

PHASE TRANSITIONS IN A COUPLED VISCOELASTIC  
SYSTEM WITH PERIODIC INITIAL-BOUNDARY CONDITION:  
(I) EXISTENCE AND UNIFORM BOUNDEDNESS

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ABSTRACT. This paper focuses on the phase transitions of a  $2 \times 2$  system of mixed type for viscosity-capillarity with periodic initial-boundary condition in a viscoelastic material. By the Liapunov functional method, we prove the existence, uniqueness, regularity and uniform boundedness of the solution. The results are correct even for *large* initial data.

**1. Introduction and main theorem.** This work is concerned with the viscous-capillarity system of mixed type in the viscoelastic material dynamics (resp. the compressible van der Waals fluids):

$$\begin{cases} v_t - u_x = \varepsilon_1 v_{xx}, \\ u_t - \sigma(v)_x = \varepsilon_2 u_{xx}, \end{cases} \quad (x, t) \in R \times R_+. \quad (1)$$

We study the coupled system with the initial condition

$$(v, u)|_{t=0} = (v_0, u_0)(x), \quad x \in (-\infty, \infty) \quad (2)$$

and the  $2L$ -periodic boundary condition

$$(v, u)(x, t) = (v, u)(x + 2L, t), \quad (x, t) \in (-\infty, \infty) \times (0, \infty) \quad (3)$$

where  $L > 0$  is a given constant. Note that from the compatibility condition, we have

$$v_0(x) = v_0(x + 2L), \quad u_0(x) = u_0(x + 2L). \quad (4)$$

Here  $v(x, t)$  is the strain (resp. specific volume),  $u(x, t)$  the velocity,  $\varepsilon_1 > 0$  and  $\varepsilon_2 > 0$  the viscous constants,  $\sigma(v)$  the stress function (resp. pressure function), which is assumed to be sufficiently smooth and non-monotonic. The stationary

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solutions of (1)-(4) has been reported in [24], in which the nonlinearity is taken as the simplest prototype:

$$\sigma(v) = v^3 - v. \quad (5)$$

This function captures the basic features for the phase transition models. For such a stress function  $\sigma(v)$ , there are only two critical points  $\pm \frac{1}{\sqrt{3}}$  such that  $\sigma'(\pm \frac{1}{\sqrt{3}}) = 0$ , and  $\sigma'(v) > 0$  for  $v \in (-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$ ,  $\sigma'(v) < 0$  for  $v \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ . Physically, this determines three phases, for example, soft material, hard material, and soft-hard mixed material in the viscoelastic dynamics; or water, gas, and water-gas mixture phases in the van der Waals fluids (in this case, the pressure is taken as  $-\sigma(v) = \frac{R\theta}{v-b} - \frac{a}{v^2}$  with positive constants  $R$ ,  $\theta$ ,  $a$  and  $b$  satisfying  $R\theta b/a < (2/3)^3$  and  $v > b$ ). Mathematically, Eq.(1) with  $\varepsilon_1 = \varepsilon_2 = 0$  is hyperbolic in  $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$  and elliptic in  $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ , therefore,  $v = \pm \frac{1}{\sqrt{3}}$  are the two phase boundaries.

Since the periodic solutions  $(v, u)(x, t)$  of (1)-(3) in the entire space  $(-\infty, \infty)$  can be regarded as  $2L$ -periodic extensions of that on  $[0, 2L]$ , we only need to consider the system (1) on the bounded interval  $[0, 2L]$ . Integrating (1) over  $[0, 2L] \times [0, t]$  and using the periodic boundary condition (3), we obtain

$$\int_0^{2L} v(x, t) dx = \int_0^{2L} v_0(x) dx, \quad \int_0^{2L} u(x, t) dx = \int_0^{2L} u_0(x) dx. \quad (6)$$

Let

$$m_0 := \frac{1}{2L} \int_0^{2L} v_0(x) dx, \quad m_1 := \frac{1}{2L} \int_0^{2L} u_0(x) dx, \quad (7)$$

then

$$\int_0^{2L} [v(x, t) - m_0] dx = 0, \quad \int_0^{2L} [u(x, t) - m_1] dx = 0. \quad (8)$$

The solution to system (1) with different types of initial or initial-boundary values has been widely studied, see [1]-[9], [11]-[16], [18]-[30], [33] and the references therein. Among them, Eden et al [6] studied (1) with the periodic initial-boundary value for the van der Waals model with pressure function:  $p(v) = -\sigma(v) = \frac{R\theta}{v-b} - \frac{a}{v^2}$ , and proved the global existence and boundedness of the solution in the weak sense. To prevent the pressure to blow up, namely, to guarantee the solution  $v(x, t) > b$ , they required that the initial data  $v_0$  and  $u_0$  are sufficiently small and that the non-convexity of the state function  $p(v)$  is not too strong. Under the assumption of small initial data, by using the Galerkin approximation method, it is standard to show the existence, uniqueness and boundedness of the solution  $(v, u)(x, t)$ . The role of the smallness of the initial data is essential for obtaining the global existence as well as the stability. See also the stability of traveling waves with small initial perturbation in [16], [18]-[23]. However, for any large initial data, the uniform boundedness of the solution with respect to the given initial datum usually are not expected, even for the global existence of the solution. Hence, the present study on the solution with large initial data is of significant interest to researchers in the mathematics and physics community. In this paper, by considering the nonlinearity of the form  $\sigma(v) = v^3 - v$ , we show that, even for any large initial data, the global solution  $(v, u)(x, t)$  to (1)-(3) in the strong sense exists uniquely and is uniformly bounded. This result improves the previous work [6]. To obtain the uniform boundedness of the solution for a given large initial data, we find that the standard energy method together with the Liapunov functional cannot be directly applied to the

original equations (1), because  $\sigma'(v)$  changes signs in different regions (hyperbolic and elliptic). In order to overcome this difficulty, we first transform the original periodic IBVP (1)-(3) into a new system (see (16) and (20)), and then show the uniform boundedness of the solution by selecting a suitable Liapunov functional for the new system.

