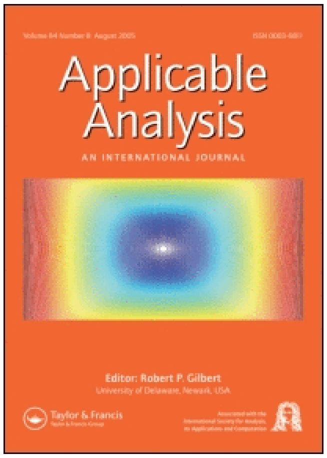

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Long-Time Behavior of Solution for Rosenau-Burgers Equation (II)

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ABSTRACT

This paper establishes more results on the decay of solutions for the Rosenau-Burgers equation $u_t + u_{xxxxt} - \alpha u_{xx} + \beta u_x + \phi(u)_x = 0$. Some new results on the asymptotic behavior of the solutions have been developed by the Fourier transform method with energy estimates, which improve our previous work [6]. Furthermore, the stability of the stationary travelling wave solutions and the exponential time-decay rate are established by the technical weighted energy method.

AMS: 35Q53, 35B40, 35L65

KEY WORDS: Rosenau-Burgers equation, stationary travelling wave, stability, decay rates.

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1. Introduction

Subsequent to our recent work [6], we continuously investigate the asymptotic behavior of the solution for the Rosenau-Burgers equation (R-B) in the form

$$u_t + u_{xxxxt} - \alpha u_{xx} + \beta u_x + \phi(u)_x = 0, \quad x \in R^1, t > 0, \tag{1.1}$$

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with initial data

$$u|_{t=0} = u_0(x) \to u_\pm, \quad \text{as } x \to \pm\infty, \tag{1.2}$$

where $\alpha > 0$, $\beta \in R$ are any fixed constants, the nonlinear function $\phi(u)$ is suitably smooth, say $\phi(u) \in C^2$, $u_\pm$ are the given constants called as the state end points.

Equation (1.1) with $\alpha = 0$ is called as the Rosenau equation proposed by P. Rosenau [11] for treating the dynamics of dense discrete systems in order to overcome the shortcomings by the KdV equation, since the KdV equation describes unidimensional propagation of waves, but wave-wave and wave-wall interactions cannot be treated by it. Such a model were studied by Park [10] and by Chung and Ha [1] for the global existence of the solution to the IBVP. Equation (1.1) with $\alpha > 0$ is called the Rosenau-Burgers equation (R-B E) and somehow corresponds to the KdV-B equation and the BBM-B equation, but it is given neither by Rosenau nor by Burgers. In fact, from both mathematical and physical point of view, the Rosenau equation with the dissipative term $-\alpha u_{xx}$, or say, the Rosenau-Burgers equation (1.1), is proposed if a good predictive power is desired. Such problems arise in some natural phenomena as, for example, in bore propagation and in water waves. For the Rosenau-Burgers equation ($\alpha > 0$), in the case $u_+ = u_- = 0$, we studied the asymptotic behavior of the solution for the Cauchy problem (1.1) and (1.2) in [6]. Where, we proved time decay rates like $\|u(t)\|_{L^2} = O(t^{-1/4})$ for $|\phi'(u)| \le C|u|^{p-1}$ with $p > 7/2$, and $\|u(t)\|_{L^\infty} = O(t^{-1/2})$ for $|\phi'(u)| \le C|u|^{p-1}$ with $p > 2$. However, these decay rates in the previous work [6] are not optimal. In the present paper, we shall improve them by the approach of the Fourier transform together with the energy estimates developed in the author's work [7] for the BBM-B equation via Zhang [13,14]. Another purpose in this paper is to prove the stability of the stationary travelling wave of (1.1) and the exponential time decay rate by the technical weighted energy method, which is used in Kawashima and Matsumura [3], see also Matsumura and Nishihara [5] and the author's works [8,9] for the generalized Burgers equation. Precisely, when $u_+ = u_- = \bar{u} \in R$, we show that the solution of (1.1) and (1.2) tends toward the constant $\bar{u}$ in the forms $\|(u-\bar{u})(t)\|_{L^2} = O(t^{-3/4})$ and $\|(u-\bar{u})(t)\|_{L^\infty} = O(t^{-1})$ for *any* nonlinearity $\phi(u) \in C^2$, which improves our previous work [6]. For the details, see Theorem 2.1 and the Remark below. When $u_+ \neq u_-$, i.e, the shock strength is positive, under some restrictions on the state constants $u_\pm$, there exists a stationary travelling wave solution of (1.1) in the form $u(x-st) = U(x)$, $U(\pm\infty) = u_\pm$, with the zero speed $s = 0$, such a stationary travelling wave is unique up to shift. In this case, we prove that it is nonlinearly stable for both the nondegenerate shock condition $\phi'(u_+) < -\beta < \phi'(u-)$ and the degenerate shock condition $\phi'(u_+) = -\beta < \phi'(u_-)$ or $\phi'(u_+) < -\beta = \phi'(u_-)$ or $\phi'(u_\pm) = -\beta$. Especially, in the nondegenerate case $\phi'(u_+) < -\beta < \phi'(u_-)$, we show that the solution $u(t,x)$ of (1.1) and (1.2) asymptotically converges to the travelling wave $U(x)$ at the decay rate $O(e^{-\theta t})$ for some $\theta > 0$, if the initial perturbation is suitably small and decays like $O(e^{-\eta|x|})$ for some $\eta > 0$.

Our plan in this paper is the following. After stating the notations below, we give our main theorems in Section 2. Section 3 proves the time decay of the solution to $\bar{u}$ in the case of $u_+ = u_- = \bar{u}$. For the case of $u_+ \neq u_-$, the stability of the stationary travelling wave of (1.1) and the exponential time decay rate are shown in Section 4.

Notations. We first make some notations for simplicity. C always denotes some positive constant without confusion. H^k ($k \geq 0$ integer) and $W^{k,p}$ denote the usual Sobolev spaces with the norm $\|\cdot\|_k$ and $\|\cdot\|_{W^{k,p}}$, respectively. L^2 denotes the square integrable space with the norm $\|\cdot\|$, and L^∞ is the essential bounded space with the norm $\|\cdot\|_\infty$. Supposing that $f(x) \in L^1 \cap L^2(R)$, we define the Fourier transforms of $f(x)$ as follows:

$$F[f](\xi) \equiv \hat{f}(\xi) = \int_R f(x)e^{-ix\xi}dx.$$

L^2_w denotes the space of measurable functions on R which satisfy $w(x)^{1/2}f \in L^2$, $w(x) > 0$ is the weight function, with the norm

$$|f|_w = \left(\int_{-\infty}^{\infty} w(x)|f(x)|^2dx\right)^{1/2},$$

also the weighted Sobolev's spaces H_w^k with the norm

$$|f|_{k,w}^2 = \sum_{i=0}^{k} |\partial_x^i f|_w^2.$$

We set $\langle x\rangle = \sqrt{1+x^2}$ and

$$\langle x\rangle_+ = \begin{cases} \sqrt{1+x^2}, & x > 0 \\ 1, & x \le 0, \end{cases} \qquad \langle x\rangle_- = \begin{cases} \sqrt{1+x^2}, & x < 0 \\ 1, & x \ge 0. \end{cases}$$

Note that $L^2 = H^0 = L_0^2$ and $\|\cdot\| = \|\cdot\|_0 \sim |\cdot|_w$ for $C^{-1} \le w(x) \le C$.

