DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **7**, Number **4**, June **2007**

Website: http://AIMsciences.org

pp. **839–857**

PHASE TRANSITIONS IN A COUPLED VISCOELASTIC SYSTEM WITH PERIODIC INITIAL-BOUNDARY CONDITION: (II) CONVERGENCE

Ming Mei[†,‡,1], Yau Shu Wong[♭] and Liping Liu[♮]

[†]Department of Mathematics and Statistics, McGill University
Montreal, Quebec, H3A 2K6, Canada, and
[‡]Department of Mathematics, Champlain College
St.-Lambert, Quebec, J4P 3P2, Canada

[♭]Department of Mathematical and Statistical Sciences, University of Alberta
Edmonton, Alberta, T6G 2G1, Canada

[♮]Department of Mathematics, University of Texas–Pan American
Edinburg, TX 78539-2999, USA

(Communicated by Yanping Lin)

Abstract. We present some new results on the asymptotic behavior of the periodic solution to a 2×2 mixed-type system of viscosity-capillarity in a viscoelastic material. We prove that the solution converges to a certain stationary solution as time approaches to infinity, in particular, when the viscosity is large enough or the mean of the initial datum is in the hyperbolic regions, the solution converges exponentially to the trivial stationary solution with *any large* initial datum. The location of the initial datum and the amplitude of viscosity play a key role for the phase transitions. Furthermore, we obtain the convergence rate to the stationary solutions. Finally, we carry out numerical simulations to confirm the theoretical predictions.

1. Introduction.

The viscous-capillarity system in the viscoelastic material dynamics (resp. the compressible van der Waals fluids) can be written as a system of 2×2 viscous conservation laws of mixed type:

$$\begin{cases} v_t - u_x = \varepsilon_1 v_{xx}, \\ u_t - \sigma(v)_x = \varepsilon_2 u_{xx}, \end{cases} \qquad (x,t) \in R \times R_+. \tag{1}$$

Here, we study the coupled system with the initial condition

$$(v,u)|_{t=0} = (v_0,u_0)(x), \quad x \in (-\infty,\infty) \tag{2}$$

and the $2L$-periodic boundary condition

$$(v,u)(x,t) = (v,u)(x+2L,t), \quad (x,t) \in (-\infty,\infty)\times(0,\infty) \tag{3}$$

2000 *Mathematics Subject Classification.* Primary: 35R10, 35B40, 34K30; Secondary: 58D25.
Key words and phrases. Phase transitions, viscoelasticity, periodic initial-boundary value problem, stationary solutions, asymptotic convergence.
[1]Corresponding author.

where $L > 0$ is a given constant. Note that from the compatibility condition, we have

$$v_0(x) = v_0(x+2L), \quad u_0(x) = u_0(x+2L), \tag{4}$$

where $v(x,t)$ is the strain (resp. specific volume), $u(x,t)$ the velocity, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ the viscous constants, $\sigma(v)$ the stress function (resp. pressure function), which is assumed to be sufficiently smooth and non-monotonic. As a prototype, $\sigma(v)$ is given in the simplest form:

$$\sigma(v) = v^3 - v. \tag{5}$$

This function captures the basic features for the phase transition models. For such a stress function $\sigma(v)$, it has only two critical points $\pm\frac{1}{\sqrt{3}}$ such that $\sigma'(\pm\frac{1}{\sqrt{3}}) = 0$, and $\sigma'(v) > 0$ for $v \in (-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$, $\sigma'(v) < 0$ for $v \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. Physically, this determines three phases, for example, water, gas, and water-gas mixture phases in the van der Waals fluids, and soft-material, hard-material and soft-hard mixture phases in the viscoelastic dynamics. Mathematically, Eq.(1) with $\varepsilon_1 = \varepsilon_2 = 0$ is hyperbolic in $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$ and elliptic in $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, therefore, $v = \pm\frac{1}{\sqrt{3}}$ are the two phase boundaries. While in the case of the van der Waals fluids, the pressure is exactly given by $-\sigma(v) = \frac{R\theta}{v-b} - \frac{a}{v^2}$ with positive constants R, θ, a and b satisfying $R\theta b/a < (2/3)^3$ and $v > b$. It is known that there are also two critical points of $\sigma(v)$, says v_1 and v_2, such that $\sigma'(v_1) = \sigma'(v_2) = 0$, $\sigma'(v) > 0$ for $v \in (b, v_1) \cup (v_2, \infty)$ and $\sigma'(v) < 0$ for $v \in (v_1, v_2)$. The region (b, v_1) is the water-region, (v_2, ∞) is the vapor-region, and (v_1, v_2) is the water-vapor mixture region, c.f. [1, 6, 17, 25, 26].

Since the periodic solutions $(v,u)(x,t)$ of (1)-(3) in the entire space $(-\infty, \infty)$ can be regarded as $2L$-periodic extensions of that on $[0, 2L]$, let us focus the system (1) on the bounded interval $[0, 2L]$. Integrating (1) over $[0, 2L] \times [0, t]$ and using the periodic boundary condition (3), we obtain

$$\int_0^{2L} v(x,t)dx = \int_0^{2L} v_0(x)dx, \qquad \int_0^{2L} u(x,t)dx = \int_0^{2L} u_0(x)dx. \tag{6}$$

Let

$$m_0 := \frac{1}{2L}\int_0^{2L} v_0(x)dx, \qquad m_1 : \frac{1}{2L}\int_0^{2L} u_0(x)dx, \tag{7}$$

then

$$\int_0^{2L} [v(x,t) - m_0]dx = 0, \qquad \int_0^{2L} [u(x,t) - m_1]dx = 0. \tag{8}$$

The corresponding stationary problem of (1)-(3) is given by

$$\begin{cases} -U_x = \varepsilon_1 V_{xx}, \\ -\sigma(V)_x = \varepsilon_2 U_{xx}, \\ (V,U)(x) = (V,U)(x+2L), \\ \frac{1}{2L}\int_0^{2L} V(x)dx = m_0, \\ \frac{1}{2L}\int_0^{2L} U(x)dx = m_1, \end{cases} \tag{9}$$

where $(V,U) = (V,U)(x)$. Substituting the first equation of (9) into the second equation and integrating the resultant equation over $[0, x]$, we obtain the stationary equation for $V(x)$ as follows

$$\begin{cases} \varepsilon_1\varepsilon_2 V_{xx} = \sigma(V) - a \\ V(x) = V(x+2L) \\ \frac{1}{2L}\int_0^{2L} V(x)dx = m_0, \end{cases} \tag{10}$$

where $a = \sigma(V(0)) - \varepsilon_1\varepsilon_2 V_{xx}(0)$ is a constant. In fact, integrating Eq. (10) over $[0, 2L]$ and noticing the periodic boundary condition $V(x) = V(x+2L)$, one verifies

$$a = \frac{1}{2L}\int_0^{2L} \sigma(V(x))dx. \tag{11}$$

The study of the phase transition problem has received considerable interests in both the mathematics and physics communities, and it has been an active research topic in recent years. For example, see [1]-[43] and the references therein. When $\sigma(v)$ is non-monotone, the system (1) without viscosity (i.e., $\varepsilon_1 = \varepsilon_2 = 0$) is ill-posed due to the sign changes of $\sigma'(v)$: in the elliptic region, one would have to specify the value of $v(t, x)$ at $t = T$ as another "boundary" condition for some positive constant T. To overcome this difficulty, non-zero viscosities $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are introduced into the system (1), so that the system is well-posed. Such an approach was also employed in the previous studies, e.g. [1, 2, 10, 17, 43]. For the 2×2 viscous-capillarity system in the compressible van der Waals fluids, Affouf and Caflisch [1] and Hsien and Wang [17] numerically showed the phase transition solution to the Cauchy problem and the periodic initial-boundary value problem, respectively. In [7], Eden *et al* established the existence of the local attractors and the exponential attractors of finite fractal dimension for the periodic initial-boundary value problem to the van der Waals fluid model by means of semigroup method, and showed the stability of trivial stationary solution in the weak sense. However, they required the initial datum to be small sufficiently. In [30], Oh and Zumbrun used the method of Evan function to study the spectral stability of the periodic traveling waves in the stable regions for the p-system of viscosity-capillarity, and later in [31], they further investigated the linearized stability of the periodic waves. The linear stability was later nontrivially extended by D. Serre [32] to the case of the long-waves. The linear orbital stability of the travelling wave-front to the Cauchy problem of p-system with viscosity-capillarity was also obtained by Zumbrun in [43]. For the other study with a strongly hyperbolic background, such as the construction of shock and wave curves, as well as the vanishing viscosity approach, see [2]-[6], [9], [10], [13]-[18] and [33]-[40] and the references therein.