The existence of the stationary solutions to (1)-(3) have been studied in our previous paper [24] by use of the energy method in [31] and the transversality arguments in [10] for Cahan-Hilliard equation. We proved that when the product of the viscosities  $\varepsilon_1\varepsilon_2$  is large, there exists only one trivial solution (no phase transition), and when  $\varepsilon_1\varepsilon_2$  is small, there exists some non-trivial solutions, in which phase transitions occur through the three phases periodically. Furthermore, we found that the number of non-trivial solutions depends on the size of  $m_0$ . However, an interesting and important continuation on this topic are to study the stability of these stationary solutions, i.e., the convergence of the solution of (1)-(3). This will be reported in our following paper [25]. In order to show the convergence by the compactness arguments and the energy method, it is essential for us firstly to establish the existence, the uniqueness, and in particular, the uniform boundedness of the solution  $(v, u)(x, t)$  of (1)-(3), which is the goal and the contribution of the present study.

**Notation.** Before stating our main results, we introduce the following notations. Throughout the paper,  $C > 0$  denotes a generic constant, while  $C_i > 0$  ( $i = 0, 1, 2, \dots$ ) represents a specific constant,  $R = (-\infty, \infty)$ . Since solutions  $(v, u)(x, t)$  of (1)-(3) are periodic, we introduce spaces of periodic functions. Letting  $p$  denote the period, we introduce the Hilbert space  $L^2_{per}(R)$  of locally square integrable functions that are periodic with period  $p$ ,

$$L^2_{per}(R) = \left\{ v(x) \mid v(x) = v(x+p) \text{ for all } x \in R, \text{ and } v(x) \in L^2(0, p) \text{ for } x \in [0, p] \right\},$$

with the norm given by the integral over  $[0, p]$  (or over any other interval of length  $p$ ), denoted by  $\|\cdot\|$ ,

$$\|v\| = \left( \int_0^p v^2(x) dx \right)^{1/2}.$$

We define the Sobolev space  $H^k_{per}(R)$  ( $k \geq 0$ ) to be the space of functions  $v(x)$  in  $L^2_{per}(R)$  whose derivatives  $\partial_x^i v$ ,  $i = 1, \dots, k$  belong to  $L^2_{per}(R)$  with the norm denoted by  $\|\cdot\|_k$ ,

$$\|v\|_k = \left( \sum_{i=0}^k \int_0^p |\partial_x^i v(x)|^2 dx \right)^{1/2},$$

where  $\|v\|_0 = \|v\|$ . We use the simplified notation  $\|(f, g)\|^2 = \|f\|^2 + \|g\|^2$  and  $\|(f, g)\|_k^2 = \|f\|_k^2 + \|g\|_k^2$ . The periodic spaces  $L^\infty_{per}(R)$  and  $L^k_{per}(R)$ , where  $k$  is a positive integer, are similarly defined. Let  $T > 0$  be a number and  $B$  be a Banach space. The space of  $B$ -valued continuous functions on  $[0, T]$  is denoted by  $C^0([0, T]; B)$ . The corresponding spaces of  $B$ -valued functions on  $[0, \infty)$  are defined similarly.

We now state our main result.

**Theorem 1.1** (Existence, Uniqueness, Uniform Boundedness). *Let  $(v_0, u_0)(x) \in H^2_{per}(R)$ . Then there exists a unique and global solution  $(v, u)(x, t) \in C([0, \infty); H^2_{per}(R))$  for the periodic IBVP (1)-(3). In particular, the solution  $(v, u)(x, t)$  is*

uniformly bounded in  $H_{per}^2$ , i.e.,

$$\|(v, u)(t)\|_2 \leq C_0, \quad t \geq 0 \tag{9}$$

for some constant  $C_0 > 0$ .

**2. Existence, uniqueness and uniform boundedness.** In this section, we prove the global existence and uniqueness of the solution  $(v, u)$  to system (1)-(3). It is important to note that there is no restriction on the amplitude of the initial data. The uniform boundedness of the solution  $(v, u)(x, t)$  plays a key role in the study, and it leads to the global existence, in particular, the convergence to certain steady-state solution (see [24, 25]).

First, by using the fixed-point iteration, we can prove the local existence. The detail of the proof is omitted.

**Proposition 2.1** (Local Existence). *Given  $(v_0, u_0)(x) \in H_{per}^2(R)$ , there exists  $t_0 = t_0(v_0, u_0) > 0$  such that system (1)-(3) has a unique solution  $(v, u)(x, t) \in C^0([0, t_0]; H_{per}^2(R))$ .*

Let  $[0, T_{max})$  be the life-span of the solution  $(v, u)(x, t)$ , we then have the following well-known alternative result (for example, see [17, 32]).

