Petersson Inner Product of Theta Series

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Abstract

In this thesis, we are interested in many aspects of the Petersson inner product of theta series attached to imaginary quadratic fields. In the first part, we find closed formulas for the Petersson norm of theta series attached to unramified Hecke characters of varying infinity type. Using these, we find formulas for the Petersson inner product of theta series attached to ideals. We then see how this generalizes a remark made by Stark in [Sta75] about the connection between the Petersson norm of the theta series attached to certain Artin representations and the value at $s = 1$ of the corresponding Artin L-functions. In the second part of the thesis, we show that the Petersson inner product of theta series can be p-adically interpolated when p splits in the imaginary quadratic field. We then evaluate the constructed p-adic analytic function outside the range of interpolation and show how the expression obtained can be seen as a p-adic analogue of the corresponding classical expression. In the third and last part of the thesis, we present some computations to give examples, illustrate the theory and support conjectures made in the text.

Abrégé

Cette thèse se penche sur plusieurs aspects du produit scalaire de Petersson des fonctions thêta associées aux corps quadratiques imaginaires. Dans la première partie, on donnera des formules explicites pour le produit scalaire de Petersson des fonctions thêta associées à une famille de charactères de Hecke. Ces formules serviront ensuite à trouver des formules pour le produit scalaire des fonctions thêta associées à des idéaux fractionaires. On verra ensuite comment ces formules permettent de généraliser une remarque faite par Stark dans [Sta75] à propos d'une relation entre le produit scalaire de Petersson des fonctions thêta et la valeur en $s = 1$ de certaines fonctions L de Artin. Dans la seconde partie de la thèse, on montrera que le produit scalaire de Petersson d'une certaine collection de fonctions thêta s'interpole p-adiquement pour certain nombres premiers p. On verra ensuite qu'en évaluant la fonction p-adique obtenue à un entier non interpolé, on obtient une expression analogue à l'expression classique correspondante. Enfin, la dernière partie de la thèse contient plusieurs exemples de calculs qui illustrent la théorie et supportent les conjectures faites dans le texte.

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Introduction

L-functions are central objects in number theory. They can be attached to a variety of mathematical objects, like Dirichlet characters, modular forms, algebraic varieties, Galois representations, etc. and contain a surprisingly large amount of information about the object to which they are attached.

For example, take the simplest possible L-function, the Riemann zeta function, which is defined for $\Re(s) > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The simple fact that it has a pole at $s = 1$ implies that there are infinitely many primes. With more work, one can also obtain the prime number theorem about the distribution of primes using the fact that $\zeta(s)$ does not vanish close to the line $\Re(s) = 1$. It is also known that one could obtain the best possible error bound on the Prime Number Theorem by knowing the Riemann Hypothesis, which conjectures that the non-trivial zeroes of $\zeta(s)$ lie on the line $\Re(s) = \frac{1}{2}$.

As another example, let χ be a Dirichlet character modulo N and consider the twist of $\zeta(s)$ by χ:

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for $\Re(s) > 0$. When χ is not the trivial character modulo N, this function extends to an analytic function of $s \in \mathbb{C}$. In this case, the knowledge that $L(\chi, 1)$ is not zero implies that there are infinitely many primes congruent to $a \pmod N$ for any a coprime to N.

The above examples are all instances of Artin L-functions, which are attached to Galois representations. Indeed, a Dirichlet character modulo N can be seen as a one dimensional representation of the Galois group of the cyclotomic field $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive Nth root of unity. As the above examples suggest, L-functions should contain some arithmetic information at the point $s = 1$. This simple observation was turned into a set of deep and relatively explicit conjectures by Stark in the seventies (in [Sta75] and the other papers in the series). Those conjectures can be thought of as vast generalizations of the well-known Class Number Formula, which states that

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \operatorname{Reg}(F)}{w_F\sqrt{|D_F|}},$$

where $r_1, 2r_2, h_F, \operatorname{Reg}(F), w_F$ and D_F are, respectively, the number of real embeddings, the number of complex embeddings, the class number, the regulator, the number of roots of unity and the discriminant of the number field F. They are known to be true in two important cases: the case of Galois representations of finite abelian extensions of $\mathbb{Q}$ or of an imaginary quadratic field K. The point that the two base fields $\mathbb{Q}$ and K have in common is that one has an explicit way of generating abelian extensions of these (using cyclotomic units for $\mathbb{Q}$, and Siegel units or elliptic units for K).

A remark of Stark

Throughout this thesis, let K be an imaginary quadratic field of discriminant $D < 0$. Let F/K be an abelian extension and let $\psi : \operatorname{Gal}(F/K) \longrightarrow \mathbb{C}^\times$ be a one-dimensional Galois representation. Then, by class field theory, the field F is contained in the ray class field $\bmod \mathfrak{f}$ for some fractional ideal $\mathfrak{f}$ of K. It follows that ψ can be seen as a ray class character $\bmod \mathfrak{f}$ (or Hecke character $\bmod \mathfrak{f}$). The Artin L-function of this representation is then the same as the Hecke L-function attached to ψ. By the work of Deligne-Serre, this L-function (or, more precisely, the L-function of the induced representation from K to $\mathbb{Q}$)

is also known to be modular. The modular form which corresponds to this representation is a theta series.

In [Sta75], Stark noticed a connection between the special values of these L-functions at $s = 1$ and the Petersson norm of the corresponding theta series when $K = \mathbb{Q}(\sqrt{-23})$ (a number field of class number 3). Let ψ be one of the two non-trivial character of the class group of K. Then ψ can be seen as a one dimensional representation of the Galois group of H/K, where H is the Hilbert class field of K. There is a unique newform

$$\theta_\psi \in S_1(\Gamma_0(23), \chi_{-23}),$$

where χ_{-23} is the Kronecker symbol, such that

$$L(\theta_\psi, s) = L(\psi, s).$$

As Stark noticed, the Petersson norm of θ_ψ is

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{3\sqrt{23}}{2\pi} L(\operatorname{Ind}_K^{\mathbb{Q}} \psi, 1) = 3 \log \varepsilon,$$

where $\operatorname{Ind}_K^{\mathbb{Q}} \psi$ is the representation of $\mathrm{Gal}(H/\mathbb{Q})$ induced from ψ and ε is the real root of

$$x^3 - x - 1,$$

which generates the Hilbert class field of K. In this thesis, we study the Petersson inner product of a collection of theta series which generalize the ones considered by Stark. In particular, we see if and how Stark's observation generalizes to other imaginary quadratic fields.

Content and structure of the thesis

This thesis is divided in three parts. In the first one, we find explicit formulas for the Petersson inner product of some theta series. In the second one, we use those formulas to show that the Petersson inner product of theta series can be p-adically interpolated when p splits in K. In the last one, we treat the computational and experimental aspects of the project.

Part I: complex formulas

Let ψ be a Hecke character of conductor $\mathcal{O}_K$ as above which is not a genus character (i.e. its order does not divide 2). In Chapter 4, we prove that

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{-h_K}{3w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \log N(\mathcal{A})^6 |\Delta(\mathcal{A})|,$$

where Δ is the weight 12 and level 1 newform. As we discuss in Chapter 5, this is a direct generalization of what Stark noticed.

More generally, one could consider higher weight theta series attached to (type A_0) Hecke characters ψ of conductor $\mathcal{O}_K$ and infinity type $(2\ell, 0)$ for some $\ell > 0$. Letting θ_ψ be such a theta series, we prove in Chapter 4 that

$$\langle \theta_\psi, \theta_\psi \rangle = (|D|/4)^\ell \frac{4h_K}{w_K^2} \sum_{\mathcal{A} \in \mathrm{Cl}_K} \psi^2(\mathcal{A}) \delta^{2\ell-1} E_2(\mathcal{A}).$$

Using those formulas, one can then compute the Petersson inner product of the theta series attached to a fractional ideals of K and $\ell > 0$:

$$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle = 4(|D|/4)^\ell \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_\mathfrak{c} \mathcal{O}_K} \lambda_\mathfrak{c}^{2\ell} \delta^{2\ell-1} E_2(\mathfrak{c}),$$

where the sum is over all ideal classes representatives $\mathfrak{c}$ such that $\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2$ is principal and generated by $\lambda_\mathfrak{c} \in K^\times$. This formula follows directly from the simple relation between the two sets of theta series.

To fix notation and terminology, the classical theory of modular forms over $\mathbb{C}$ is introduced in Chapter 1. In particular, the many Eisenstein series that come into play are introduced. Chapter 2 is devoted to the task of computing the Petersson inner product in general using the Rankin-Selberg method. In Chapter 3, the relevant results of the theory of Complex Multiplication are introduced. The last two chapters of the first part are devoted to finding the explicit formulas above and discussing the weight one case in more detail.

Part II: p-adic interpolation

For $\ell = 0$, it is well-known that one can attach weight one theta series to fractional ideals of K. Since those theta series are not cuspidal, it makes no sense *a priori* to compute their Petersson inner product. However, one could still try to use the formula for the Petersson norm of θ_ψ when $\ell = 0$ to compute it. This gives *formally*:

$$\langle \theta_{\mathfrak{a},0}, \theta_{\mathfrak{b},0} \rangle = \frac{-1}{3} \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K} \log N(\mathfrak{c})^6 |\Delta(\mathfrak{c})|.$$

In the second part, we show that the formulas for $\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$ when $\ell > 0$ can be given sense p-adically and that, when p splits in K, there exists a p-adic analytic function on weight space

$$F : \mathcal{W} \longrightarrow \mathbb{C}_p$$

which p-adically interpolates those values. By evaluating this function at 0, which lies outside the range of interpolation, one obtains a p-adic analogue of the above expression for $\langle \theta_{\mathfrak{a},0}, \theta_{\mathfrak{b},0} \rangle$. This formula can be seen as a p-adic analogue of Kronecker's First Limit formula.

The techniques used to p-adically interpolate the above quantities use the same techniques that Katz used to construct the p-adic L-function attached to Hecke characters of imaginary quadratic fields in which p splits (see [Kat76]). Those techniques rely on the theory of p-adic modular forms. The definition of these objects and the transition from the classical theory modular forms over $\mathbb{C}$ to the theory of algebraic and eventually p-adic modular forms is done in Chapter 6. We then re-visit the theory of complex multiplication from an algebraic point of view in Chapter 7. Finally, the theory introduced in those two chapters is used to p-adically interpolate the Petersson inner product of theta series.

Part III: computations

A significant part of my research time was devoted to doing computations and finding algorithms to compute the Petersson inner product of theta series, and so I felt it was relevant to include some computations in this thesis. The computations are related to some of the sections of this thesis. The sections which are marked with the symbol ★ have computations attached to them (for example, Section 2.3 is one of them). The last chapter of this thesis, Chapter 13, is exploratory and contains a series of experiments which lead to various observations, conjectures and open questions.

To maximize the fun, the reader should read this last part with a computer nearby to reproduce the computations in PARI/GP and run his or her own experiments!

Part I

Complex formulas

CHAPTER 1
Modular forms over $\mathbb{C}$

In this first chapter, we give an overview of the classical theory of modular forms over $\mathbb{C}$, while in the next part we give an overview of the algebraic and p-adic theories. This is done mostly to fix notation. Indeed, the notation is not standard (at least for Eisenstein series and Poincaré series) and we need results from a few references (namely [DS05], [RBvdG$^+$08], [Shi10], [MM06] and [Coh07]).