Recently, as shown in [12, 41] for Cahn-Hilliard equation, we considered in [27] the corresponding stationary system (9), and proved that when $\varepsilon_1\varepsilon_2 > \frac{L^2}{\pi^2}(1-\frac{3}{4}m_0^2)$, there exists only one trivial stationary solution (m_0, m_1), and when $\varepsilon_1\varepsilon_2 \le \frac{L^2}{\pi^2}(1 - 3m_0^2)$, there exist multiple nontrivial solutions. These solutions $(V, U)(x)$ may lead to phase transitions. The number of these solutions is counted, which is related to the size of m_0. When $\frac{L^2}{\pi^2}(1 - 3m_0^2) \le \varepsilon_1\varepsilon_2 \le \frac{L^2}{\pi^2}(1 - \frac{3}{4}m_0^2)$, the stationary system (9) may have only one trivial solution or multiple nontrivial solutions due to the size of $m_0 \in [0, \frac{2}{\sqrt{3}}]$. In [28], we further proved the existence, the uniqueness and the uniform boundedness of the solution to (1)-(3) for any given initial datum, even for the large initial datum. In this part, we study the relationship between the original solution $(v, u)(x, t)$ of (1)-(3) and the stationary solutions $(V, U)(x)$ of (9). We prove that, the solution $(v, u)(x, t)$ converges to a certain stationary solution

$(V,U)(x)$, even if the viscosity $\varepsilon_1\varepsilon_2$ is very small and the initial datum $(v_0,u_0)(x)$ is in the elliptic region. In particular, when $\varepsilon_1\varepsilon_2 > \frac{L^2}{\pi^2}(1-\frac{3}{4}m_0^2)$, for any given initial datum, we show the stability of the trivial stationary solution (m_0,m_1), and establish an exponential decay rate. This is a stability result for the large initial perturbation. No phase transition for the solution occurs after a short initial time. Finally, numerical simulations are carried to confirm the theoretical predictions.

Notation. Throughout the paper, $C>0$ denotes a generic constant, while $C_i>0$ $(i=0,1,2,\cdots)$ represents a specific constant, $R=(-\infty,\infty)$. Since solutions $(v,u)(x,t)$ of (2) and (3) are periodic, we introduce spaces of periodic functions. Letting $p=2L$ denote the period, we introduce the Hilbert space $L^2_{per}(R)$ of locally square integrable functions which are periodic with period p,

$$L^2_{per}(R)=\Big\{v(x)\mid v(x)=v(x+p) \text{ for all } x\in R, \text{ and } v(x)\in L^2(0,p) \text{ for } x\in[0,p]\Big\},$$

with the norm given by integral over $[0,p]$ (or over any other interval of length p), denoted by $\|\cdot\|$,

$$\|v\|=\Big(\int_0^p v^2(x)dx\Big)^{1/2}.$$

We define the Sobolev space $H^k_{per}(R)$ $(k\geq 0)$ to be the space of functions $v(x)$ in $L^2_{per}(R)$ whose derivatives $\partial_x^i v$, $i=1,\cdots,k$ also belong to $L^2_{per}(R)$ with the norm denoted by $\|\cdot\|_k$,

$$\|v\|_k=\Big(\sum_{i=0}^k\int_0^p|\partial_x^i v(x)|^2dx\Big)^{1/2},$$

where $\|v\|_0=\|v\|$. We use the simplified notation $\|(f,g)\|^2=\|f\|^2+\|g\|^2$ and $\|(f,g)\|_k^2=\|f\|_k^2+\|g\|_k^2$. The periodic spaces $L^\infty_{per}(R)$ and $L^k_{per}(R)$, where k is a positive integer, are similarly defined. Let $T>0$ be a number and B be a Banach space. We denote by $C^0([0,T];B)$ the space of B-valued continuous functions on $[0,T]$. The corresponding spaces of B-valued functions on $[0,\infty)$ are defined similarly.

2. **Main results.** Based on the study on the existence and the uniform boundedness of the solution (v,u) of (1)-(3) and the stationary solution (V,U) of (9) already reported in [27, 28], we now establish the convergence of (v,u) to the stationary solutions (V,U) stated as follows.

Theorem 2.1 (Convergence). *Let $(v_0,u_0)(x)\in H^2_{per}(R)$. Then the solution of the periodic IBVP (1)-(3) converges to a certain stationary solution $(V,U)(x)$ in $H^1_{per}(R)$:*

$$\|(v-V,u-U)(t)\|_1\to 0, \quad as \ \ t\to\infty. \tag{12}$$

Remark 1. Theorem 2.1 implies that, even for very small viscosity $\varepsilon_1\varepsilon_2$ and the initial average m_0 in the elliptic region $(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$, the solution $(v,u)(x,t)$ of (1)-(3) converges to a nontrivial stationary solution $(V,U)(x)$ of (9) time-asymptotically. As we remarked in [27], such a nontrivial stationary solution usually exhibits phase transitions. Since $(U,V)(x)$ is the asymptotic profile of the original solution (v,u) (x,t), so the solution $(v,u)(x,t)$ must also occur phase transitions after long time.

Theorem 2.2 (Convergence Rates). *Let $(v_0,u_0)(x)\in H^2_{per}(R)$ and*

$$\varepsilon_1\varepsilon_2>\frac{L^2}{\pi^2}(1-\frac{3}{4}m_0^2). \tag{13}$$

Then the solution of the periodic IBVP (1)-(3) converges to the trivial stationary solution (m_0, m_1) *with an exponential rate. Precisely,*

1. If m_0 *is in the hyperbolic region* $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$*, then*

$$\|(v - m_0, u - m_1)(t)\|_1 \le C_1 \|(v_0 - m_0, u_0 - m_1)\|_1 e^{-\gamma_1 t}, \quad t \ge 0 \tag{14}$$

where $C_1 > 0$ *is a constant, and*

$$0 < \gamma_1 < \frac{\pi^2}{L^2} \min\{\varepsilon_1, \varepsilon_2\}. \tag{15}$$

Moreover, there exists

$$t_1 = \max\Big\{0, \frac{1}{\gamma_1} \ln \frac{C_1 \|(v_0 - m_0, u_0 - m_1)\|_1}{m_0 - \frac{1}{\sqrt{3}}}\Big\}$$

in the case $m_0 > \frac{1}{\sqrt{3}}$*, or*

$$t_1 = \max\Big\{0, \frac{1}{\gamma_1} \ln \frac{C_1 \|(v_0 - m_0, u_0 - m_1)\|_1}{|m_0 + \frac{1}{\sqrt{3}}|}\Big\}$$

in the case $m_0 < -\frac{1}{\sqrt{3}}$*, such that, when* $t > t_1$*, the solution* $v(x,t)$ *stays in the same hyperbolic region and no phase transition occurs, i.e.,*

$$v(x,t) > \frac{1}{\sqrt{3}} \;\; (or \;\; < -\frac{1}{\sqrt{3}}) \;\; for\ t > t_1. \tag{16}$$

2. If m_0 *is in the elliptic region* $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$*, then*

$$\|(v - m_0, u - m_1)(t)\|_1 \le C_2 \|(v_0 - m_0, u_0 - m_1)\|_1 e^{-\gamma_2 t}, \quad t \ge 0 \tag{17}$$

where $C_2 > 0$ *is a constant, and*

$$0 < \gamma_2 < \frac{\pi^2}{L^2} \min\Big\{(1-\eta)\varepsilon_2, \varepsilon_1 - \frac{L^2(1 - 3m_0^2)}{\eta \varepsilon_2 \pi^2}\Big\}, \tag{18}$$

η *is a positive constant satisfying*

$$1 > \eta > \frac{L^2(1 - 3m_0^2)}{\varepsilon_1 \varepsilon_2 \pi^2}.$$

Moreover, there exists

$$t_1 = \max\Big\{0, \; \frac{1}{\gamma_2} \ln \frac{C_2 \|(v_0 - m_0, u_0 - m_1)\|_1}{m_0 + \frac{1}{\sqrt{3}}}, \; \frac{1}{\gamma_2} \ln \frac{C_2 \|(v_0 - m_0, u_0 - m_1)\|_1}{\frac{1}{\sqrt{3}} - m_0}\Big\},$$

such that, when $t > t_1$*, the solution* $v(x,t)$ *stays in the elliptic region and no phase transition occurs*

$$-\frac{1}{\sqrt{3}} < v(x,t) < \frac{1}{\sqrt{3}} \;\; for\ t > t_1. \tag{19}$$

Remark 2. 1). The condition $\varepsilon_1 \varepsilon_2 > \frac{L^2}{\pi^2}(1 - \frac{4}{3} m_0^2)$ means, either m_0 is in the hyperbolic regions $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$ and not close to the phase boundaries $v = \pm\frac{1}{\sqrt{3}}$ (in fact, $m_0 > \frac{2}{\sqrt{3}}$ or $m_0 < -\frac{2}{\sqrt{3}}$), or m_0 is in the elliptic region $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ but the viscosity $\varepsilon_1 \varepsilon_2$ must be large enough (i.e., strong parabolicity for the equations (1)).

2). Theorem 2.2 shows the stability of the trivial stationary solution (m_0, m_1) as well as the exponential decay rates for any *large* initial perturbation. This is different from the usual case of small initial perturbation.

Theorem 2.3 (Improved Convergence Rates)**.** *For the viscosities ε_1 and ε_2 satisfying*

$$\frac{L^2}{\pi^2}(1-3m_0^2) < \varepsilon_1\varepsilon_2 \le \frac{L^2}{\pi^2}(1-\frac{3}{4}m_0^2), \tag{20}$$

there exists a positive constant δ_1, such that when $\|(v_0-m_0, u_0-m_1)\|_1 \le \delta_1$, the convergence rates stated in Theorem 2.2 still hold.

Remark 3. Theorem 2.3 implies that, when $\frac{L^2}{\pi^2}(1-3m_0^2) < \varepsilon_1\varepsilon_2 \le \frac{L^2}{\pi^2}(1-\frac{3}{4}m_0^2)$, if the initial perturbation around the trivial stationary solution (m_0, m_1) is small enough, then the solution $(v,u)(x,t)$ is still exponentially convergent to (m_0, m_1).

3. **Convergence and convergent rates.** In this section, we present the proofs for Theorems 2.1, 2.2 and 2.3.

3.1. **Proof of Theorem 2.1.** We first prove that, for any given initial datum in $H^2_{per}(R)$, the solution will converge to a certain stationary solution in $H^1_{per}(R)$. No restriction is imposed on the amplitude of the initial perturbation around the stationary solution. The approach we adopt is the method of ω-limit (see [3] and the references therein).