**Proposition 2.2** (Alternative). *Let  $[0, T_{max})$  be the maximal interval of the existence and  $(v, u)(x, t) \in C^0([0, T_{max}); H_{per}^2(R))$ , then either  $T_{max} = +\infty$ ; or  $T_{max} < +\infty$  and  $\lim_{t \rightarrow T_{max}^-} \|(v, u)(t)\|_2 = \infty$ .*

Now we are going to prove  $T_{max} = \infty$ . The most important step is to show the boundedness of the solution  $(v, u)(x, t)$ . In fact, we will prove that not only the solution is bounded but it also is uniformly bounded for any given initial datum.

It has been already shown that, the solution of (1)-(3) is bounded when the initial datum is sufficiently small (see [5, 6, 16],[18]-[23]). But for large initial datum, it is not trivial to get such a uniform boundedness directly from the equations (1) because  $\sigma'(v)$  changes signs in different regions. To overcome the difficulty, we transform the original system, so that we could find a suitable Liapunov functional for the equivalent new system and the uniform boundedness can then be proved.

Let

$$\bar{v}(x, t) := v(x, t) - m_0, \quad \bar{u}(x, t) := u(x, t) - m_1. \tag{10}$$

Then the system (1)-(3) can be rewritten as

$$\begin{cases} \bar{v}_t - \bar{u}_x = \varepsilon_1 \bar{v}_{xx}, \\ \bar{u}_t - \bar{\sigma}(\bar{v})_x = \varepsilon_2 \bar{u}_{xx}, \\ (\bar{v}, \bar{u})|_{t=0} = (v_0(x) - m_0, u_0(x) - m_1) =: (\bar{v}_0, \bar{u}_0)(x), \\ (\bar{v}, \bar{u})(x, t) = (\bar{v}, \bar{u})(x + 2L, t), \\ \int_0^{2L} (\bar{v}, \bar{u})(x, t) dx = (0, 0), \end{cases} \tag{11}$$

where

$$\bar{\sigma}(\bar{v}) = \sigma(\bar{v} + m_0) - \sigma(m_0).$$

We now technically set up

$$\begin{cases} (a(t), b(t)) := \left( \frac{1}{2L} \int_0^{2L} \int_0^x \bar{v}(y, t) dy dx, \frac{1}{2L} \int_0^{2L} \int_0^x \bar{u}(y, t) dy dx \right), \\ (\phi(x, t), \psi(x, t)) := \left( \int_0^x \bar{v}(y, t) dy - a(t), \int_0^x \bar{u}(y, t) dy - b(t) \right). \end{cases} \tag{12}$$

It is easy to verify

$$(\phi_x, \psi_x)(x, t) = (\bar{v}, \bar{u})(x, t), \quad \int_0^{2L} \phi(x, t) dx = 0, \quad \int_0^{2L} \psi(x, t) dx = 0. \quad (13)$$

We have also the periodic condition

$$(\phi, \psi)(x + 2L, t) = (\phi, \psi)(x, t). \quad (14)$$

In fact, noting the zero-average of  $\bar{v}$  in (11), i.e.,  $\int_x^{x+2L} \bar{v}(y, t) dy = 0$ , we have

$$\begin{aligned} \phi(x + 2L, t) &= \int_0^{x+2L} \bar{v}(y, t) dy - a(t) \\ &= \int_0^x \bar{v}(y, t) dy + \int_x^{x+2L} \bar{v}(y, t) dy - a(t) \\ &= \int_0^x \bar{v}(y, t) dy - a(t) \\ &= \phi(x, t). \end{aligned}$$

Similarly,  $\psi(x + 2L, t) = \psi(x, t)$  can be proved.

Thus, substituting  $(\bar{v}, \bar{u})(x, t) = (\phi_x, \psi_x)(x, t)$  into (11), we have

$$\begin{cases} \phi_{xt} - \psi_{xx} = \varepsilon_1 \phi_{xxx}, \\ \psi_{xt} - \bar{\sigma}(\phi_x)_x = \varepsilon_2 \psi_{xxx}, \\ (\phi, \psi)|_{t=0} = \left( \int_0^x \bar{v}_0(y) dy - a(0), \int_0^x \bar{u}_0(y) dy - b(0) \right) =: (\phi_0, \psi_0)(x), \\ (\phi, \psi)(x, t) = (\phi, \psi)(x + 2L, t), \\ \int_0^{2L} (\phi, \psi)(x, t) dx = (0, 0), \end{cases} \quad (15)$$

where

$$\bar{\sigma}(\phi_x) = \sigma(\phi_x + m_0) - \sigma(m_0) = \phi_x^3 + 3m_0\phi_x^2 + (3m_0^2 - 1)\phi_x.$$

