{n^2} - \frac{1}{n_-^2} \Big) dx \\
&\quad + \int_{-\infty}^{-L_0} \frac{3\tilde{n}^2}{2\tilde{\theta}} \zeta (\hat{J}^2 + 2\hat{J}\tilde{J}) \Big(\frac{1}{n^2} - \frac{1}{n_+^2} \Big) dx \\
&\quad + \int_{-L_0}^{L_0} \frac{3\tilde{n}^2}{2\tilde{\theta}} \zeta \Big[\frac{\hat{J}^2 + 2\hat{J}\tilde{J}}{n^2} - \frac{(1 - g(x))}{n_-^2} ((J^-(t))^2 - \tilde{J}^2) \\
&\qquad\qquad - \frac{g(x)}{n_+^2} ((J^+(t))^2 - \tilde{J}^2) \Big] dx \\
&\le C\delta \|(\zeta, e_x)(t)\|^2 + C\delta e^{-\nu_0 t} \int_{-\infty}^{-L_0} (\tilde{n} - n_-)^2 dx \\
&\quad + C\delta e^{-\nu_0 t} \int_{L_0}^{\infty} (\tilde{n} - n_+)^2 dx + C\delta e^{-\nu_0 t} \\
&\le C\delta \|(\zeta, e_x)(t)\|^2 + C\delta e^{-\nu_0 t}. \qquad (112)
\end{aligned}
$$

Using (112), we obtain

$$\begin{aligned}\int_{\mathbb{R}} R_7\zeta\frac{3\tilde{n}^2}{2\tilde{\theta}}dx &= \int_{\mathbb{R}}\Big(\frac{(-e_t+\hat{J}+\tilde{J})^2}{n^2}-\frac{\tilde{J}^2}{\tilde{n}^2}-h(x,t)\Big)\zeta\frac{3\tilde{n}^2}{2\tilde{\theta}}dx\\ &= \int_{\mathbb{R}}\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta\Big(\frac{e_t^2}{n^2}-\frac{2e_t(\tilde{J}+\hat{J})}{n^2}\Big)dx+\int_{\mathbb{R}}\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta\tilde{J}^2\Big(\frac{1}{n^2}-\frac{1}{\tilde{n}^2}\Big)dx\\ &\quad+\int_{\mathbb{R}}\frac{3\tilde{n}^2}{2\tilde{\theta}}\Big(\frac{\hat{J}^2}{n^2}+\frac{2\hat{J}\tilde{J}}{n^2}-h(x,t)\Big)\zeta dx\\ &\le C\delta\|(\zeta,e_x,e_t)(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}\tag{113}$$

Substituting (107)-(110) and (113) into (106), we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{R}}\frac{3\tilde{n}^2}{4\tilde{\theta}}\zeta^2dx\Big)+2C_0\|(\zeta,\zeta_x)(t)\|^2-\int_{\mathbb{R}}\tilde{n}e_t\zeta_xdx\\ &\le C(\delta+\varepsilon)\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}\tag{114}$$

Combining (99)+μ_1(×(105)+(114)), and taking μ_1 suitably large, we obtain Lemma 3.3. ☐

Lemma 3.4. *Let μ_2 be constant and suitably large, then it holds that*

$$\frac{d}{dt}F_2(t)+C_2F_2(t)+C_2\|\zeta_{xx}(t)\|^2\le C\|(\zeta,e,e_x,e_t)(t)\|^2+C\delta e^{-\nu_0 t},\tag{115}$$

where

$$\begin{cases}F_2(t)=\dfrac{d}{dt}\displaystyle\int_{\mathbb{R}}\Big(e_{xt}e_x+\frac{1}{2}e_x^2+\frac{\mu_2}{2}\Big[e_{xt}^2+\tilde{n}e_x^2+\Big(\tilde{\theta}+\frac{J^2}{n^2}\Big)e_{xx}^2\Big]\\ \qquad\qquad\qquad\qquad+\dfrac{3\mu_2\tilde{n}^2}{4\tilde{\theta}}\zeta_x^2\Big)dx\\ C_{21}\|(\zeta_x,e_x,e_{xx},e_{xt})(t)\|^2\le F_2(t)\le C_{22}\|(\zeta_x,e_x,e_{xx},e_{xt})(t)\|^2\end{cases}\tag{116}$$

provided $\varepsilon+\delta\ll 1$.

Proof. Similarly to Lemma 3.3, integrating

$$(82)_{1x}\times(e_x+\mu_2e_{xt})+(82)_{3x}\times\frac{3\mu_2\tilde{n}^2}{2\tilde{\theta}}\zeta_x$$

over $(-\infty,+\infty)$, where μ_2 is a suitably large constant, one easily obtains

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\Big(e_{xt}e_x+\frac{1}{2}e_x^2+\frac{\mu_2}{2}\Big[e_{xt}^2+\tilde{n}e_x^2+\Big(\tilde{\theta}+\frac{J^2}{n^2}\Big)e_{xx}^2\Big]+\frac{3\mu_2\tilde{n}^2}{4\tilde{\theta}}\zeta_x^2\Big)dx\\ &\quad+C_0\|(e_{xx},e_{xt},\zeta_x,\zeta_{xx})(t)\|^2\\ &\le C\|(e,e_x,e_t,\zeta)(t)\|^2+C\delta e^{-\nu_0 t},\end{aligned}\tag{117}$$

where μ_2 suitably large, then we obtain Lemma 3.4. ☐

Lemma 3.5. *It holds that*

$$\begin{aligned}&\frac{d}{dt}F_3(t)+C_3F_3(t)+C_3\|\zeta_{xxx}(t)\|^2\\ &\le C\|(\zeta,e,e_x,\zeta_x,e_t,e_{xx},e_{xt})(t)\|^2+C\delta e^{-\nu_0 t},\end{aligned}\tag{118}$$

where

$$\begin{cases} F_3(t) = \dfrac{d}{dt}\displaystyle\int_{\mathbb{R}} \Big(e_{xxt}e_{xx} + \frac{1}{2}e_{xx}^2 \\ \qquad\qquad + \dfrac{\lambda_3}{2}\Big[e_{xxt}^2 + \tilde{n}e_{xx}^2 + \Big(\tilde{\theta} + \zeta + \hat{\theta} + \dfrac{J^2}{n^2}\Big)e_{xxx}^2 + \dfrac{3\tilde{n}^2}{4\tilde{\theta}}\zeta_{xx}^2 \Big]\Big) dx \\ C_{31}\|(\zeta_{xx}, e_{xx}, e_{xxx}, e_{xxt})(t)\|^2 \le F_3(t) \le C_{32}\|(\zeta_{xx}, e_{xx}, e_{xxx}, e_{xxt})(t)\|^2 \end{cases} \tag{119}$$

provided $\varepsilon + \delta \ll 1$.

Proof. Similarly to Lemma 3.3, $(82)_{1xx} \times e_{xx}$ over $(-\infty, +\infty)$, one easily obtains

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \Big(e_{xxt}e_{xx} + \frac{1}{2}e_{xx}^2 \Big) dx + C_0\|(e_{xx}, e_{xxx})(t)\|^2 \\ &\le C\|(e, e_x, \zeta_x, e_t, e_{xx}, e_{xt}, e_{xxt})(t)\|^2 + C\|(\zeta, \zeta_x, \zeta_{xx})\|^2 + C\delta e^{-\nu_0 t}. \end{aligned} \tag{120}$$

Integrating $(82)_{1xx} \times e_{xxt}$ over $\mathbb{R}$, one easily obtains,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}e_{xxt}^2 dx + \|e_{xxt}(t)\|^2 - \int_{\mathbb{R}} (\tilde{\theta}e_x + \tilde{n}\zeta)_{xxx}e_{xxt}dx + \int_{\mathbb{R}} (\tilde{n}e)_{xx}e_{xxt}dx \\ &= \int_{\mathbb{R}} (R_{1xxx} - R_{2xx} + R_{3xxx})e_{xxt}dx. \end{aligned} \tag{121}$$

The third term of left hand side can be estimated as

$$\begin{aligned} &-\int_{\mathbb{R}} (\tilde{\theta}e_x + \tilde{n}\zeta)_{xxx}e_{xxt}dx \\ &= \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\tilde{\theta}e_{xxx}^2 dx - \int_{\mathbb{R}} (2\tilde{\theta}_x e_{xx} + \tilde{\theta}_{xx}e_x)_x e_{xxt}dx \\ &\quad - \int_{\mathbb{R}} (\tilde{n}_{xxx}\zeta + 2\tilde{n}_{xx}\zeta_x + 2\tilde{n}_x\zeta_{xx} + \tilde{n}\zeta_{xxx})e_{xxt}dx \\ &\ge \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\tilde{\theta}e_{xxx}^2 dx - \frac{1}{16}\|e_{xxt}(t)\|^2 - C(\delta + \varepsilon)\|(e_{xxx}, \zeta_{xx})(t)\|^2 \\ &\quad - C\|(\zeta, e_x, e_{xx}, \zeta_x)(t)\|^2 - \int_{\mathbb{R}} \tilde{n}\zeta_{xxx}e_{xxt}dx - C\delta e^{-\nu_0 t}, \end{aligned} \tag{122}$$

and

$$\begin{aligned} &\int_{\mathbb{R}} (\tilde{n}e)_{xx}e_{xxt}dx \\ &= \int_{\mathbb{R}} (\tilde{n}_{xx}e + 2\tilde{n}_x e_x + \tilde{n}e_{xx})e_{xxt}dx \\ &\ge \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\tilde{n}e_{xx}^2 dx - \frac{1}{32}\|e_{xxt}(t)\|^2 - C\|(e, e_x)(t)\|^2 - C\delta e^{-\nu_0 t}. \end{aligned} \tag{123}$$