Thanks to Theorem 1.1 in [28], for any given $(v_0, u_0) \in H^2_{per}(R)$, the system (1)-(3) admits a unique and global solution $(v,u) \in C([0,\infty), H^2_{per}(R))$, which defines a strongly continuous nonlinear semigroup $(v,u) = T(t)(v_0,u_0)$ in $H^2_{per}(R)$.

For the solution $(v,u)(x,t) = T(t)(v_0,u_0)$, we define the ω-limit as follows

$$\begin{aligned}\omega(v_0,u_0) = \{(V,U)(x) \quad | \quad & \text{there exists } t_n, \text{ such that} \\ & T(t_n)(v_0,u_0) \to (V,U)(x) \text{ in } H^1_{per}(R)\}.\end{aligned} \tag{21}$$

We recall a well-known result from the elementary topological dynamics as follows (for example, see Proposition 2.1 and its proof in [3]).

Proposition 1 ([3])**.** *Let $T(t)$ be a continuous semigroup on $H^1_{per}(R)$. If the orbit $\gamma(v_0,u_0) := \cup_{t>0}T(t)(v_0,u_0)$, through some point $(v_0,u_0) \in H^1_{per}(R)$, is relatively compact in $H^1_{per}(R)$, then there exists one element $(V,U)(x) \in \omega(v_0,u_0)$ such that*

$$\lim_{t\to\infty} \|(v,u)(\cdot,t) - (V,U)(\cdot)\|_1 = 0, \tag{22}$$

where $\omega(v_0,u_0)$ is a compact connected set of $H^1_{per}(R)$. Furthermore, $\omega(v_0,u_0)$ is positive invariant under $T(t)$, i.e., $T(t)\omega(v_0,u_0) \subset \omega(v_0,u_0)$ for any $t \ge 0$.

According to the definition of the ω-limit set, we have

Proposition 2. *Each element $(V,U)(x)$ in the ω-limit set $\omega(v_0,u_0)$ is a stationary solution of* (9)*.*

Proof of Theorem 2.1. As we know, to prove the relatively compact of the orbit $\cup_{t\ge0}T(t)(v_0,u_0)$ in $H^1_{per}(R)$, it suffices to prove the uniform boundedness of $(v,u)(x,t)$ in $H^2_{per}(R)$ for all time $t \ge 0$. Thanks to Theorem 1.1 in [28], $(v,u)(x,t)$ is indeed uniformly bounded in H^2_{per}. Thus, Proposition 1, and Proposition 2 imply Theorem 2.1 immediately. □

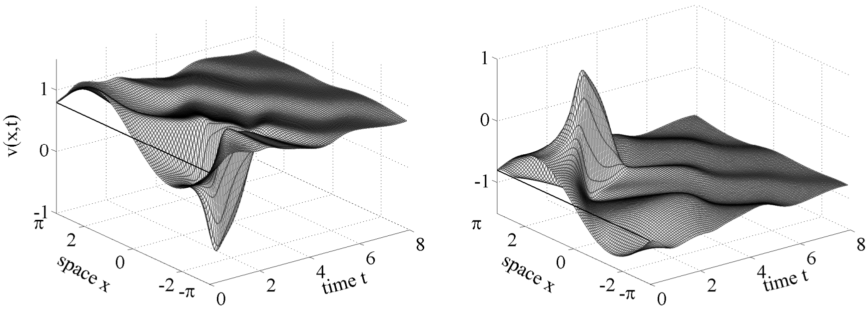


FIGURE 1. (Case 1) Hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$. (Case 2) Hyperbolic region $(-\infty, -\frac{1}{\sqrt{3}})$.

3.2. **Proof of Theorem 2.2.** Let $\varepsilon_1\varepsilon_2 > \frac{L^2}{\pi^2}(1 - \frac{3}{4}m_0^2)$. For any given initial datum $(v_0, u_0) \in H^2_{per}(R)$, from Convergence Theorem 2.1 we know that the solution $(v, u)(x, t)$ of (1)-(3) converges to a certain stationary solution $(V, U)(x)$ in $H^1_{per}(R)$. When $\varepsilon_1\varepsilon_2 > \frac{\pi^2}{L^2}(1 - \frac{3}{4}m_0^2)$, according to Theorem 1.1 in [27], there exists only one stationary solution (m_0, m_1). Therefore, the solution $(v, u)(x, t)$ must converge only to (m_0, m_1) in $H^1_{per}(R)$. Namely,

Lemma 3.1. *It holds*

$$\lim_{t\to\infty} \|(v - m_0, u - m_1)(t)\|_1 = 0. \tag{23}$$

According to Lemma 3.1, for a given small $\eta_0 > 0$, there exists a sufficiently large number $t_* > 0$ such that

$$\sup_{x\in[0,2L]} |(v - m_0, u - m_1)(x, t)| \le \eta_0 \ll 1, \text{ for } t \ge t_*. \tag{24}$$

Since t_* is a finite number, we can easily have the following estimate

$$\|(v - m_0, u - m_1)(t)\|_1^2 \le C_{t_*}\|(v_0 - m_0, u_0 - m_1)\|_1^2 e^{-2\gamma_1 t_*}, \quad 0 \le t \le t_*, \tag{25}$$

for example, we may take

$$C_{t_*} = \frac{\sup_{0\le t\le t_*} \|(v - m_0, u - m_1)(t)\|_1^2 e^{2\lambda_1 t_*}}{\|(v_0 - m_0, u_0 - m_1)\|_1^2}.$$

Let $\bar{v}(x, t) := v(x, t) - m_0$ and $\bar{u}(x, t) := u(x, t) - m_1$. With (8), the original problem (1)-(3) is then rewritten as

$$\begin{cases} \bar{v}_t - \bar{u}_x = \varepsilon_1 \bar{v}_{xx}, \\ \bar{u}_t - \sigma'(m_0)\bar{v}_x = \varepsilon_2\bar{u}_{xx} + F_x, \quad x \in (0, 2L), t > 0 \\ \bar{v}|_{t=0} = v_0(x) - m_0 =: \bar{v}_0(x), \ \bar{u}|_{t=0} = u_0(x) - m_1 =: \bar{u}_0(x), \\ \bar{v}(x, t) = \bar{v}(x + 2L, t), \ \bar{u}(x, t) = \bar{u}(x + 2L, t), \\ \int_0^{2L} \bar{v}(x, t)dx = 0, \ \int_0^{2L} \bar{u}(x, t)dx = 0, \end{cases} \tag{26}$$

where

$$F = \sigma(\bar{v} + m_0) - \sigma(m_0) - \sigma'(m_0)\bar{v}.$$

By the Taylor's expansion, we have

$$|F| \le C\bar{v}^2, \quad |F_x| \le C|\bar{v}\bar{v}_x| \le C(\bar{v}^2 + \bar{v}_x^2). \tag{27}$$

Case 1: m_0 is in the hyperbolic region. In this case, we have $\sigma'(m_0) > 0$. Multiplying the first equation of (26) by $\sigma'(m_0)\bar{v}e^{2\gamma_1 t}$ and the second by $\bar{u}e^{2\gamma_1 t}$, then by adding and integrating over $[0, 2L] \times [t_*, t]$ $(t > t_*)$, we obtain

$$\begin{aligned}
& e^{2\gamma_1 t}\sigma'(m_0)\|\bar{v}(t)\|^2 + e^{2\gamma_1 t}\|\bar{u}(t)\|^2 - 2\gamma_1\sigma'(m_0)\int_{t_*}^{t} e^{2\gamma_1\tau}\|\bar{v}(\tau)\|^2 d\tau \\
& -2\gamma_1 \int_{t_*}^{t} e^{2\gamma_1\tau}\|\bar{u}(\tau)\|^2 d\tau + 2\varepsilon_1\sigma'(m_0)\int_{t_*}^{t} e^{2\gamma_1\tau}\|\bar{v}_x(\tau)\|^2 d\tau \\
& +2\varepsilon_2 \int_{t_*}^{t} e^{2\gamma_1\tau}\|\bar{u}_x(\tau)\|^2 d\tau \\
= \;& e^{2\gamma_1 t_*}(\sigma'(m_0)\|\bar{v}(t_*)\|^2 + \|\bar{u}(t_*)\|^2) + 2\int_{t_*}^{t}\int_0^{2L} F_x \bar{u}(x,\tau)e^{2\gamma_1\tau}\, dx d\tau. \quad (28)
\end{aligned}$$