Integrating (15) over  $[0, x]$ , we then have

$$\begin{cases} \phi_t - \psi_x = \varepsilon_1 \phi_{xx} + c(t), \\ \psi_t - \bar{\sigma}(\phi_x) = \varepsilon_2 \psi_{xx} + d(t), \\ (\phi, \psi)|_{t=0} = \left( \int_0^x \bar{v}_0(y) dy - a(0), \int_0^x \bar{u}_0(y) dy - b(0) \right) =: (\phi_0, \psi_0)(x), \\ (\phi, \psi)(x, t) = (\phi, \psi)(x + 2L, t), \\ \int_0^{2L} (\phi, \psi)(x, t) dx = (0, 0), \end{cases} \quad (16)$$

where  $c(t)$  and  $d(t)$  are integral constants

$$\begin{aligned} c(t) &= \phi_t(0, t) - \psi_x(0, t) - \varepsilon_1 \phi_{xx}(0, t), \\ d(t) &= \psi_t(0, t) - \bar{\sigma}(\phi_x(0, t)) - \varepsilon_2 \psi_{xx}(0, t). \end{aligned}$$

**Lemma 2.3.** *It holds*

$$c(t) \equiv 0, \quad d(t) \equiv 0, \quad \text{for all } t \geq 0. \quad (17)$$

*Proof.* Integrating the equations of (16) with respect to  $x$  over  $[0, 2L]$ , and noting the periodicity, we have

$$\begin{cases} \int_0^{2L} \phi_t(x, t) dx = 2Lc(t), \\ \int_0^{2L} \psi_t(x, t) dx = 2Ld(t). \end{cases} \quad (18)$$

By using the zero-averages  $\int_0^{2L} (\phi, \psi)(x, t)dx = (0, 0)$ , we prove

$$\begin{cases} c(t) = \frac{1}{2L} \frac{d}{dt} \int_0^{2L} \phi(x, t)dx = 0, \\ d(t) = \frac{1}{2L} \frac{d}{dt} \int_0^{2L} \psi(x, t)dx = 0. \end{cases} \tag{19}$$

The proof is complete. □

Thus, we finally have

$$\begin{cases} \phi_t - \psi_x = \varepsilon_1 \phi_{xx}, \\ \psi_t - \bar{\sigma}(\phi_x) = \varepsilon_2 \psi_{xx}, \\ (\phi, \psi)|_{t=0} = \left( \int_0^x \bar{v}_0(y)dy - a(0), \int_0^x \bar{u}_0(y)dy - b(0) \right) =: (\phi_0, \psi_0)(x), \\ (\phi, \psi)(x, t) = (\phi, \psi)(x + 2L, t), \\ \int_0^{2L} (\phi, \psi)(x, t)dx = (0, 0). \end{cases} \tag{20}$$

Differentiating the second equation of (20) with respect to  $x$  and substituting the first equation of (20) to this resultant equation, we obtain a scalar equation on  $\phi$

$$\begin{cases} \phi_{tt} - (\varepsilon_1 + \varepsilon_2)\phi_{xxt} - \bar{\sigma}(\phi_x)_x + \varepsilon_1 \varepsilon_2 \phi_{xxxx} = 0, \\ (\phi, \phi_t)|_{t=0} = (\phi_0, \psi_{0x} + \varepsilon_1 \phi_{0xx})(x), \\ \phi(x, t) = \phi(x + 2L, t), \\ \int_0^{2L} \phi(x, t)dx = 0. \end{cases} \tag{21}$$

Before proving the uniform boundedness of the solution, we introduce the following Poincaré inequality.

**Lemma 2.4.** *Let  $\phi(x, t)$  be the solution of (21). It holds for all  $t \geq 0$  that*

$$\|(\partial_x^k \phi, \partial_x^k \psi)(t)\| \leq \frac{L}{\pi} \|(\partial_x^{k+1} \phi, \partial_x^{k+1} \psi)(t)\|, \quad k = 0, 1, 2, \dots \tag{22}$$

*Proof.* Consider the following eigenvalue problem

$$\begin{cases} -\tilde{v}_{xx} = \beta^2 \tilde{v}, \\ \tilde{v}(x) = \tilde{v}(x + 2L), \\ \int_0^{2L} \tilde{v}(x)dx = 0. \end{cases} \tag{23}$$

The eigenvalues are

$$\beta_k = \frac{k\pi}{L}, \quad k = 1, 2, 3, \dots, \tag{24}$$

and the corresponding eigenfunctions are

$$\tilde{v}_{1,k}(x) = \frac{1}{\sqrt{L}} \sin \beta_k x, \quad \tilde{v}_{2,k}(x) = \frac{1}{\sqrt{L}} \cos \beta_k x, \quad k = 1, 2, 3, \dots \tag{25}$$

which satisfy

$$\langle \tilde{v}_{i,k}, \tilde{v}_{j,l} \rangle \begin{cases} 1, & i = j, k = l \\ 0, & \text{otherwise,} \end{cases} \tag{26}$$

where

$$\langle \tilde{v}_{i,k}, \tilde{v}_{j,l} \rangle = \int_0^{2L} \tilde{v}_{i,k}(x) \tilde{v}_{j,l}(x) dx$$

is the inner product of  $L^2_{per}$ . It is known that the sequence  $\{\tilde{v}_{i,k}(x)\}$  ( $i = 1, 2$  and  $k = 1, 2, 3, \dots$ ) forms an orthonormal basis for the space

$$L^2_{per,0} = \left\{ \tilde{v}(x) \mid \tilde{v}(x) \in L^2_{per} \text{ and } \int_0^{2L} \tilde{v}(x) dx = 0 \right\}.$$