We assume that the reader is familiar with the basic theory of classical modular forms. If this is not the case, good references are [RBvdG$^+$08, Part 1] and [DS05].

1.1 The spaces GL^+, $\mathcal{H}$, $\mathcal{L}_\Gamma$ and $Y(\Gamma)$

As in [Kat76, Sec.1.0], let

$$\mathrm{GL}^+ = \{(\omega_1;\omega_2) \in \mathbb{C}^2 : \Im(\omega_1/\omega_2) > 0\} \tag{1.1}$$

be the set of positively oriented $\mathbb{R}$-bases of $\mathbb{C}$. The notation $(a;b)$ stands for a 2×1 column vector (the semi-colon indicates a change of row, instead of column). This set is naturally equipped with a structure of a smooth manifold by inclusion into $\mathbb{C}^2$.

As usual, let $\mathrm{SL}_2(\mathbb{Z})$ denote the group of 2×2 matrices with integer entries and determinant 1. This group acts on GL^+ on the left as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a\omega_1 + b\omega_2 \\ c\omega_2 + d\omega_2 \end{pmatrix}. \tag{1.2}$$

The group $\mathbb{C}^\times$ also acts on GL^+ by scaling. Finally, GL^+ is equipped with an involution ρ defined as

$$(\omega_1;\omega_2)^\rho = (\bar{\omega}_1; -\bar{\omega}_2), \tag{1.3}$$

where the bar denotes complex conjugation. The involution ρ is essentially complex conjugation, just slightly modified to preserve the orientation of bases.

The map

$$(\omega_1;\omega_2) \mapsto \omega_1/\omega_2 \tag{1.4}$$

from GL^+ to the complex upper-half plane

$$\mathcal{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$$

is surjective and induces a bijection

$$\mathrm{GL}^+/\mathbb{C}^\times \simeq \mathcal{H}. \tag{1.5}$$

Note that this map also has a natural section

$$\tau \mapsto (\tau;1) : \mathcal{H} \longrightarrow \mathrm{GL}^+/\mathbb{C}^\times. \tag{1.6}$$

The action of $\mathrm{SL}_2(\mathbb{Z})$ on GL^+ descends to the quotient by $\mathbb{C}^\times$ and the induced action on $\mathcal{H}$ via the above bijection corresponds to the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}. \tag{1.7}$$

The involution ρ on $\mathcal{H}$ sends τ to $-\bar{\tau}$.

The quotient of GL^+ by $\mathrm{SL}_2(\mathbb{Z})$ is the set of lattices in $\mathbb{C}$, denoted $\mathcal{L}$. More generally, let N be a strictly positive integer and let

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod N \text{ and } b, c \equiv 0 \pmod N \right\} \tag{1.8}$$

be the kernel of the reduction $\mod N$ map on $\mathrm{SL}_2(\mathbb{Z})$. A *congruence subgroup* is a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ which contains the subgroup $\Gamma(N)$ for some N. The two most common examples of congruence subgroups are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\} \tag{1.9}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod N \text{ and } c \equiv 0 \pmod N \right\}. \tag{1.10}$$

For a general congruence subgroup Γ, define

$$\mathcal{L}_\Gamma = \Gamma \setminus \mathrm{GL}^+. \tag{1.11}$$

The group $\mathbb{C}^\times$ and the involution ρ act on $\mathcal{L}_\Gamma$ in the obvious way. Note that ρ sends a lattice $L \in \mathcal{L}_\Gamma$ to $\bar{L}$, as one might expect.

The quotient of GL^+ by the action of a congruence subgroup Γ and $\mathbb{C}^\times$ is denoted $Y(\Gamma)$ and is called the *open modular curve* for Γ. It has the structure of a (non-compact) Riemann surface (see [DS05, Ch.2]).

To summarize, the four spaces introduced in this section fit into the diagram

$$\begin{tikzcd} & \mathrm{GL}^+ \arrow[dl] \arrow[dr] & \\ \mathcal{H} \arrow[dr] & & \mathcal{L}_\Gamma \arrow[dl] \\ & Y(\Gamma) & \end{tikzcd} \tag{1.12}$$

where all the arrows are surjective and compatible with the actions of $\mathrm{SL}_2(\mathbb{Z})$, $\mathbb{C}^\times$ and ρ on the various spaces.

1.2 Weight and q-expansions

Let $(k, s) \in \mathbb{Z} \times \mathbb{C}$. As in [Kat76, Sec.1.1], a C^∞ function $F : \mathrm{GL}^+ \longrightarrow \mathbb{C}$ is said to be of *weight* (k, s) if

$$F(\lambda(\omega_1; \omega_2)) = \lambda^{-k}|\lambda|^{-2s}F((\omega_1; \omega_2)) \qquad \text{for all } \lambda \in \mathbb{C}^\times. \tag{1.13}$$

The notion of weight also applies to C^∞ functions $f : \mathcal{L}_\Gamma \longrightarrow \mathbb{C}$, since they can be viewed as Γ-invariant functions on GL^+. Such functions on $\mathcal{L}_\Gamma$ are usually called *homogeneous* of weight (k, s). A C^∞ function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is said to be of *weight* (k, s) *for* Γ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k|c\tau + d|^{2s}f(\tau) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{1.14}$$

For simplicity, a function of weight $(k, 0)$ is said to be of weight k.

Now let F be a C^∞ homogeneous function of weight (k, s) on $\mathcal{L}_\Gamma$. Then F induces a C^∞ function f on $\mathcal{H}$ which is of weight (k, s) for Γ by letting

$$f(\tau) = F((\tau; 1)). \tag{1.15}$$

Conversely, any C^∞ function f on $\mathcal{H}$ of weight (k,s) for Γ induces a C^∞ function F on $\mathcal{L}_\Gamma$ of weight (k,s) by defining

$$F([\omega_1,\omega_2]) = \omega_2^{-k}|\omega_2|^{-2s}f(\omega_1/\omega_2). \tag{1.16}$$

Here, $[\omega_1,\omega_2]$ denotes the Γ-equivalence class of $(\omega_1;\omega_2) \in \mathrm{GL}^+$ (if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then $[\omega_1,\omega_2]$ corresponds to the lattice spanned by ω_1 and ω_2 in $\mathbb{C}$). One can check that those maps induce a bijection between the set of C^∞ homogeneous functions of weight (k,s) on $\mathcal{L}_\Gamma$ and the set of C^∞ functions of weight (k,s) for Γ on $\mathcal{H}$.

Now let $f : \mathcal{H} \longrightarrow \mathbb{C}$ be a *holomorphic* function of weight k for $\Gamma \supset \Gamma(N)$. Then

$$f(\tau + N) = f(\tau),$$

since

$$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N) \subseteq \Gamma.$$

It follows that $f(\tau)$ can be expressed as a holomorphic function of $q = e^{2\pi i\tau/N}$ on the open unit disc in $\mathbb{C}$ with the origin removed. This function of q, still denoted f, has a Laurent expansion

$$f(q) = \sum_n a_n(f)q^n \tag{1.17}$$

around the origin. This expression is called the *q-expansion* (or sometimes *Fourier expansion*) of f.[1] The complex numbers $a_n(f)$ are called the *Fourier coefficients* of f. Using the above correspondence, one can also define the q-expansion of a holomorphic function of weight k on $\mathcal{L}_\Gamma$.

[1] Note that this definition differs from the one given in [Kat76, Sec.1.1], but agrees with the one in [DS05]. The convention adopted here is more common these days.

1.3 Modular forms

In the first part of this thesis, modular forms over $\mathbb{C}$ will be viewed in two equivalent ways.

Modular forms as functions on the upper half-plane

A *weakly holomorphic modular form* of weight k for a congruence subgroup Γ is a holomorphic function $f : \mathcal{H} \longrightarrow \mathbb{C}$ of weight k for Γ whose translates by elements of $\mathrm{SL}_2(\mathbb{Z})$ have meromorphic q-expansions (i.e. f is holomorphic on $\mathcal{H}$ and meromorphic at the cusps). Note that this definition makes sense since the translate of f by an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is holomorphic and of weight k for the congruence subgroup $\gamma^{-1}\Gamma\gamma$.

If the q-expansions of the $\mathrm{SL}_2(\mathbb{Z})$-translates of f are holomorphic, we call f a *(holomorphic) modular form* of weight k for Γ. If those q-expansions all have no constant term, we call f a *cusp form*. The $\mathbb{C}$-vector spaces of modular forms (resp. cusp forms) of weight k for Γ are denoted

$$M_k(\Gamma) \text{ (resp. } S_k(\Gamma)).$$

If f is just C^∞ (i.e. not necessarily holomorphic on $\mathcal{H}$) and of weight (k, s) for Γ, we call f a C^∞ *(or non-holomorphic) modular form* of weight (k, s) for Γ.

Modular forms as functions on $\mathcal{L}_\Gamma$

Using the correspondence of the previous section, one can equivalently define modular forms as weight k functions on $\mathcal{L}_\Gamma$ satisfying certain analyticity conditions. In particular, modular forms for $\mathrm{SL}_2(\mathbb{Z})$ can be viewed as functions on lattices.

Some important examples

A first simple example of a C^∞ modular form of weight $(0, -1)$ for $\mathrm{SL}_2(\mathbb{Z})$ is

$$a(L) = \mathrm{area}(\mathbb{C}/L).$$

As a function on GL^+, it is given by

$$a((\omega_1;\omega_2)) = \Im(\omega_1\bar{\omega}_2) = \frac{\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2}{2i}.$$

As a function on $\mathcal{H}$, it is simply

$$a(\tau) = a((\tau;1)) = \Im(\tau).$$

A second less trivial example is given by the Δ function, which is easier to define via its q-expansion as

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{1.18}$$

This holomorphic function on $\mathcal{H}$ is a weight 12 cusp form for $\mathrm{SL}_2(\mathbb{Z})$ (see [RBvdG+08, Prop.7]).

An example of weakly holomorphic modular form is given by $1/\Delta$, since Δ does not vanish on $\mathcal{H}$.

The Eisenstein series, introduced in the next section, give a lot of examples of modular forms of various weight on $\mathrm{SL}_2(\mathbb{Z})$. In particular, they can be used to construct other modular forms. For example,

$$1728\Delta = (240E_4)^3 - (504E_6)^2, \tag{1.19}$$

where E_4 and E_6 are the weight 4 and weight 6 Eisenstein series, respectively. They also enter in the construction of the most important modular function on $\mathrm{SL}_2(\mathbb{Z})$, the *j-invariant*, which is defined as

$$\begin{aligned} j(q) &= \frac{(240E_4(q))^3}{\Delta(q)} \\ &= q^{-1} + 744 + 196884q + 21493760q^2 + O(q^3) \end{aligned}$$

(see [RBvdG+08, Sec.2.4]).

The q-expansion principle

The *q-expansion principle* says that modular forms are determined by their q-expansions. This is obvious for modular forms over $\mathbb{C}$, since the can be viewed as function on $\mathcal{H}$ and q is given by $e^{2\pi i\tau}$ for $\tau \in \mathcal{H}$. However, for algebraic modular forms, defined in the second part, this is a less trivial and very useful fact.

