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# REMARK ON CRITICAL SPEED OF TRAVELING WAVEFRONTS FOR NICHOLSON'S BLOWFLIES EQUATION WITH DIFFUSION*

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**Abstract** This note is devoted to the study on the traveling wavefronts to the Nicholson's blowflies equation with diffusion, a time-delayed reaction-diffusion equation. For the critical speed of traveling waves, we give a detailed analysis on its location and asymptotic behavior with respect to the mature age.

**Key words** Nicholson's blowflies equation; traveling wavefronts; critical wave-speed

**2000 MR Subject Classification** 35R10; 35B40; 34K30; 58D25

## 1 Introduction

The study on how to eliminate the blowflies has been being an interesting spot for scientists, because the flies are usually fatal to sheep a lot. The blowflies lay their eggs on sheep, and soon the eggs become the maggots, which feed on the host. As the results, the injured sheep may die. So, in order to eliminate the blowflies, it is interesting to investigate their population. In 1940s, Nicholson [6,7] had the pioneer study on the distribution of blowflies' population. Based on Nicholson's experimental data, Gurney et al. [1] established a dynamical model, the so-called Nicholson's blowflies equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + du(t) = pf(u(t-r)), \tag{1}$$

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where $u(t)$ is the total mature population of the blowflies at time $t$, $d > 0$ is the death rate of the mature population, $r > 0$ is the mature age, the (delayed) time required for a newborn to become matured, $p > 0$ is the impact of the death on the immature population, and

$$f(u(t-r)) = u(t-r)\mathrm{e}^{-au(t-r)} \tag{2}$$

is the Nicholson's birth function, where $a > 0$ is a constant. It is necessary to consider the distribution of blowflies in space. This leads us naturally to study a time-delayed reaction-diffusion equation as follows:

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + du(t,x) = pu(t-r,x)\mathrm{e}^{-au(t-r,x)}, \tag{3}$$

where $-u_{xx}(t,x)$ is the diffusion in space. The first study on eq. (3) was [10] by So and Yang in 1998. Since then, there were a number of deep research works on this model, for example, see [2, 3, 4, 5, 8, 9, 11, 13] and the references therein, see also the great textbooks [12, 14, 16]. For other related research, for example, we refer to [17, 18] and so on.

Eq. (3) has two constant equilibria

$$u_- = 0 \quad \text{and} \quad u_+ = \frac{1}{a}\ln\frac{p}{d}. \tag{4}$$

When $p > d$, we have $u_- < u_+$. A traveling wavefront is a monotone solution of $u(t,x) = \phi(x+ct)$ to eq. (3) connecting with two states $u_\pm$, where $c > 0$ is the speed of the wavefront. Namely, the traveling wave solution $\phi(x+ct)$ satisfies

$$c\phi'(\xi) - \phi''(\xi) + d\phi(\xi) = p\phi(\xi - cr)\mathrm{e}^{-a\phi(\xi-cr)}, \qquad \phi(\pm\infty) = u_\pm, \tag{5}$$

where $\xi = x + ct$ and $' = \frac{\mathrm{d}}{\mathrm{d}\xi}$. In [11], So and Zou proved the existence of the traveling wave to eq. (5) by means of the method of the upper-lower solutions. For the generalized birth rate function, the existence of the traveling waves was studied by Liang and Wu in [2].

**Lemma 1** [11] If $1 < \frac{\varepsilon p}{d} \le \mathrm{e}$, then there exists a critical number $c_* \ge 0$, such that for every $c > c_*$, eq. (5) has a traveling wavefront solution $\phi(\xi)$ connecting with $u_\pm$, with $\phi'(\xi) > 0$ and $u_- < \phi(\xi) < u_+$ for all $\xi \in (-\infty, \infty)$. Here, the critical speed $c_*$ is the unique solution of

$$\Delta_{c_*}(\lambda_*) = 0, \qquad \frac{\partial}{\partial\lambda}\Delta_{c_*}(\lambda)\Big|_{\lambda=\lambda_*} = 0, \tag{6}$$

where $\Delta_c(\lambda)$ is defined by

$$\Delta_c(\lambda) = p\mathrm{e}^{-\lambda cr} - [c\lambda + d - \lambda^2]. \tag{7}$$

Regarding the stability of wavefronts, by use the technical weighted energy method, Mei et al. [4] proved that the wavefront is asymptotically stable in time when the wave speed is large as $c > 2\sqrt{p-d}$. Later then, Lin and Mei [3] improved the stability of the wavefront to the case of $c > \sqrt{2(p-d)}$. For the nonlocal reaction-diffusion equations, in [15], we investigated the asymptotic behavior of the critical speed of traveling waves.