Therefore, as a periodic function satisfying  $\phi(x, t) = \phi(x+2L, t)$  and  $\int_0^{2L} \phi(x, t) dx = 0$ ,  $\phi(x, t)$  is in the space  $L^2_{per,0}$ , and can be expressed in the Fourier form

$$\phi(x, t) = \sum_{k=1}^{\infty} (A_k(t)\tilde{v}_{1,k}(x) + B_k(t)\tilde{v}_{2,k}(x)),$$

where the coefficients  $A_k(t)$  and  $B_k(t)$  are determined by

$$A_k(t) = \langle \phi(x, t), \tilde{v}_{1,k}(x) \rangle, \quad B_k(t) = \langle \phi(x, t), \tilde{v}_{2,k}(x) \rangle.$$

The derivative of  $\phi(x, t)$  in  $x$  is

$$\phi_x(x, t) = \sum_{k=1}^{\infty} \beta_k [A_k(t)\tilde{v}_{2,k}(x) - B_k(t)\tilde{v}_{1,k}(x)].$$

Making the inner products leads to

$$\begin{aligned} \|\phi(t)\|^2 &= \int_0^{2L} \phi^2(x, t) dx \\ &= \int_0^{2L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} [A_k(t)\tilde{v}_{1,k}(x) + B_k(t)\tilde{v}_{2,k}(x)] \\ &\quad \cdot [A_l(t)\tilde{v}_{1,l}(x) + B_l(t)\tilde{v}_{2,l}(x)] dx \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} [A_k(t)A_l(t) \int_0^{2L} \tilde{v}_{1,k}(x)\tilde{v}_{1,l}(x) dx \\ &\quad + A_k(t)B_l(t) \int_0^{2L} \tilde{v}_{1,k}(x)\tilde{v}_{2,l}(x) dx \\ &\quad + B_k(t)A_l(t) \int_0^{2L} \tilde{v}_{2,k}(x)\tilde{v}_{1,l}(x) dx \\ &\quad + B_k(t)B_l(t) \int_0^{2L} \tilde{v}_{2,k}(x)\tilde{v}_{2,l}(x) dx] \\ &= \sum_{k=1}^{\infty} [A_k^2(t) + B_k^2(t)] \end{aligned} \tag{27}$$

and

$$\begin{aligned} \|\phi_x(t)\|^2 &= \int_0^{2L} \phi_x^2(x, t) dx \\ &= \int_0^{2L} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \beta_k \beta_l [A_k(t)\tilde{v}_{2,k}(x) - B_k(t)\tilde{v}_{1,k}(x)] \\ &\quad \cdot [A_l(t)\tilde{v}_{2,l}(x) - B_l(t)\tilde{v}_{1,l}(x)] dx \\ &= \sum_{k=1}^{\infty} \beta_k^2 [A_k^2(t) + B_k^2(t)]. \end{aligned} \tag{28}$$

Since  $\beta_k \geq \beta_1 = \frac{\pi}{L}$  for  $k = 1, 2, \dots$ , then (27) and (28) imply

$$\|\phi_x(t)\|^2 = \sum_{k=1}^{\infty} \beta_k^2 [A_k^2(t) + B_k^2(t)] \geq \beta_1^2 \sum_{k=1}^{\infty} [A_k^2(t) + B_k^2(t)] \frac{\pi^2}{L^2} \|\phi(t)\|^2,$$

namely,

$$\|\phi(t)\| \leq \frac{L}{\pi} \|\phi_x(t)\|. \tag{29}$$

This can be extended to

$$\|\partial_x^k \phi(t)\| \leq \frac{L}{\pi} \|\partial_x^{k+1} \phi_x(t)\|, \quad k = 0, 1, 2, \dots \tag{30}$$

Similarly, we can show for the periodic function  $\psi(x, t)$ :

$$\|\partial_x^k \psi(t)\| \leq \frac{L}{\pi} \|\partial_x^{k+1} \psi_x(t)\|, \quad k = 0, 1, 2, \dots \tag{31}$$

Combining (30) and (31), we prove (22). □

Instead of dealing with the original system (1)-(3), we focus on the new system (20) (or equivalently, (21)). Let  $(\phi, \psi)(x, t) \in C([0, T_{max}); H_{per,0}^3(R))$  and  $\phi_t(x, t) \in C([0, T_{max}); H_{per,0}^2(R))$ , where  $H_{per,0}^q(R)$  (the integer  $q \geq 0$ ) is the space whose functions are in  $H_{per}^q$  with the zero-average  $\int_0^{2L} \phi(x) dx = 0$ , i.e.,

$$H_{per,0}^q(R) = \left\{ \phi(x) \in H^q(R) \mid \phi(x) = \phi(x + 2L), \int_0^{2L} \phi(x) dx = 0 \right\},$$

we then have the following uniform boundedness.

**Lemma 2.5 (Key).** *It holds uniformly*

$$\|\phi_t(t)\| \leq C, \quad \|\phi_x(t)\|_1 \leq C, \quad \|\psi_x(t)\| \leq C, \quad \text{for } t \in [0, T_{max}), \tag{32}$$

where  $C$  is a positive constant independent of  $T_{max}$ .