Noticing $(82)_2$, we have

$$|\zeta_t| \le C|\zeta_{xx}| + C(\delta + \varepsilon). \tag{124}$$

Using (124) and Cauchy's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}} R_{1xxx}e_{xxt}dx &= \int_{\mathbb{R}} [e_x(\zeta + \hat{\theta}) + \hat{n}\theta + \tilde{n}\hat{\theta}]_{xxx}e_{xxt}dx \\ &\le -\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}(\zeta + \hat{\theta})e_{xxx}^2 dx + \frac{1}{64}\|e_{xxt}\|^2 \\ &\quad + C(\delta + \varepsilon)\|(\zeta_{xx}, \zeta_{xxx}, e_{xxx})\|^2 + C\delta e^{-\nu_0 t}. \end{aligned} \tag{125}$$

It is easy to obtain

$$\int_{\mathbb{R}}(-R_{2xx}+R_{3xxx})e_{xxt}dx \le -\frac{d}{dt}\int_{\mathbb{R}}\frac{J^2}{2n^2}e_{xxx}^2dx+\frac{1}{64}\|e_{xxt}\|^2+C(\delta+\varepsilon)\|e_{xxx}\|^2 \\ +C\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2. \quad (126)$$

Substitute (122), (123), (125) and (126) into (121), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big(e_{xxt}^2+\tilde{n}e_{xx}^2+(\tilde{\theta}+\zeta+\hat{\theta}+\frac{J^2}{n^2})e_{xxx}^2\Big)dx\\&+\frac{7}{8}\|e_{xxt}(t)\|^2-\int_{\mathbb{R}}\tilde{n}\zeta_{xxx}e_{xxt}dx\\&\le C(\delta+\varepsilon)\|(e_{xxx},\zeta_{xxx})(t)\|^2+C\|(e,e_x,e_t,e_{xx},e_{xt})(t)\|^2\\&\quad+C\|(\zeta,\zeta_x,\zeta_{xx})(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}\quad (127)$$

Integrating $(82)_{2xx}\times\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}$ with respect to x over $\mathbb{R}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big(\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}^2\Big)dx+C_0\|\zeta_{xx}(t)\|^2\\&=-\int_{\mathbb{R}}\Big(\frac{2}{3\tilde{n}}\zeta_{xx}\Big)_x\Big(\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}\Big)_xdx\\&\quad-\int_{\mathbb{R}}(R_{4x}+R_{5x}+R_{6x}+R_{7x})\Big(\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}\Big)_xdx.\end{aligned}\quad (128)$$

The right hand side of (128) can be estimated as follows

$$-\int_{\mathbb{R}}\Big(\frac{2}{3\tilde{n}}\zeta_{xx}\Big)_x\Big(\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}\Big)_xdx\le-\int_{\mathbb{R}}\frac{\tilde{n}}{\tilde{\theta}}\zeta_{xxx}^2dx+C(\delta+\varepsilon)\|\zeta_{xx}(t)\|^2,\quad (129)$$

and

$$\begin{aligned}&-\int_{\mathbb{R}}(R_{4x}+R_{5x}+R_{6x}+R_{7x})\Big(\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}\Big)_xdx\\&\le C(\delta+\varepsilon)\|(e_x,e_t,e_{xx},e_{xt},e_{xxx},e_{xxt},\zeta,\zeta_x,\zeta_{xx},\zeta_{xxx})(t)\|^2\\&\quad-\int_{\mathbb{R}}\tilde{n}\zeta_{xx}e_{xxt}dx+C\delta e^{-\nu_0 t}.\end{aligned}\quad (130)$$

Substitute (129) and (130) into (128), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big(\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}^2\Big)dx+C_0\|(\zeta_{xx},\zeta_{xxx})(t)\|^2+\int_{\mathbb{R}}\tilde{n}\zeta_{xx}e_{xxt}dx\\&\le C(\delta+\varepsilon)\|(e_x,e_t,e_{xx},e_{xt},e_{xxx},e_{xxt},\zeta,\zeta_x)(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}\quad (131)$$

Taking (127) + (130), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big[e_{xxt}^2+\tilde{n}e_{xx}^2+\Big(\tilde{\theta}+\zeta+\hat{\theta}+\frac{J^2}{n^2}\Big)e_{xxx}^2+\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}^2\Big]dx\\&\quad+C_0\|(\zeta_{xx},\zeta_{xxx},e_{xxt})(t)\|^2\\&\le\int_{\mathbb{R}}\Big(\tilde{n}-\frac{\tilde{n}^2}{n}\Big)\zeta_{xx}e_{xxt}dx\\&\quad+C(\delta+\varepsilon)\|(e_x,e_t,e_{xx},e_{xt},e_{xxx},e_{xxt},\zeta,\zeta_x)(t)\|^2+C\delta e^{-\nu_0 t}\\&\le C(\delta+\varepsilon)\|(e_x,e_t,e_{xx},e_{xt},e_{xxx},e_{xxt},\zeta,\zeta_x,\zeta_{xx})(t)\|^2+C\delta e^{-\nu_0 t},\end{aligned}\quad (132)$$

namely,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left[e_{xxt}^2+\tilde{n}e_{xx}^2+\left(\tilde{\theta}+\zeta+\hat{\theta}+\frac{J^2}{n^2}\right)e_{xxx}^2+\frac{3\tilde{n}^2}{2\tilde{\theta}}\zeta_{xx}^2\right]dx\\&\quad+C_0\|(\zeta_{xx},\zeta_{xxx},e_{xxt})(t)\|^2\\&\le C(\delta+\varepsilon)\|(e_x,e_t,e_{xx},e_{xt},e_{xxx},e_{xxt},\zeta,\zeta_x)(t)\|^2+C\delta e^{-\nu_0 t}.\end{aligned}\tag{133}$$

Let μ_3 suitably large, and take (120) $+\,\mu_3\times$ (133), we complete Lemma 3.5. □

Proof of Theorem 3.2. Let N_1, N_2, N_3 be suitably large, and take $N_1\times(91)+N_2\times(115)+N_3\times(118)$, we obtain

$$\frac{d}{dt}F(t)+C_4F(t)+C_4\|\zeta_{xxx}(t)\|^2\le C\delta e^{-\nu_0 t},\tag{134}$$

where

$$\begin{cases}F(t)=N_1F_1(t)+N_2F_2(t)+N_3F_3(t),\\C_{41}(\|e\|_{H^3}^2+\|e_t\|_{H^2}^2+\|\zeta(t)\|_{H^2}^2)\le F(t)\\\qquad\qquad\le C_{42}(\|e\|_{H^3}^2+\|e_t\|_{H^2}^2+\|\zeta(t)\|_{H^2}^2).\end{cases}\tag{135}$$

Gronwall's inequality and (134) imply

$$\|e\|_{H^3}^2+\|e_t\|_{H^2}^2+\|\zeta(t)\|_{H^2}^2\le C(\delta+\Phi_0)e^{-\nu_1 t},\tag{136}$$

where ν_1 is a positive constant, then Theorem 3.2 is completed. □

4. **Convergence to planar stationary waves in m-D space.** In this section, for simplicity, we are going to study the 3-D nonisentropic Euler-Poisson equations for the unipolar hydrodynamic model of semiconductors:

$$\begin{cases}n_t+\mathrm{div}(n\mathbf{u})=0,\\(n\mathbf{u})_t+\mathrm{div}(n\mathbf{u}\otimes\mathbf{u})+\nabla p(n,\theta)=n\nabla\omega-n\mathbf{u},\\\theta_t+\mathbf{u}\nabla\theta+\frac{2}{3}\theta\mathrm{div}\mathbf{u}=\frac{2}{3}\frac{\kappa}{n}\Delta\theta+\frac{1}{3}|\mathbf{u}|^2-(\theta-\theta^0),\\\Delta\omega=n-b(x_1),\end{cases}\tag{137}$$

with the initial data

$$\begin{cases}n(x,0)=n_0(x)\to n_\pm,\\\mathbf{u}(x,0)=\mathbf{u}_0(x)\to(u_\pm,0,0),\quad\text{as } x_1\to\pm\infty\\\theta(x,0)=\theta_0(x)\to\theta_\pm,\end{cases}\tag{138}$$

and the boundary condition at far field

$$\lim_{x_1\to-\infty}\nabla\omega(x,t)=\lim_{x_1\to-\infty}(\partial_{x_1}\omega,\partial_{x_2}\omega,\partial_{x_3}\omega)=(E_-,0,0).\tag{139}$$

Here, we assume $b(x)=b(x_1)$ through out this section, because, by taking the perturbation of the m-D equations around the 1-D steady-state equations, we need $b(x)-b(x_1)=0$ under consideration of the stability of planar diffusion waves.