In this note, we are particularly interested in the critical wave speed $c_*$, because the critical wave is the slowest wave, and plays an important role in the wave study. Usually, when the wave is faster than the critical wave, it may be proved to be asymptotically stable in time, while for the critical wave, it may not be stable anymore. So, to study such a wave like its location of the speed, and the asymptotic behaviors with respect to the time delay $r$ (the mature age), is quite significant.

Now we are going to state our main result.

**Theorem 1** Let $1 < \frac{p}{d} \le \mathrm{e}$. Then, the critical wave speed $c_*$ satisfies:

1) Upper and lower bounds of $c_*$:

$$0 \le c_* \le \min\Big\{2\sqrt{p-d},\ \sqrt{\frac{1}{r}\ln\frac{p}{d}}\ \Big\}. \tag{8}$$

2) Asymptotic behavior of $c_*$ with respect to the mature age $r$:

Let $r$ be free, and the other parameters $p$, $d$ and $a$ be fixed, then

$$\lim_{r\to 0^+} c_* = 2\sqrt{p-d}, \tag{9}$$

$$\lim_{r\to+\infty}\Big|c_* - \frac{A}{r}\Big| = 0, \tag{10}$$

where the positive constant $A$ is given by

$$pA^2\mathrm{e}^{1-\sqrt{1+A^2d}} = 2\sqrt{1+A^2d}-2. \tag{11}$$

**Remarks** 1) Asymptote (9) implies that $c_* = 2\sqrt{p-d}$ is the critical wave speed for the corresponding reaction-diffusion equation (3) without time-delay (i.e., $r=0$).

2) When $r\to+\infty$, the decay rate $c_* = O\big(\frac{1}{r}\big)$ for the local reaction-diffusion equation (3) is faster than the rate $c_* = O\Big(\sqrt{\frac{1}{r}}\Big)$ for the nonlocal reaction-diffusion equation as shown in [15]. But both of the rates in the local and nonlocal cases are optimal.

## 2 Proof of Main Theorem

Let

$$F_c(\lambda) := p\mathrm{e}^{-\lambda cr}, \qquad G_c(\lambda) := c\lambda + d - \lambda^2, \tag{12}$$

then $\Delta_c(\lambda) = F_c(\lambda) - G_c(\lambda)$ and the critical point $(c_*, \lambda_*)$ is the unique tangent point touched by the two surfaces $F_c(\lambda)$ and $G_c(\lambda)$. Obviously, $F_{c_*}(\lambda)$ is always above $G_{c_*}(\lambda)$ except the touched point $\lambda_*$, see Figure 1.

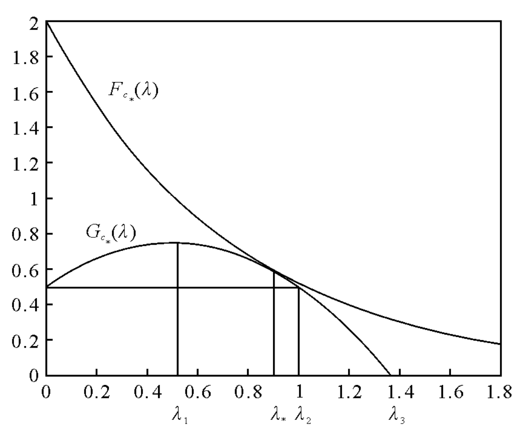


Fig.1

Let

$$\lambda_1 = \frac{c_*}{2}, \qquad \lambda_2 = c_*, \qquad \lambda_3 = \frac{c_* + \sqrt{c_*^2 + 4d}}{2}, \tag{13}$$