Let T and B be a positive constant and a Banach space, respectively. $C^k(0,T; B)$ $(k \ge 0)$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$, and $L^2(0,T;B)$ denotes the space of B-valued L^2-functions on $[0,T]$. The corresponding spaces of B-valued function on $[0,\infty)$ are defined similarly.

2. Main Theorems

2.1. Case: $u_+ = u_- = \bar{u}$

We firstly state the results on the decay of the solution for (1.1) and (1.2) in the case of $u_+ = u_- = \bar{u}$. Suppose that

$$\int_{-\infty}^{\infty} (u_0(x) - \bar{u})dx = 0. \tag{2.1}$$

We denote a new unknown function as follows

$$v(x,t) = \int_{-\infty}^{x} (u(y,t) - \bar{u})dy, \tag*{(2.2)$_1$}$$

namely,

$$u(x,t) = v_x(x,t) + \bar{u}, \tag*{(2.2)$_2$}$$

and let

$$v_0(x) = \int_{-\infty}^{x} (u_0(y) - \bar{u})dy, \tag*{(2.2)$_3$}$$

then the equation (1.1) can be reduced to the "integrated" equation

$$v_t + v_{xxxxt} - \alpha v_{xx} + (\beta + \phi'(\bar{u}))v_x + F(v_x) = 0, \tag{2.3}$$

with the initial data

$$v|_{t=0} = v_0(x), \tag{2.4}$$

where $F(v_x) \equiv \phi(\bar{u}+v_x) - \phi(\bar{u}) - \phi'(\bar{u})v_x$ satisfying $|F(v_x)| \le C|v_x|^2$. One of the main theorems is stated as follows.

Theorem 2.1($u_+ = u_- = \bar{u}$). *Suppose that (2.1) and $v_0(x) \in W^{3,1}$ hold, then there exists a positive constant ε_1 such that when $\|v_0\|_{W^{3,1}} < \varepsilon_1$, then (1.1) and (1.2) has a unique global solution $u(x,t)$ satisfying*

$$u(x,t) - \bar{u} \in C(0,\infty; H^2 \cap W^{1,\infty}),$$

and the asymptotic decay rates

$$\|v(t)\| \le C(1+t)^{-1/4}, \quad \|v_x(t)\| \le C(1+t)^{-3/4}, \tag{2.5}$$

$$\|v(t)\|_\infty \le C(1+t)^{-1/2}, \quad \|v_x(t)\|_\infty \le C(1+t)^{-1}, \tag{2.6}$$

for all $t \ge 0$.

Remark. In [6], when $\bar{u} = 0$ and $|\phi'(u)| \le C|u|^{p-1}$, we get the decay rates as $\|u(t)\| \le C(1+t)^{-1/4}$ for $p > 7/2$ and $\|u(t)\|_\infty \le C(1+t)^{-1/2}$ for $p > 2$. However, note $(2.2)_1$ or $(2.2)_2$, our new results in (2.5) and (2.6) for $\bar{u} = 0$ give us that $\|u(t)\| \le C(1+t)^{-3/4}$ and $\|u(t)\|_\infty \le C(1+t)^{-1}$ for *any* $\phi \in C^2$. Therefore, we here get much stronger decay rates with much weaker conditions than those in the former work [6].

2.2. Case: $u_+ \neq u_-$

Secondly, we consider the asymptotic stability of the stationary travelling wave of (1.1) in the case of $u_+ \neq u_-$. We first recall what the stationary travelling wave solution means, and state its existence. $U(x)$ is called a stationary travelling wave solution of (1.1), and this means that it is a solution of (1.1) of the form

$u(x-st)=U(x)$, $U\pm\infty)=u_\pm$, with the zero speed $s=0$, exactly, which is the solution of the following ordinary differential equation:

$$\begin{cases} -\alpha U_{xx}+\beta U_x+\phi(U)_x=0 \\ U(\pm\infty)=u_\pm. \end{cases} \tag{2.7}$$

The speed of propagation $s=0$ and the state constants $u_\pm$ satisfy the Rankine-Hugniot condition

$$s=0=\beta+\frac{\phi(u_+)-\phi(u_-)}{u_+-u_-}, \tag{2.8}$$

and the Oleinik's entropy condition

$$f(u)\equiv[\beta(u-u_\pm)+\phi(u)-\phi(u_\pm)]\begin{cases} <0, & \text{if} \quad u_+<u<u_- \\ >0, & \text{if} \quad u_-<u<u_+. \end{cases} \tag{2.9}$$

Such an entropy condition (2.9) implies that

$$\phi'(u_+)<-\beta<\phi'(u_-), \tag*{(2.10)$_1$}$$

or

$$\phi'(u_+)=-\beta<\phi'(u_-) \quad \text{or} \quad \phi'(u_+)<-\beta=\phi'(u_-) \quad \text{or} \quad \phi(u_\pm)=-\beta. \tag*{(2.10)$_2$}$$

The condition $(2.10)_1$ is the well-known Lax's shock condition, we call it the *non-degenerate shock condition*, or the *noncontact shock condition*. For each case in $(2.10)_2$, we call it the *degenerate shock condition*, or the *contact shock condition*. Integrating (2.7) over $(\pm\infty,x)$ yields

$$\alpha U_x=\beta(U-u_\pm)+\phi(U)-\phi(u_\pm)\equiv: f(U). \tag{2.11}$$

Thanks to Kawashima and Matsumura [3], see also Matsumura and Nishihara [5], we have the existence of travelling wave solutions from the sufficient and necessary conditions (2.8) and (2.9). In the degenerate cases $(2.10)_2$, we note $k_\pm=n$ if $f'(u_\pm)=\cdots=f^{(n)}(u_\pm)=0$ and $f^{(n+1)}(u_\pm)\neq 0$. Due to the well-known arguments in [3,5], we state the existence of the stationary travelling wave without proof as follows.