Let $(n,\mathbf{u},\omega)(x,t)$ be the solution of the multi-dimensional system (137), and let $\bar{\mathbf{u}}(x_1,t)=(\bar{u}(x_1,t),0,0)$, $\bar{\mathbf{E}}(x_1,t)=(\bar{E}(x_1,t),0,0)$, where $\bar{u}(x_1,t)=\frac{\bar{J}(x_1,t)}{\bar{n}(x_1,t)}$, and $(\bar{n},\bar{J},\bar{\theta},\bar{E})(x_1,t)$ be the scalar solution of (6) in 1-D with the following initial boundary data

$$\begin{cases}(\bar{n},\bar{J},\bar{\theta})(x_1,0)=(\bar{n}_0,\bar{J}_0,\bar{\theta}_0)(x_1)\to(n_\pm,J_\pm,\theta_\pm)\ \text{ as } x\to\pm\infty,\\\lim\limits_{x_1\to-\infty}\bar{E}(x_1,t)=E_-.\end{cases}\tag{140}$$

Now we define

$$\begin{cases} \phi(x,t) := n(x,t) - \bar{n}(x_1,t), \\ \mathbf{\Psi}(x,t) := \mathbf{u}(x,t) - \bar{\mathbf{u}}(x_1,t), \\ \zeta(x,t) := \theta(x,t) - \bar{\theta}(x,t). \end{cases} \tag{141}$$

From (6) and (1), we can reduce the system to

$$\begin{cases} \phi_t + \text{div } (\bar{n}\mathbf{\Psi} + \phi\mathbf{\Psi} + \phi\bar{\mathbf{u}}) = 0, \\ \mathbf{\Psi}_t + \mathbf{\Psi} + \dfrac{1}{\bar{n}+\phi}\nabla(\bar{\theta}\phi + \bar{n}\zeta) - (\nabla\omega - \bar{\mathbf{E}}) = -R_1, \\ \zeta_t + \zeta = \dfrac{2\kappa}{3(\bar{n}+\phi)}\Delta\zeta + R_2 - R_3 - R_4 + R_5, \\ \text{div}(\nabla\omega - \bar{E}) = \phi, \end{cases} \tag{142}$$

where

$$\begin{cases} R_1 = (\dfrac{1}{\bar{n}+\phi} - \frac{1}{\bar{n}})\nabla(\bar{n}\bar{\theta}) + \bar{\mathbf{u}}\nabla\mathbf{\Psi} + \mathbf{\Psi}\nabla\bar{\mathbf{u}} + \mathbf{\Psi}\nabla\mathbf{\Psi}, \\ R_2 = \dfrac{2\kappa}{3}(\dfrac{1}{\bar{n}+\phi} - \dfrac{1}{\bar{n}})\Delta\bar{\theta}, \\ R_3 = (\bar{\mathbf{u}} + \mathbf{\Psi})\nabla\zeta + \mathbf{\Psi}\nabla\bar{\theta}, \\ R_4 = \dfrac{2}{3}(\bar{\theta}+\zeta)\text{div}\mathbf{\Psi} + \dfrac{2}{3}\zeta\text{div}\bar{\mathbf{u}}, \\ R_5 = \mathbf{\Psi}\mathbf{\Psi} + 2\bar{\mathbf{u}}\mathbf{\Psi}. \end{cases} \tag{143}$$

Notice that

$$\text{curl}(\nabla\omega) = \mathbf{0} \quad \text{and} \quad \text{curl}(\bar{\mathbf{E}}) = \mathbf{0} \quad \text{which imply} \quad \text{curl}(\nabla\omega - \bar{\mathbf{E}}) = \mathbf{0}, \tag{144}$$

so there exists a function $H(x,t)$ such that

$$\nabla H = \nabla\omega - \bar{\mathbf{E}}. \tag{145}$$

Thus, we can reduce (142) into

$$\begin{cases} \phi_t + \text{div } (\bar{n}\mathbf{\Psi} + \phi\mathbf{\Psi} + \phi\bar{\mathbf{u}}) = 0, \\ \mathbf{\Psi}_t + \mathbf{\Psi} + \frac{1}{\bar{n}+\phi}\nabla(\bar{\theta}\phi + \bar{n}\zeta) - (\nabla\omega - \bar{\mathbf{E}}) = -R_1, \\ \zeta_t + \zeta = \frac{2\kappa}{3(\bar{n}+\phi)}\Delta\zeta + R_2 - R_3 - R_4 + R_5, \\ \Delta H = \phi, \end{cases} \tag{146}$$

with the initial data

$$\begin{cases} \phi(x,0) = n_0(x) - \bar{n}_0(x_1,0) =: \phi_0(x) \in H^3(\mathbb{R}^3), \\ \mathbf{\Psi}(x,0) = \mathbf{u}_0(x) - \bar{\mathbf{u}}_0(x_1,0) =: \mathbf{\Psi}_0(x) \in H^3(\mathbb{R}^3), \\ \zeta(x,0) = \zeta_0(x) - \bar{\theta}(x_1,0) =: \zeta_0(x) \in H^3(\mathbb{R}^3), \end{cases} \tag{147}$$

and the boundary condition

$$|\nabla H(x,t)| \to 0 \quad \text{as} \quad |x| \ \to +\infty. \tag{148}$$

We also define

$$\begin{cases} \Delta H_0(x) := n_0(x) - \bar{n}_0(x_1), \\ H_0(x) \to 0, \quad \text{as } |x| \to +\infty, \end{cases} \tag{149}$$

and

$$\begin{cases} \eta := \|(\phi_0, \mathbf{\Psi}_0, \theta_0, \nabla H_0)\|_{H^3(\mathbb{R}^3)}, \\ \delta_1 := \delta + \|e(0)\|_{H^6} + \|(e_t, \zeta)(0)\|_{H^5(\mathbb{R}^3)}, \end{cases} \tag{150}$$

where $\bar{e}, \bar{\zeta}$ is defined in (84) similarly.

Now we are ready to state the stability results for the planar stationary waves in $\mathbb{R}^3$ as follows.

Theorem 4.1. *Let $b(x) = b(x_1)$ satisfy (20). if $\delta_1 + \eta \ll 1$, where δ_1 is defined in (150). Then there exists a unique global smooth solution $(n, \mathbf{u}, \theta, \nabla\omega)$ for 3-D unipolar hydrodynamic model for semiconductors system (137)(138) and satisfies*

$$n - \bar{n}, \mathbf{u} - \bar{\mathbf{u}}, \theta - \bar{\theta}, \nabla\omega - \bar{\mathbf{E}} \in C([0,\infty), H^3(\mathbb{R}^3)) \tag{151}$$

and

$$\|(n - \bar{n}, \mathbf{u} - \bar{\mathbf{u}}, \theta - \bar{\theta}, \nabla\omega - \bar{\mathbf{E}})(t)\|_{H^3(\mathbb{R}^3)} \le C(\delta_1 + \eta)e^{-\tilde{\mu}t} \tag{152}$$

for some constant $\tilde{\mu} > 0$.

From Theorem 4.1 and Corollary 1, one can immediately obtain the following stability of the planar stationary wave.

Corollary 2 (Convergence to Planar Stationary Waves)**.** *Under the conditions of Theorem 4.1, the solutions $(n, \mathbf{u}, \theta, \nabla\omega)$ for 3-D system (137) and (138) converges to its planar stationary wave $(\tilde{n}, \tilde{u}, \tilde{\theta}, \tilde{E})(x_1)$ (the steady-state solution of (16) and (17)) as follows*

$$\begin{cases} \|(n - \tilde{n})(t)\|_{L^\infty(\mathbb{R}^3)} \le O(1)e^{-\bar{\mu}t}, \\ \|(\mathbf{u} - \tilde{\mathbf{u}})(t)\|_{L^\infty(\mathbb{R}^3)} \le O(1)e^{-\bar{\mu}t}, \\ \|(\theta - \tilde{\theta})(t)\|_{L^\infty(\mathbb{R}^3)} \le O(1)e^{-\bar{\mu}t}, \\ \|(\nabla\omega - \tilde{\mathbf{E}})(t)\|_{L^\infty(\mathbb{R}^3)} \le O(1)e^{-\bar{\mu}t}, \end{cases} \tag{153}$$

where $\bar{\mu}$ is a positive constant, $\tilde{\mathbf{u}}(x_1) = (\tilde{u}(x_1), 0, 0)$, $\tilde{\mathbf{E}}(x_1) = (\tilde{E}(x_1), 0, 0)$.

4.1. **A priori estimates.** In order to prove Theorem 4.1 by the energy method with the continuous extension argument, the crucial step is to establish the *a priori* estimates for the solution. This will be our main target in this section.

Let $T \in (0, +\infty]$, we define the solution space, for $0 \le t \le T$,

$$X(T) = \Big\{(\phi, \mathbf{\Psi}, \zeta, \nabla H)(x,t)\Big|(\phi, \mathbf{\Psi}, \zeta, \nabla H)(x,t) \in C(0,T; H^3(\mathbb{R}^3)),\Big\} \tag{154}$$

with the norm

$$N(T)^2 =: \sup_{0\le t\le T} \|(\phi, \mathbf{\Psi}, \zeta, \nabla H)(t)\|^2_{H^3(\mathbb{R}^3)}. \tag{155}$$

Here, without confusion, we still denote the 3-D solution space by $X(T)$ and its norm by $N(T)$.