where $\lambda_1$ is the point, such that at which $G_{c_*}(\lambda)$ arrives at the maximum $G_{c_*}(\lambda_1) = d + \frac{c_*^2}{4}$, $\lambda_2$ is the non-zero root of the equation $G_{c_*}(\lambda) = d$, and $\lambda_3$ is the positive root of the equation $G_{c_*}(\lambda) = 0$ (for the detail, we refer to Figure 1). Since

$$F_{c_*}(\lambda_1) \geq G_{c_*}(\lambda_1) \qquad \text{and} \qquad F_{c_*}(\lambda_2) \geq G_{c_*}(\lambda_2), \tag{14}$$

namely,

$$p\mathrm{e}^{-c_*{}^2 r/2} \geq \frac{c_*{}^2}{4} + d \qquad \text{and} \qquad p\mathrm{e}^{-c_*{}^2 r} \geq d.$$

This is equivalent to

$$c_*{}^2 \leq 4(p\mathrm{e}^{-c_*{}^2 r/2} - d) \leq 4(p-d) \qquad \text{and} \qquad c_*{}^2 \leq \frac{1}{r}\ln\frac{p}{d},$$

which immediately imply the boundedness of $c_*$ in (8):

$$0 \leq c_* \leq \min\left\{2\sqrt{p-d},\ \sqrt{\frac{1}{r}\ln\frac{p}{d}}\ \right\}.$$

To prove (9) as $r \to 0^+$, let $c_{*0} := \lim\limits_{r\to 0^+} c_*$ and $\lambda_{*0} := \lim\limits_{r\to 0^+} \lambda_*$. Since $c_*$ and $\lambda_*$ are bounded by

$$0 \leq c_* \leq 2\sqrt{p-d}, \qquad 0 < \lambda_* < \lambda_3,$$

respectively, and $\lambda_3$ is bounded by

$$\lambda_3 = \frac{c_* + \sqrt{c_*^2 + 4d}}{2} \leq \frac{2\sqrt{p-d} + \sqrt{(2\sqrt{p-d})^2 + 4d}}{2} = \sqrt{p-d} + \sqrt{p},$$

then the limits of $c_{*0}$ and $\lambda_{*0}$ are also bounded. Thus,

$$\lim_{r\to 0^+} \mathrm{e}^{-c_*\lambda_* r} = \mathrm{e}^{-c_{*0}\cdot\lambda_{*0}\cdot 0} = 1. \tag{15}$$

Notice that $(c_*, \lambda_*)$ satisfies equations (6), namely,

$$p\mathrm{e}^{-c_*\lambda_* r} = c_*\lambda_* + d - \lambda_*^2, \qquad -pc_* r\mathrm{e}^{-c_*\lambda_* r} = c_* - 2\lambda_*. \tag{16}$$

Taking limits of the above equations as $r \to 0^+$, and applying (15), we then have

$$p = c_{*0}\lambda_{*0} + d - \lambda_{*0}^2, \qquad 0 = c_{*0} - 2\lambda_{*0},$$

which gives

$$\lambda_{*0} = \frac{c_{*0}}{2}, \qquad c_{*0} = 2\sqrt{p-d},$$

i.e.,

$$\lim_{r\to 0^+} c_* = 2\sqrt{p-d}.$$

This proves (9).

Now, we are going to prove the asymptotic behavior (10) as $r \to +\infty$. From

$$0 \leq c_* \leq \sqrt{\frac{1}{r}\ln\frac{p}{d}},$$

we should have $c_* = O(r^{-\alpha}) \to 0$ as $r \to +\infty$, with $\alpha \geq \frac{1}{2}$. In what follows, we shall determine that $\alpha = 1$.