1.4 Holomorphic and C^∞ Eisenstein series

Eisenstein series are very useful in the theory of modular forms in general and they play a central role in this thesis. For the rest of this section, let $k \geq 0$ be an integer. The simplest example of an Eisenstein series is

$$G_k(L) = {\sum_{\lambda \in L}}' \frac{1}{\lambda^k} \qquad \text{for } k > 2, \tag{1.20}$$

where $L \in \mathcal{L}$ is a lattice and the $'$ symbol means that the term corresponding to $\lambda = 0$ is excluded. Then G_k is a weight k modular form on $\mathrm{SL}_2(\mathbb{Z})$. To prove this, it is preferable to first write G_k as a function on $\mathcal{H}$

$$G_k(\tau) = {\sum_{m,n \in \mathbb{Z}}}' \frac{1}{(m\tau + n)^k} \qquad \text{for } k > 2, \tag{1.21}$$

where the $'$ symbol means that the term $(m,n) = (0,0)$ is excluded. For $k > 2$ this sum converges absolutely. This proves that $G_k(\tau)$ is holomorphic on $\mathcal{H}$ and, by rearranging the sum, that it is of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Finally, a direct application of Lipschitz's formula shows that G_k has q-expansion

$$G_k(q) = 2\zeta(k) + 2\frac{(2\pi i)^k}{\Gamma(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1.22}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and $\zeta(s)$ is the Riemann zeta function (see [RBvdG+08, Prop.5], keeping in mind that our notation differ). It is convenient to normalize G_k in such a way that its q-expansion has rational coefficients. Define

$$E_k = \frac{1}{2}\frac{\Gamma(k)}{(2\pi i)^k} G_k, \tag{1.23}$$

so that

$$E_k(q) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1.24}$$

where B_k is the k Bernoulli number (note that Euler's evaluation of $\zeta(k)$ for $k > 0$ even in terms of Bernoulli numbers is used). For example,

$$E_4(q) = \frac{1}{240} + q + 9q^2 + 28q^3 + \dots \tag{1.25}$$

and

$$E_6(q) = -\frac{1}{504} + q + 33q^2 + 244q^3 + \dots. \tag{1.26}$$

Those Eisenstein series can be generalized in many ways. In this text, the following three generalizations will be needed:

1. Eisenstein series of weight $k = 2$;
2. C^∞ Eisenstein series of weight (k, s);
3. Eisenstein series of higher level.

C^∞ **Eisenstein series for** $\mathbf{SL}_2(\mathbb{Z})$

When $k = 2$, the series defining G_k in (1.21) diverges. One way to define a weight 2 Eisenstein series is to introduce a parameter $s \in \mathbb{C}$ in the summation defining G_k:

$$G_{k,s}(L) = \sum_{\lambda \in L}{}' \frac{1}{\lambda^k |\lambda|^{2s}} \qquad \text{for } \Re(2s) + k > 2. \tag{1.27}$$

Then $G_{k,s}$ is a C^∞ modular form of weight (k, s) for $\mathrm{SL}_2(\mathbb{Z})$. As above, this can be proven by writing $G_{k,s}(L)$ as an absolutely convergent sum over the upper half-plane

$$G_{k,s}(\tau) = \sum_{m,n \in \mathbb{Z}}{}' \frac{1}{(m\tau + n)^k |m\tau + n|^{2s}} \qquad \text{for } \Re(2s) + k > 2. \tag{1.28}$$

When $k > 2$, one can let s be 0 in (1.27) or (1.28) and recover the Eisenstein series defined above:

$$G_{k,0} = G_k \qquad \text{for } k > 2. \tag{1.29}$$

When $k = 2$, the modular form $G_{2,s}$ is not defined at $s = 0$. However, one can define $G_2(\tau)$ as the limit of $G_{2,\varepsilon}(\tau)$ as ε tends to 0 from the right. Then one can prove that

$$G_2(\tau) = -8\pi^2 \left(\frac{-1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n \right) - \frac{\pi}{\Im(\tau)} \tag{1.30}$$

(see [RBvdG+08, Prop.6], keeping in mind again that our notation differ). This function G_2 is a C^∞ weight 2 modular form for $\mathrm{SL}_2(\mathbb{Z})$. Using the normalization introduced in (1.23), one is lead to define

$$E_2(\tau) = \frac{1}{2} \frac{1}{(2\pi i)^2} G_2(\tau), \tag{1.31}$$

which has "q-expansion"

$$\frac{1}{8\pi\Im(\tau)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n. \tag{1.32}$$

Note that since E_2 is a modular form for $\mathrm{SL}_2(\mathbb{Z})$, it can be evaluated on $\mathcal{L}$ using the correspondence (1.16).

The C^∞ Eisenstein series $G_{k,s}$ can also be completed by introducing some factors at infinity as follows:

$$G^*_{k,s}(L) = \frac{1}{2}\Gamma(s+k) \left(\frac{a(L)}{\pi} \right)^s G_{k,s}(L) \qquad \text{for } \Re(2s) + k > 2. \tag{1.33}$$

Equivalently,

$$G^*_{k,s}(\tau) = \frac{1}{2}\Gamma(s+k) \left(\frac{\Im(\tau)}{\pi} \right)^s G_{k,s}(\tau) \qquad \text{for } \Re(2s) + k > 2. \tag{1.34}$$

Then $G^*_{k,s}$ is a C^∞ modular form of weight k (recall that $a(L)$ and $\Im(\tau)$ have weight $(0,-1)$) and

$$E_k = (2\pi i)^{-k} G^*_{k,0} \qquad \text{for } k \geq 2. \tag{1.35}$$

Moreover, the following Theorem holds.

Theorem 1. *For every $k \geq 0$ and $\tau \in \mathcal{H}$, the C^∞ modular form $G^*_{k,s}(\tau)$ can be continued to a meromorphic function of $s \in \mathbb{C}$, which is entire if $k > 0$, and has poles with residues $-\frac{1}{2}$ and $\frac{1}{2}$ at $s = 0$ and $s = 1$, respectively, if $k = 0$. Moreover, $G^*_{k,s}(\tau)$ satisfies the following functional equation:*

$$G^*_{k,s}(\tau) = G^*_{k,1-k-s}(\tau).$$

Proof. See [Shi10, Theorem 9.7]. Note that

$$G^*_{k,s}(\tau) = \frac{1}{2} Z^1_k(\tau, s; 0, 0)$$

in Shimura's notation. □

When $k = 0$, Kronecker's first limit formula gives the constant term of the Taylor expansion of $G^*_{0,s}$ at $s = 1$.

Theorem 2 (Kronecker's First Limit Formula)**.** *Around $s = 1$,*

$$G^*_{0,s}(L) = \frac{1/2}{s-1} + (\gamma - \log 2) - \frac{1}{12} \log(a(L)^6 |\Delta(L)|) + O(s-1),$$

where γ is Euler's constant.

Proof. This is proved in [Coh07, Thm.10.4.6], for example. □

C^∞ **Eisenstein series on** $\Gamma_0(N)$

Let $N \geq 1$ be an integer. One can define a C^∞ Eisenstein series for $\Gamma_0(N)$ as

$$\begin{aligned} G_{k,s,N}(\tau) &= \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} \sum_{(m,n) \equiv (0,t) \pmod N}{}' \frac{1}{(m\tau+n)^k |m\tau+n|^{2s}} \\ &= \sum_{m,n \in \mathbb{Z}}{}' \frac{1_N(n)}{(mN\tau+n)^k |mN\tau+n|^{2s}} \qquad \text{for } \Re(2s)+k>2, \end{aligned}$$

where $1_N(n) = 1$ if $\gcd(n, N) = 1$ and 0 otherwise (this is the *trivial character* $\pmod N$).

A straightforward computation shows that $G_{k,s,N}$ is a C^∞ weight (k, s) modular form for $\Gamma_0(N)$. Define

$$G^*_{k,s,N}(\tau) = \frac{1}{2}\Gamma(k+s)\left(\frac{\Im(\tau)}{\pi}\right)^s G_{k,s,N}(\tau) \qquad \text{for } \Re(2s)+k>2. \tag{1.36}$$

Then one has the following

Theorem 3. *For $k \geq 0$ and $N > 1$, the C^∞ modular form $G^*_{k,s,N}$ can be continued to a meromorphic function of $s \in \mathbb{C}$, which is entire if $k > 0$, and has a pole with residue $\frac{\varphi(N)}{2N^2}$ at $s = 1$ if $k = 0$. Here, $\varphi(N)$ is Euler's phi function.*

Proof. This is [Shi10, Theorem 9.7] again, using the fact that

$$G^*_{k,s,N}(\tau) = \frac{1}{2} \sum_{t \in (\mathbb{Z}/N\mathbb{Z})^\times} Z_k^N(\tau, s; 0, t)$$

in Shimura's notation. □

Recall that $\varphi(N)$ has the simple expression

$$\varphi(N) = N \prod_{p|N} (1 - p^{-1}). \tag{1.37}$$

We conclude this section by introducing Poincaré series, which are used in the Rankin-Selberg method. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, define the *automorphy factor* $j(\gamma, \tau)$ as

$$j(\gamma, \tau) = c\tau + d. \tag{1.38}$$

Then the *Poincaré series* $P_{k,s,N}$ is defined as

$$P_{k,s,N}(\tau) = \Im(\tau)^s \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} j(\gamma,\tau)^{-k} |j(\gamma,\tau)|^{-2s} \qquad \text{for } \Re(2s) + k > 2, \tag{1.39}$$

where

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

Taking the set

$$\left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} : ad - bcN = 1, d > 0 \right\}$$

as a set of representatives for the quotient $\Gamma_\infty \setminus \Gamma_0(N)$ (see [MM06, Lemma 7.1.6]), we see that

$$P_{k,s,N}(\tau) = \Im(\tau)^s \sum_{\gcd(cN,d)=1, d>0} \frac{1}{(cN\tau + d)^k |cN\tau + d|^{2s}}.$$

It follows that

$$\frac{1}{2}\Im(\tau)^s G_{k,s,N}(\tau) = \zeta_N(2s+k) P_{k,s,N}(\tau), \tag{1.40}$$

where

$$\zeta_N(s) = \prod_{p|N} (1 - p^{-s}) \zeta(s).$$

When $k = 0$, one has

$$P_{0,s,N}(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \Im(\gamma\tau)^s \qquad \text{for } \Re(s) > 1, \tag{1.41}$$

since

$$\Im(\gamma\tau) = |j(\gamma,\tau)|^{-2}\Im(\tau) \tag{1.42}$$

(note that this equation is just saying that $\Im(\tau)$ has weight $(0,-1)$). Using relation (1.40), one sees that $P_{0,s,N}$ extends to a meromorphic function of $s \in \mathbb{C}$. Completing $G_{0,s,N}$ in (1.40) and taking residues at $s = 1$, one sees using Theorem 3 when $N > 1$, Theorem 1 when $N = 1$ and (1.37) that

$$\frac{1}{2N}\prod_{p|N}(1-p^{-1}) = \frac{\varphi(N)}{2N^2} = \mathrm{res}_{s=1}G^{*}_{0,s,N}(\tau) = \frac{\zeta_N(2)}{\pi}\mathrm{res}_{s=1}P_{0,s,N}(\tau).$$

It follows that

$$\mathrm{res}_{s=1}P_{0,s,N}(\tau) = \frac{3}{N\pi}\prod_{p|N}(1+p^{-1})^{-1} = \mathrm{Vol}(\Gamma_0(N)\setminus\mathcal{H})^{-1}, \tag{1.43}$$

where the volume is computed in the hyperbolic measure on $\mathcal{H}$ (see [RBvdG+08, Sec.1.3]).