*Proof.* Define an energy functional, the so-called Liapunov functional

$$E(t) = \int_0^{2L} \left[ \frac{1}{2} \phi_t^2 + H(\phi_x) + \frac{1}{2} \varepsilon_1 \varepsilon_2 \phi_{xx}^2 \right] dx, \tag{33}$$

where

$$H(\phi_x) = \int_0^{\phi_x} \bar{\sigma}(s) ds = \frac{1}{4} \phi_x^4 + m_0 \phi_x^3 + \frac{1}{2} (3m_0^2 - 1) \phi_x^2.$$

Differentiating  $E(t)$  with respect to  $t$ , and using integration by parts and the equation of (21), we have

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_0^{2L} [\phi_{tt} \phi_t + \bar{\sigma}(\phi_x) \phi_{xt} + \varepsilon_1 \varepsilon_2 \phi_{xx} \phi_{xxt}] dx \\ &= \int_0^{2L} [\phi_{tt} - \bar{\sigma}(\phi_x)_x + \varepsilon_1 \varepsilon_2 \phi_{xxx}] \phi_t dx \\ &= \int_0^{2L} (\varepsilon_1 + \varepsilon_2) \phi_{xxt} \phi_t dx \\ &= - \int_0^{2L} (\varepsilon_1 + \varepsilon_2) \phi_{xt}^2 dx \leq 0. \end{aligned} \tag{34}$$

Integrating (34) with respect to  $t$  over  $[0, t]$  yields  $E(t) \leq E(0) =: C_3$ , i.e.,

$$\|\phi_t(t)\|^2 + 2 \int_0^{2L} H(\phi_x) dx + \varepsilon_1 \varepsilon_2 \|\phi_{xx}(t)\|^2 \leq 2C_3. \tag{35}$$

Notice that

$$\int_0^{2L} H(\phi_x) dx = \int_0^{2L} \left[ \frac{1}{4} \phi_x^4 + m_0 \phi_x^3 + \frac{3m_0^2 - 1}{2} \phi_x^2 \right] dx, \tag{36}$$

and by the Cauchy-Schwartz inequality ( $ab \leq \eta a^2 + (1/4\eta)b^2$  for any  $\eta > 0$ )

$$\begin{aligned} \left| m_0 \int_0^{2L} \phi_x^3 dx \right| &= \left| \int_0^{2L} (m_0 \phi_x) \phi_x^2 dx \right| \\ &\leq \int_0^{2L} \left( 2m_0^2 + \frac{1}{8} \phi_x^2 \right) \phi_x^2 dx \\ &= \frac{1}{8} \int_0^{2L} \phi_x^4 dx + 2m_0^2 \int_0^{2L} \phi_x^2 dx, \end{aligned} \tag{37}$$

then substituting (37) on (36), we obtain

$$\int_0^{2L} H(\phi_x) dx \geq \frac{1}{8} \|\phi_x(t)\|_{L^4_{per}}^4 - \frac{1}{2} (m_0^2 + 1) \|\phi_x(t)\|^2. \tag{38}$$

In the same way, we obtain

$$\begin{aligned} -\frac{1}{2} (m_0^2 + 1) \|\phi_x(t)\|^2 &= -\int_0^{2L} \phi_x^2 \cdot \frac{1}{2} (m_0^2 + 1) dx \\ &\geq -\frac{1}{16} \int_0^{2L} \phi_x^4 dx - 4 \int_0^{2L} \left( \frac{1}{2} (m_0^2 + 1) \right)^2 dx \\ &= -\frac{1}{16} \|\phi_x(t)\|_{L^4_{per}}^4 - 2L (m_0^2 + 1)^2. \end{aligned} \tag{39}$$

Thus, (39) and (38) give

$$\int_0^{2L} H(\phi_x) dx \geq \frac{1}{16} \|\phi_x(t)\|_{L^4_{per}}^4 - 2L (m_0^2 + 1)^2. \tag{40}$$

Applying (40) into (35), we finally prove the boundedness of  $\phi$  in the form

$$\|\phi_t(t)\|^2 + \frac{1}{8} \|\phi_x(t)\|_{L^4_{per}}^4 + \varepsilon_1 \varepsilon_2 \|\phi_{xx}(t)\|^2 \leq C_4, \tag{41}$$

where  $C_4 = 2C_3 + 4L(m_0^2 + 1)^2$ . Using the Poincaré inequality

$$\|\phi_x(t)\| \leq \frac{L}{\pi} \|\phi_{xx}(t)\| \tag{42}$$

and (41), we then prove the uniform boundedness for  $\phi$  in the form

$$\|\phi_t(t)\| \leq C, \quad \|\phi_x(t)\|_1 \leq C, \quad t \in [0, T_{max}).$$

Furthermore, from the first equation of (20), i.e.,  $\phi_t - \psi_x = \varepsilon_1 \phi_{xx}$ , we can easily prove

$$\|\psi_x(t)\| \leq \|\phi_t(t)\| + \varepsilon_1 \|\phi_{xx}(t)\| \leq C, \quad t \in [0, T_{max}).$$

Thus, the proof of this Lemma is completed. □

**Lemma 2.6 (Key).** *It holds uniformly*

$$\|\phi_{xt}(t)\|_1 \leq C, \quad \|\phi_x(t)\|_3 \leq C, \quad \|\psi_x(t)\|_2 \leq C, \quad \text{for } t \in [0, T_{max}), \quad (43)$$

where  $C > 0$  is a constant independent of  $T_{max}$ .