Let $N(T)^2 \le \varepsilon^2$, where ε is sufficiently small which will be determined later. It is noted that, (155) with Sobolev inequality $\|f\|_{L^\infty(R^3)} \le C\|f\|^{1/4}\|\nabla^2 f\|^{3/4}$ gives

$$\sum_{k=0}^{1} \|\nabla^k(\phi, \mathbf{\Psi}, \zeta, \nabla H)(t)\|_{L^\infty(\mathbb{R}^3)} \le C\varepsilon. \tag{156}$$

It is easy to verify from (29) and (156) that, there exists a positive constant c such that

$$0 < \frac{1}{c} \le n = \phi + \bar{n} \le c. \tag{157}$$

Remark 3. Before we deal with the *a priori* estimates, we can get an estimate about $H(x,t)$, which will be used later. Notice that $\Delta H(x,t), \nabla H(x,t) \in L^2(\mathbb{R}^3)$, we can easily obtain

$$\begin{aligned} |H(x,t)| &= \left| \int_{\mathbb{R}^3} \frac{1}{|x-y|} \phi(y,t)dy + C \right| \\ &= \left| \int_{\mathbb{R}^3} \frac{1}{|x-y|} \Delta H(y,t)dy + C \right| \\ &= \left| \int_{\mathbb{R}^3} \nabla \left(\frac{1}{|x-y|} \right) \nabla H(y,t)dy + C \right| \le \tilde{C} < +\infty. \end{aligned} \tag{158}$$

Now we are going to establish the *a priori* estimates.

Proposition 2 (A priori Estimate)**.** *It holds that*

$$\|(\nabla H, \phi, \zeta, \mathbf{\Psi})(t)\|^2_{H^3(\mathbb{R}^3)} \le C \|(\nabla H_0, \phi_0, \zeta_0, \mathbf{\Psi}_0)\|^2_{H^3(\mathbb{R}^3)} e^{-\mu t}, \tag{159}$$

provided $\varepsilon + \delta \ll 1$.

In order to prove Proposition 2, we are going to establish the L^2-energy estimate for the solution first, then to establish for the first, the second, and the third derivatives of the solution.

Now we prove our first L^2-energy estimate. Multiplying $(146)_2$ by $-(\bar{n}+\phi)\nabla H$ and integrating the resultant equation over $\mathbb{R}^3$ with respect to x, we obtain

$$\begin{aligned} &- \int_{\mathbb{R}^3} (\bar{n}+\phi) \mathbf{\Psi}_t \nabla H dx - \int_{\mathbb{R}^3} (\bar{n}+\phi) \mathbf{\Psi} \nabla H dx \\ &- \int_{\mathbb{R}^3} \nabla(\bar{\theta}\phi + \bar{n}\zeta) \nabla H dx + \int_{\mathbb{R}^3} (\bar{n}+\phi)|\nabla H|^2 dx \\ &= \int_{\mathbb{R}^3} (\bar{n}+\phi) R_1 \nabla H dx. \end{aligned} \tag{160}$$

Differentiating $(146)_4$ with respect to t twice, and utilizing $(38)_1$, we have

$$\Delta H_{tt} = -\mathrm{div}((\bar{n}+\phi)\mathbf{\Psi}_t + \phi_t \mathbf{\Psi} + \bar{n}_t \mathbf{\Psi} + \phi_t \bar{\mathbf{u}} + \phi \bar{\mathbf{u}}_{\mathbf{t}}). \tag{161}$$

Next multiplying (161) by H and integrating it by parts with respect to x over R^3 and using (158), we obtain

$$\begin{aligned} &- \int_{\mathbb{R}^3} (\bar{n}+\phi) \mathbf{\Psi}_t \nabla H dx \\ &= \int_{\mathbb{R}^3} \mathrm{div}((\bar{n}+\phi)\mathbf{\Psi}_t) H dx \\ &= \int_{\mathbb{R}^3} \nabla H_{tt} \nabla H dx + \int_{\mathbb{R}^3} (\phi_t \mathbf{\Psi} + \bar{n}_t \mathbf{\Psi} + \phi_t \bar{\mathbf{u}} + \phi \bar{\mathbf{u}}_{\mathbf{t}}) \nabla H dx \\ &\ge \frac{d}{dt} \left(\int_{\mathbb{R}^3} \nabla H \nabla H_t dx \right) - \|\nabla H_t(t)\|^2 \\ &\quad -C(\delta+\varepsilon)\|(\nabla\phi, \nabla\mathbf{\Psi}, \nabla H, \phi, \mathbf{\Psi})(t)\|^2, \end{aligned} \tag{162}$$

where, in order to prove the last estimate in (162), namely,

$$\int_{\mathbb{R}^3} (\phi_t \mathbf{\Psi} + \bar{n}_t \mathbf{\Psi} + \phi_t \bar{\mathbf{u}} + \phi \bar{\mathbf{u}}_{\mathbf{t}}) \nabla H dx \ge -C(\delta+\varepsilon)\|(\nabla\phi, \nabla\mathbf{\Psi}, \nabla H, \phi, \mathbf{\Psi})(t)\|^2, \tag{163}$$

we have used the Cauchy-Schwarz inequality $|ab| \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, the *a priori* assumption (156), and the following smallness

$$|\partial_t^i \partial_{x_1}^j (\bar{n}, \bar{\mathbf{u}})| \le C\delta, \quad \text{for } i = 0,1; j = 0,1,2,3,4, \text{and } \ i+j \le 4, \tag{164}$$

which will be frequently used later. Such a smallness (164) can be easily obtained from Propositions 1 and Corollary 1. Here, in the estimate of (163), the following estimate is needed too:

$$\|\phi_t\|^2 \le C\|\nabla \mathbf{\Psi}\|^2 + C(\delta+\varepsilon)\|(\nabla\phi, \phi, \mathbf{\Psi})\|^2, \tag{165}$$

which can be easily obtained from $(146)_1$ and (164).

Differentiating $(146)_4$ with respect to t, and utilizing $(146)_1$, we have

$$\Delta H_t = -\mathrm{div}((\bar{n}+\phi)\mathbf{\Psi} + \phi\bar{\mathbf{u}}). \tag{166}$$

Integrating $(166)\cdot H$ by parts with respect to x over R^3 and using (158), we obtain

$$\begin{aligned} -\int_{\mathbb{R}^3} (\bar{n}+\phi)\mathbf{\Psi}\nabla H dx &= -\int_{\mathbb{R}^3} \Delta H_t H dx - \int_{\mathbb{R}^3} \mathrm{div}(\phi\bar{\mathbf{u}}) H dx \\ &\ge \frac{d}{dt}\Big(\int_{\mathbb{R}^3} \frac{|\nabla H|^2}{2} dx\Big) - C(\delta+\varepsilon)\|(\phi, \nabla H)(t)\|^2. \end{aligned} \tag{167}$$

Notice $(146)_4$, we obtain

$$\begin{aligned} -\int_{\mathbb{R}^3} \nabla(\bar{\theta}\phi + \bar{n}\zeta)\nabla H dx &= \int_{\mathbb{R}^3} (\bar{\theta}\phi + \bar{n}\zeta)\phi dx \\ &\ge 2C_0\|\phi(t)\|^2 - C(\delta+\varepsilon)\|\zeta(t)\|^2, \end{aligned} \tag{168}$$

and

$$\int_{\mathbb{R}^3} (\bar{n}+\phi)|\nabla H|^2 dx \ge 2C_0\|\nabla H(t)\|^2. \tag{169}$$

Using Cauchy inequality and $(143)_1$ as well as the smallness results (164), we obtain

$$\int_{\mathbb{R}^3} (\bar{n}+\phi) R_1 \nabla H dx \le C(\delta+\varepsilon)\|(\nabla\mathbf{\Psi}, \nabla H, \phi, \mathbf{\Psi})(t)\|^2, \tag{170}$$

then, substituting (162), (167)-(170) into (160) and noticing the smallness of δ and ε, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_{\mathbb{R}^3} \Big(\nabla H\nabla H_t + \frac{|\nabla H|^2}{2}\Big)dx\Big\} + C_0\|(\nabla H, \nabla^2 H)(t)\|^2 \\ &\le \|\nabla H_t(t)\|^2 + C(\delta+\varepsilon)\|(\nabla\mathbf{\Psi}, \nabla\phi, \phi, \mathbf{\Psi})(t)\|^2. \end{aligned} \tag{171}$$

From $(146)_4$, we can easily obtain

$$\|\nabla^2 H(t)\|^2 = \|\phi(t)\|^2, \tag{172}$$

and multiplying (166) by H_t and integrating it by parts, we obtain

$$\|\nabla H_t(t)\|^2 \le C\|\mathbf{\Psi}(t)\|^2 + C\delta\|\phi(t)\|^2. \tag{173}$$

Combining (172) and (173), we reduce (171) into

$$\begin{aligned} &\frac{d}{dt}M_{11}(t) + C_0\|(\nabla H, \phi, \nabla^2 H)(t)\|^2 \\ &\le C\|(\mathbf{\Psi}, \zeta)(t)\|^2 + C(\delta+\varepsilon)\|(\nabla\mathbf{\Psi}, \nabla\phi)(t)\|^2, \end{aligned} \tag{174}$$

where

$$M_{11}(t) = \int_{\mathbb{R}^3} \Big[\nabla H\nabla H_t + \frac{|\nabla H|^2}{2}\Big] dx. \tag{175}$$