From (16), it is reduced to $-c_*^2 r\lambda_* - c_* rd + c_* r\lambda_*^2 = c_* - 2\lambda_*$, which can be solved in $\lambda_*$ as

$$\begin{aligned}\lambda_* &= \frac{c_*^2 r - 2 + \sqrt{(c_*^2 r - 2)^2 + 4c_*^2 r(rd+1)}}{2c_* r} \\ &= \frac{c_*}{2} - \frac{1}{c_* r} + \frac{1}{2}\sqrt{\Big(c_* - \frac{2}{c_* r}\Big)^2 + 4d + \frac{4}{r}}.\end{aligned}$$

Notice that $c_* = O(r^{-\alpha})$ as $r \to +\infty$, then the above equation for $\lambda_*$ is reduced to

$$\begin{aligned}\lambda_* &\approx O(r^{-\alpha}) - O(r^{-(1-\alpha)}) + \frac{1}{2}\sqrt{[O(r^{-\alpha}) - O(r^{-(1-\alpha)})]^2 + 4d + 4r^{-1}} \\ &\approx O(1) \quad \text{for} \quad \alpha = 1, \quad \text{or} \\ &\quad O(r^{\alpha-1}) \quad \text{for} \quad \alpha > 1, \quad \text{or} \\ &\quad O(1) \quad \text{for} \quad 1/2 \leq \alpha < 1, \quad \text{as} \quad r \to +\infty.\end{aligned}$$

It is also verified that

$$\begin{aligned}c_* r &\approx O(1) \quad \text{for} \quad \alpha = 1, \quad \text{or} \\ &\quad O(r^{-(\alpha-1)}) \quad \text{for} \quad \alpha > 1, \quad \text{or} \\ &\quad O(r^{1-\alpha}) \quad \text{for} \quad 1/2 \leq \alpha < 1 \quad \text{as} \quad r \to +\infty,\end{aligned}$$

and

$$\begin{aligned}c_*\lambda_* r &\approx O(1) \quad \text{for} \quad \alpha = 1, \quad \text{or} \\ &\quad O(1) \quad \text{for} \quad \alpha > 1, \quad \text{or} \\ &\quad O(r^{1-\alpha}) \quad \text{for} \quad 1/2 \leq \alpha < 1 \quad \text{as} \quad r \to +\infty.\end{aligned}$$

Now, letting $r \to +\infty$ and applying the above equations to the second equation of (16), we obtain

$$|-pc_* r\mathrm{e}^{-c_*\lambda_* r}| \approx O(r^{-(\alpha-1)}) \to 0, \quad \text{as } r \to +\infty,$$

and

$$|c_* - 2\lambda_*| \approx O(r^{\alpha-1}) \to +\infty, \quad \text{as} \quad r \to +\infty.$$

This proves that the second equation of (16) doesn't match the order of $r$ for both the left and right hand sides, so we cannot have $\alpha > 1$.

Similarly, when $\frac{1}{2} \leq \alpha < 1$, then

$$|-pc_* r\mathrm{e}^{-c_*\lambda_* r}| \approx O(r^{1-\alpha}\mathrm{e}^{-O(1)r^{1-\alpha}}) \to 0, \quad \text{as} \quad r \to +\infty,$$

and

$$|c_* - 2\lambda_*| \approx O(1), \quad \text{as} \quad r \to +\infty.$$

This shows also that the orders of $r$ as $r \to +\infty$ in both the left and right hand sides of the second equation of (16) do not match. So, we cannot allow $\frac{1}{2} \leq \alpha < 1$. Therefore, we prove that the unique possibility for $\alpha$ is $\alpha = 1$.

As shown above, we obtain $c_* = O(r^{-1})$ and $\lambda_* = O(1)$ as $r \to +\infty$. Let us assume

$$\lim_{r\to+\infty}\left|c_* - \frac{A}{r}\right| = 0 \qquad \text{and} \qquad \lim_{r\to+\infty}\lambda_* = B,$$

for some positive constants $A$ and $B$. Now, we are going to determine $A$ and $B$.

As $r \to +\infty$, taking limits of (16), and using that $\lim\limits_{r\to+\infty} c_* r = A$ and $\lim\limits_{r\to+\infty}\lambda_* = B$, we obtain

$$p\mathrm{e}^{-AB} = d - B^2, \qquad -pA\mathrm{e}^{-AB} = -2B.$$

Solving the above equations gives

$$B = \frac{-1+\sqrt{1+A^2d}}{A},$$

and $A$ is given by

$$pA^2\mathrm{e}^{1-\sqrt{1+A^2d}} = 2\sqrt{1+A^2d} - 2.$$

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