By $|F_x| \le C(|\bar{v}|^2 + |\bar{v}_x|^2)$ (Talyor's formula, see (27)) and (24), i.e., $\sup_{t\ge t_*} \|\bar{u}(t)\|_{L^\infty_{per}} < \eta_0 \ll 1$), we have

$$\begin{aligned}
& 2\int_{t_*}^{t}\int_0^{2L} F_x \bar{u}(x,\tau)e^{2\gamma_1\tau}\, dx d\tau \\
\le\; & C\int_{t_*}^{t}\int_0^{2L} (|\bar{v}(x,\tau)|^2 + |\bar{v}_x(x,\tau)|^2)|\bar{u}(x,\tau)|e^{2\gamma_1\tau}\, dx d\tau \\
\le\; & C(\sup_{t\ge t_*}\|\bar{u}(t)\|_{L^\infty_{per}})\int_{t_*}^{t} (\|\bar{v}(\tau)\|^2 + \|\bar{v}_x(\tau)\|^2)e^{2\gamma_1\tau}\, d\tau \\
\le\; & C_3\eta_0 \int_{t_*}^{t} (\|\bar{v}(\tau)\|^2 + \|\bar{v}_x(\tau)\|^2)e^{2\gamma_1\tau}\, d\tau, \quad (29)
\end{aligned}$$

for some positive constant C_3. Note Lemma 2.4 (Poincaré inequalities) in [28], i.e.,

$$\|\bar{v}\| \le \frac{L}{\pi}\|\bar{v}_x\|, \quad \|\bar{u}\| \le \frac{L}{\pi}\|\bar{u}_x\|,$$

substituting (29) into (28) yields

$$\begin{aligned}
& e^{2\gamma_1 t}\sigma'(m_0)\|\bar{v}(t)\|^2 + e^{2\gamma_1 t}\|\bar{u}(t)\|^2 \\
& +A_1(t)\int_{t_*}^{t} e^{2\gamma_1\tau}\|\bar{v}_x(\tau)\|^2 d\tau + B_1(t)\int_{t_*}^{t} e^{2\gamma_1\tau}\|\bar{u}_x(\tau)\|^2 d\tau \\
\le\; & e^{2\gamma_1 t_*}[\sigma'(m_0)\|\bar{v}(t_*)\|^2 + \|\bar{u}(t_*)\|^2], \quad (30)
\end{aligned}$$

where

$$\begin{cases} A_1(t) := 2\varepsilon_1\sigma'(m_0) - \frac{2L^2\sigma'(m_0)}{\pi^2}\gamma_1 - \frac{2C_3L^2}{2\pi^2}\eta_0 - \frac{2C_3}{2}\eta_0, \\ B_1(t) := 2\varepsilon_2 - \frac{2L^2}{\pi^2}\gamma_1. \end{cases} \quad (31)$$

Let γ_1 be such that

$$0 < \gamma_1 < \frac{\pi^2}{L^2}\min\{\varepsilon_1, \varepsilon_2\},$$

and η_0 be small such that

$$0 < \eta_0 < \sigma'(m_0)\Big(\varepsilon_1 - \frac{L^2\gamma_1}{\pi^2}\Big)\Big/\Big(\frac{C_3L^2}{2\pi^2} + \frac{C_3}{2}\Big).$$

Thus, we have

$$A_1(t) \ge C_4 > 0, \quad B_1(t) \ge C_5 > 0$$

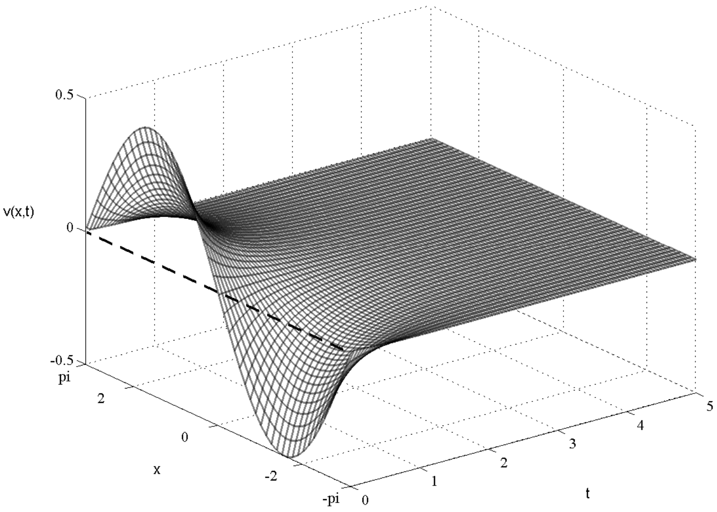


FIGURE 2. (Case 3) m_0 is in the elliptic phase but with a big viscosity

for some positive constants C_4 and C_5. Therefore, (30) and (25) imply

$$\begin{aligned} & e^{2\gamma_1 t}\|(\bar{v},\bar{u})(t)\|^2 + \int_{t_*}^t e^{2\gamma_1\tau}\|(\bar{v}_x,\bar{u}_x)(\tau)\|^2 d\tau \\ \le\; & Ce^{2\gamma_1 t_*}\|(\bar{v},\bar{u})(t_*)\|^2 \le C\|(\bar{v}_0,\bar{u}_0)\|_1^2, \quad t \ge t_*. \end{aligned} \tag{32}$$

Combining (25) and (32), we have

$$\|(\bar{v},\bar{u})(t)\| \le C\|(\bar{v}_0,\bar{u}_0)\|e^{-\gamma_1 t}, \quad t \ge 0. \tag{33}$$

For a higher order estimate, differentiating the equations in (26) with respect to x, multiplying the first equation by $\sigma'(m_0)e^{2\gamma_1 t}\bar{v}_x$ and the second by $e^{2\gamma_1 t}\bar{u}_x$, by adding and integrating the resulting equation over $[0,2L]\times[t_*,t]$, then using $\|\bar{v}_x\|^2 \le \frac{L^2}{\pi^2}\|\bar{v}_{xx}\|^2$ and $\|\bar{u}_x\|^2 \le \frac{L^2}{\pi^2}\|\bar{u}_{xx}\|^2$, the following estimate can be similarly obtained

$$e^{2\gamma_1 t}\|(\bar{v}_x,\bar{u}_x)(t)\|^2 + \int_{t_*}^t e^{2\gamma_1\tau}\|(\bar{v}_{xx},\bar{u}_{xx})(t)\|^2 d\tau \le C\|(\bar{v}_0,\bar{u}_0)\|_1^2, \qquad t \ge t_*. \tag{34}$$

Combining (34) with (25) yields

$$\|(\bar{v}_x,\bar{u}_x)(t)\| \le C\|(\bar{v}_0,\bar{u}_0)\|_1 e^{-\gamma_1 t}, \quad t \ge 0. \tag{35}$$

Thus, (33) and (35) give (14)

$$\|(\bar{v},\bar{u})(t)\|_1 \le C_1\|(\bar{v}_0,\bar{u}_0)\|_1 e^{-\gamma_1 t}, \quad t \in [0,\infty)$$

for some $C_1 > 0$.

Finally, it remains to prove (16), i.e., no phase transition after time t_1. We only prove the case $m_0 > \frac{1}{\sqrt{3}}$, since the case $m_0 < -\frac{1}{\sqrt{3}}$ can be similarly treated. Using the Sobolev's inequality, we have

$$\|\bar{v}(t)\|_{L^\infty_{per}}^2 \le 2\|\bar{v}(t)\|\|\bar{v}_x(t)\| \le \|\bar{v}(t)\|_1^2 \le C_1^2\|(\bar{v}_0,\bar{u}_0)\|_1^2 e^{-2\gamma_1 t}.$$

This leads to

$$\begin{aligned} v(x,t) &= m_0 + (v(x,t) - m_0) \geq m_0 - \|\bar v(t)\|_{L^\infty_{per}} \\ &\geq m_0 - C_1\|(\bar v_0, \bar u_0)\|_1 e^{-\gamma_1 t} \\ &= \frac{1}{\sqrt{3}} + \Big[(m_0 - \frac{1}{\sqrt{3}}) - C_1\|(\bar v_0, \bar u_0)\|_1 e^{-\gamma_1 t}\Big] \\ &\geq \frac{1}{\sqrt{3}}, \qquad \text{for } t \geq t_1, \end{aligned}$$

where

$$t_1 = \max\Big\{0, \frac{1}{\gamma_1}\ln\frac{C_1\|(\bar v_0, \bar u_0)\|_1}{m_0 - \frac{1}{\sqrt{3}}}\Big\}.$$

Thus, this proves the first part of Theorem 2.2.

Case 2: m_0 is in the elliptic region. In this case, $\sigma'(m_0) < 0$. Multiplying the first equation of (26) by $|\sigma'(m_0)|\bar v e^{2\gamma_2 t}$ and the second by $\bar u e^{2\gamma_2 t}$, then by adding and integrating over $[0, 2L] \times [t_*, t]$, we have

$$\begin{aligned} & e^{2\gamma_2 t}|\sigma'(m_0)|\|\bar v(t)\|^2 + e^{2\gamma_2 t}\|\bar u(t)\|^2 - 2\gamma_2|\sigma'(m_0)|\int_{t_*}^t e^{2\gamma_2\tau}\|\bar v(\tau)\|^2 d\tau \\ & -2\gamma_2\int_{t_*}^t e^{2\gamma_2\tau}\|\bar u(\tau)\|^2 d\tau - 4|\sigma'(m_0)|\int_{t_*}^t\int_0^{2L} e^{2\gamma_2\tau}\bar v(x,\tau)\bar u_x(x,\tau)\ dx\ d\tau \\ & +2\varepsilon_1|\sigma'(m_0)|\int_{t_*}^t e^{2\gamma_2\tau}\|\bar v_x(\tau)\|^2 d\tau + 2\varepsilon_2\int_{t_*}^t e^{2\gamma_2\tau}\|\bar u_x(\tau)\|^2 d\tau \\ = \ & e^{2\gamma_2 t_*}(|\sigma'(m_0)|\|\bar v(t_*)\|^2 + \|\bar u(t_*)\|^2) \\ & +2\int_{t_*}^t\int_0^{2L} F_x\bar u(x,\tau)e^{2\gamma_2\tau}\ dxd\tau, \quad t \geq t_* \end{aligned} \tag{36}$$