Claim. *There exists a stationary travelling wave solution $U(x)$ of (1.1) with $U(\pm\infty) = u_\pm$, unique up to a shift, if and only if the Rankine-Hugoniot condition (2.8) and Oleinik's condition (2.9) hold.*

Moreover, such a stationary travelling wave $U(x)$ satisfies

$$u_+ \lessgtr U(x) \lessgtr u_-, \quad \textit{and} \quad f(U) \lessgtr 0, \quad \textit{for} \quad u_+ \lessgtr u_-, \tag{2.12}$$

and is such that, as $x \to \pm\infty$,

$$|U(x) - u_\pm| \sim exp(-c_\pm|x|), \quad if \quad \phi'(u_+) < -\beta < \phi'(u_-), \tag{2.13}$$

$$|U(x) - u_\pm| \sim |x|^{-1/k_\pm}, \quad if \quad -\beta = \phi'(u_+) \text{ or } \phi'(u_-), \tag{2.14}$$

where $c_\pm = |\beta + \phi'(u_\pm)|/\alpha$.

Without loss generality, we focus on

$$u_+ < u_-,$$

which implies $f(U) < 0$ for $U \in (u_+, u_-)$ from (2.12). Defining the weight functions, cf [5,9]

$$w_1(U) = \frac{(U - u_+)(U - u_-)}{f(U)}, \quad w_2(U) = -\frac{(U - u_+)^{1/2}(u_- - U)^{1/2}}{f(U)}, \tag{2.15}$$

we know that $w_i(U) > 0$ and in C^2 $(i = 1, 2)$ due to $u_+ < U < u_-$ and $f(U) < 0$, and that

$$w_1(U) \sim C \quad \text{for} \quad \phi'(u_+) < -\beta < \phi'(u_-), \tag{2.16}$$

$$w_1(U) \sim \langle x\rangle_+, \text{or } \langle x\rangle_-, \text{ or } \langle x\rangle, \tag{2.17}$$

for $\phi'(u_+) = -\beta < \phi'(u_-)$, or $\phi'(u_+) < -\beta = \phi'(u_-)$, or $\phi'(u_+) = -\beta = \phi'(u_-)$, and also

$$w_2(U) \sim \begin{cases} e^{c_1|x|}, & \text{for } x > 0 \\ e^{c_2|x|}, & \text{for } x \leq 0 \end{cases} \qquad \text{for } \phi'(u_+) < -\beta < \phi'(u_-), \tag{2.18}$$

with some positive constants c_1 and c_2.

Suppose that

$$\int_{-\infty}^{\infty}(u_0(x)-U(x))dx=0 \tag{2.19}$$

and define

$$\psi_0(x)=\int_{-\infty}^{x}(u_0(y)-U(y))dy. \tag{2.20}$$

Then the other two main theorems on the stability of the stationary travelling wave are stated as follows.

Theorem 2.2($u_+\neq u_-$: Stability of Travelling Wave). *Suppose that (2.8), (2.9) and (2.19) hold. $U(x)$ is the stationary travelling wave solution of (1.1) connecting $u_\pm$.*

(i) When $\phi'(u_+)<-\beta<\phi'(u_-)$, assume $\psi_0(x)\in H^4$, then there exists a positive constant ε_2 such that when $\|\psi_0\|_4+|u_+-u_-|<\varepsilon_2$, then (1.1) and (1.2) has a unique global solution $u(x,t)$ satisfying

$$u(x,t)-U(x)\in C(0,\infty;H^3(R))\cap L^2(0,\infty;H^2(R))$$

and the asymptotic stability holds such that

$$\sup_{x\in R}|\partial_x^l u(x,t)-\partial_x^l U(x)|\to 0,\quad l=0,1,2,\ \text{as } t\to\infty, \tag{2.21}$$

where $\partial_x^l u=\partial^l u/\partial x^l$.

(ii) When $\phi'(u_+)=-\beta<\phi'(u_-)$, or $\phi'(u_+)<-\beta=\phi'(u_-)$ or $\phi'(u_+)=-\beta=\phi'(u_-)$, assume $\psi_0(x)+\partial_x^4\psi_0\in H^2_{w_1}$, where $w_1(U)\sim\langle x\rangle_+$, or $\langle x\rangle_-$ or $\langle x\rangle$ corresponding to the above shock conditions. Then there exists a positive constant ε_3 such that when $|\psi_0+\partial_x^4|_{2,w}+|u_+-u_-|<\varepsilon_3$, then (1.1) and (1.2) has a unique global solution $u(x,t)$ satisfying

$$u-U+\partial_x^4 u-\partial_x^4 U\in C(0,\infty;H^1_{w_1}),$$

$$u(x,t)-U(x)\in L^2(0,\infty;H^4_{w_1}).$$

Moreover, the asymptotic stability (2.21) holds.

Theorem 2.3($u_+ \neq u_-$: Decay rate). *Suppose that (2.8), (2.9) and (2.19) hold. $U(x)$ is the stationary travelling wave solution of (1.1) connecting $u_\pm$. When $\phi'(u_+) < -\beta < \phi'(u_-)$, assume $\psi_0 + \partial_x^4\psi_0 \in H^2_{w_2}$, then there exist the positive constants ε_4 and θ such that when $|\psi_0 + \partial_x^4\psi_0|_{2,w_2} + |u_+ - u_-| < \varepsilon_4$, then (1.1) and (1.2) has a unique global solution $u(x,t)$ satisfying*

$$u - U + \partial_x^4 u - \partial_x^4 U \in C(0,\infty; H^1_{w_2}),$$

$$u - U \in L^2(0,\infty; H^4_{w_2}),$$

and the time decay rate

$$\sup_{x\in R} |(u - U + \partial_x^4 u - \partial_x^4 U)(t)| \leq Ce^{-\theta t} \tag{2.22}$$

holds for all $t \geq 0$.

3. The Case: $u_+ = u_- = \bar{u}$

In this section, when $u_+ = u_- = \bar{u}$, we are going to prove that the Cauchy problem (1.1) and (1.2) has a unique global solution, which asymptotically converges to the trivial constant solution $\bar{u}$ in the forms $0(t^{-3/4})$ of L^2-sense and $0(t^{-1})$ of L^∞-sense as $t \to \infty$.