On the other hand, by taking $(\bar n+\phi)(146)_2\cdot\mathbf{\Psi}$ and integrating the resultant equation, we get

$$\begin{aligned}&\int_{\mathbb{R}^3}(\bar n+\phi)\mathbf{\Psi}_t\mathbf{\Psi}dx+\int_{\mathbb{R}^3}(\bar n+\phi)|\mathbf{\Psi}|^2dx\\&\quad+\int_{\mathbb{R}^3}\nabla(\bar\theta\phi+\bar n\zeta)\mathbf{\Psi}dx\ -\int_{\mathbb{R}^3}(\bar n+\phi)\nabla H\mathbf{\Psi}dx\\&=-\int_{\mathbb{R}^3}(\bar n+\phi)R_1\mathbf{\Psi}dx.\end{aligned}\tag{176}$$

Furthermore, we can prove

$$\int_{\mathbb{R}^3}(\bar n+\phi)\mathbf{\Psi}_t\mathbf{\Psi}dx\geq\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac12(\bar n+\phi)|\mathbf{\Psi}|^2dx\Big)-C(\delta+\varepsilon)\|\mathbf{\Psi}(t)\|^2,\tag{177}$$

and

$$\int_{\mathbb{R}^3}(\bar n+\phi)|\mathbf{\Psi}|^2dx\geq C_0\|\mathbf{\Psi}(t)\|^2,\tag{178}$$

and using $(146)_1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla(\bar\theta\phi+\bar n\zeta)\mathbf{\Psi}dx\quad\geq\quad&-\int_{\mathbb{R}^3}\frac{\bar\theta}{\bar n+\phi}\phi\mathrm{div}((\bar n+\phi)\mathbf{\Psi})dx+\int_{\mathbb{R}^3}\bar n\nabla\zeta\mathbf{\Psi}dx\\&-C(\delta+\varepsilon)\|(\zeta,\phi,\nabla\phi,\nabla\zeta,\mathbf{\Psi})\|^2.\end{aligned}\tag{179}$$

Similarly to (167), we have

$$-\int_{\mathbb{R}^3}(\bar n+\phi)\mathbf{\Psi}\nabla Hdx\geq\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{|\nabla H|^2}{2}dx\Big)-C(\delta+\varepsilon)\|(\phi,\nabla H)(t)\|^2,\tag{180}$$

and

$$-\int_{\mathbb{R}^3}(\bar n+\phi)R_1\mathbf{\Psi}dx\leq C(\delta+\varepsilon)\|(\phi,\mathbf{\Psi})(t)\|^2.\tag{181}$$

Substituting (177)-(181) into (176) and noticing the smallness of δ and ε, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big\{\int_{\mathbb{R}^3}\Big(\frac{|\nabla H|^2}{2}+\frac{\bar\theta}{2(\bar n+\phi)}\phi^2+\frac12(\bar n+\phi)|\mathbf{\Psi}|^2\Big)dx\Big\}\\&\quad+C_0\|\mathbf{\Psi}(t)\|^2+\int_{\mathbb{R}^3}\bar n\nabla\zeta\mathbf{\Psi}dx\\&\leq C(\delta+\varepsilon)\|(\nabla\zeta,\nabla H,\nabla\phi,\mathbf{\Psi},\phi)(t)\|^2,\end{aligned}\tag{182}$$

Taking $(\bar n+\phi)(146)_3\times\frac{3(\bar n+\phi)}{2\bar\theta}\zeta$ and integrating the resultant equation, we get

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{3(\bar n+\phi)}{2\bar\theta}\zeta_t\zeta dx+\int_{\mathbb{R}^3}\frac{3(\bar n+\phi)}{2\bar\theta}\zeta^2dx\\&=\int_{\mathbb{R}^3}\frac{\kappa}{\bar\theta}\zeta\Delta\zeta dx+\int_{\mathbb{R}^3}\frac{3(\bar n+\phi)}{2\bar\theta}(R_2-R_3-R_4+R_5)\zeta dx.\end{aligned}\tag{183}$$

Furthermore, we can prove

$$\int_{\mathbb{R}^3}\frac{3(\bar n+\phi)}{2\bar\theta}\zeta_t\zeta dx\geq\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{3(\bar n+\phi)}{4\bar\theta}\zeta^2dx\Big)-C(\delta+\varepsilon)\|\zeta(t)\|^2,\tag{184}$$

and

$$\int_{\mathbb{R}^3}\frac{3(\bar n+\phi)}{2\bar\theta}\zeta^2dx\geq C_0\|\zeta(t)\|^2,\tag{185}$$

and

$$\int_{\mathbb{R}^3} \frac{\kappa}{\bar{\theta}} \zeta \Delta \zeta dx \leq -C_0 \|\nabla \zeta(t)\|^2 + C(\delta+\varepsilon)\|\zeta(t)\|^2, \tag{186}$$

and

$$-\int_{\mathbb{R}^3} \frac{3(\bar{n}+\phi)}{2\bar{\theta}} R_4 \zeta dx \leq \int_{\mathbb{R}^3} \bar{n} \nabla \zeta \boldsymbol{\Psi} dx + C(\delta+\varepsilon)\|(\zeta, \nabla\zeta)(t)\|^2, \tag{187}$$

and

$$\int_{\mathbb{R}^3} \frac{3(\bar{n}+\phi)}{2\bar{\theta}} (R_2 - R_3 + R_5)\zeta dx \leq C(\delta+\varepsilon)\|(\zeta, \nabla\zeta, \phi, \boldsymbol{\Psi})(t)\|^2. \tag{188}$$

Substituting (184)-(188) into (183) and noticing the smallness of δ and ε, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\int_{\mathbb{R}^3} \frac{3(\bar{n}+\phi)}{4\bar{\theta}} \zeta^2 dx\right) + C_0\|(\zeta, \nabla\zeta)(t)\|^2 - \int_{\mathbb{R}^3} \bar{n}\nabla\zeta\boldsymbol{\Psi} dx \\ &\leq C(\delta+\varepsilon)\|(\zeta, \nabla\zeta, \phi, \boldsymbol{\Psi})(t)\|^2. \end{aligned} \tag{189}$$

Then, from (174) and (182), we have established the first energy estimate as follows.

Lemma 4.2. *Let N_1 be a positive and large number. By taking* $(174) + N_1((182) + (189))$*, we obtain*

$$\frac{d}{dt} M_1(t) + C_0\|(\phi, \zeta, \boldsymbol{\Psi}, \nabla\zeta, \nabla H, \nabla^2 H)(t)\|^2 \leq C(\delta+\varepsilon)\|(\nabla\phi, \nabla\boldsymbol{\Psi})(t)\|^2, \tag{190}$$

where

$$\begin{cases} M_1(t) = M_{11} + N_1 M_{12}, \\ M_{12}(t) = \displaystyle\int_{\mathbb{R}^3} \Big[\frac{|\nabla H|^2}{2} + \frac{\bar{\theta}}{2(\bar{n}+\phi)}\phi^2 + \frac{1}{2}(\bar{n}+\phi)|\boldsymbol{\Psi}|^2 + \frac{3(\bar{n}+\phi)}{4\bar{\theta}}\zeta^2\Big] dx \end{cases} \tag{191}$$

provided $\delta + \varepsilon \ll 1$.

Now we are going to establish the energy estimates for the solution with the first and second derivatives, the proofs are similar to Lemma 4.2, we omit the details for the following two lemma.

Let β_1 be nonnegative multi-index, and $|\beta_1| = 1$. Taking

$$\sum_{|\beta_1|=1} \int_{\mathbb{R}^3} \partial_x^{\beta_1}((\bar{n}+\phi)(38)_2) \times (-\partial_x^{\beta_1}\nabla H) dx,$$

we have

$$\frac{d}{dt} M_{21}(t) + C_0\|(\nabla\phi, \phi)(t)\|^2 \leq C\|(\nabla\boldsymbol{\Psi}, \nabla\zeta)(t)\|^2 + C(\delta+\varepsilon)\|(\nabla H, \boldsymbol{\Psi})(t)\|^2, \tag{192}$$

where

$$M_{21}(t) = \int_{\mathbb{R}^3} \Big[\nabla^2 H \nabla^2 H_t + \frac{|\nabla^2 H|^2}{2}\Big] dx. \tag{193}$$

Taking $\sum_{|\beta_1|=1} \int_{\mathbb{R}^3} \partial_x^{\beta_1}(146)_3 \times \partial_x^{\beta_1} \boldsymbol{\Psi} dx$, we obtain

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_{\mathbb{R}^3} \Big(\frac{\bar{\theta}}{2(\bar{n}+\phi)^2}|\nabla\phi|^2 + \frac{1}{2}|\nabla\boldsymbol{\Psi}|^2\Big) dx\Big\} + \frac{3}{4}\|\nabla\boldsymbol{\Psi}(t)\|^2 \\ &\quad + \sum_{|\beta_1|=1} \int_{\mathbb{R}^3} \nabla\partial_x^{\beta_1}\zeta\partial_x^{\beta_1}\boldsymbol{\Psi} dx \\ &\leq C\|\phi(t)\|^2 + C(\delta+\varepsilon)\|(\zeta, \nabla\zeta, \nabla^2\zeta, \nabla\phi, \boldsymbol{\Psi})(t)\|^2. \end{aligned} \tag{194}$$

On the other hand side, taking $\sum_{|\beta_1|=1}\int_{\mathbb{R}^3}\partial_x^{\beta_1}(146)_2\times\frac{3}{2\bar\theta}\partial_x^{\beta_1}\zeta dx$, we obtain

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{3}{4\bar\theta}|\nabla\zeta|^2dx\Big)+C_0\|(\nabla\zeta,\nabla^2\zeta)(t)\|^2-\sum_{|\beta_1|=1}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_1}\zeta\partial_x^{\beta_1}\mathbf{\Psi}dx \\ \le C(\delta+\varepsilon)\|(\phi,\nabla\mathbf{\Psi},\nabla\zeta,\nabla^2\zeta)(t)\|^2. \tag{195}$$

Then, from (192), (195) and (194), we have established our second energy estimate as follows.