Applying the Poincaré Lemma 2.4 in [28], i.e.,

$$\|(\bar v, \bar u)(t)\| \leq \frac{L}{\pi}\|(\bar v_x, \bar u_x)(t)\|,$$

we can estimate

$$2\gamma_2|\sigma'(m_0)|\int_{t_*}^t e^{2\gamma_2\tau}\|\bar v(\tau)\|^2 d\tau \leq \frac{2L^2\gamma_2|\sigma'(m_0)|}{\pi^2}\int_{t_*}^t e^{2\gamma_2\tau}\|\bar v_x(\tau)\|^2 d\tau, \tag{37}$$

$$2\gamma_2\int_{t_*}^t e^{2\gamma_2\tau}\|\bar u(\tau)\|^2 d\tau \leq \frac{2L^2\gamma_2}{\pi^2}\int_{t_*}^t e^{2\gamma_2\tau}\|\bar u_x(\tau)\|^2 d\tau. \tag{38}$$

By $|F_x| \leq C(\bar v^2 + \bar v_x^2)$ and the Sobolev's inequality

$$\sup_{t\geq t_*}\|\bar u(t)\|_{L^\infty_{per}} \leq \sup_{t\geq t_*}\|\bar u(t)\|_1 < \eta_0$$

to the last term of (36), we have

$$\begin{aligned} & 2\int_{t_*}^t\int_0^{2L} F_x\bar u(x,\tau)e^{2\gamma_2\tau}\ dxd\tau \\ \leq\ & C(\sup_{t\geq t_*}\|\bar u(t)\|_{L^\infty_{per}})\int_{t_*}^t e^{2\gamma_2\tau}[\|\bar v(\tau)\|^2 + \|\bar v_x(\tau)\|^2]\ d\tau \\ \leq\ & C_6\eta_0\Big(1 + \frac{L^2}{\pi^2}\Big)\int_{t_*}^t e^{2\gamma_2\tau}\|\bar v_x(\tau)\|^2\ d\tau, \quad t \geq t_*, \end{aligned} \tag{39}$$

for some positive constant C_6. Similarly, we can control, for any $\eta > 0$

$$\begin{aligned}
&4|\sigma'(m_0)|\int_{t_*}^{t}\int_0^{2L} e^{2\gamma_2\tau}\bar{v}(x,\tau)\bar{u}_x(x,\tau)\ dx\ d\tau\\
\le\ & 2\eta\varepsilon_2\int_{t_*}^{t}\int_0^{2L} e^{2\gamma_2\tau}\bar{u}_x^2(x,\tau)dxd\tau + \frac{2|\sigma'(m_0)|^2}{\eta\varepsilon_2}\int_{t_*}^{t}\int_0^{2L} e^{2\gamma_2\tau}\bar{v}^2(x,\tau)dxd\tau\\
\le\ & 2\eta\varepsilon_2\int_{t_*}^{t} e^{2\gamma_2\tau}\|\bar{u}_x(\tau)\|^2 d\tau + \frac{2L^2|\sigma'(m_0)|^2}{\eta\varepsilon_2\pi^2}\int_{t_*}^{t} e^{2\gamma_2\tau}\|\bar{v}_x(\tau)\|^2 d\tau. \qquad (40)
\end{aligned}$$

Substituting (37)-(40) into (36), we obtain

$$\begin{aligned}
& e^{2\gamma_2 t}|\sigma'(m_0)|\|\bar{v}(t)\|^2 + e^{2\gamma_2 t}\|\bar{u}(t)\|^2\\
& +A_2(t)\int_{t_*}^{t} e^{2\gamma_2\tau}\|\bar{v}_x(\tau)\|^2 d\tau + B_2(t)\int_{t_*}^{t} e^{2\gamma_2\tau}\|\bar{u}_x(\tau)\|^2 d\tau\\
\le\ & e^{2\gamma_2 t_*}(|\sigma'(m_0)|\|\bar{v}(t_*)\|^2 + \|\bar{u}(t_*)\|^2), \quad t \ge t_*, \qquad (41)
\end{aligned}$$

where

$$\begin{cases} A_2(t) := 2\varepsilon_1|\sigma'(m_0)| - \frac{2L^2|\sigma'(m_0)|^2}{\eta\varepsilon_2\pi^2} - \frac{2L^2|\sigma'(m_0)|}{\pi^2}\gamma_2 - \frac{C_6L^2}{\pi^2}\eta_0 - C_6\eta_0,\\ B_2(t) := 2(1-\eta)\varepsilon_2 - \frac{2L^2}{\pi^2}\gamma_2. \end{cases} \qquad (42)$$

By the assumption (13), i.e.,

$$\varepsilon_1 > \frac{L^2}{\varepsilon_2\pi^2}(1-\frac{3}{4}m_0^2) \ge \frac{L^2}{\varepsilon_2\pi^2}(1-3m_0^2) = \frac{L^2}{\varepsilon_2\pi^2}|\sigma'(m_0)|,$$

let η be such that

$$1 > \eta > \frac{L^2}{\varepsilon_1\varepsilon_2\pi^2}|\sigma'(m_0)|,$$

and γ_2 be such that

$$0 < \gamma_2 < \frac{\pi^2}{L^2}\min\Big\{(1-\eta)\varepsilon_2, \varepsilon_1 - \frac{L^2}{\eta\varepsilon_2\pi^2}|\sigma'(m_0)|\Big\},$$

and η_0 be sufficiently small, such that

$$0 < \eta_0 < \Big(2\varepsilon_1|\sigma'(m_0)| - \frac{2L^2|\sigma'(m_0)|^2}{\eta\varepsilon_2\pi^2} - \frac{2L^2|\sigma'(m_0)|}{\pi^2}\gamma_2\Big)\Big/\Big(\frac{C_6L^2}{\pi^2} + C_6\Big).$$

Thus, we have

$$A_2(t) \ge C > 0, \quad B_2(t) \ge C > 0.$$

Therefore, (41) and (25) imply

$$\begin{aligned}
& e^{2\gamma_2 t}\|\bar{v}(t)\|^2 + e^{2\gamma_2 t}\|\bar{u}(t)\|^2 + \int_{t_*}^{t} e^{2\gamma_2\tau}[\|\bar{v}_x(t)\|^2 + \|\bar{u}_x(t)\|^2]d\tau\\
\le\ & Ce^{2\gamma_2 t_*}\|(\bar{v},\bar{u})(t_*)\|^2 \le C\|(\bar{v}_0,\bar{u}_0)\|^2, \quad t \ge t_*. \qquad (43)
\end{aligned}$$

Combing (43) and (25) yield

$$\|(\bar{v},\bar{u})(t)\| \le C\|(\bar{v}_0,\bar{u}_0)\|e^{-\gamma_2 t}, \quad t \ge 0. \qquad (44)$$

Similarly, we can show the energy estimate for the higher order derivatives of the solution

$$\|(\bar{v}_x,\bar{u}_x)(t)\| \le C\|(\bar{v}_0,\bar{u}_0)\|_1 e^{-\gamma_2 t}, \quad t \in [0,\infty). \qquad (45)$$

The detail of the proof is omitted. Hence, (44) and (45) imply (17) for some positive constant C_2.

Finally, the time t_1 given in the second part of Theorem 2.2, as well as (19) for no phase transition exhibiting after t_1, can be also proved as shown in Case 1. We omit the detail. Thus, the proof is complete.

3.3. **Proof of Theorem 2.3.** Let $\frac{\pi^2}{L^2}(1-3m_0^2) < \varepsilon_1\varepsilon_2 \le \frac{\pi^2}{L^2}(1-\frac{3}{4}m_0^2)$. We now prove that the sufficient condition for the convergence rates to the trivial solution (m_0, m_1) can be relaxed to $\frac{\pi^2}{L^2}(1-3m_0^2) < \varepsilon_1\varepsilon_2 \le \frac{\pi^2}{L^2}(1-\frac{3}{4}m_0^2)$. In this case, unlike the previous study, there is no information from Theorem 2.1 to guarantee the solution $(v,u)(x,t)$ to converge to the trivial stationary solution (m_0, m_1) only. However, if the initial perturbation $(v_0 - m_0, u_0 - m_1)$ is small enough in $H^1_{per}(R)$, we can prove that the solution $(v,u)(x,t)$ still converges to (m_0, m_1) with the same decay rate as shown in Theorem 2.2.

For any given $T > 0$, let $(\bar{v}, \bar{u}) \in C([0,T]; H^1_{per})$ be a solution of (26), and

$$N(t) = \sup_{x\in[0,2L]} \|(\bar{v},\bar{u})(t)\|_1.$$

As reported in [19, 21, 25], the key step is to establish the *a priori* estimates for the solution.

When m_0 is in the hyperbolic region, we multiply the first equation of (26) by $\sigma'(m_0)\bar{v}e^{2\gamma_1 t}$ and the second by $\bar{u}e^{2\gamma_1 t}$, then by adding and integrating over $[0,2L]\times[0,t]$, we get

$$\begin{aligned} & e^{2\gamma_1 t}\sigma'(m_0)\|\bar{v}(t)\|^2 + e^{2\gamma_1 t}\|\bar{u}(t)\|^2 \\ & +A_3(t)\int_0^t e^{2\gamma_1\tau}\|\bar{v}_x(\tau)\|^2 d\tau + B_3(t)\int_0^t e^{2\gamma_1\tau}\|\bar{u}_x(\tau)\|^2 d\tau \\ \le\ & \sigma'(m_0)\|\bar{v}_0\|^2 + \|\bar{u}_0\|^2, \quad t\ge 0 \end{aligned} \tag{46}$$

where

$$\begin{cases} A_3(t) := 2\varepsilon_1\sigma'(m_0) - \frac{2L^2\sigma'(m_0)}{\pi^2}\gamma_1 - \frac{2CL^2}{2\pi^2}N(t) - \frac{2C}{2}N(t), \\ B_3(t) := 2\varepsilon_2 - \frac{2L^2}{\pi^2}\gamma_1. \end{cases} \tag{47}$$

So, when $\frac{\pi^2}{L^2}(1-3m_0^2) < \varepsilon_1\varepsilon_2 \le \frac{\pi^2}{L^2}(1-\frac{3}{4}m_0^2)$, let γ_1 be chosen as before, and $N(t) \ll 1$, then

$$A_3(t) \ge C > 0, \qquad B_3(t) \ge C > 0.$$

Thus, we prove the first *a priori* estimate

Lemma 3.2. *It holds*

$$\|(\bar{v},\bar{u})(t)\|^2 \le C\|(\bar{v}_0,\bar{u}_0)\|^2 e^{-2\gamma_1 t}, \quad t\ge 0 \tag{48}$$

provided $N(t) \ll 1$.