We take the Fourier transform to (2.3) to obtain

$$\hat{v}_t + (i\xi)^4\hat{v}_t - \alpha(i\xi)^2\hat{v} + (\beta + \phi'(\bar{u}))i\xi\hat{v} + \widehat{F(v_x)} = 0, \tag{3.1}$$

which means that

$$\hat{v}(\xi,t) = e^{-A(\xi)t}\hat{v}_0(\xi) - \int_0^t e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}}{1+\xi^4}ds, \tag{3.2}$$

where

$$A(\xi) = B(\xi) + \frac{(\beta + \phi'(\bar{u}))i\xi}{1+\xi^4} \equiv \frac{\alpha\xi^2 + (\beta + \phi'(\bar{u}))i\xi}{1+\xi^4}. \tag{3.3}$$

Taking the inverse Fourier transform of (3.2) yields

$$
\begin{aligned}
v(t,x) =&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)t}\hat{v}_0(\xi)d\xi \\
&-\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4}d\xi ds.
\end{aligned}
\tag{3.4}
$$

Differentiating (3.4) with respect to x, we obtain

$$
\begin{aligned}
v_x(t,x) =&\frac{1}{2\pi}\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)t}\hat{v}_0(\xi)d\xi \\
&-\frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} i\xi e^{i\xi x}e^{-A(\xi)(t-s)}\frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4}d\xi ds.
\end{aligned}
\tag{3.5}
$$

For a positive constant δ, define a Banach space as follows

$$X_\delta = \{v \in C(0,\infty; H^1 \cap W^{1,\infty}) | M(v,v_x) \le \delta\}$$

with the distance

$$
\begin{aligned}
M(v,v_x) = \sup_{0\le t<\infty} &\{(1+t)^{1/4}\|v(t)\| + (1+t)^{3/4}\|v_x(t)\| \\
&+(1+t)^{1/2}\|v(t)\|_\infty + (1+t)\|v_x(t)\|_\infty\}.
\end{aligned}
$$

Rewriting (3.4) as the form $v = Sv$, we want to prove that there exists the positive constant δ_1, such that the operator S maps X_{δ_1} into itself and has a unique fixed point in X_{δ_1}, namely, such a fixed point $v(x,t)$ is the solution of (3.4). To this end, we need several preliminary results.

Lemma 3.1. *Suppose that $a > 0$ and $b > 0$, and $\max(a,b) > 1$, then*

$$\int_0^t (1+s)^{-a}(1+t-s)^{-b}ds \le C(1+t)^{-\min(a,b)}. \tag{3.6}$$

For the proof of Lemma 3.1 we refer to Segal [12] and Matsumura [4].

Lemma 3.2. *The following results hold*

$$\int_{-\infty}^{\infty} \frac{e^{-B(\xi)t}}{1+\xi^4} d\xi \le C(1+t)^{-1/2}, \tag{3.7}$$

$$\int_{-\infty}^{\infty} |\xi| \frac{e^{-B(\xi)t}}{1+\xi^4} d\xi \le C(1+t)^{-1}, \tag{3.8}$$

$$\int_{-\infty}^{\infty} |\xi|^2 \frac{e^{-2B(\xi)t}}{1+\xi^6} d\xi \le C(1+t)^{-3/2}, \tag{3.9}$$

for all $t \ge 0$.

Lemma 3.3. *The following results hold*

$$\left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi) d\xi \right\| \le C\|v_0\|_{W^{2,1}} (1+t)^{-1/4}, \tag{3.10}$$

$$\left\| \int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi) d\xi \right\| \le C\|v_0\|_{W^{3,1}} (1+t)^{-3/4}, \tag{3.11}$$

$$\left\| \int_{-\infty}^{\infty} e^{ix\xi} e^{-A(\xi)t} \hat{v}_0(\xi) d\xi \right\|_{\infty} \le C\|v_0\|_{W^{3,1}} (1+t)^{-1/2}, \tag{3.12}$$

$$\left\| \int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi) d\xi \right\|_{\infty} \le C\|v_0\|_{W^{3,1}} (1+t)^{-1}, \tag{3.13}$$

for all $t \ge 0$.

Lemmas 3.2 and 3.3 have been proved in our previous work [6].

Lemma 3.4. *Suppose that* $v(x,t) \in X_\delta$, *then*

$$\left\| \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4} d\xi \right\| \le C\delta^2 (1+s)^{-\frac{3}{2}} (1+t-s)^{-\frac{1}{4}}, \tag{3.14}$$

$$\left\| \int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4} d\xi \right\| \le C\delta^2 (1+s)^{-\frac{3}{2}} (1+t-s)^{-\frac{3}{4}}, \tag{3.15}$$

$$\left\| \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4} d\xi \right\|_{\infty} \le C\delta^2 (1+s)^{-\frac{3}{2}} (1+t-s)^{-\frac{1}{2}}, \tag{3.16}$$

$$\left\| \int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4} d\xi \right\|_{\infty} \leq C\delta^2(1+s)^{-\frac{3}{2}}(1+t-s)^{-1}, \qquad (3.17)$$

hold for all $t \geq s \geq 0$.

Proof. By the Parseval's equality and (3.7) in Lemma 3.2, we have

$$\begin{aligned}
&\left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4} d\xi \right\| \\
&= \left\| e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}}{1+\xi^4} \right\| \\
&= \left\{ \int_{-\infty}^{\infty} \left| e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}}{1+\xi^4} \right|^2 d\xi \right\}^{1/2} \\
&\leq \sup_{\xi \in R} |\widehat{F(v_x)}(\xi,s)| \left\{ \int_{-\infty}^{\infty} \frac{e^{-2B(\xi)(t-s)}}{(1+\xi^4)^2} d\xi \right\}^{1/2} \\
&\leq \sup_{\xi \in R} |\widehat{F(v_x)}(\xi,s)| \left\{ \int_{-\infty}^{\infty} \frac{e^{-B(\xi)(t-s)}}{(1+\xi^4)} d\xi \right\}^{1/2} \\
&\leq C(1+t-s)^{-1/4} \sup_{\xi \in R} |\widehat{F(v_x)}(\xi,s)|. \qquad (3.18)
\end{aligned}$$

According to the definitions of $\widehat{F(v_x)}$ and X_δ, and noting $|F| \leq C|v_x|^2$, we have

$$\begin{aligned}
&\sup_{\xi \in R} |\widehat{F(v_x)}| = \sup_{\xi \in R} \left| \int_{-\infty}^{\infty} e^{-ix\xi} F(v_x)(x,s) dx \right| \\
&\leq \int_{-\infty}^{\infty} |F(v_x)| dx \leq C\|v_x(s)\|^2 \leq C\delta^2(1+s)^{-\frac{3}{2}}. \qquad (3.19)
\end{aligned}$$

Therefore, substituting (3.19) into (3.18) yields (3.14).

Similarly, using the Parseval's equality and (3.9),(3.19), we can prove (3.15). The details are omitted.