Lemma 4.3. *Let N_2 be a positive and suitably large number. By taking (192) + N_2((194) + (195)), we obtain*

$$\frac{d}{dt}M_2(t)+C_0\|(\phi,\nabla\phi,\nabla\zeta,\nabla\mathbf{\Psi},\nabla^2\zeta)(t)\|^2\le C(\delta+\varepsilon)\|(\phi,\mathbf{\Psi},\nabla H)(t)\|^2, \tag{196}$$

where

$$\begin{cases}M_2(t)=M_{21}(t)+N_2M_{22}(t),\\ M_{22}(t)=\int_{\mathbb{R}^3}\Big[\frac{\bar\theta}{2(\bar n+\phi)^2}|\nabla\phi|^2+\frac{1}{2}|\nabla\mathbf{\Psi}|^2+\frac{3}{4\bar\theta}|\nabla\zeta|^2\Big]dx\end{cases} \tag{197}$$

provided $\delta+\varepsilon\ll 1$.

Let β_2 be nonnegative multi-index, and $|\beta_2|=2$. Taking

$$\sum_{|\beta_2|=2}\int_{\mathbb{R}^3}\partial_x^{\beta_2}((\bar n+\phi)(146)_2)\times(-\partial_x^{\beta_1}\nabla H)dx,$$

we have

$$\frac{d}{dt}M_{31}(t)+C_0\|\nabla^2\phi(t)\|^2 \\ \le C\|(\nabla^2\mathbf{\Psi},\nabla^2\zeta,\nabla H,\phi)(t)\|^2+C(\delta+\varepsilon)\|(\mathbf{\Psi},\zeta,\nabla\phi,\nabla\zeta,\nabla\mathbf{\Psi})(t)\|^2, \tag{198}$$

where

$$M_{31}(t)=\int_{\mathbb{R}^3}\Big[\nabla^3H\nabla^3H_t+\frac{|\nabla^3H|^2}{2}\Big]dx. \tag{199}$$

Taking

$$\sum_{|\beta_2|=2}\int_{\mathbb{R}^3}\partial_x^{\beta_2}(146)_3\times\partial_x^{\beta_2}\mathbf{\Psi}dx,$$

we obtain

$$\frac{d}{dt}\Big\{\int_{\mathbb{R}^3}\Big(\frac{\bar\theta}{2(\bar n+\phi)^2}|\nabla^2\phi|^2+\frac{1}{2}|\nabla^2\mathbf{\Psi}|^2\Big)dx\Big\} \\ +\frac{3}{4}\|\nabla^2\mathbf{\Psi}(t)\|^2+\sum_{|\beta_2|=2}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_2}\zeta\partial_x^{\beta_2}\mathbf{\Psi}dx \\ \le C\|\nabla\phi(t)\|^2+C(\delta+\varepsilon)\|(\nabla^3\zeta,\nabla^2\phi,\nabla^2\zeta,\nabla\zeta,\nabla\mathbf{\Psi},\phi,\mathbf{\Psi})(t)\|^2. \tag{200}$$

On the other hand side, taking $\sum_{|\beta_2|=2}\int_{\mathbb{R}^3}\partial_x^{\beta_2}(38)_3\times\frac{3}{2\bar\theta}\partial_x^{\beta_2}\zeta dx$, we obtain

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{3}{4\bar\theta}|\nabla^2\zeta|^2dx\Big)+C_0\|(\nabla^2\zeta,\nabla^3\zeta)(t)\|^2-\sum_{|\beta_2|=2}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_2}\zeta\partial_x^{\beta_2}\mathbf{\Psi}dx \\ \le C(\delta+\varepsilon)\|(\phi,\mathbf{\Psi},\zeta,\nabla\phi,\nabla\mathbf{\Psi},\nabla\zeta,\nabla^2\phi,\nabla^2\mathbf{\Psi},\nabla^2\zeta,\nabla^3\zeta)(t)\|^2. \tag{201}$$

Then, from (198), (200) and (201), we have established our third energy estimate as follows.

Lemma 4.4. *Let N_3 be a positive and suitably large number. By taking (198) + N_3((200) + (201)), we obtain*

$$\frac{d}{dt}M_3(t) + C_0\|(\nabla^2\phi, \nabla^2\zeta, \nabla^2\mathbf{\Psi}, \nabla^3\zeta)(t)\|^2 \\ \leq C\|(\phi, \mathbf{\Psi}, \zeta, \nabla H, \nabla\phi, \nabla\zeta, \nabla\mathbf{\Psi})(t)\|^2, \tag{202}$$

where

$$\begin{cases} M_3(t) = M_{31}(t) + N_3 M_{32}(t), \\ M_{32}(t) = \int_{\mathbb{R}^3} \Big[\frac{\bar{\theta}}{2(\bar{n}+\phi)^2}|\nabla^2\phi|^2 + \frac{1}{2}|\nabla^2\mathbf{\Psi}|^2 + \frac{3}{4\bar{\theta}}|\nabla^2\zeta|^2\Big]dx \end{cases} \tag{203}$$

provided $\delta + \varepsilon \ll 1$.

Finally, we are going to establish the fourth energy estimates for the solution with the third derivatives, and here we need to used the *Gagliardo-Nirenberg* inequality to complete our proofs.

Let β_3 be nonnegative multi-index, and $|\beta_3| = 3$. Taking

$$\sum_{|\beta_3|=3}\int_{\mathbb{R}^3} \partial_x^{\beta_3}((\bar{n}+\phi)(146)_2) \times (-\partial_x^{\beta_3}\nabla H)dx,$$

we obtain

$$\frac{d}{dt}M_{41}(t) + C_0\|\nabla^3\phi(t)\|^2 \\ \leq C\|(\nabla^3\mathbf{\Psi}, \nabla^3\zeta, \nabla^2\mathbf{\Psi}, \nabla^2\phi, \nabla^2\zeta, \nabla\mathbf{\Psi}, \nabla\phi, \nabla\zeta, \nabla H, \mathbf{\Psi}, \phi, \zeta)(t)\|^2, \tag{204}$$

where

$$M_{41}(t) = \int_{\mathbb{R}^3}\Big[\nabla^4 H\nabla^4 H_t + \frac{|\nabla^4 H|^2}{2}\Big]dx. \tag{205}$$

Taking $\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\partial_x^{\beta_3}(146)_2 \cdot \partial_x^{\beta_3}\mathbf{\Psi}dx$, we obtain

$$\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{|\nabla^3\mathbf{\Psi}|^2}{2}dx\Big) + \|\nabla^3\mathbf{\Psi}(t)\|^2 \\ + \sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\partial_x^{\beta_3}\Big(\frac{1}{\bar{n}+\phi}\nabla(\bar{\theta}\phi + \bar{n}\zeta)\Big)\partial_x^{\beta_3}\mathbf{\Psi}dx - \sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_3}H\partial_x^{\beta_3}\mathbf{\Psi}dx \\ = -\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\partial_x^{\beta_3}R_1\partial_x^{\beta_3}\mathbf{\Psi}dx. \tag{206}$$

On the other hand, Cauchy inequality and Gagliardo-Nirenberg inequality imply that

$$\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\partial_x^{\beta_3}\Big(\frac{1}{\bar{n}+\phi}\nabla(\bar{\theta}\phi + \bar{n}\zeta)\Big)\partial_x^{\beta_3}\mathbf{\Psi}dx \\ \geq \sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_3}\phi\partial_x^{\beta_3}\mathbf{\Psi}dx - C\int_{\mathbb{R}^3}|\nabla^2\phi|^2|\nabla^3\mathbf{\Psi}|dx \\ -C\int_{\mathbb{R}^3}|\nabla^2\zeta\nabla^2\phi\nabla^3\mathbf{\Psi}|dx - \sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\frac{\bar{\theta}}{\bar{n}+\phi}\partial_x^{\beta_3}\phi\partial_x^{\beta_3}\mathrm{div}\mathbf{\Psi}dx \\ -C(\delta+\varepsilon)\|(\nabla^3\zeta, \nabla^3\phi, \nabla^3\mathbf{\Psi}, \nabla^2\phi, \nabla^2\mathbf{\Psi}, \nabla\phi, \nabla\mathbf{\Psi}, \phi, \mathbf{\Psi})(t)\|^2. \tag{207}$$