Using (48) and Eq. (26), we further prove the second *a priori* estimate

Lemma 3.3. *It holds*

$$\|(\bar{v}_x,\bar{u}_x)(t)\|^2 \le \|(\bar{v}_0,\bar{u}_0)\|_1^2 e^{-2\lambda_1 t}, \quad t\ge 0 \tag{49}$$

provided $N(t) \ll 1$.

It is known that, $N(t) \ll 1$ implies the smallness of the initial datum $\|(\bar{v}_0,\bar{u}_0)\|_1 \ll 1$, and by the continuity argument [19, 21, 25], the two *a priori* estimates (48) and (49) can be used to guarantee Theorem 2.3 in the case of m_0 in the hyperbolic region for $\frac{\pi^2}{L^2}(1-3m_0^2) < \varepsilon_1\varepsilon_2 \le \frac{\pi^2}{L^2}(1-\frac{3}{4}m_0^2)$.

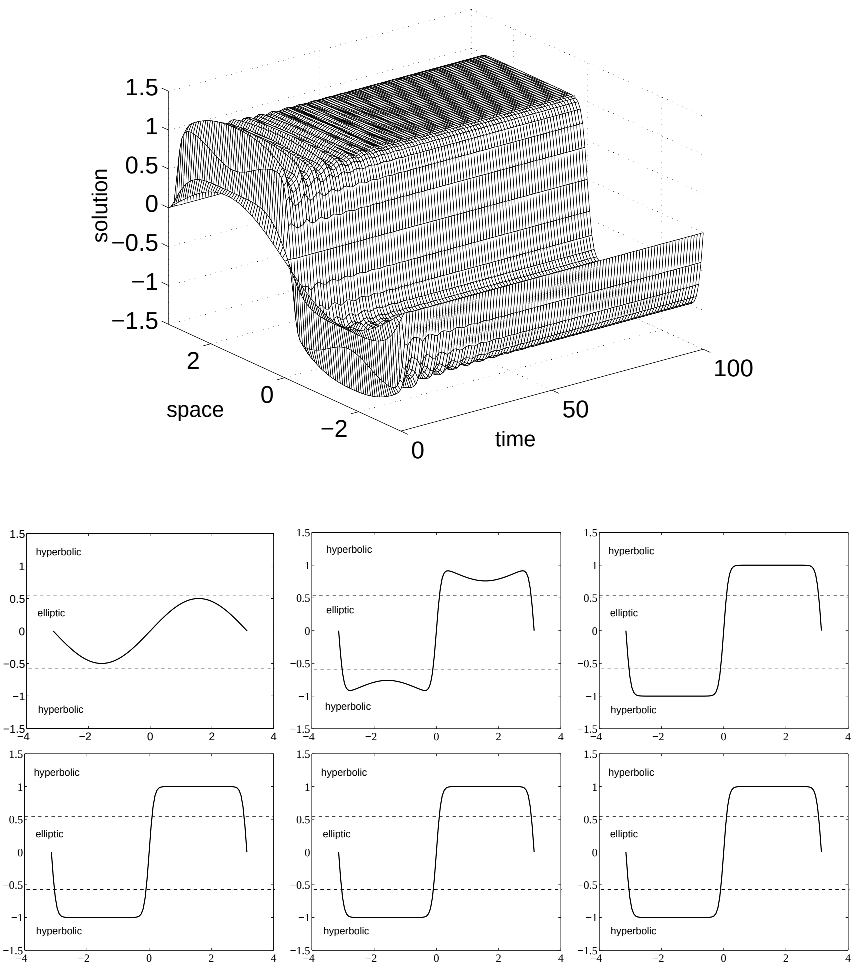


FIGURE 3. (Case 4) $3D$-figure of $v(x,t)$ and $2D$-figure of $v(t,x)$ at different time: $t = 0$, 10, 100, 300, 600, and 1000

On the other hand, when m_0 is in the elliptic region $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, the convergence rate in the second part of Theorem 2.2 can be similarly proved, if $N(t) \ll 1$.

4. **Numerical simulations.** In this section, we carry out numerical simulations to confirm the theoretical results presented in Section 2. The numerical scheme is based on the explicit Euler scheme:

$$\begin{cases} v_i^{j+1} & = v_i^j + \frac{l}{h}(u_{i+1}^j - u_i^j) + \frac{\varepsilon_1 l}{h^2}(v_{i+1}^j - 2v_i^j + v_{i-1}^j) \\ u_i^{j+1} & = u_i^j + \frac{l}{h}(\sigma(v)_{i+1}^j - \sigma(v)_i^j) + \frac{\varepsilon_2 l}{h^2}(u_{i+1}^j - 2u_i^j + u_{i-1}^j) \end{cases} \tag{50}$$

where h denotes the space step, l the time step. The i-index and the j-index are for space and time, respectively, e.g. $v_i^j = v(x_0 + ih, t_0 + jl)$. Applying the Taylor

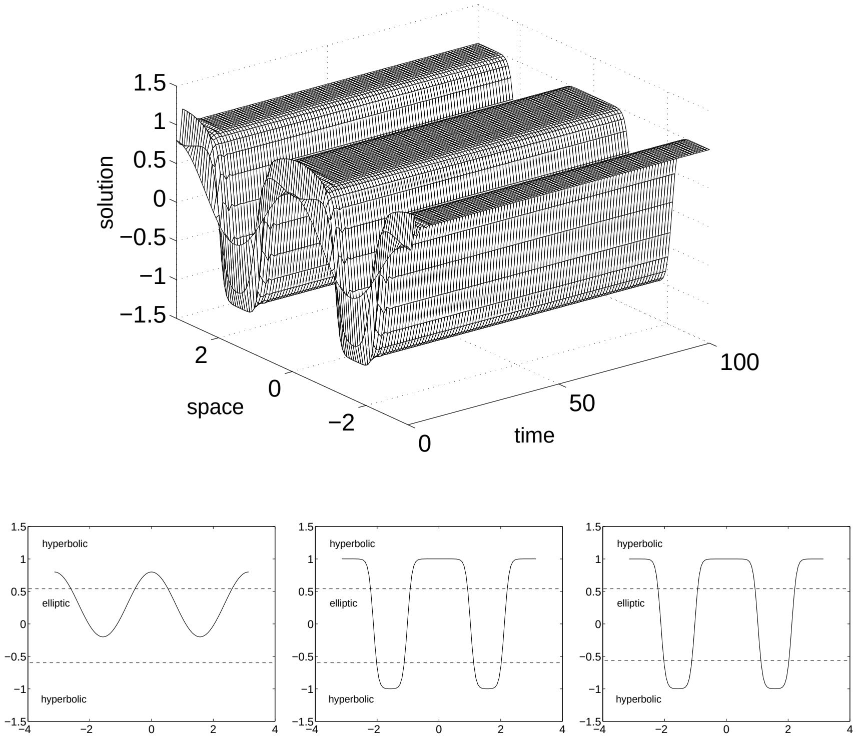


FIGURE 4. (Case 5) $3D$-figure of $v(x,t)$ and $2D$-figure of $v(t,x)$ at different time: $t = 0$, 10, and 1000

expansion at u_i^j and v_i^j in (50), we have

$$\begin{cases} v_t - u_x & = \varepsilon_1 v_{xx} + \frac{h}{2} u_{xx} + O(l) + O(h^2) \\ u_t - \sigma(v)_x & = \varepsilon_2 u_{xx} + \frac{h}{2} \sigma(v)_{xx} + O(l) + O(h^2) \end{cases} \tag{51}$$

Therefore, the scheme is first order in time and space, and it introduces artificial viscosities with coefficients given by $\frac{h}{2}$. In the computation, the space step h must be sufficiently small compared to ε_1 and ε_2 to ensure an accurate computed solution.