To prove (3.16), note (3.7) and (3.19), then we have

$$\left| \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4} d\xi \right|$$

$$\leq \int_{-\infty}^{\infty} e^{-B(\xi)(t-s)} \frac{|\widehat{F(v_x)}(\xi,s)|}{1+\xi^4} d\xi$$

$$\leq \sup_{\xi\in R} |\widehat{F(v_x)}(\xi,s)| \int_{-\infty}^{\infty} \frac{e^{-B(\xi)(t-s)}}{1+\xi^4} d\xi$$

$$\leq C\delta^2(1+s)^{-3/2}(1+t-s)^{-1/2}, \tag{3.20}$$

which implies (3.16).

By the same approach, using (3.8) and (3.19), we can easily prove (3.17). The details are also omited here. □

Proof of Theorem 2.1. We are going to prove that there exists a positive constant δ_1 such that the operator S is a contraction mapping from X_{δ_1} into X_{δ_1}.

Step 1. $S : X_\delta \to X_\delta$. For any $v_1(x,t) \in X_\delta$, and denote $v = Sv_1$, we now want to prove that $v = Sv_1 \in X_\delta$. Indeed, using (3.10) in Lemma 3.3, (3.14) in Lemma 3.4, and (3.6) in Lemma 3.1, we have

$$\|v(t)\| = \|Sv_1\|$$

$$\leq \frac{1}{2\pi}\Big\| \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi) d\xi \Big\|$$

$$+ \frac{1}{2\pi} \int_0^t \Big\| \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi,s)}{1+\xi^4} d\xi \Big\| ds$$

$$\leq C\|v_0\|_{W^{2,1}}(1+t)^{-\frac{1}{4}} + C\delta^2 \int_0^t (1+s)^{-\frac{3}{2}}(1+t-s)^{-\frac{1}{4}} ds$$

$$\leq C\|v_0\|_{W^{2,1}}(1+t)^{-\frac{1}{4}} + C\delta^2(1+t)^{-\frac{1}{4}}. \tag{3.21}$$

Similarly, in the same way, we have due to (3.11), (3.15) and (3.6)

$$\|v_x(t)\| = \|\partial_x Sv_1\|$$

$$\le \frac{1}{2\pi}\Big\| \int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)t} \hat{v}_0(\xi) d\xi \Big\|$$

$$+ \frac{1}{2\pi} \int_0^t \Big\| \int_{-\infty}^{\infty} i\xi e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_x)}(\xi, s)}{1+\xi^4} d\xi \Big\| ds$$

$$\le C\|v_0\|_{W^{3,1}}(1+t)^{-\frac{3}{4}} + C\delta^2 \int_0^t (1+s)^{-\frac{3}{2}}(1+t-s)^{-\frac{3}{4}} ds$$

$$\le C\|v_0\|_{W^{3,1}}(1+t)^{-\frac{3}{4}} + C\delta^2(1+t)^{-\frac{3}{4}}. \tag{3.22}$$

In the same fashion, we can prove that

$$\|Sv_1(t)\|_\infty \le C\|v_0\|_{W^{3,1}}(1+t)^{-\frac{1}{2}} + C\delta^2(1+t)^{-\frac{1}{2}} \tag{3.23}$$

from (3.12),(3.16) and (3.6), as well as

$$\|\partial_x Sv_1(t)\|_\infty \le C\|v_0\|_{W^{3,1}}(1+t)^{-1} + C\delta^2(1+t)^{-1} \tag{3.24}$$

from (3.13),(3.17) and (3.6). Thus, (3.21) $-$ (3.24) imply that

$$M(v, v_x) \le c_3(\|v_0\|_{W^{3,1}} + \delta^2) \tag{3.25}$$

for some positive constant c_3 independent of $\|v_0\|_{W^{3,1}}$ and δ.

Thus, there exists some small constant $\delta_3 > 0$, and letting $\|v_0\|_{W^{3,1}} \le \delta_3/2c_3$, and $\delta \le \min\{\delta_3,\ 1/2c_3\}$, we have proved $M(v, v_x) \le \delta$ for some small δ, namely, $S : X_\delta \to X_\delta$ for some small δ.

Step 2. S is contraction in X_δ. Suppose that $v_1(x,t), v_2(x,t) \in X_\delta$, and noting the fact that

$$\sup_{\xi\in R} |\widehat{F(v_{1x})} - \widehat{F(v_{2x})}|$$

$$\le \int_{-\infty}^{\infty} |F(v_{1x}) - F(v_{2x})| dx$$

$$\le C(\|v_{1x}(s)\| + \|v_{2x}(s)\|)\|(v_{1x} - v_{2x})(s)\|$$

$$\le C\delta M(v_1 - v_2, v_{1x} - v_{2x})(1+s)^{-\frac{3}{2}}, \tag{3.26}$$

then we have by the Parseval's equality and Lemma 3.1

$$
\begin{aligned}
&\|Sv_1(t) - Sv_2(t)\| \\
&\le \frac{1}{2\pi}\int_0^t \left\| \int_{-\infty}^{\infty} e^{i\xi x} e^{-A(\xi)(t-s)} \frac{\widehat{F(v_{1x})} - \widehat{F(v_{2x})}}{1+\xi^4} d\xi \right\| ds \\
&\le C \int_0^t (1+t-s)^{-\frac{1}{4}} \sup_{\xi\in R} |(\widehat{F(v_{1x})} - \widehat{F(v_{2x})})(\xi, s)| ds \\
&\le C\delta M(v_1 - v_2, v_{1x} - v_{2x}) \int_0^t (1+s)^{-\frac{3}{2}} (1+t-s)^{-\frac{1}{4}} ds \\
&\le C\delta M(v_1 - v_2, v_{1x} - v_{2x})(1+t)^{-\frac{1}{4}}.
\end{aligned} \tag{3.27}
$$

We also have by the same way in (3.27)

$$\|(Sv_1(t) - Sv_2)_x(t)\| \le C\delta M(v_1 - v_2, v_{1x} - v_{2x})(1+t)^{-\frac{3}{4}}, \tag{3.28}$$

$$\|(Sv_1(t) - Sv_2)(t)\|_\infty \le C\delta M(v_1 - v_2, v_{1x} - v_{2x})(1+t)^{-\frac{1}{2}}, \tag{3.29}$$

$$\|(Sv_1(t) - Sv_2)_x(t)\|_\infty \le C\delta M(v_1 - v_2, v_{1x} - v_{2x})(1+t)^{-1}, \tag{3.30}$$

where the details are omitted. Therefore, from (3.27) to (3.30), we obtain

$$M(Sv_1 - Sv_2, (Sv_1 - Sv_2)_x) \le c_4\delta M(v_1 - v_2, v_{1x} - v_{2x}) \tag{3.31}$$

for some positive constant c_4 independent of δ and $M(v_1 - v_2, v_{1x} - v_{2x})$.