*Proof.* Differentiating (21) with respect to  $x$ , multiplying it by  $\phi_{xt} + \lambda_3\phi_x$ , where  $\lambda_3 = (\varepsilon_1 + \varepsilon_2)\pi^2/(4L^2)$ , and then integrating the resultant equation over  $[0, 2L]$  with respect to  $x$ , we obtain

$$\frac{d}{dt}E_1(t) + E_2(t) = 0, \quad (44)$$

where

$$\begin{aligned} E_1(t) &= \frac{1}{2}\|\phi_{xt}(t)\|^2 + \frac{\varepsilon_1\varepsilon_2}{2}\|\phi_{xxx}(t)\|^2 \\ &\quad + \lambda_3 \int_0^{2L} \phi_{xt}\phi_x dx + \frac{\varepsilon_1 + \varepsilon_2}{2}\lambda_3\|\phi_{xx}(t)\|^2, \end{aligned} \quad (45)$$

$$\begin{aligned} E_2(t) &= (\varepsilon_1 + \varepsilon_2)\|\phi_{xxt}(t)\|^2 - \lambda_3\|\phi_{xt}(t)\|^2 + \varepsilon_1\varepsilon_2\lambda_3\|\phi_{xxx}(t)\|^2 \\ &\quad + \int_0^{2L} \bar{\sigma}(\phi_x)_x\phi_{xxt} dx + \lambda_3 \int_0^{2L} \bar{\sigma}(\phi_x)_x\phi_{xx} dx. \end{aligned} \quad (46)$$

By using the Sobolev’s inequality and Lemma 2.5

$$\|\phi_x(t)\|_{L_{per}^\infty} \leq \|\phi_x(t)\|_1 \leq C, \quad t \in [0, T_{max}).$$

Using the Cauchy-Schwartz inequality and the above inequality, we have

$$\begin{aligned} &\left| \int_0^{2L} \bar{\sigma}(\phi_x)_x\phi_{xxt} dx \right| \\ &\leq \frac{\varepsilon_1 + \varepsilon_2}{2}\|\phi_{xxt}(t)\|^2 + \frac{1}{2(\varepsilon_1 + \varepsilon_2)} \int_0^{2L} [\bar{\sigma}(\phi_x)_x]^2 dx \\ &= \frac{\varepsilon_1 + \varepsilon_2}{2}\|\phi_{xxt}(t)\|^2 + \frac{1}{2(\varepsilon_1 + \varepsilon_2)} \int_0^{2L} [3(\phi_x + m_0)^2 - 1]^2\phi_{xx}^2 dx \\ &\leq \frac{\varepsilon_1 + \varepsilon_2}{2}\|\phi_{xxt}(t)\|^2 + \frac{1}{2(\varepsilon_1 + \varepsilon_2)} [3(\|\phi_x(t)\|_{L_{per}^\infty} + m_0)^2 + 1]^2\|\phi_{xx}(t)\|^2 \\ &\leq \frac{\varepsilon_1 + \varepsilon_2}{2}\|\phi_{xxt}(t)\|^2 + C. \end{aligned} \quad (47)$$

Similarly, we can prove

$$\begin{aligned} \left| \lambda_3 \int_0^{2L} \bar{\sigma}(\phi_x)_x\phi_{xx} dx \right| &= \lambda_3 \left| \int_0^{2L} \bar{\sigma}'(\phi_x)\phi_{xx}^2 dx \right| \\ &\leq \lambda_3 [3(\|\phi_x\|_{L_{per}^\infty} + m_0)^2 + 1]\|\phi_{xx}\|^2 \leq C. \end{aligned} \quad (48)$$

Substituting (47) and (48) into (46), using the following Poincaré inequality

$$\|\phi_{xt}(t)\|^2 \leq \frac{L^2}{\pi^2}\|\phi_{xxt}(t)\|^2,$$

and noting  $\lambda_3 = (\varepsilon_1 + \varepsilon_2)\pi^2/(4L^2)$ , we get

$$E_2(t) \geq \frac{\varepsilon_1 + \varepsilon_2}{4}\|\phi_{xxt}(t)\|^2 + \varepsilon_1\varepsilon_2\lambda_3\|\phi_{xxx}(t)\|^2 - C. \quad (49)$$

By substituting (49) into (44), it yields

$$\frac{d}{dt}E_1(t) + \frac{\varepsilon_1 + \varepsilon_2}{4}\|\phi_{xxt}(t)\|^2 + \varepsilon_1\varepsilon_2\lambda_3\|\phi_{xxx}(t)\|^2 \leq C. \quad (50)$$

Multiplying (50) by  $e^{\lambda_4 t}$ , where  $\lambda_4 > 0$  is a small constant which will be determined later, we obtain

$$\begin{aligned} & \frac{d}{dt}\{e^{\lambda_4 t} E_1(t)\} - \lambda_4 e^{\lambda_4 t} E_1(t) \\ & + e^{\lambda_4 t} \left( \frac{\varepsilon_1 + \varepsilon_2}{4} \|\phi_{xxt}(t)\|^2 + \varepsilon_1 \varepsilon_2 \lambda_3 \|\phi_{xxx}(t)\|^2 \right) \\ & \leq C e^{\lambda_4 t}. \end{aligned} \tag{51}$$