Gagliardo-Nirenberg inequality and Young inequality imply

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\nabla^2\phi|^2|\nabla^3\mathbf{\Psi}|dx &\leq C\|\nabla^2\phi\|_{L^4}^2\|\nabla^3\mathbf{\Psi}\|_{L^2} \\
&\leq C\|\nabla^2\phi\|^{\frac{1}{2}}\|\nabla^3\phi\|^{\frac{3}{2}}\|\nabla^3\mathbf{\Psi}\| \\
&\leq C\varepsilon\|(\nabla^3\mathbf{\Psi},\nabla^3\phi)\|^2, \qquad (208)\\
\int_{\mathbb{R}^3} |\nabla^2\zeta\nabla^2\phi\nabla^3\mathbf{\Psi}|dx &\leq C\varepsilon\|(\nabla^3\mathbf{\Psi},\nabla^3\phi,\nabla^3\zeta)\|^2. \qquad (209)
\end{aligned}
$$

We note that for index α, by using $(146)_1$,

$$
\begin{aligned}
\partial_x^\alpha \operatorname{div}\mathbf{\Psi} = -\frac{1}{\bar{n}+\phi} \Bigg\{ & \partial_x^\alpha\phi_t + \sum_{|\gamma|=1,\gamma\leq\alpha}^{|\alpha|} C_\gamma\partial_x^\gamma\bar{n}\partial_x^{\alpha-\gamma}\operatorname{div}\mathbf{\Psi} \\
& + \sum_{|\gamma|=1,\gamma\leq\alpha}^{|\alpha|} C_\gamma\partial_x^\gamma\phi\partial_x^{\alpha-\gamma}\operatorname{div}\mathbf{\Psi} \\
& + \partial_x^\alpha[\phi\operatorname{div}\bar{\mathbf{u}} + \bar{\mathbf{u}}\nabla\phi + \mathbf{\Psi}\nabla\bar{n} + \mathbf{\Psi}\nabla\phi]\Bigg\}. \qquad (210)
\end{aligned}
$$

Using (210) and Gagliardo-Nirenberg inequality again, we can control the fourth term of right-hand side of (207) as follows:

$$
\begin{aligned}
& -\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\frac{\bar{\theta}}{\bar{n}+\phi}\partial_x^{\beta_3}\phi\partial_x^{\beta_3}\operatorname{div}\mathbf{\Psi}dx \\
&\geq \frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{\bar{\theta}}{2(\bar{n}+\phi)^2}|\nabla^3\phi|^2dx\Big) \\
&\quad -C(\delta+\varepsilon)\|(\nabla^3\phi,\nabla^3\mathbf{\Psi},\nabla^2\phi,\nabla^2\mathbf{\Psi},\nabla\phi,\nabla\mathbf{\Psi},\phi,\mathbf{\Psi})(t)\|^2. \qquad (211)
\end{aligned}
$$

Applying (211), (208) and (209) into (207), we obtain

$$
\begin{aligned}
& \sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\partial_x^{\beta_3}\Big(\frac{1}{\bar{n}+\phi}\nabla(\bar{\theta}\phi+\bar{n}\zeta)\Big)\partial_x^{\beta_3}\mathbf{\Psi}dx \\
&\geq \frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{\bar{\theta}}{2(\bar{n}+\phi)^2}|\nabla^3\phi|^2dx\Big) + \sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_3}\zeta\partial_x^{\beta_3}\mathbf{\Psi}dx \\
&\quad -C(\delta+\varepsilon)\|(\nabla^4\zeta,\nabla^3\zeta,\nabla^3\phi,\nabla^3\mathbf{\Psi},\nabla^2\phi,\nabla^2\mathbf{\Psi},\nabla\phi,\nabla\mathbf{\Psi},\phi,\mathbf{\Psi})(t)\|^2, \quad (212)
\end{aligned}
$$

Since Cauchy inequality implies

$$
\begin{aligned}
-\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_3}H\partial_x^{\beta_3}\mathbf{\Psi}dx &\leq \frac{1}{8}\|\nabla^3\mathbf{\Psi}(t)\|^2 + C\|\nabla^4 H(t)\|^2 \\
&= \frac{1}{8}\|\nabla^3\mathbf{\Psi}(t)\|^2 + C\|\nabla^2\phi(t)\|^2, \qquad (213)
\end{aligned}
$$

and Cauchy inequality and Gagliardo-Nirenberg inequality imply that

$$
\begin{aligned}
& -\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\partial_x^{\beta_3}R_1\partial_x^{\beta_3}\mathbf{\Psi}dx \\
&\leq C(\delta+\varepsilon)\|(\nabla^3\phi,\nabla^3\mathbf{\Psi},\nabla^2\phi,\nabla^2\mathbf{\Psi},\nabla\phi,\nabla\mathbf{\Psi},\phi,\mathbf{\Psi})(t)\|^2, \qquad (214)
\end{aligned}
$$

by substituting (212)-(214) into (206), and noticing the smallness of $\delta+\varepsilon$, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big\{\int_{\mathbb{R}^3}\Big(\frac{1}{2}|\nabla^3\mathbf{\Psi}|^2+\frac{\bar\theta}{2(\bar n+\phi)^2}|\nabla^3\phi|^2\Big)dx\Big\}\\&\quad+\frac{3}{4}\|\nabla^3\mathbf{\Psi}(t)\|^2+\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_3}\zeta\partial_x^{\beta_3}\mathbf{\Psi}dx\\&\leq C\|\nabla^2\phi(t)\|^2+C(\delta+\varepsilon)(\|\nabla^4\zeta\|^2+\|(\mathbf{\Psi},\phi,\zeta)(t)\|^2_{H^3}).\end{aligned}\tag{215}$$

On the other hand side, taking $\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\partial_x^{\beta_3}(146)_2\times\frac{3}{2\bar\theta}\partial_x^{\beta_3}\zeta dx$, then Cauchy inequality and Gagliardo-Nirenberg inequality imply

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{3}{4\bar\theta}|\nabla^3\zeta|^2dx\Big)+C_0\|(\nabla^3\zeta,\nabla^4\zeta)(t)\|^2-\sum_{|\beta_3|=3}\int_{\mathbb{R}^3}\nabla\partial_x^{\beta_3}\zeta\partial_x^{\beta_3}\mathbf{\Psi}dx\\&\leq C(\delta+\varepsilon)\|(\phi,\mathbf{\Psi},\zeta,\nabla\phi,\nabla\mathbf{\Psi},\nabla\zeta,\nabla^2\phi,\nabla^2\mathbf{\Psi},\nabla^2\zeta,\nabla^3\mathbf{\Psi})(t)\|^2.\end{aligned}\tag{216}$$

Thus, from (215), (204) and (216), we have established the fourth energy estimates for the solution with the third derivatives as follows.

Lemma 4.5. *Let N_4 be a positive and suitably large number. By taking (204) + N_4((216) + (215)), we obtain*

$$\begin{aligned}&\frac{d}{dt}M_4(t)+C_0\|(\nabla^4\zeta,\nabla^3\zeta,\nabla^3\phi,\nabla^3\mathbf{\Psi})(t)\|^2\\&\leq C\|(\nabla^2\mathbf{\Psi},\nabla^2\phi,\nabla^2\zeta,\nabla\mathbf{\Psi},\nabla\phi,\nabla\zeta,\nabla H,\mathbf{\Psi},\phi,\zeta)(t)\|^2,\end{aligned}\tag{217}$$

where

$$\begin{cases}M_4(t)=M_{41}(t)+N_4M_{42}(t),\\ M_{42}(t)=\displaystyle\int_{\mathbb{R}^3}\Big[\frac{\bar\theta}{2(\bar n+\phi)^2}|\nabla^3\phi|^2+\frac{1}{2}|\nabla^3\mathbf{\Psi}|^2+\frac{3}{4\bar\theta}|\nabla^3\zeta|^2\Big]dx\end{cases}\tag{218}$$

provided $\delta+\varepsilon\ll 1$.

Proof of Proposition 2. Let $\delta+\varepsilon\ll1$ and λ_1, λ_2 and λ_3 be suitably large numbers, and apply Lemmas 4.2–4.5, we then obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\lambda_3\Big[\lambda_2(\lambda_1F_1(t)+F_2(t))+F_3(t)\Big]+F_4(t)\Big)\\&+C_4\|(\nabla^4H,\nabla^3\phi,\nabla^3\mathbf{\Psi},\nabla^3H,\nabla^2\phi,\nabla^2\mathbf{\Psi},\nabla^2H,\nabla\phi,\nabla\mathbf{\Psi},\nabla H,\phi,\mathbf{\Psi})(t)\|^2\\&\leq 0\end{aligned}\tag{219}$$

and

$$\begin{aligned}C_{51}\|(\nabla H,\phi,\mathbf{\Psi},\zeta)(t)\|^2_{H^3(\mathbb{R}^3)}&\leq\lambda_3\Big[\lambda_2(\lambda_1F_1(t)+F_2(t))+F_3(t)\Big]+F_4(t)\\&\leq C_{52}\|(\nabla H,\phi,\mathbf{\Psi},\zeta)(t)\|^2_{H^3(\mathbb{R}^3)},\end{aligned}\tag{220}$$

where C_{51} and C_{52}, and we have used (173) and

$$\|\nabla^2H_t(t)\|^2\leq C\|\nabla\mathbf{\Psi}(t)\|^2+C(\delta+\varepsilon)\|(\nabla\phi,\phi,\mathbf{\Psi})(t)\|^2,\tag{221}$$

and

$$\|\nabla^3H_t(t)\|^2\leq C\|\nabla^2\mathbf{\Psi}(t)\|^2+C(\delta+\varepsilon)\|(\nabla^2\phi,\nabla\phi,\nabla\mathbf{\Psi},\phi,\mathbf{\Psi})(t)\|^2,\tag{222}$$

and

$$\|\nabla^4H_t(t)\|^2\leq C\|\nabla^3\mathbf{\Psi}(t)\|^2+C(\delta+\varepsilon)\|(\nabla^3\phi,\nabla^2\phi,\nabla^2\mathbf{\Psi},\nabla\phi,\nabla\mathbf{\Psi},\phi,\mathbf{\Psi})(t)\|^2.$$