Let us choose $\delta \le \delta_4 < 1/c_4$, we have proved

$$M(Sv_1 - Sv_2, (Sv_1 - Sv_2)_x) < M(v_1 - v_2, v_{1x} - v_{2x}),$$

i.e. $S : X_\delta \to X_\delta$ is contraction for some small δ.

Thanks to Steps 1 and 2, let $\delta_1 < \min\{\delta_3, \delta_4\}$, we have proved that the operator S is contraction from X_{δ_1} to X_{δ_1}. By the Banach's fixed point theorem, we see that S has a unique fixed point $v(x,t)$ in X_{δ_1}. This means the integral equation (3.4) has a unique global solution $v(x,t)$, which satisfies (2.5) and (2.6). Thus, we have completed the proof of Theorem 2.1. $\square$

4. The Case: $u_+ \neq u_-$

This section proves the asymptotic stability of the stationary travelling wave and the exponential time decay rate for the small initial perturbation with spatial exponential decay.

Let $U(x)$ be the stationary travelling wave solution of (1.1). Without loss of generality, we assume $u_+ < u_-$. So, $\alpha U_x = f(U) < 0$ and $u_+ < U < u_-$ by the Claim in Section 2. Set

$$\psi_x(x,t) = u(x,t) - U(x), \tag{4.1}$$

then the initial problem (1.1) and (1.2) is reduced to a "integrated" equation

$$\psi_t + \psi_{xxxxt} - \alpha\psi_{xx} + f'(U)\psi_x = G \tag{4.2}$$

with the initial data

$$\psi(x,0) = \psi_0(x), \tag{4.3}$$

where $G = -\phi(U+\psi_x) + \phi(U) + \phi'(U)\psi_x$ satisfies $|G| \leq C|\psi_x|^2$.

We define the solution spaces of (4.2) and (4.3) for any $T \in [0,\infty]$ as follows

$$Y_1(0,T) = \{\psi \in C(0,T;H^4), \psi_x \in L^2(0,T;H^2\}$$

$$Y_2(0,T) = \{\psi + \partial_x^4\psi \in C(0,T;H^2_{w_1}), \psi_x \in L^2(0,T;H^4_{w_1}\}$$

$$Y_3(0,T) = \{\psi + \partial_x^4\psi \in C(0,T;H^{2}_{w_2}), \psi \in L^2(0,T;H^5_{w_2}\}$$

and set

$$N_1(t) = \sup_{0\leq s\leq t} \|\psi(s)\|_4, \quad N_2(t) = \sup_{0\leq s\leq t} |(\psi + \partial_x^4\psi)(s)|_{2,w_1},$$

$$N_3(t) = \sup_{0\leq s\leq t} |(\psi + \partial_x^4\psi)(s)|_{2,w_2}.$$

Then (4.2) and (4.3) can solved globally in time as follows.

Theorem 4.1. *Under the assumptions in Theorem 2.2.*

(i). When $\phi'(u_+) < -\beta < \phi'(u_-)$, then there exists $\delta_5 > 0$ such that if $\|\psi_0\|_4 + |u_+ - u_-| < \delta_5$, then (4.2) and (4.3) has a unique global solution $\psi \in Y_1(0,\infty)$ satisfying the following estimate for all $t \geq 0$

$$\|\psi(t)\|_4^2 + \int_0^t \|\psi_x(s)\|_2^2 ds \leq C\|\psi_0\|_4^2. \tag{4.4}$$

(ii). When $\phi'(u_+) = -\beta < \phi'(u_-)$, or $\phi'(u_+) < -\beta = \phi'(u_-)$ or $\phi'(u_+) = -\beta = \phi'(u_-)$, then there exists $\delta_6 > 0$ such that if $|\psi_0 + \partial_x^4\psi_0|_{2,w_1} + |u_+ - u_-| < \delta_6$, then (4.2) and (4.3) has a unique global solution $\psi \in Y_2(0,\infty)$ satisfying the following estimate for all $t \geq 0$

$$|(\psi + \partial_x^4\psi)(t)|_{2,w_1}^2 + \int_0^t |\psi_x(s)|_{4,w_1}^2 ds \leq C|\psi_0 + \partial_x^4\psi_0|_{2,w_1}^2. \tag{4.5}$$

(iii). When $\phi'(u_+) < -\beta < \phi'(u_-)$, then there exists $\delta_7 > 0$ such that if $|\psi_0 + \partial_x^4\psi_0|_{2,w_2} + |u_+ - u_-| < \delta_7$, then (4.2) and (4.3) has a unique global solution $\psi \in Y_3(0,\infty)$ satisfying the following estimates for all $t \geq 0$

$$|(\psi + \partial_x^4\psi)(t)|_{2,w_2}^2 + 2\theta\int_0^t |\psi(s)|_{5,w_2}^2 ds \leq |\psi_0 + \partial_x^4\psi_0|_{2,w_2}^2, \tag{4.6}$$

$$|(\psi + \partial_x^4\psi)(t)|_{1,w_2} \leq Ce^{-\theta t}. \tag{4.7}$$

Since Theorem 4.1 implies Theorems 2.2 and 2.3, the proof of Theorem 4.1 is our main purpose in the remainds of this paper. Theorem 4.1 can be proved by the weighted energy method, together the local existence with the *a priori* estimates.

Proposition 4.2 (Local existence). *Suppose the conditions in Theorem 3.1 hold. Then, for any given $\delta_0 > 0$, there is a positive constant T_0 depending on δ_0 such that the problem (4.2),(4.3) has a unique solution $\phi(t,\xi) \in Y_i(0,T_0)$ $(i = 1,2,3)$ corresponding to $\|\psi_0\|_4$ or $|\psi_0 + \partial_x^4\psi_0|_{2,w_1}$ or $|\psi_0 + \partial_x^4\psi_0|_{2,w_1} < \delta_0$.*

Proposition 4.3 (*A priori* estimate). *Let T be a positive constant, and $\phi(t,\xi)$ be a solution of the problem (4.2),(4.3).*

(i). When $\phi'(u_+) < -\beta < \phi'(u_-)$, then there exists $\delta_8 > 0$ such that if $\|\psi_0\|_4 < \delta_8$, the solution $\psi \in Y_1(0,T)$ satisfies (4.4).

(ii). When $\phi'(u_+) = -\beta < \phi'(u_-)$, or $\phi'(u_+) < -\beta = \phi'(u_-)$ or $\phi'(u_+) = -\beta = \phi'(u_-)$, then there exists $\delta_9 > 0$ such that if $|\psi_0 + \partial_x^4\psi_0|_{2,w_1} < \delta_9$, then the solution $\psi \in Y_2(0,T)$ satisfies (4.5).