Similarly, applying the Cauchy-Schwartz inequality and the Poincaré inequality, we can estimate

$$-\lambda_4 e^{\lambda_4 t} E_1(t) \geq -\lambda_4 e^{\lambda_4 t} (\bar{C}_1 \|\phi_{xxt}(t)\|^2 + \bar{C}_2 \|\phi_{xxx}(t)\|^2), \tag{52}$$

where

$$\bar{C}_1 = \left(\frac{1}{2} + \frac{\lambda_3}{2}\right) \frac{L^2}{\pi^2}, \quad \bar{C}_2 = \frac{\varepsilon_1 \varepsilon_2}{2} + \frac{\lambda_3 L^4}{2\pi^4} + \frac{(\varepsilon_1 + \varepsilon_2) \lambda_3 L^2}{2\pi^2}.$$

Substituting (52) into (51), we have

$$\begin{aligned} & \frac{d}{dt}\{e^{\lambda_4 t} E_1(t)\} \\ & + e^{\lambda_4 t} \left[ \left( \frac{\varepsilon_1 + \varepsilon_2}{4} - \lambda_4 \bar{C}_1 \right) \|\phi_{xxt}(t)\|^2 + (\varepsilon_1 \varepsilon_2 \lambda_3 - \lambda_4 \bar{C}_2) \|\phi_{xxx}(t)\|^2 \right] \\ & \leq C e^{\lambda_4 t}. \end{aligned} \tag{53}$$

Now let  $\lambda_4$  be sufficiently small such that

$$\frac{\varepsilon_1 + \varepsilon_2}{4} - \lambda_4 \bar{C}_1 > 0 \quad \text{and} \quad \varepsilon_1 \varepsilon_2 \lambda_3 - \lambda_4 \bar{C}_2 > 0,$$

(53) is then reduced to

$$\frac{d}{dt}\{e^{\lambda_4 t} E_1(t)\} \leq C e^{\lambda_4 t}. \tag{54}$$

Integrating (54) over  $[0, t]$  ( $t < T_{max}$ ), we have

$$e^{\lambda_4 t} E_1(t) \leq E_1(0) + C \int_0^t e^{\lambda_4 s} ds = E_1(0) + \frac{C}{\lambda_4} (e^{\lambda_4 t} - 1),$$

i.e.,

$$E_1(t) \leq E_1(0) e^{-\lambda_4 t} + \frac{C}{\lambda_4} (1 - e^{-\lambda_4 t}) \leq C, \quad t \in [0, T_{max}]. \tag{55}$$

Since the Cauchy-Schwartz inequality gives

$$\left| \lambda_3 \int_0^{2L} \phi_{tx} \phi_x dx \right| \leq \frac{1}{4} \|\phi_{xt}\|^2 + \lambda_3^2 \|\phi_x\|^2 \leq \frac{1}{4} \|\phi_{xt}\|^2 + C,$$

then  $E_1(t)$  (see (45)) can be estimated as follows

$$E_1(t) \geq \frac{1}{4} \|\phi_{xt}(t)\|^2 + \frac{\varepsilon_1 \varepsilon_2}{2} \|\phi_{xxx}(t)\|^2 + \frac{\varepsilon_1 + \varepsilon_2}{2} \lambda_3 \|\phi_{xx}(t)\|^2 - C.$$

Substituting the above inequality into (55), we finally prove

$$\frac{1}{4} \|\phi_{xt}(t)\|^2 + \frac{\varepsilon_1 \varepsilon_2}{2} \|\phi_{xxx}(t)\|^2 + \frac{\varepsilon_1 + \varepsilon_2}{2} \lambda_3 \|\phi_{xx}(t)\|^2 \leq C.$$

Hence, we prove

$$\|\phi_{xt}(t)\| \leq C, \quad \|\phi_{xxx}(t)\| \leq C, \quad t \in [0, T_{max}]. \tag{56}$$

Notice that  $\psi_x = \phi_t + \varepsilon_1 \phi_{xx}$ , we then further prove

$$\|\psi_{xx}(t)\| \leq C, \quad t \in [0, T_{max}]. \tag{57}$$

Furthermore, differentiating (21) with respect to  $x$  twice, multiplying it by  $\phi_{xxt} + \lambda_3 \phi_{xx}$ , and integrating the resultant equation over  $[0, 2L] \times [0, t]$ , and then using (56), we can similarly prove

$$\|\phi_{xxt}(t)\| \leq C, \quad \|\phi_{xxxx}(t)\| \leq C, \quad \|\psi_{xxx}(t)\| \leq C, \quad t \in [0, T_{max}]. \quad (58)$$

Thus, combining (56), (57) and (58), we finally prove (43).  $\square$

Noting  $v - m_0 = \bar{v} = \phi_x$  and  $u - m_1 = \bar{u} = \psi_x$ , from Lemma 2.5 and Lemma 2.6 we obtain immediately the uniform boundedness of  $(v, u)$  in  $H_{per}^2(R)$  as follows.

**Proposition 2.7.** *It holds uniformly*

$$\|(v, u)(t)\|_2 \leq C \quad \text{for } t \in [0, T_{max}]. \quad (59)$$

*Proof of Theorem 1.1.* By Proposition 2.1, Proposition 2.2 and Proposition 2.7, we immediately prove  $T_{max} = \infty$ , and  $\|(v, u)(t)\|_2 \leq C$  for all  $t \geq 0$ .  $\square$

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