1.5 Operators on $M_k(\Gamma_0(N),\chi)$

A detailed exposition of the material of this section can be found in [DS05, Ch.5], for example.

A basic, but non-trivial property of the spaces $M_k(\Gamma_1(N))$ is that they are finite dimensional over $\mathbb{C}$ (see [RBvdG+08, Prop.3]). For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, one can show that the Eisenstein series E_4 and E_6 introduced above generate the graded $\mathbb{C}$-algebra of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ (see [RBvdG+08, Prop.4]):

$$\bigoplus_{k\geq 0} M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4,E_6]. \tag{1.44}$$

It follows that,

$$M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C} - \text{span of } \{E_4^a E_6^b : 4a+6b=k\},$$

which shows in particular that $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is finite dimensional for every $k \geq 0$. For example,

1. $M_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}$;
2. $M_2(\mathrm{SL}_2(\mathbb{Z})) = 0$;
3. $M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_4, \mathbb{C}E_6, \mathbb{C}E_4^2, \mathbb{C}E_4E_6$ for $k = 4, 6, 8$ and 10, respectively;
4. $M_{12}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_4^3 \oplus \mathbb{C}E_6^2$ and $S_{12}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}\Delta$.

Throughout this section, the q-expansion of a modular form $f \in M_k(\Gamma_1(N))$ is denoted

$$f(q) = \sum_{n=0}^{\infty} a_n q^n.$$

The diamond operators and the space $M_k(\Gamma_0(N), \chi)$

Let f be a modular form of weight k on $\Gamma_1(N)$. The homomorphism

$$\Gamma_0(N) \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \tag{1.45}$$

which sends a matrix $\gamma \in \Gamma_0(N)$ to its lower right entry is surjective and has $\Gamma_1(N)$ as kernel. It follows that $\Gamma_0(N)$ normalizes $\Gamma_1(N)$, and so $\Gamma_0(N)$ acts on $M_k(\Gamma_1(N))$ by precomposition (i.e. $(f\gamma)(\tau) = f(\gamma\tau)$ for $\gamma \in \Gamma_0(N)$). Since $\Gamma_1(N)$ acts trivially on that space by definition, one gets an induced action of

$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times \tag{1.46}$$

on that space. To each $d \in (\mathbb{Z}/N\mathbb{Z})$ one attaches a linear operator $\langle d \rangle$, called a diamond operator, defined as 0 if $\gcd(d, N) > 1$ and as

$$(\langle d \rangle f)(\tau) = f(\gamma_d \tau), \tag{1.47}$$

where γ_d is any preimage of $d \pmod N$ under the isomorphism (1.46), otherwise.

Now let

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

be any Dirichlet character, extended to $\mathbb{Z}/N\mathbb{Z}$ in the usual way and define

$$M_k(\Gamma_0(N), \chi) = \{f \in M_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f, \forall d \in \mathbb{Z}/N\mathbb{Z}\}. \tag{1.48}$$

An element of $M_k(\Gamma_0(N), \chi)$ is called a modular form of weight k, level N and character χ. By definition, if f is such a modular form, it satisfies the functional equation

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{1.49}$$

The space $S_k(\Gamma_0(N), \chi)$ is defined in a similar way (just note that the diamond operators preserve $S_k(\Gamma_1(N))$).

The U_p operator on $M_k(\Gamma_0(N), \chi)$

Let p be a prime number and let $f \in M_k(\Gamma_0(N), \chi)$. The U_p operator is defined on q-expansions as

$$(U_p f)(q) = \sum_{n=0}^{\infty} a_{pn} q^n. \tag{1.50}$$

Viewing f as a function on the upper half-plane, one can express U_p as

$$(U_p f)(\tau) = \frac{1}{p} \sum_{j=1}^{p} f\left(\frac{\tau + j}{p}\right). \tag{1.51}$$

If $p|N$, the operator U_p preserves the spaces $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ (see [DS05, Prop.5.2.1]). However, if $p \nmid N$, it *does not.*

The V_m operator on $M_k(\Gamma_0(N),\chi)$

Let m be an integer and $f \in M_k(\Gamma_0(N),\chi)$ be as above. The V_m operator is defined on q-expansions as

$$(V_m f)(q) = \sum_{n=0}^{\infty} a_n q^{mn}. \tag{1.52}$$

Equivalently, V_m has the simple expression

$$(V_m f)(\tau) = f(m\tau). \tag{1.53}$$

The operator V_m never preserves the space $M_k(\Gamma_0(N),\chi)$; it sends f to a form of level mN (this is exercise 1.2.11 in [DS05]).

As operators on q-expansions, the V_m operators satisfy the following relations:

$$V_m = \prod_{p^r \| m} (V_p)^r.$$

The Hecke operator T_m on $M_k(\Gamma_0(N),\chi)$

First, let p be a prime number and let f be as above. The *Hecke operator* T_p of weight k on $M_k(\Gamma_0(N),\chi)$ is defined as

$$T_p = U_p + \chi(p)p^{k-1}V_p. \tag{1.54}$$

Here the character χ is extended to all $\mathbb{Z}/N\mathbb{Z}$ in the usual way, so that

$$T_p = U_p \qquad \text{if } p|N.$$

The operator T_1 is defined as the identity and T_{p^r}, for $r \geq 2$, is defined recursively as

$$T_{p^r} = T_p T_{p^{r-1}} - \chi(p)p^{k-1}T_{p^{r-2}}. \tag{1.55}$$

Note that this equation simplifies to

$$T_{p^r} = (U_p)^r$$

if $p|N$. Finally, for positive integers m, define

$$T_m = \prod_{p^r \| m} T_{p^r}. \tag{1.56}$$

The definition of T_m can be written more succinctly as an equality between formal power series in X as

$$\sum_{m=1}^{\infty} T_m X^m = \prod_{p|N} (1 - T_p X)^{-1} \prod_{p\nmid N} (1 - T_p X + \chi(p) p^{k-1} X^2)^{-1}. \tag{1.57}$$

The Hecke operators have the following explicit expression on q-expansions:

$$(T_m f)(q) = \sum_{n=0}^{\infty} \left(\sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2} \right) q^n, \tag{1.58}$$

where the inner sum is taken over positive common divisors of m and n (see [DS05, Prop.5.3.1]).

An important fact is that the Hecke operators T_m of weight k preserve the spaces $M_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ (see [DS05, Prop.5.2.2]), and that they all commute with each other (see [DS05, Prop.5.2.4]).

The Hecke operators also have nice expressions on modular forms considered as functions on $\mathcal{L}_\Gamma$. For example, in level one (i.e. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$) one has

$$(T_m f)(L) = m^{k-1} \sum_{L' \subseteq L : [L:L'] = m} f(L'), \tag{1.59}$$

where the sum is taken over all sub-lattices of index m in L (see [RBvdG$^+$08, Sec.4.1]).

The ρ operator on $M_k(\Gamma_0(N), \chi)$

It was shown in Section 1.1 that the spaces GL^+, $\mathcal{L}_\Gamma$ and $\mathcal{H}$ could be equipped with an action of complex conjugation, which was denoted ρ. One can also define an action of ρ on $f \in M_k(\Gamma_0(N), \chi)$ by letting

$$f^\rho(q) = \sum_{n=0}^{\infty} \overline{a_n} q^n. \tag{1.60}$$

Then f^ρ is a modular form in $M_k(\Gamma_0(N), \overline{\chi})$, where as usual $\overline{\chi}(n) = \overline{\chi(n)} = \chi(n)^{-1}$ for any n prime to the level (this is in [Shi76, Sec.2], for example). Equivalently,

$$f^\rho(\tau) = \overline{f(\tau^\rho)} = \overline{f(-\bar{\tau})}, \tag{1.61}$$

when $\tau \in \mathcal{H}$ or

$$f^\rho(L) = \overline{f(L^\rho)}, \tag{1.62}$$

when $L \in \mathcal{L}_{\Gamma_1(N)}$.

Using (1.24), it is clear that

$$E_k^\rho = E_k \qquad \text{for } k > 2. \tag{1.63}$$

In weight 2, it follows from (1.32) that

$$E_2(-\bar{\tau}) = \overline{E_2(\tau)}. \tag{1.64}$$

It follows that

$$E_k(\bullet^\rho) = \overline{E_k(\bullet)} \qquad \text{for } k \geq 2, \tag{1.65}$$

where $\bullet \in \mathcal{L}$ or $\mathcal{H}$.

1.6 Petersson inner product and Atkin-Lehner Theory

The Petersson inner product

Let x and y be the standard coordinates on the upper half-plane, so that

$$\tau = x + iy,$$

and let

$$d\mu = \frac{dxdy}{y^2}$$

be the $\mathrm{SL}_2(\mathbb{Z})$-invariant hyperbolic measure on $\mathcal{H}$. For any two cusp forms $f, g \in S_k(\Gamma_0(N), \chi)$, the function

$$F(\tau) = f(\tau)\overline{g(\tau)}\Im(\tau)^k$$

is a C^∞ modular form of weight 0 for $\Gamma_0(N)$. Since $f(\tau)$ goes to 0 exponentially fast as τ approaches the cusps, so does $F(\tau)$. The *Petersson inner product* of f and g is defined as

$$\langle f, g\rangle = \iint_{\Gamma_0(N)\backslash\mathcal{H}} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu, \tag{1.66}$$

which is a well-defined integral by the properties of F (see [DS05, Sec.5.4]). This defines a Hermitian inner product on $S_k(\Gamma_0(N), \chi)$.

The adjoint of the Hecke operator T_m with respect to the Petersson inner product is, by definition, the operator T_m^* on $S_k(\Gamma_0(N), \chi)$ such that

$$\langle T_m f, g\rangle = \langle f, T_m^* g\rangle.$$

Proposition 1. *If $p \nmid N$, the adjoint of T_p under the Petersson inner product is*

$$T_p^* = \bar{\chi}(p)T_p.$$

It follows that the Hecke operators T_m for m prime to N are normal (i.e. they commute with their adjoint).

Proof. [DS05, Theorem 5.5.3]. $\square$

Newforms

Since the Hecke operators T_m, for $\gcd(m, N) = 1$, form a commuting family of normal operators on the space $S_k(\Gamma_0(N), \chi)$, this space has a basis of orthogonal eigenvectors for all T_m with $\gcd(m, N) = 1$. Those eigenvectors are called *eigenforms*.

Fix an integer m prime to the level, let f be an eigenform and let λ_m be such that

$$T_m f = \lambda_m f.$$

Then

$$a_1(T_m f) = \lambda_m a_1(f).$$

Using (1.58), one sees that

$$a_1(T_m f) = a_m(f).$$

It follows that

$$a_m(f) = a_1(f)\lambda_m, \tag{1.67}$$

which implies that the Fourier coefficients with index prime to the level of an eigenform are determined by its eigenvalues and the leading term of its q-expansion.