(iii). When $\phi'(u_+) < -\beta < \phi'(u_-)$, then there exists $\delta_{10} > 0$ such that if $|\psi_0 + \partial_x^4\psi_0|_{2,w_2} < \delta_{10}$, then the solution $\psi \in Y_3(0,T)$ satisfies (4.6) and (4.7).

Since Proposition 4.2 can be proved in the standard way, we omit its proof. Once Proposition 4.3 is proved, using the continuation arguments based on Propositions 4.2 and 4.3, we can show Theorem 4.1. We are going to prove Proposition 4.3.

Multiplying (4.2) by $2w(U)(\psi + \partial_x^4\psi)$, and using $\alpha U_x = f(U)$, we have

$$\begin{aligned}
&\{w(\psi + \partial_x^4\psi)^2\}_t - \{\cdots\}_x + 2\alpha w\psi_{xxx}^2 - U_x(wf)''(U)\psi^2 \\
&+ [2\alpha w + (wf')_{xxx}]\psi_x^2 - U_x[(wf)''(U) + 2(wf')'(U)]\psi_{xx}^2 \\
&= Gw(\psi + \partial_x^4\psi),
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
\{\cdots\}_x =&\{2\alpha w\psi\psi_x - \alpha w_x\psi^2 - f'w\psi^2 \\
&+ 2\alpha w\psi_{xx}\psi_{xxx} - \alpha w_x\psi_{xx}^2 - 2f'w\psi_x\psi_{xxx} \\
&+ f'w\psi_{xx}^2 + 2f'w_x\psi_x\psi_{xx} - f'w_{xx}\psi_x^2\}_x
\end{aligned}$$

which will disappear after integration with respect to x over $(-\infty, \infty)$. Since $\alpha U_x = f(U) < 0$, we first see from (2.15)

$$-U_x(w_1f)''(U) = 2|U_x| > 0$$

for all shock conditions, and that

$$-U_x(w_2f)''(U) \geq c_5w_2(U)$$

for $\phi'(u_+) < -\beta < \phi'(u_-)$, where $c_5 = (u_- - u_+)^2/(4\alpha c_6)$, $c_6 = \max\limits_{[u_+,u_-]} w_1(U)^2 > 0$. To see this, since $|f(U)| \sim |U - u_\pm|$ as $U \to u_\pm$, and $w_2(U) \sim |U - u_\pm|^{-1/2}$ for $\psi'(u_+) < -\beta < \psi'(u_-)$, we compute it directly from (2.15)

$$-\frac{U_x(w_2 f)''(U)}{w_2(U)} = -\frac{f(U)^2((U-u_+)^{1/2}(u_- - U)^{1/2})''}{\alpha(U-u_+)^{1/2}(u_- - U)^{1/2}}$$

$$= \frac{f(U)^2(u_- - u_+)^2}{4\alpha(U-u_+)^2(u_- - U)^2} \geq c_5.$$

Secondly, we can also check that

$$|U_x[(w_i f)''(U) + 2(w_i f')'(U)]/w_i(U)| \leq C|u_- - u_+|, \tag{4.9}$$

$$|(w_i f')_{xxx}(U)/w_i(U)| \leq C|u_- - u_+|, \tag{4.10}$$

for $i = 1, 2$. For the proof of (4.9), by using $|\alpha U_x| = |f(U)| \leq C|u_+ - u_-|^2$, $|f''(U)| \leq C$ and $|f'(U)| = |f'(U) - f'(u_*)| \leq C|u_+ - u_-|$, where u_* denotes the point in (u_+, u_-) such that $f'(u_*) = 0$, also by (2.15)–(2.18), then we prove the following result for all shock conditions

$$|U_x[(w_1 f)''(U) + 2(w_1 f')'(U)]/w_1(U)|$$

$$= 2\frac{|f(U)|}{\alpha w_1}|1 + w_1(U)f''(U) + w_1'(U)f'(U)|$$

$$\leq \frac{2}{\alpha}\Big(\frac{|f(U)|}{w_1(U)} + |f''(U)f(U)| + \frac{|f'(U)(2U - u_+ - u_-)|}{w_1(U)} + |f'(U)|^2\Big)$$

$$\leq C|u_+ - u_-|, \quad \text{for } |u_+ - u_-| \ll 1. \tag{4.11}$$

Similar to (4.11), by simple but trivial calculations, we can easily prove (4.9) for $i = 1$, i.e., $w_1(U)$, as well as (4.10) for $i = 1, 2$, i.e., $w_1(U)$ and $w_2(U)$. Here we omit the proofs. Integrating (4.8) over $[0, t] \times R$, and using (4.9) and (4.10), we have proved the following basic energy estimates.

Lemma 4.4. *Let $\psi(x,t)$ be the solution of (4.2) and (4.3).*

(i). When $\phi'(u_+) < -\beta < \phi'(u_-)$ and $\psi \in Y_1(0,T)$, then ψ satisfies

$$\|(\psi+\partial_x^4\psi)(t)\|^2 + (1-CN_1(t))\int_0^t \|\psi_x(s)\|^2 + \|\partial_x^3\psi(s)\|^2 ds$$

$$\le C\|\psi_0+\partial_x^4\psi_0\|^2 + C|u_+-u_-|\int_0^t \|\psi_{xx}(s)\|^2 ds. \tag{4.12}$$

(ii). When $\phi'(u_+) = -\beta < \phi'(u_-)$, or $\phi'(u_+) < -\beta = \phi'(u_-)$ or $\phi'(u_+) = -\beta = \phi'(u_-)$, and $\psi \in Y_2(0,T)$, then we have

$$|(\psi+\partial_x^4\psi)(t)|_{w_1}^2 + (1-CN_2(t))\int_0^t |\psi_x(s)|_{4,w_1}^2 ds$$

$$\le C|\psi_0+\partial_x^4\psi_0|_{w_1}^2 + C|u_+-u_-|\int_0^t |\psi_{xx}(s)|_{w_1}^2 ds. \tag{4.13}$$

(iii). When $\phi'(u_+) < -\beta < \phi'(u_-)$, then there exists $C_1 > 0$ such that the solution $\psi \in Y_3(0,T)$ satisfying

$$|(\psi+\partial_x^4\psi)(t)|_{w_2}^2 + 2C_1\int_0^t |\psi_x(s)|_{1,w_2}^2 + |\partial_x^3\psi(s)|_{w_2}^2 ds$$

$$\le |\psi_0+\partial_x^4\psi_0|_{w_2}^2 + CN_3(t)\int_0^t |\psi_x(s)|_{w_2}^2 + C|u_+-u_-|\int_0^t |\psi_{xx}(s)|_{w_2}^2 ds. \tag{4.14}$$