Thus, applying Gronwall inequality to (219), we obtain

$$\|(\nabla H, \phi, \mathbf{\Psi})(t)\|^2_{H^3(\mathbb{R}^3)} \le C\|(\nabla H_0, \phi_0, \mathbf{\Psi}_0)\|^2_{H^3(\mathbb{R}^3)} e^{-\tilde{\mu} t} \tag{223}$$

for some positive constant $\tilde{\mu} > 0$, provided with $\delta + \varepsilon \ll 1$. □

Proof of Theorem 4.1. Theorem 4.1 is immediately proved from Proposition 2. □

REFERENCES

[1] G. Ali, *Global existence of smooth solutions of the N-dimensional Euler-Possion model*, SIAM J. Math. Anal., **35** (2003), 389–422.

[2] G. Ali, D. Bini and D. Rionero, *Global existence and relaxation limit for smooth solutions to the Euler-Possion model for semiconductors*, SIAM J. Math. Anal., **32** (2000), 572–587.

[3] G. Ali and A. Jüngel, *Global smooth solutions to the multi-dimensional hydrodynamic model for two-carrier plasmas*, J. Differential Equations, **190** (2003), 663–685

[4] P. Degond and P. A. Markowich, *On a one-dimensional steady-state hydrodynamic model*, Appl. Math. Lett., **3** (1990), 25–29.

[5] P. Degond and P. A. Markowich, *A steady-state potential flow model for semiconductors*, Ann. Math. Pure Appl., **4** (1993), 87–98.

[6] W. Fang and K. Ito, *Steady-state solutions of a one-dimensional hydrodynamic model for semiconductors*, J. Differential Equations, **133** (1997), 224–244.

[7] I. Gamba, *Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductor*, Comm. Partial Diff. Eqns, **17** (1992), 553–577.

[8] I. Gasser, L. Hsiao and H.-L. Li, *Large time behavior of solutions of the bipolar hydrodynamical model for semiconductors*, J. Differential Equations, **192** (2003), 326–359.

[9] I. Gasser and R. Natalini, *The energy transport and the drift diffusion equations as relaxation limits of the hydrodynamic model for semiconductors*, Quart. Appl. Math., **57** (1996), 269–282.

[10] Y. Guo and W. Strauss, *Stability of semiconductor states with insulating and contact boundary conditions*, Arch. Rational Mech. Anal., **179** (2005), 1–30.

[11] L. Hsiao and T.-P. Liu, *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping*, Comm. Math. Phys., **143** (1992), 599–605.

[12] L. Hsiao, P. A. Markowich and S. Wang, *The asymptotic behavior of globally smooth solutions of the multidimensional isentropic hydrodynamic model for semiconductors*, J. Differential Equations, **192** (2003), 111–133.

[13] F.-M. Huang, M. Mei and Y. Wang, *Large time behavior of solutions to n-dimensional bipolar hydrodynamic model for semiconductors*, SIAM J. Math. Anal., **43** (2011), 1595–1630.

[14] F.-M. Huang, M. Mei, Y. Wang and T. Yang, *Long-time behavior of solutions for bipolar hydrodynamic model of semiconductors with boundary effects,* SIAM J. Math. Anal., in press.

[15] F.-M. Huang, M. Mei, Y. Wang and H. Yu, *Asymptotic convergence to stationary waves for unipolar hydrodynami model of semiconductors*, SIAM J. Math. Anal., **43** (2011), 411–429.

[16] F.-M. Huang, M. Mei, Y. Wang and H. Yu, *Asymptotic convergence to planar stationary waves for multi-dimensional unipolar hydrodynamic model of semiconductors*, J. Differential Equations, **251** (2011), 1305–1331.

[17] A. Jüngel, "Quasi-hydrodynamic Semiconductor Equations," Progress in Nonlinear Differential Equations and their Applications, Vol 41, Birkhäuser Verlag, Besel-Boston-Berlin, 2001.

[18] L. Hsiao and S. Wang, *Asymptotic behavior of global smooth solutions to the full 1D hydrodynamic model for semiconductors*, Math. Models Methods Appl. Sci., **12** (2002), 777–796.

[19] L. Hsiao and K. Zhang, *The global weak solution and relaxation limits of the initial boundary value problem to the bipolar hydrodynamic model for semiconductors*, Math. Models Methods Appl. Sci., **10** (2000), 1333–1361.

[20] L. Hsiao and K. Zhang, *The relaxation of the hydrodynamic model for semiconductors to drift-diffusion equations*, J. Differential Equations, **165** (2000), 315–354.

[21] F.-M. Huang and Y.-P. Li, *Large time behavior and quasineutral limit of solutions to a bipolar hydrodynamic model with large data and vacuum*, Discrete Contin. Dyn. Syst., **24** (2009), 455–470.

[22] H.-L. Li, P. Markowich and M. Mei, *Asymptotic behavior of solutions of the hydrodynamic model of semiconductors*, Proc. Royal Soc. Edinburgh, Sect. A, **132** (2002), 359–378.

[23] H.-L. Li, P. Markowich and M. Mei, *Asymptotic behavior of subsonic entropy solutions of the isentropic Euler-Poisson equations*, Quart. Appl. Math., **60** (2002), 773–796.

[24] Y.-P. Li, *Global existence and asymptotic behavior for a multidimensional nonisentropic hydrodynamic semiconductor model with the heat source*, J. Differential Equations, **225** (2006), 134–167.

[25] Y.-P. Li, *Diffusion relaxation limit of a nonisentropic hydrodynamic model for semiconductors*, Math. Methods Appl. Sci., **30** (2007), 2247–2261.

[26] C.-K. Lin, C.-T. Lin and M. Mei, *Asymptotic behavior of solution to nonlinear damped p-system with boundary effect,* Int. J. Numer. Anal. Model. Ser. B, **1** (2010), 70–92.

[27] T. Luo, R. Natalini and Z. Xin, *Large time behavior of the solutions to a hydrodynamic model for semiconductors*, SIAM J. Appl. Math., **59** (1998), 810–830.

[28] P. A. Markowich, C. A. Ringhofer and C. Schmeiser, "Semiconductor Equations," Springer-Verlag, Vienna, 1990.

[29] P. Marcati and R. Natalini, *Weak solutions to a hydrodynamic model for semiconductors: the Cauchy problem*, Proc. Roy. Soc. Edinburgh, Sect. A, **125** (1995), 115–131

[30] P. Marcati and R. Natalini, *Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation*, Arch. Rational Mech. Anal., **129** (1995), 129–145.

[31] A. Matsumura, *Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equation with the first-order dissipation*, Publ. Res. Inst. Math. Sci., **13** (1977), 349–379.

[32] A. Matsumura and M. Mei, *Convergence to travelling fronts of solutions of the p-system with viscosity in the presence of a boundary*, Arch. Rational Mech. Anal., **146** (1999), 1–22.

[33] M. Mei, *Best asymptotic profile for hyperbolic p-sytem with damping*, SIAM J. Math. Anal., **42** (2010), 1–23.

[34] M. Mei, *Nonlinear diffusion waves for hyperbolic p-system with nonlinear damping*, J. Differentoial Equations, **247** (2009), 1275–1269.

[35] R. Natalini, *The bipolar hydrodynamic model for semiconductors and the drift-diffusion equations*, J. Math. Anal. Appl., **198** (1996), 262–281.

[36] S. Nishibata and M. Suzuki, *Asymptotic stability of a stationary solution to a hydrodynamic model of semiconductors*, Osaka J. Math., **44** (2007), 639–665.

[37] S. Nishibata and M. Suzuki, *Asymptotic stability of a stationary solution to a thermal hydrodynamic model for semiconductors*, Arch. Rational Mech. Anal., **192** (2009), 187–215.

[38] F. Poupaud, M. Rascle and J.-P. Vila, *Global solutions to the isothermal Euler-Poisson system with arbitrarily large data*, J. Differential Equations, **123** (1995), 93–121.

[39] A. Sitenko and V. Malnev, "Plasma Physics Theory," Applied Mathematics and Mathematical Computation, 10. Chapman & Hall, London, 1995.

[40] B. Zhang, *Convergence of the Godunov scheme for a simplified one-dimensional hydrodynamic model for semiconductor devices*, Comm. Math. Phys., **157** (1993), 1–22.

[41] C. Zhu and H. Hattori, *Stability of steady state solutions for an isentropic hydrodynamic model of semiconductors of two species*, J. Differential Equations, **166** (2000), 1–32.

[42] C. Zhu and H. Hattori, *Asymptotic behavior of the solution to a nonisentropic hydrodynamic model of semiconductors,* J. Differential Equations, **144** (1998), 353–389.

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