Define the space of *oldforms* $S_k^{\mathrm{old}}(\Gamma_0(N), \chi)$ as the $\mathbb{C}$-vector-space spanned by the modular forms $V_d g$ for $d|N/M$, $d > 1$, $M|N$ and $g \in S_k(\Gamma_0(M), \chi)$, where the modular form $V_d g$ of level Md is viewed as a modular form of level N under the natural inclusion

$$S_k(\Gamma_0(dM), \chi) \subseteq S_k(\Gamma_0(N), \chi).$$

The *new subspace* of $S_k(\Gamma_0(N), \chi)$, denoted $S_k^{\mathrm{new}}(\Gamma_0(N), \chi)$, is defined as the orthogonal complement (under the Petersson inner product) of $S_k^{\mathrm{old}}(\Gamma_0(N), \chi)$.

Proposition 2. *The subspaces* $S_k^{old}(\Gamma_0(N),\chi)$ *and* $S_k^{new}(\Gamma_0(N),\chi)$ *are stable under the Hecke operators.*

Proof. [DS05, Proposition 5.6.2]. □

If f is in the old subspace, it is clear that $a_n(f) = 0$ whenever $\gcd(n, N) = 1$. The Main Lemma in Atkin-Lehner theory says that the converse also holds.

Theorem 4 (Main Lemma)**.** *If* $f \in S_k(\Gamma_0(N),\chi)$ *has Fourier expansion* $f(q) = \sum_{n=1}^{\infty} a_n(f)q^n$ *with* $a_n(f) = 0$ *whenever* $\gcd(n, N) = 1$*, then* $f \in S_k^{old}(\Gamma_0(N),\chi)$*.*

Proof. [DS05, Theorem 5.7.1]. □

If f is in an eigenform in the new subspace, $a_1(f) \neq 0$ by the Main Lemma and (1.67). Therefore it makes sense to define a *newform* as an eigenform which is normalized in such a way that $a_1(f) = 1$. Using the Main Lemma and (1.67) again, one can then prove the following

Proposition 3. *Let* f *be a newform. Then* f *is an eigenvector for all Hecke operators with eigenvalue*

$$T_m f = a_m(f)f.$$

In particular, *all* the Fourier coefficients of a newform, hence the newform itself, are determined by the eigenvalues of that newform under the action of the Hecke operators.

1.7 Differential operators on modular forms

Define the D operator as

$$D = \frac{1}{2\pi i}\frac{\partial}{\partial\tau} = q\frac{d}{dq}, \tag{1.68}$$

so that

$$D\left(\sum_{n=0}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} n a_n q^n \tag{1.69}$$

on q-expansions. If $f \in M_k(\Gamma_1(N))$ is a modular form, Df is holomorphic, but does not behave well under the action of $\Gamma_1(N)$. In fact,

$$Df\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2}Df(\tau) + \frac{k}{2\pi i}c(c\tau+d)^{k+1}f(\tau) \tag{1.70}$$

(see [RBvdG+08, Eq.52]). Note that if there was no second term, Df would be a modular form of weight $k+2$. To get rid of the second term, one can introduce a "correction factor". This leads to the *Shimura-Maass operator*

$$\delta_k = D - \frac{k}{4\pi\Im(\tau)} = \frac{1}{2\pi i}\left(\frac{\partial}{\partial\tau} + \frac{k}{\tau-\bar{\tau}}\right). \tag{1.71}$$

Using (1.42), one can show that $\delta_k f$ has weight $k+2$ under $\Gamma_1(N)$ (see [RBvdG+08, after Eq.55]). However, it is not holomorphic anymore (but still C^∞). Note that the D operator extends to C^∞ functions by letting

$$\frac{\partial}{\partial\tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{1.72}$$

One can therefore iterate this operator. Define

$$\delta_k^r = \delta_{k+2r-2} \circ \ldots \delta_{k+2} \circ \delta_k.$$

The Shimura-Maass operator on modular forms on $\mathbf{SL}_2(\mathbb{Z})$

The Shimura-Maass operator on $M_k(\mathrm{SL}_2(\mathbb{Z}))$ can expressed in terms of the Eisenstein series E_2, E_4 and E_6 as follows:

$$\delta_2 E_2 = \frac{5}{6}E_4 - 2E_2^2 \tag{1.73}$$

$$\delta_4 E_4 = \frac{7}{10}E_6 - 8E_2E_4 \tag{1.74}$$

$$\delta_6 E_6 = \frac{400}{7}E_4^2 - 12E_2E_6. \tag{1.75}$$

To prove the second equation, for example, first compute

$$(\delta_4 E_4 + 8E_2E_4)(\tau) = \left(DE_4(\tau) - \frac{E_4(\tau)}{\pi\Im(\tau)}\right) + \left(\frac{E_4(\tau)}{\pi\Im(\tau)} - \frac{(PE_4)(\tau)}{3}\right) = DE_4(\tau) - \frac{(PE_4)(\tau)}{3},$$

where

$$P(\tau) = E_2(\tau) - \frac{1}{8\pi\Im(\tau)} = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n.$$

Since the left hand side is of weight 6 for $\mathrm{SL}_2(\mathbb{Z})$ and the right hand side is holomorphic, $\delta_4 E_4 + 8E_2E_4$ is a modular form of weight 6 for $\mathrm{SL}_2(\mathbb{Z})$, so it is a multiple of E_6.

It follows from the above formulas that the graded $\mathbb{C}$-algebra generated by E_2, E_4 and E_6 is preserved by the Shimura-Maass operator (by acting on each graded piece). A weight k element of this algebra is called a *nearly holomorphic modular form* of weight k on $\mathrm{SL}_2(\mathbb{Z})$.

The Shimura-Maass operator on C^∞ modular forms on $\mathrm{SL}_2(\mathbb{Z})$

Recall that the C^∞ Eisenstein series $G^*_{k,s}$ has weight k. Therefore is makes sense to apply the Shimura-Maass operator to it. Using term-by-term differentiation, one shows that

$$\delta_k G^*_{k,s} = (2\pi i)^{-2} G^*_{k+2,s-1} \tag{1.76}$$

for $\Re(s)$ large enough and then for all s (see [Shi10, Section 9.4]).

1.8 The L-function of modular forms

Let $f = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(N), \chi)$ be a modular form. One can naturally attach a Dirichlet series to it:

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{1.77}$$

This function of $s \in \mathbb{C}$ converges for $\Re(s) > k/2 + 1$ if f is a cusp form and for $\Re(s) > k$ otherwise. When f is a newform, it is an L-function.

Theorem 5. *Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N), \chi)$ be a newform and define*

$$L^*(f,s) = N^{s/2}\Gamma_{\mathbb{C}}(s)L(f,s) \qquad \textit{for } \Re(s) > k/2 + 1, \tag{1.78}$$

where $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. *Then*

1. $L^*(f, s)$ *extends to a holomorphic function on* $\mathbb{C}$*;*
2. $L^*(f, s)$ *satisfies the functional equation*

$$L^*(f, s) = \varepsilon L^*(f, k - s),$$

   where $\varepsilon = \pm 1$*;*

3. *For* $\Re(s) > k/2 + 1$, $L(f, s)$ *can be expressed as an Euler product*

$$L(f, s) = \prod_p (1 - a_p p^{-s} + \chi(p) p^{k-1} p^{-2s})^{-1}.$$

Proof. See [DS05, Theorem 5.10.2] for the first two points and [DS05, Theorem 5.9.2] for the third point. □

CHAPTER 2
Computing the Petersson inner product of cusp forms

In this chapter, the main tools and formulas to compute the Petersson inner product of modular forms are introduced, in decreasing order of generality.

2.1 The Rankin-Selberg method

The Rankin-Selberg method is a simple but clever manipulation of integrals that leads to an explicit relation between the Petersson inner product of two modular forms and a residue of a Dirichlet series. Another exposition of this method is given in [Shi76, Sec.2], for example.

Let $f, g \in S_k(\Gamma_0(N))$ be two modular forms with q-expansions $\sum a_n q^n$ and $\sum b_n q^n$, respectively. Recall that the Poincaré series $P_{0,s,N}$ of weight $(0,0)$ for $\Gamma_0(N)$ can be expressed as

$$P_{0,s,N}(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \Im(\gamma\tau)^{-s}$$

for $\Re(s)$ large enough. Since the function

$$F(\tau) = f(\tau)\overline{g(\tau)}\Im(\tau)^k$$

is also of weight $(0,0)$ for $\Gamma_0(N)$, the integral

$$I(s) = \iint_{\Gamma_0(N)\backslash\mathcal{H}} F(\tau) P_{0,s,N}(\tau) d\mu(\tau) \tag{2.1}$$

converges for $\Re(s) > 1$. The idea of the Rankin-Selberg method is to compute the residue at $s = 1$ of $I(s)$ in two different ways.

On the one hand, since $P_{0,s,N}(\tau)$ has a simple pole at $s = 1$ whose residue *does not depend on* τ, it follows that

$$\mathrm{res}_{s=1} I(s) = \mathrm{Vol}(\Gamma_0(N) \setminus \mathcal{H})^{-1} \langle f, g \rangle, \tag{2.2}$$

by (1.43) and the definition of the Petersson inner product of f and g.

On the other hand, for $\Re(s)$ large enough

$$\begin{aligned}
\iint_{\Gamma_0(N)\setminus\mathcal{H}} F(\tau) P_{0,s,N}(\tau) d\mu(\tau) &= \iint_{\Gamma_0(N)\setminus\mathcal{H}} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} F(\tau) \Im(\gamma\tau)^s d\mu(\tau) \\
&= \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \iint_{\Gamma_0(N)\setminus\mathcal{H}} F(\tau) \Im(\gamma\tau)^s d\mu(\tau) \\
&= \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \iint_{\Gamma_0(N)\setminus\mathcal{H}} F(\gamma\tau) \Im(\gamma\tau)^s d\mu(\tau) \\
&= \iint_{\Gamma_\infty\setminus\mathcal{H}} F(\tau) \Im(\tau)^s d\mu(\tau).
\end{aligned}$$

A fundamental domain for the region $\Gamma_\infty \setminus \mathcal{H}$ is given by $\{\tau \in \mathcal{H} : 0 \leq \Re(\tau) \leq 1\}$, so that the last integral can be written as

$$\iint_{\Gamma_\infty\setminus\mathcal{H}} F(\tau) \Im(\tau)^s d\mu(\tau) = \int_0^\infty \int_0^1 F(x+iy) y^{s-2} dx dy.$$

Now

$$f(\tau)\overline{g(\tau)} = \sum_{m,n=1}^{\infty} a_n \overline{b_m} e^{2\pi i n z} e^{-2\pi i m \bar{\tau}} = \sum_{m,n=1}^{\infty} a_n \overline{b_m} e^{2\pi i (n-m) x} e^{-2\pi (m+n) y},$$

so

$$\int_0^1 F(x+iy) dx = \sum_{n=1}^{\infty} a_n \overline{b_n} e^{-4\pi n y} y^k$$

and

$$\int_0^\infty \left(\int_0^1 F(x+iy) y^s dx \right) \frac{dy}{y^2} = \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{s+k-1}}.$$

Taking the residue at $s = 1$, we get

$$\text{res}_{s=1} I(s) = \frac{\Gamma(k)}{(4\pi)^k} \text{res}_{s=k} D(f, g^\rho, s), \tag{2.3}$$

where

$$D(f, g, s) = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^s}.$$

Putting (2.2) and (2.3) together, one obtains the following

Theorem 6 (Rankin-Selberg)**.** *Let $f, g \in S_k(\Gamma_0(N), \chi)$. Then*

$$\langle f, g \rangle = Vol(\Gamma_0(N) \setminus \mathcal{H}) \frac{\Gamma(k)}{(4\pi)^k} res_{s=k} D(f, g^\rho, s),$$

where

$$Vol(\Gamma_0(N) \setminus \mathcal{H}) = \frac{\pi}{3} N \prod_{p|N} (1 + p^{-1}).$$

Thanks to this theorem, the computation of the Petersson inner product is reduced to the computation of the residue of a Dirichlet series at $s = k$.