The Proof of (i) in Proposition 4.3. Differentiating (4.2) on x and multiplying it by ψ_x to yield

$$\frac{1}{2}\frac{d}{dt}[\psi_x^2 + (\partial_x^3\psi)^2] + \alpha\psi_{xx}^2 - \{-\partial_x^4\psi_t\partial_x\psi + \partial_x^3\psi_t\partial_x^2\psi + \alpha\psi_{xx}\psi_x\}_x$$

$$+f'U)_x\psi_x^2 + f'(U)\psi_{xx}\psi_x = G_x\psi_x,$$

noting

$$|f'(U)_x\psi_x^2| \le C|u_+-u_-|\psi_x^2, \quad |f'(U)\psi_{xx}\psi_x| \le C|u_+-u_-|(\psi_{xx}^2+\psi_x^2)$$

due to the Cauchy's inequality and $|f'(U)| \le C|u_+-u_-|$ and $|U_x| \le C|u_+--u_-|$, we then integrate the above resultant equitity over $[0,t]\times R$ and make use of (4.12) in Lemma 4.4 to get

$$\|\psi_x(t)\|^2 + \|\partial_x^3\psi(t)\|^2 + \int_0^t \|\psi_{xx}(s)\|^2 ds$$

$$\leq C\|\psi_0\|_4^2 + C(N_1(t) + |u_+ - u_-|)\int_0^t \|\psi_x(s)\|_1^2 ds. \tag{4.15}$$

Similarly, multiplying (4.2) by $\partial_x^4\psi$ and integrating it over $[0,t]\times R$, and then using (4.12), we have

$$\|\psi_{xx}(t)\|^2 + \|\partial_x^4\psi(t)\|^2 + \int_0^t \|\partial_x^3\psi(s)\|^2 ds$$

$$\leq C\|\psi_0\|_4^2 + C(N_1(t) + |u_+ - u_-|)\int_0^t \|\psi_x(s)\|_3^2 ds. \tag{4.16}$$

Thus, combining (4.12), (4.15) and (4.16) yields

$$\|(\psi, \psi_x, \partial_x^3\psi, \partial_x^4\psi)(t)\|^2 + \int_0^t \|\psi_x(s)\|_3^2 ds \leq C\|\psi_0\|_4^2, \tag{4.17}$$

for $N_1(t) \ll 1$ and $|u_+ - u_-| \ll 1$.

Furthermore, taking $\int_0^t\int_{-\infty}^{\infty} \partial_x^2(4.2) \times \psi_{xx} dx d\tau$ and noting (4.17) yield

$$\|(\partial_x^2\psi, \partial_x^3\psi)(t)\|^2 + \int_0^t \|\partial_x^3\psi(s)\|^2 ds \leq C\|\psi_0\|_4^2, \tag{4.18}$$

for $N_1(t) \ll 1$ and $|u_+ - u_-| \ll 1$.

Thus, combining (4.17) and (4.18) impies (4.4). The proof is complete. □

The Proof of (ii) in Proposition 4.3. We here give an outline of the proof, since it is similar to the proof of (i) in Proposition 4.3. We first differentiate (4.2) with respect to x and multiply it by $w_1(U)(\psi_x + \partial_x^5\psi)$, after integrating the resulting equality and using (4.13), we have

$$|(\psi + \partial_x^4\psi)(t)|_{1,w_1}^2 + \int_0^t |\psi_x(s)|_{3,w_1}^2 ds \leq C|\psi_0 + \partial_x^4\psi_0|_{1,w_1}^2, \tag{4.19}$$

for $N_2(t) \ll 1$ and $|u_+ - u_-| \ll 1$.

Secondly, by $\int_0^t\int_{-\infty}^{\infty} \partial_x^2(4.2) \times w_1(U)(\psi_{xx} + \partial_x^6\psi) dx d\tau$, and noting (4.13) and (4.19), we can prove

$$|(\psi + \partial_x^4\psi)(t)|_{2,w_1}^2 + \int_0^t |\psi_x(s)|_{4,w_1}^2 ds \leq C|\psi_0 + \partial_x^4\psi_0|_{2,w_1}^2 \tag{4.20}$$

for $N_2(t) \ll 1$ and $|u_+ - u_-| \ll 1$. The proof is complete. □

The Proof of (iii) in Proposition 4.3. By the same procedure mantioned above, firstly, taking $\partial_x^1(4.2) \times w_2(U)(\psi_x + \partial_x^5\psi)$ and integrating it over $[0,t] \times R$, in particular, and noting the positivity of the coefficient $C_1 > 0$ of the second term in the left-hand side of (4.14), we can prove that

$$|(\psi + \partial_x^4\psi)(t)|_{1,w_2}^2 + 2\theta_1 \int_0^t |\psi(s)|_{4,w_2}^2 ds \le |\psi_0 + \partial_x^4\psi_0|_{1,w_2}^2, \tag{4.21}$$

for some constant $\theta_1 > 0$. Here the conditions $N_3(t) \ll 1$ and $|u_+ - u_-| \ll 1$ are also necessary.

Secondly we focus $\partial_x^2(4.2) \times w_2(U)(\psi_{xx} + \partial_x^6\psi)$, and integrate it over $[0,t] \times R$, noting (4.14) and (4.21), then there exists a constant $\theta_2 > 0$, such that

$$|(\psi + \partial_x^4\psi)(t)|_{2,w_2}^2 + 2\theta_2 \int_0^t |\psi(s)|_{5,w_2}^2 ds \le |\psi_0 + \partial_x^4\psi_0|_{2,w_2}^2, \tag{4.22}$$

for $N_3(t) \ll 1$ and $|u_+ - u_-| \ll 1$, which gives us (4.6).

Moreover, since

$$|\psi(t)|_{5,w_2}^2 \ge C|(\psi + \partial_x^4\psi)(t)|_{1,w_2}^2,$$

there exists a positive constant $\theta = \min\{\theta_1, \theta_2\}$, we can obtain from (4.22) such that

$$|(\psi + \partial_x^4\psi)(t)|_{1,w_2}^2 + 2\theta \int_0^t |(\psi + \partial_x^4\psi)(s)|_{1,w_2}^2 ds \le |\psi_0 + \partial_x^4\psi_0|_{2,w_2}^2, \tag{4.23}$$

provided $N_3(t) \ll 1$ and $|u_+ - u_-| \ll 1$.

Using Gronwall's inequality in (4.23), we have

$$|(\psi + \partial_x^4\psi)(t)|_{1,w_2} \le Ce^{-\theta t} \tag{4.24}$$

for $t \in [0,T]$. We have completed the proof of (iii) in Proposition 4.3. □

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