In order to investigate the stability condition, the system (50) is reformulated as

$$\begin{cases} V^{j+1} & = A(\varepsilon_1) V^j + BU^j \\ U^{j+1} & = A(\varepsilon_2) U^j - BV^j + \frac{l}{h} (V^3)^j \end{cases} \tag{52}$$

Here

$$A(\varepsilon_1) = \begin{pmatrix} 1-2r_1 & r_1 & 0 & \cdots & 0 & r_1 \\ r_1 & 1-2r_1 & r_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r_1 & 0 & 0 & \cdots & r_1 & 1-2r_1 \end{pmatrix}, \quad r_1 = \frac{\varepsilon_1 l}{h^2}$$

$A(\varepsilon_2)$ is defined by replacing ε_1 by ε_2 in $A(\varepsilon_1)$,

$$B = \frac{l}{h}\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}, \qquad (V^3)^j = \begin{pmatrix} (v_0^j)^3 \\ (v_1^j)^3 \\ (v_2^j)^3 \\ \vdots \\ (v_n^j)^3 \end{pmatrix}.$$

V and U represent the vector variables for the partition points

$$V^j = \begin{pmatrix} v_0^j \\ v_1^j \\ v_2^j \\ \vdots \\ v_n^j \end{pmatrix}, \qquad U^j = \begin{pmatrix} u_0^j \\ u_1^j \\ u_2^j \\ \vdots \\ u_n^j \end{pmatrix}.$$

Combining the two vector variables, the numerical scheme can be expressed in the matrix form

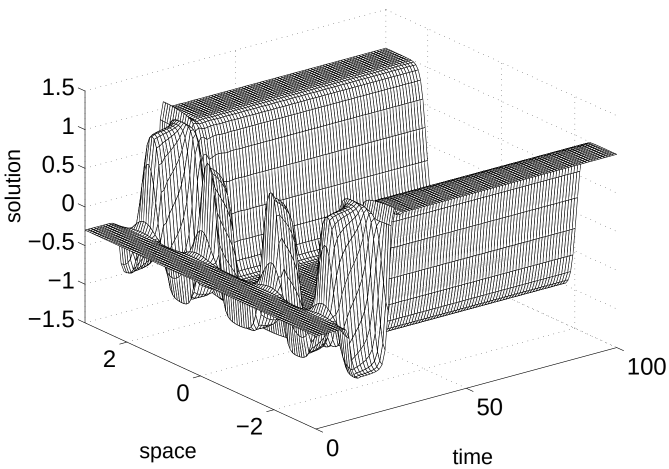
$$\begin{pmatrix} V \\ U \end{pmatrix}^{j+1} = \begin{pmatrix} A(\varepsilon_1) & B \\ -B & A(\varepsilon_2) \end{pmatrix} \begin{pmatrix} V \\ U \end{pmatrix}^j + \begin{pmatrix} 0 \\ \frac{l}{h}B(V^3) \end{pmatrix}^j \tag{53}$$

Therefore, the stability of the scheme is determined by the eigenvalues of the matrix $D = \begin{pmatrix} A(\varepsilon_1) & B \\ -B & A(\varepsilon_2) \end{pmatrix}$. The analysis of the eigenvalues leads to the stability condition: $r_1 < \frac{1}{2}$, $r_2 < \frac{1}{2}$ and $\beta = \frac{l}{h} < \min(r_1\ , r_2)$. The considered stress function is chosen as $\sigma(v) = v^3 - v$, and the periodic boundary is chosen as $v(x - \pi, t) = v(x + \pi, t)$ (i.e., $L = \pi$). According to the sign of $\sigma'(v)$, the phase is divided to $(-\infty, -\frac{1}{\sqrt{3}})$ as the hyperbolic region, $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ as the elliptic region, and $(\frac{1}{\sqrt{3}}, \infty)$ as another hyperbolic region. The two phase boundaries are defined by $v = \pm\frac{1}{\sqrt{3}}$. To confirm the theoretical results, seven case studies are reported in Table 1.

Table 1. Case Studies

Cases	ε_1	ε_2	$v_0(x)$	$u_0(x)$	h	l
1	0.07	0.07	$0.8 + 0.5\sin x$	$0.01\sin 10x$	10^{-3}	10^{-6}
2	0.07	0.07	$-0.8 + 0.5\sin x$	$0.01\sin 10x$	10^{-3}	10^{-6}
3	2.5	2.5	$0.5\sin x$	$0.01\sin 10x$	0.125	10^{-3}
4	0.01	0.01	$0.5\sin x$	$0.01\sin 10x$	10^{-3}	10^{-7}
5	0.01	0.01	$0.3 + 0.5\cos 2x$	$0.01\sin 10x$	10^{-3}	10^{-7}
6	0.01	0.01	-0.3	$0.01\sin 10x$	10^{-3}	10^{-7}
7	0.01	0.01	$\begin{cases} -0.8 & x \in (-\pi,\ -\pi/2) \\ 0.8 & x \in (-\pi/2,\ \pi/2) \\ -0.8 & x \in (\pi/2,\ \pi) \end{cases}$	$0.01\sin 10x$	10^{-3}	10^{-7}

The size of the time and space steps is crucial to ensure stable and accurate numerical solutions. The computational scheme generated artificial viscosity must be controlled so that the numerical viscosity is smaller than the model viscosity in the original system. For simplification, the same values in the viscosities ε_1 and ε_2 are considered, i.e. $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $r_1 = r_2 = r$. Assuming $h = \alpha\varepsilon$, we have $l = \beta h$, $r = \varepsilon l/h^2 = \beta/\alpha$, i.e. $\alpha = \beta/r$. Therefore the stability condition $\beta < r < 0.5$ is easy



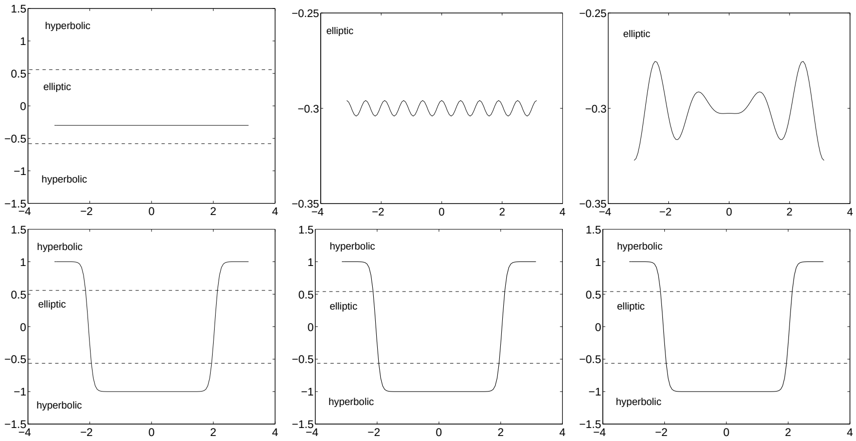


FIGURE 5. (Case 6) $3D$-figure of $v(x,t)$ and $2D$-figure of $v(t,x)$ at different time: $t = 0$, 0.1, 10, 300, 600, and 1000

to satisfy. However, to minimize the effects introduced by the numerical viscosity, we require $h \ll \varepsilon$, i.e. α should be very small. As $l < h$, the values of h and l have to be very small if the model viscosity is small. The values of the space step h and the time step l used in our computations for each case are given in Table 1.

In Case 1, some part of $v_0(x)$ is in the hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$ and the other part is in the elliptic region. But the average of the initial datum $m_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_0(x)dx = 0.8 > \frac{1}{\sqrt{3}}$ is in the hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$. The numerical solution for Case 1 is displayed in Figure 1: Case 1, where the solid straight line denotes the average of the initial datum $m_0 = 0.8$. The results demonstrates that, although the viscosity $\varepsilon_1 = \varepsilon_2 = 0.07$ is very small, the average of the initial value $m_0 = 0.8$ implies automatically the condition (20), such a strong hyperbolicity of the system ensures that the solution $v(x,t)$ of (1)-(3) converges to the constant $m_0 = 0.8$, and stays in the same hyperbolic region $(\frac{1}{\sqrt{3}}, \infty)$ after a short initial

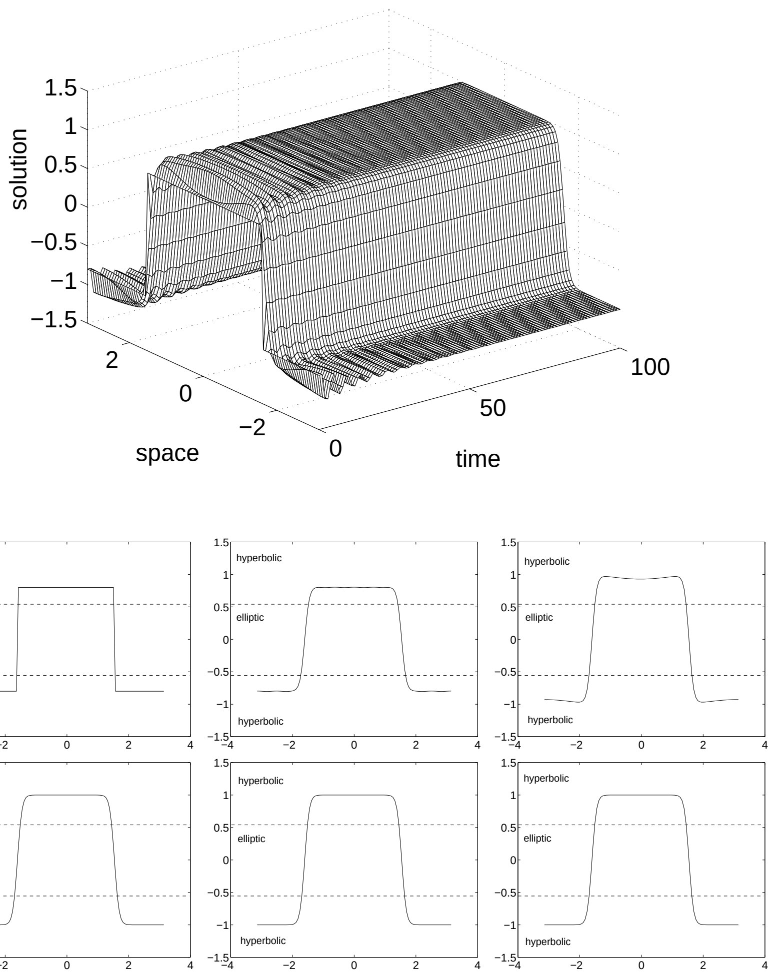


FIGURE 6. (Case 7) $3D$-figure of $v(x,t)$ and $2D$-figure of $v(t,x)$ at different time: $t = 0,\ 0.1,\ 10,\ 300,\ 600$, and 1000

oscillation. This indicates the phenomenon of no phase transition. A similar numerical result in another hyperbolic region $(-\infty, -\frac{1}{\sqrt{3}})$ is displayed in Figure 1: Case 2. The results in Figure 1 confirm Theorem 2.3 numerically.