2.2 Rankin- Selberg convolution and symmetric square L-functions

Suppose now that f and g are common eigenforms for all Hecke operators with $a_1 = b_1 = 1$. Those conditions hold when f and g are newforms, for example. Let

$$L(f, s) = \prod_p (1 - a_p p^{-s} + \chi(p) p^{k-1} p^{-2s})^{-1} = \prod_p ((1 - \alpha_p p^{-s})(1 - \beta_p p^{-s}))^{-1}$$

be the Euler product of $L(f, s)$. Recall that $\chi(p) = 0$ if $p|N$. By convention, let $\beta_p = 0$ if $p|N$. It follows that

$$\alpha_p + \beta_p = a_p \tag{2.4}$$

and

$$\alpha_p \beta_p = \begin{cases} \chi(p) p^{k-1} & \text{if } p \nmid N \\ 0 & \text{if } p|N \end{cases}.$$

The numbers α_p and β_p are called the *roots* of the Hecke polynomial of f at p. Similarly, let α'_p and β'_p be the roots of g at p.

Twisted modular forms

Let ω be a *primitive* Dirichlet character $\mod M$ and let χ_0 be the primitive Dirichlet character corresponding to χ. Then the q-expansion

$$\sum_{n=1}^{\infty} \omega(n) a_n q^n \tag{2.5}$$

is the q-expansion of a modular form of level $\operatorname{lcm}(N, M^2, M\operatorname{cond}(\chi_0))$, character $\chi_0\omega^2$ and weight k (see [Shi71][Prop 3.64]). This modular form is called the *twist of f by ω* and is denoted f^ω. Here $\chi_0\omega^2$ is viewed as a $\operatorname{lcm}(N, M^2, M\operatorname{cond}(\chi_0))$-periodic function on $\mathbb{Z}$ (in particular, it may not be a primitive character).

It is not hard to see that the roots of f^ω at p are $\omega(p)\alpha_p$ and $\omega(p)\beta_p$.

Twisted symmetric square L-function

The *symmetric square* L-function of f twisted by ω is defined by the Euler product

$$L(\operatorname{Sym}^2 f, \omega, s) = \prod_p [(1-\omega(p)\alpha_p^2 p^{-s})(1-\omega(p)\alpha_p\beta_p p^{-s})(1-\omega(p)\beta_p^2 p^{-s})]^{-1} \tag{2.6}$$

for $\Re(s)$ large enough.

The following theorem of Shimura is very useful when dealing with $L(\operatorname{Sym}^2 f, \omega, s)$.

Theorem 7. *Let $f \in M_k(\Gamma_0(N), \chi)$ be a common eigenfunction for all Hecke operators and let $L(\operatorname{Sym}^2 f, \omega, s)$ be the symmetric square L-function of f twisted by ω. Then the function*

$$R(\operatorname{Sym}^2 f, \omega, s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+2-k-\delta)L(\operatorname{Sym}^2 f, \omega, s),$$

where δ is 0 or 1 according as $\chi(-1)\omega(-1) = 1$ or -1 can be continued to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic except for possible simple poles at $s = k$ and

$s = k-1$. *Moreover, the function* $R(\mathrm{Sym}^2 f, \omega, s)$ *has a pole at* $s = k$ *if and only if the following conditions are satisfied:*

1. $\chi\omega$ *is a non-trivial character of order* 2*;*
2. $\int_\Phi f\bar{g}\Im(\tau)^k d\mu \neq 0$, *where* $g(\tau) = \sum_{n=1}^{\infty} \bar{\omega}(n)\overline{a_n}e^{2\pi i n\tau}$ *and* Φ *is a fundamental domain for* $\Gamma_0(NM^2) \setminus \mathcal{H}$.

Proof. This is a combination of Theorems 1 and 2 of [Shi75]. □

An identity between Dirichlet series

The following lemma gives an expression for the Euler product of $D(f, g, s)$ in terms of the roots of f and g.

Lemma 1. *Suppose we have formally*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p ((1-\alpha_p p^{-s})(1-\beta_p p^{-s}))^{-1},$$

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \prod_p ((1-\alpha'_p p^{-s})(1-\beta'_p p^{-s}))^{-1}.$$

Then

$$\sum_{n=1}^{\infty} \frac{a_n b_n}{n^s} = \prod_p (1-\alpha_p\beta_p\alpha'_p\beta'_p p^{-2s})((1-\alpha_p\alpha'_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\beta_p\alpha'_p p^{-s})(1-\beta_p\beta'_p p^{-s}))^{-1}.$$

Proof. See [Shi76][Sec.3, Lemma 1]. □

If $L(s)$ is a Dirichlet series which has an Euler product expansion, the expression $L_N(s)$ denotes the Dirichlet series with the Euler factors at the primes dividing N removed. For example,

$$L(1_N, s) = \zeta_N(s),$$

where $L(1_N, s)$ is the Dirichlet series attached to 1_N, the trivial Dirichlet character $\bmod N$.

Proposition 4. *Let $f \in M_k(\Gamma_0(N), \chi)$ be a newform and let ω be a primitive Dirichlet character of conductor M. Then*

$$L_{MN}(\chi^2\omega^2, 2s+2-2k)D(f, f^\omega, s) = L_{MN}(\chi\omega, s+1-k)L(\mathrm{Sym}^2 f, \omega, s)$$

for all $s \in \mathbb{C}$.

Proof. For $\Re(s)$ large enough, this follows easily from computing the Euler factors at all primes p of the four Dirichlet series involved (using Lemma 1, in particular). The result then follows for all s by Theorem 7. □

By the Rankin-Selberg method, the Petersson norm of f can be expressed in terms of the residue of $D(f, f^\rho, s)$ at $s = k$. On the other hand, the last proposition establishes a relation between $D(f, f^\omega, s)$ and $L(\mathrm{Sym}^2 f, \omega, s)$. In the next two sections, relations between $D(f, f^\rho, s)$ and $D(f, f^\omega, s)$, for some ω, are found.

2.3 ★ Petersson norm of newforms and special values of their symmetric square L-functions

This section is based on the argument of [Hid81][Sec.5].

The general case

From now on, suppose that f is a newform. Using the fact that the adjoint of the Hecke operator T_n under the Petersson inner product is $\bar{\chi}(n)T_n$, for n coprime to N, one sees immediately that

$$\bar{a}_n = \bar{\chi}(n)a_n \qquad \text{for } \gcd(n, N) = 1. \tag{2.7}$$

It follows that

$$D_N(f, f^\rho, s) = D_N(f, f^{\bar{\chi}_0}, s),$$

where $\bar{\chi}_0$ is the primitive Dirichlet character attached to $\bar{\chi}$. Since f is *new* of level N, Equation (2.7) also holds for $\gcd(n, N') = 1$, where N' is the conductor of χ. It follows that

$$D_{N'}(f, f^{\rho}, s) = D_{N'}(f, f^{\bar{\chi}_0}, s). \tag{2.8}$$

Let N'_p and N_p be the p-part of N' and N, respectively (note that $N'_p|N_p$ for all primes p). Then

$$a_p\bar{a}_p = p^{k-1} \qquad \text{if } p|N \text{ and } N_p = N'_p, \tag{2.9}$$

$$a_p\bar{a}_p = p^{k-2} \qquad \text{if } p|N \text{ and } N_p = p, N'_p = 1, \tag{2.10}$$

$$a_p = 0 \qquad \text{if } p^2|N \text{ and } N_p = N'_p \tag{2.11}$$

(see [Hid81][Eqn.5.10]). It follows that the Euler factors of the Dirichlet series in (2.8) differ only at the primes for which $N'_p = N_p$. This proves the following

Proposition 5. *Let $f \in S_k(\Gamma_0(N), \chi)$ be a newform, let N' be the conductor of χ and let N'_p and N_p be the p-parts of N' and N, respectively. Then*

$$D(f, f^{\rho}, s) = \prod_{p:N'_p=N_p} (1-p^{k-1-s})^{-1} D(f, f^{\bar{\chi}_0}, s).$$

Using this, one can now prove the following

Theorem 8. *Let $f \in S_k(\Gamma_0(N), \chi)$ be a newform and let N' be the conductor of χ. Then*

$$\langle f, f \rangle = \left(\frac{\pi}{2}\frac{\phi(N)}{NN'\phi(N/N')}\frac{(4\pi)^k}{(k-1)!}\right)^{-1} L(\mathrm{Sym}^2 f, \bar{\chi}_0, k),$$

where ϕ is the Euler totient function.

Proof. Using Proposition 4 with $\psi = \bar{\chi}_0$, Proposition 5 and by comparing residues at $s = k$, one sees that

$$\prod_{p:N'_p=N_p} (1-p^{-1})\zeta_N(2)\,\mathrm{res}_{s=k}\, D(f, f^{\rho}, s) = \mathrm{res}_{s=1}\,\zeta_N(s) L(\mathrm{Sym}^2 f, \bar{\chi}_0, k)$$

$$\Leftrightarrow \prod_{p|(N/N')}(1-p^{-1})^{-1}\prod_{p|N}(1-p^{-2})\frac{\pi^2}{6}\operatorname{res}_{s=k} D(f,f^\rho,s) = L(\operatorname{Sym}^2 f,\bar{\chi}_0,k).$$

Using the Rankin-Selberg method to relate $D(f,f^\rho,s)$ and $\langle f,f\rangle$, we get

$$\prod_{p|(N/N')}(1-p^{-1})^{-1}\prod_{p|N}(1-p^{-2}))\frac{\pi^2}{6}\left(\frac{3}{\pi}N^{-1}\prod_{p|N}(1+p^{-1})^{-1}\frac{(4\pi)^k}{(k-1)!}\langle f,f\rangle\right) = L(\operatorname{Sym}^2 f,\bar{\chi}_0,k)$$

$$\Leftrightarrow \frac{\pi}{2}\frac{1}{N}\prod_{p|N}(1-p^{-1})\prod_{p|(N/N')}(1-p^{-1})^{-1}\frac{(4\pi)^k}{(k-1)!}\langle f,f\rangle = L(\operatorname{Sym}^2 f,\bar{\chi}_0,k).$$

The theorem follows since

$$\phi(N) = N\prod_{p|N}(1-p^{-1}).$$

$\square$

The special case where f has trivial character

In the special case where the character of f is trivial, i.e. $f \in S_k(\Gamma_0(N))$, the formula of Theorem 8 simplifies to

$$\langle f,f\rangle = \left(\frac{\pi}{2N}\frac{(4\pi)^k}{(k-1)!}\right)^{-1} L(\operatorname{Sym}^2 f,1,k). \tag{2.12}$$

In this case, the completed symmetric square L-function

$$L^*(\operatorname{Sym}^2 f,1,s) = (N^2)^s\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+2-k)L(\operatorname{Sym}^2 f,1,s)$$

satisfies the functional equation

$$L^*(\operatorname{Sym}^2 f,1,2k-1-s) = L^*(\operatorname{Sym}^2 f,1,s).$$

This information allows efficient computations with PARI/GP.

In particular, this formula applies to the newforms

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$
$$\Delta_5(q) = q\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{5n})^4 \text{ and}$$
$$\Delta_{11}(q) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$$

in $S_{12}(\mathrm{SL}_2(\mathbb{Z})), S_4(\Gamma_0(5))$ and $S_2(\Gamma_0(11))$, respectively.

2.4 ★ Petersson norm of newforms with real Fourier coefficients

In the special case where f is twisted by the trivial character, the formula of Proposition 4 gives

$$L_N(\chi^2, 2s+2-2k)D(f,f,s) = L_N(\chi, s+1-k)L(\mathrm{Sym}^2 f, 1, s).$$

Suppose now that f has real Fourier coefficients and that χ is not the trivial character mod N (since this case was treated in the previous section). Then, using the formula of Theorem 6, one sees that $L(\mathrm{Sym}^2 f, 1, s)$ has a pole at $s = k$ (since $L_N(\chi, s)$ is analytic for χ non-trivial). It then follows from Theorem 7 that χ has order 2. This leads to the following

Theorem 9. *Let f be a newform with real Fourier coefficients and non-trivial character. Then*

$$\langle f, f\rangle = \left(\frac{\pi}{2}\frac{\phi(N)}{N^2}\frac{(4\pi)^k}{(k-1)!}\right)^{-1} L_N(\chi,1)\operatorname{res}_{s=k} L(\mathrm{Sym}^2 f, 1, s).$$

Proof. From the previous discussion, we have

$$\zeta_N(2)\operatorname{res}_{s=k} D(f,f,s) = L_N(\chi,1)\operatorname{res}_{s=k} L(\mathrm{Sym}^2 f, 1, s).$$

Since $f = f^\rho$, Theorem 6 gives

$$\prod_{p|N}(1-p^{-2})\frac{\pi^2}{6}\left(\frac{3}{\pi}N^{-1}\prod_{p|N}(1+p^{-1})^{-1}\frac{(4\pi)^k}{(k-1)!}\langle f,f\rangle\right) = L_N(\chi,1)\,\mathrm{res}_{s=k}\,L(\mathrm{Sym}^2 f,1,s)$$

$$\Leftrightarrow \frac{\pi}{2N}\prod_{p|N}(1-p^{-1})\frac{(4\pi)^k}{(k-1)!}\langle f,f\rangle = L_N(\chi,1)\,\mathrm{res}_{s=k}\,L(\mathrm{Sym}^2 f,1,s),$$

from which the result follows. □

CHAPTER 3
Complex multiplication and modular forms over $\mathbb{C}$

The theory of complex multiplication is certainly one of the most beautiful subject in number theory. It is also vast. For this reason, only the results that are needed in this thesis are introduced.

Some notation

From now on in this thesis, let K be an imaginary quadratic field of discriminant D embedded in $\mathbb{C}$. Moreover, let H be its Hilbert class field, that is the maximal abelian unramified extension of K, which has degree h_K over K. Let also $\mathcal{O}_K$ be the ring of integers of K, I_K be the group of fractional ideals of K and Cl_K be the ideal class group of K.

3.1 CM points in $\mathcal{H}$, $\mathcal{L}$ and $Y(\mathbf{SL}_2(\mathbb{Z}))$

Let $\mathfrak{a} \in I_K$ be a fractional ideal of K. Then $\mathfrak{a}$ is a free $\mathbb{Z}$-module of rank 2 and

$$\mathfrak{a} \otimes \mathbb{R} = K \otimes \mathbb{R} = \mathbb{C},$$

so $\mathfrak{a}$ can be viewed as a lattice in $\mathbb{C}$, i.e. an element of $\mathcal{L}$. This gives a map

$$(\Omega, \mathfrak{a}) \mapsto \Omega\mathfrak{a} : \mathbb{C}^{\times} \times I_K \longrightarrow \mathcal{L}.$$

The points of $\mathcal{L}$ in the image of this map for some K are called *CM points*.

Note that one can map a fractional ideal $\mathfrak{a}$ to $\mathcal{H}$ in many ways by first fixing an oriented basis (i.e. by taking a preimage in GL^{+}) and then mapping it to $\mathcal{H}$. Since there is no canonical choice for this basis, there is no canonical map from I_K to $\mathcal{H}$. One may still call a point of $\mathcal{H}$ which can be obtained from this procedure a CM point of $\mathcal{H}$. Note that those are simply the points of $\mathcal{H}$ which are algebraic of degree 2 over $\mathbb{Q}$.

3.2 ★ CM values of modular functions

A central object in the automorphic side of complex multiplication is the modular function j introduced in the first chapter. Then one has the following important

Theorem 10. *Let $\mathfrak{a} \in I_K$ be a fractional ideal of K and let $\mathcal{A} \in Cl_K$ be an ideal class of K. Then*

1. *$j(\mathcal{A})$ is an algebraic* integer *in H,*
2. *$[K(j(\mathcal{A})) : K] = [\mathbb{Q}(j(\mathcal{A})) : \mathbb{Q}] = h_K$ and so $K(j(\mathcal{A})) = H$,*
3. *the $j(\mathcal{A})$ are Galois conjugate over K and $\mathbb{Q}$ as $\mathcal{A}$ ranges over the classes in Cl_K,*
4. *if $\mathfrak{p}$ is a prime of K, then*
$$j(\mathfrak{a})^{\left(\frac{H/K}{\mathfrak{p}}\right)} = j(\mathfrak{p}^{-1}\mathfrak{a}),$$
where $\left(\frac{H/K}{\mathfrak{p}}\right) \in \mathrm{Gal}(H/K)$ is the Artin symbol.

Proof. This is a restatement in terms of ideal classes of [Sil94][Thm.4.3] and [Sil94][Thm.6.1]. See the second part of this thesis for the correspondence between ideal classes and elliptic curves with complex multiplication. □

The values of j at CM points are called *singular moduli.*

It is a basic fact that modular functions on $\mathrm{SL}_2(\mathbb{Z})$ can be seen as rational functions in j with coefficients in $\mathbb{C}$, i.e. as element in

$$\mathbb{C}(j).$$

Then it follows from the above theorem that

Corollary 1. *Let f be a modular function on $SL_2(\mathbb{Z})$ with algebraic Fourier coefficients, i.e. an element of $\bar{\mathbb{Q}}(j)$. Then if f is defined at an ideal class $\mathcal{A}$,*

$$f(\mathcal{A}) \in \bar{\mathbb{Q}}.$$

3.3 ★ Siegel units

Siegel units form a collection of units in the Hilbert class field of K that will appear later in the formulas for the Petersson inner product of weight one theta series.

As above, let $\mathfrak{a}$ be a fractional ideal of K and define

$$\varphi_{\mathfrak{a}} = \frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{a}^{-1})}. \tag{3.1}$$

This quantity has weight 12 in the ideal $\mathfrak{a}$ in the sense that

$$\varphi_{z\mathfrak{a}} = z^{-12}\varphi_{\mathfrak{a}} \tag{3.2}$$

for all $z \in \mathbb{C}^\times$. Then one has the following

Theorem 11. *Let $\mathfrak{a}$ and $\varphi_{\mathfrak{a}}$ be as above. Then*

1. *$\varphi_{\mathfrak{a}}$ is an algebraic number in H;*
2. *$\varphi_{\mathfrak{a}}\mathcal{O}_H = \mathfrak{a}^{-12}\mathcal{O}_H$;*
3. *if $\mathfrak{p}$ is a prime of K, then*
$$\varphi_{\mathfrak{a}}^{\left(\frac{H/K}{\mathfrak{p}}\right)} = \varphi_{\mathfrak{a}\mathfrak{p}}\varphi_{\mathfrak{p}}^{-1},$$
where $\left(\frac{H/K}{\mathfrak{p}}\right) \in \mathrm{Gal}(H/K)$ is the Artin symbol.

Proof. See [DS87, Sec.2.2] or [Lan87][Ch.12, Sec.2] (the formulation is slightly different there). □

It follows directly from this theorem that

$$\delta_{\mathfrak{a}} = \alpha^{12}\varphi_{\mathfrak{a}}^{h_K} = \alpha^{12}\left(\frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{a}^{-1})}\right)^{h_K}, \tag{3.3}$$

where $\alpha\mathcal{O}_K = \mathfrak{a}^{h_K}$, is a unit in the Hilbert class field of K which depends only on the ideal class of $\mathfrak{a}$. Those units are called *Siegel units*.

Let $N(\mathfrak{a})$ denote the ideal norm of $\mathfrak{a}$. Then $N(\mu\mathfrak{a}) = |\mu|^2 N(\mathfrak{a})$ for any $\mu \in K$ and

$$N(\mathfrak{a}) = \frac{2a(\mathfrak{a})}{\sqrt{|D|}}, \tag{3.4}$$

where a is the modular form defined in Section 1.3. With α as above, we see that $N(\mathfrak{a})^{h_K} = |\alpha|^2$ and so

$$|\delta_{\mathfrak{a}}| = (N(\mathfrak{a})^6 |\varphi_{\mathfrak{a}}|)^{h_K} \tag{3.5}$$

is a unit in H which depends only on the ideal class of $\mathfrak{a}$. In fact, one can see using the second point of the previous theorem that

$$(N(\mathfrak{a})^6 |\varphi_{\mathfrak{a}}|)^2$$

is also a unit in the Hilbert class field (see [Lan87][Ch.12, Sec.2]).

3.4 ★ CM values of modular forms

If f is a modular form of weight greater than 0, the statement "f is algebraic at CM points" does not make sense. One can salvage this by introducing periods in the statements.

Proposition 6. *Let f be a modular form of weight k and level 1 with algebraic Fourier coefficients, and let K be an imaginary quadratic field. Then there exists a constant $\Omega(K)$, depending* only *on K, such that*

$$f(\mathfrak{a}) \in \Omega(K)^k \bar{\mathbb{Q}}$$

for all fractional ideals $\mathfrak{a}$ of K (equivalently, for all $\tau \in K \cap \mathcal{H}$).

Proof. This is [RBvdG+08][Ch.1, Prop.26]. □

Note that this constant $\Omega(K)$ is well-defined only up to an algebraic number. However, using the Chowla-Selberg formula, one can find an explicit expression for an $\Omega(K)$.

Proposition 7. *One can choose $\Omega(K)$ to be*

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left(\prod_{n=1}^{|D|-1} \Gamma\left(\frac{n}{|D|}\right)^{\chi_D(n)} \right)^{w_K/(4h_K)}$$

in the previous proposition.

Proof. This result is proved in [RBvdG+08][Ch.1, Formula (97)], for example, and follows directly from the Chowla-Selberg formula. □

The complex number Ω_K is called the *Chowla-Selberg period* attached to K.

3.5 ★ CM values of nearly holomorphic modular forms

Recall that the graded $\mathbb{C}$-algebra of nearly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ is generated by the Eisenstein series E_2, E_4 and E_6. From the previous section, it follows that the modular forms E_4/Ω_K^4 and E_6/Ω_K^6 take algebraic values at fractional ideals of K. The following proposition proves that a similar statement is true for E_2.