In Case 3, both $v_0(x)$ and the average of its integral $m_0 = 0$ are in the elliptic region $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The viscosities $\varepsilon_1 = \varepsilon_2 = 2.5$, thus, the sufficient condition (13) holds. Figure 2 shows that the solution $v(x,t)$ converges to $m_0 = 0$ time-asymptotically. Since the viscous coefficients ε_2 and ε_1 are large, as shown in Table 1, the system captures a strong decay feature of parabolic partial differential equations. Naturally, we have the convergence of the solution $v(x,t)$ to $m_0 = 0$ in the same elliptic region with no phase transition. This demonstrates Theorem 2.2 numerically.

The results for Cases 4-7 are displayed in Figures 3-6, respectively. With really small viscous constants $\varepsilon_1 = \varepsilon_2 = 0.01$, such that the conditions (13) and (20) no longer hold (i.e., $\varepsilon_1\varepsilon_2 < \frac{\pi^2}{L^2}(1-3m_0^2)$). In these cases, the averages m_0 of the initial datum are in the elliptic region, as shown in the Figures, the solutions $v(x,t)$ are oscillating with time. However, after certain time, they become stable and behave as one of the standing waves (the stationary solutions), which are determined by the given initial datum. In particular, they exhibit phase transitions crossing through three phases in the long time. Case 7 is for the discontinuous initial data (Riemann data) and the computed solution is illustrated in Figure 6. These results confirm Theorem 2.1, namely, the solution converges to an oscillatory stationary solution which exhibits phase transitions.

Acknowledgments. The authors would like to thank Professor Songmu Zheng for his valuable discussion. The research is supported by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

[1] M. Affouf and R. E. Caflisch, *A numerical study of Riemann problem solutions and stability for a system of viscous conservation laws of mixed type*, SIAM J. Appl. Math., **51** (1991), 605-634.

[2] X. Chen and X.-P. Wang, *Phase transition near a liquid-gas coexistence equilibrium*, SIAM J. Appl. Math., **61** (2000), 454–471.

[3] C. M. Dafermos, *Conservation laws with dissipation*, in Nonlinear Phenomena in Mathematical Sciences, V. Lakshmikantham, ed., Academic Press, New York, 1981.

[4] J. Carr, M. Gurtin and M. Slemrod, *Structured phase transitions on a finite interval*, Arch. Rational Mech. Anal., **86** (1984), 317–351.

[5] J. Carr, M. Gurtin and M. Slemrod, *One-dimensional structured phase transformations under prescribed loads*, J. Elasticity, **15** (1985), 133–142.

[6] I-L. Chern and M. Mei, *Asymptotic stability of critical viscous shock waves for a degenerate hyperbolic viscous conservation laws*, Comm. Partial Differential Equations, **23** (1998), 869–886.

[7] A. Eden, A. Milani and B. Nicolaenko, *Local exponential attractors for models of phase change for compressible gas dynamics*, Nonlinearity, **6** (1993), 93–117.

[8] B. Nicolaenko, W. Qian, *Inertial manifolds for nonlinear viscoelasticity equations*, Nonlinearity, **11** (1998), 1075–1093.

[9] H.-T. Fan, *A vanishing viscosity approach on the dynamics of phase transitions in van der Waals fluids*, J. Differential Equations, **103** (1993), 179–204.

[10] P. C. Fife and X.-P. Wang, *Periodic structures in a van der Waals fluid*, Proc. Royal Soc. Edinburgh, **128A** (1998) 235–250.

[11] M. Gander, M. Mei and G. Schmidt, *Phase transitions in a relaxation model of mixed type with periodic boundary condition*, Appl. Math. Research Express, to appear.

[12] M. Grinfeld and A. Novick-Cohen, *Counting stationary solutions of the Cahn-Hilliard equation by transversality arguments*, Proc. Royal Soc. Edinburgh, **125A** (1995), 351–370.

[13] M. E. Gurtin, *On a theory of phase transitions with interfacial energy*, Arch. Rational Mech. Anal., **87** (1985), 187–212.

[14] M. E. Gurtin and H. Matano, *On the structure of equilibrium phase transitions within the gradient theory of fluids*, Quart. Appl. Math.,**46** (1988), 301–317.

[15] R. Hagan and M. Slemrod, *The viscous-capillarity criterion for shocks and phase transitions*, Arch. Rational Mech. Anal., **83** (1983), 333–361.

[16] D. Hoff and M. Khodja, *Stability of coexisting phases for compressible van der Waals fluids*, SIAM J. Appl. Math., **53** (1993), 1–14.

[17] D.-Y. Hsieh and X.-P. Wang, *Phase transition in van der Waals fluid*, SIAM J. Appl. Math., **57** (1997), 871–892.

[18] R. D. James, *The propagation of phase boundaries in elastic bars*, Arch. Rational Mech. Anal.,**73** (1980), 125–158.

[19] S. Kawashima and A. Matsumura, *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Comm. Pure Appl. Math., **47** (1994), 1547–1569.

[20] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ., **18** (1978), 221–227.

[21] A. Matsumura and M. Mei, *Convergence to traveling fronts of solutions of the p-system with viscosity in the presence of a boundary*, Arch. Rational Mech. Anal., **146** (1999), 1–22.

[22] A. Matsumura and M. Mei, *Nonlinear stability of viscous shock profile for a non-convex system of viscoelasticity*, Osaka J. Math., **34** (1997), 589-603.

[23] A. Matsumura and K. Nishihara, *On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan Journal of Applied Mathematics, **2** (1985), 17-25.

[24] A. Matsumura and K. Nishihara, *Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas*, Japan Journal of Applied Mathematics, **3** (1986), 1-13.

[25] M. Mei, *Asymptotic behavior of solutions for a degenerate hyperbolic system of viscous conservation laws with boundary effect*, Z. Angew. Math. Phys., **50** (1999), 617–637.

[26] M. Mei, *Stability of traveling wave solutions for nonconvex equations of barotropic viscous gas*, Osaka J. Math., **34** (1997), 308-318.

[27] M. Mei, Y.-S. Wong and L. Liu, *Stationary solutions of phase transitions in a coupled viscoelastic system*, to appear.

[28] M. Mei, Y.-S. Wong and L. Liu, *Phase transitions in a coupled viscoelastic system with periodic initial-boundary condition: (I) existence and uniform boundedness*, Discrete Continuous Dynam. Systems, Series B, **7** (2007), 825-837.

[29] R. Natalini and S. Tang, *Discrete kinetic models for dynamical phase transitions*, Comm. Appl. Nonlinear Anal., **7** (2000), 1–32.

[30] M. Oh and K. Zumbrun, *Stability of periodic solutions of conservation laws with viscosity: analysis of the Evan function*, Arch. Rational Mech. Anal., **166** (2003), 99–166.

[31] M. Oh, K. Zumbrun, *Stability of periodic solutions of conservation laws with viscosity: pointwise bounds on the Green function*, Arch. Ration. Mech. Anal., **166** (2003), 167–196.

[32] D. Serre, *Spectral stability of periodic solutions of conservation laws with viscosity: large wavelength analysis*, Commun. Partial Diff. Equations, **30** (2005), 259–282.

[33] M. Shearer, *Dynamic phase transitions in a van der Waals gas*, Quart. Appl. Math., **46** (1988), 631–636.

[34] M. Shearer, *Nonuniqueness of admissible solutions of Riemann initial value problems for a system of conservation laws of mixed type*, Arch. Rational Mech. Anal., **93** (1986), 45–59.

[35] M. Shearer, *The Riemann problem for a class of conservation laws of mixed type*, J. Differential Equations, **46** (1982), 426–443.

[36] M. Slemrod, *A limiting "viscosity" approach to the Riemann problem for materials exhibiting change of phase*, Arch. Rational Mech. Anal., **105** (1989), 327–365.

[37] M. Slemrod, *Admissibility criteria for propagating phase boundaries in a van der Waals fluid*, Arch. Rational Mech. Anal., **81** (1983), 301–315.

[38] M. Slemrod, *Dynamic phase transitions in a van der Waals fluid*, J. Differential Equations, **52** (1984), 1–23.

[39] P. Sternberg and K. Zumbrun, *Connectivity of phase boundaries in a convex domains*, Arch. Rational Mech. Anal., **141** (1998), 375–400.

[40] S. Tang and H. Zhao, *Nonlinear stability for dissipative nonlinear evolution equations with ellipticity*, J. Math. Anal. Appl., **233** (1999), 336–358.

[41] S. Zheng, *Asymptotic behavior of solution to the Cahn-Hillard equation*, Applicable Analysis, **23** (1986), 164–184.

[42] S. Zheng, "Nonlinear Parabolic Equations and Hyperbolic-Parabolic Coupled Systems," Pitman Monographs and Surverys in Pure and Applied Mathematics **76**, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1995. .

[43] K. Zumbrun, *Dynamical stability of phase transitions in the p-system with viscosity-capillarity*, SIAM J. Appl. Math., **60** (2000), 1913–1924.

Received June 2006; revised January 2007.

E-mail address: `mei@math.mcgill.ca`
E-mail address: `yaushu.wong@ualberta.ca`
E-mail address: `lipingliu@utpa.edu`