Proposition 8. *Let E_2 be the C^∞ Eisenstein series of weight 2 on $SL_2(\mathbb{Z})$ and let $\mathfrak{a}$ be a fractional ideal of K. Then*

$$E_2(\mathfrak{a}) \in \Omega_K^2 \bar{\mathbb{Q}}.$$

Proof. This is [RBvdG+08][Ch.1, Prop.28]. □

Corollary 2. *Let f be a nearly-holomorphic modular form of weight k on $SL_2(\mathbb{Z})$ with algebraic Fourier coefficients, i.e. an element of $\bar{\mathbb{Q}}[E_2, E_4, E_6]$, and let $\mathfrak{a}$ be a fractional ideal of K. Then*

$$f(\mathfrak{a}) \in \Omega_K^k \bar{\mathbb{Q}}.$$

CHAPTER 4
Petersson inner product of theta functions

In this chapter, the Rankin-Selberg method is used to compute the Petersson inner product of theta series attached to K in terms of L-functions of Hecke characters, C^∞ Eisenstein series and derivatives of nearly-holomorphic Eisenstein series.

4.1 ★ Hecke characters of type A_0 with trivial conductor

Let K, $\mathcal{O}_K$, etc. be as in the previous chapter. A *Hecke character of K* of type A_0 with trivial conductor is given by the following data: a pair of integers (k_1, k_2) and a homomorphism

$$\psi : I_K \to \mathbb{C}^\times$$

such that

$$\psi((\alpha)) = \alpha^{k_1}\bar{\alpha}^{k_2}$$

for all $\alpha \in K^\times$. For simplicity, a Hecke character of type A_0 is often represented by the function ψ. The pair (k_1, k_2) is called the *infinity type* of ψ. To this pair, one can attach the function

$$\mathbf{X} : K^\times \to \mathbb{C}^\times$$

which sends α to $\alpha^{k_1}\bar{\alpha}^{k_2}$. Note that for ψ to be well defined, $\mathbf{X}$ has to be trivial on $\mathcal{O}_K^\times$.

Hecke characters of infinity type $(0,0)$ are called *class characters*, since they naturally descend to characters of the class groups $\mathrm{Cl}_K = I_K/\{(\alpha) : \alpha \in K^\times\}$. Any two Hecke characters with the same infinity type differ by a class character, since their quotient has infinity type $(0,0)$. In particular, there are only finitely many Hecke characters of a fixed infinity type.

Given a class character χ and a function $\mathbf{X}$ on $K^\times$ as above which is trivial on $\mathcal{O}_K^\times$, one can obtain a Hecke character by sending $\mathfrak{a} \in I_K$ to $\mathbf{X}(\mu)\chi(\mathfrak{a}_0)$, where $\mu \in K^\times$, $\mathfrak{a}_0$ is an integral ideal and $\mathfrak{a} = (\mu)\mathfrak{a}_0$. By the above remarks, all Hecke characters of type A_0 can be obtained in this way. This gives a simple and explicit way to construct Hecke characters, since the class group can be computed explicitly.

L-functions of Hecke characters

Let ψ be a Hecke character as above and define

$$L(\psi, s) = \sum_{\mathfrak{a}} \frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^s} \qquad \text{for } \Re(s) > \frac{k_1 + k_2}{2} + 1, \tag{4.1}$$

where the sum is taken over all integral ideals of K. In its region of absolute convergence, this Dirichlet series has an Euler product expansion

$$L(\psi, s) = \prod_{\mathfrak{p}} (1 - \psi(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}, \tag{4.2}$$

where the product is taken over all primes of K.

Theorem 12. *Let K be an imaginary quadratic field of discriminant D and let ψ be a Hecke character of type A_0 with infinity type (k_1, k_2) as above. Define the completed L-function attached to ψ as*

$$L^*(\psi, s) = |D|^{s/2} \Gamma_{\mathbb{C}}(s - \min(k_1, k_2)) L(\psi, s) \qquad \textit{for } \Re(s) > \frac{k_1 + k_2}{2} + 1,$$

where $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. Then

1. *$L^*(\psi, s)$ extends to a meromorphic function on $\mathbb{C}$, with poles at $s = 0$ and $s = 1$ if and only if ψ is the trivial character;*
2. *$L^*(\psi, s)$ satisfies the functional equation*

$$L^*(\psi, 1 + k_1 + k_2 - s) = W(\psi) L^*(\bar{\psi}, s),$$

where $W(\psi)$ is some complex number of absolute value 1 and $\bar{\psi}$ is the Hecke character defined as $\bar{\psi}(\mathfrak{a}) = \overline{\psi(\mathfrak{a})}$.

Note that $L(\psi, s)$ is simply the Dedekind zeta function of K when ψ is the trivial character.

Proof. This follows from [MM06][Thm.3.3.1]. Indeed, it suffices to note that

$$\psi(\mathfrak{a}) = \xi(\mathfrak{a}) N(\mathfrak{a})^{\frac{k_1+k_2}{2}},$$

where ξ is a Hecke character as defined in the reference. Then

$$L^*(\psi, s) = c\Lambda\left(\xi, s - \frac{k_1+k_2}{2}\right)$$

for some constant c and the theorem follows. □

The number $W(\psi)$ that appears in the functional equation can be computed explicitly in terms of Gauss sums (see [MM06][Sec. 3.3]).

4.2 ★ Theta functions attached to imaginary quadratic fields

From now on, the term Hecke character will always refer to Hecke characters of type A_0 and conductor 1, as they were defined in the previous section.

Theta functions attached to Hecke characters of conductor one

Let ψ be such a character of infinity type (k_1, k_2) and consider the following q-expansion

$$\theta_\psi(q) = c_\psi + \sum_{\mathfrak{a}}{}' \psi(\mathfrak{a}) q^{N(\mathfrak{a})}, \tag{4.3}$$

where the sum is taken over all non-trivial integral ideals of K and

$$c_\psi = \begin{cases} 0 & \text{if } \psi \text{ is non-trivial} \\ \frac{h_K}{w_K} & \text{if } \psi \text{ is the trivial character} \end{cases}.$$

For θ_ψ to be the q-expansion of a modular form, it is necessary that $k_1 = 0$ or $k_2 = 0$. To see this, suppose that θ_ψ was a modular form. Then its L-series would be

$$L(\theta_\psi, s) = L(\psi, s) \tag{4.4}$$

and then one sees that the gamma factor of $L(\psi, s)$, namely $\Gamma(s - \min(k_1, k_2))$, can be the gamma factor of a modular form only if $\min(k_1, k_2) = 0$. Furthermore, by looking at the exponential factor and the weight of $L(\psi, s)$, one sees that θ_ψ would have weight $\max(k_1, k_2) + 1$ and level $|D|$. Finally, by computing $a_p(\theta_\psi)$ and by looking at the Euler factor at p of $L(\theta_\psi, s)$, one can deduce that θ_ψ would have character the *Kronecker symbol* χ_D defined as

$$\chi_D(p) = \begin{cases} 1 & \text{if } p \text{ splits in } K \\ -1 & \text{if } p \text{ is inert in } K \\ 0 & \text{if } p \text{ is inert in } K \end{cases}.$$

As it turns out, the condition $k_1 = 0$ or $k_2 = 0$ is also *sufficient* for θ_ψ to be a modular form.

Note that there is no loss of generality in supposing that ψ has infinity type $(k_1, 0)$, since then $\bar{\psi}$ has infinity type $(0, k_1)$ and the equality $\theta_{\bar{\psi}} = \theta_\psi^\rho$ implies that

$$\theta_{\bar{\psi}} = \theta_\psi,$$

since $\theta_\psi^\rho = \theta_\psi$. To prove the last equality, compute $a_p(\theta_\psi)$ and note that

$$\psi(\bar{\mathfrak{p}}) = \bar{\psi}(\mathfrak{p})$$

for any prime ideal $\mathfrak{p}$ in K since

$$\psi(\mathfrak{p})\bar{\psi}(\mathfrak{p}) = N(\mathfrak{p})^{k_1} = \psi(N(\mathfrak{p})\mathcal{O}_K) = \psi(\mathfrak{p}\bar{\mathfrak{p}}) = \psi(\mathfrak{p})\psi(\bar{\mathfrak{p}}).$$

From now on, let ψ is a Hecke character of infinity type $(2\ell, 0)$ for some $\ell \geq 0$ (note that $k_1 + k_2$ must be even when the conductor is $\mathcal{O}_K$, since $\pm 1 \in \mathcal{O}_K^\times$).

Theorem 13. *Let ψ be a Hecke character of infinity type $(2\ell, 0)$ for some $\ell \geq 0$. Then $\theta_\psi(q)$, given by (4.3), is the q-expansion of a modular form*

$$\theta_\psi \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol introduced above. If $\ell \neq 0$, the modular form θ_ψ is a newform. If $\ell = 0$, the modular form θ_ψ is a newform unless ψ^2 is the trivial class character, in which case it is an Eisenstein series.

The class characters of order dividing 2 are called *genus characters* and the Eisenstein series θ_ψ attached to them are called *genus Eisenstein series.*

To prove Theorem 13, it is convenient to introduce another collection of theta series.

Theta functions attached to ideals

Let $\mathfrak{a}$ be a fractional ideal of K, let $\ell \geq 0$ be an integer as above and consider the following q-expansion

$$\theta_{\mathfrak{a},\ell}(q) = \sum_{\lambda \in \mathfrak{a}} \lambda^{2\ell} q^{N(\lambda)/N(\mathfrak{a})}. \tag{4.5}$$

Note that there is no loss of generality in supposing that $\mathfrak{a}$ is integral, since

$$\theta_{\mu\mathfrak{a},\ell}(q) = \mu^{2\ell}\theta_{\mathfrak{a},\ell}. \tag{4.6}$$

Note also that

$$\theta_{\mathfrak{a},\ell}^\rho = \theta_{\bar{\mathfrak{a}},\ell}. \tag{4.7}$$

The following result is well-known.

Theorem 14. *Let $\mathfrak{a}$ be an integral ideal of K and let ℓ be a positive integer. Then $\theta_{\mathfrak{a},\ell}(q)$, given by (4.5), is the q-expansion of a modular form*

$$\theta_{\mathfrak{a},\ell} \in M_{2\ell+1}(\Gamma_0(|D|), \chi_D),$$

where χ_D is the Kronecker symbol. Moreover, the modular form $\theta_{\mathfrak{a},\ell}$ is a cusp form if and only if $\ell > 0$.

Proof. First, note that $\mathfrak{a}$ is a free $\mathbb{Z}$-module of rank 2 and so $N(\lambda)/N(\mathfrak{a})$ is a binary quadratic form which has *integral* coefficients. Second, a direct computation shows that $\lambda^{2\ell}$ is a *spherical polynomial* for this quadratic form. The result then follows from the general theory of classical theta series (see [Iwa97, Thm.10.9], for example). □

The relation between the theta functions attached to ideals and those attached to Hecke characters is given in the following

Proposition 9. *Let ψ be a Hecke character of infinity type $(2\ell, 0)$ and let $\mathfrak{a}_1, \dots \mathfrak{a}_{h_K}$ be a set of integral ideal class representatives of Cl_K. Then*

$$\theta_\psi(q) = \frac{1}{w_K} \sum_{j=1}^{h_K} \psi^{-1}(\mathfrak{a}_j)\theta_{\mathfrak{a}_j,\ell}(q)$$

and

$$\theta_{\mathfrak{a},\ell} = \frac{w_K}{h_K} \sum_{\psi} \psi(\mathfrak{a})\theta_\psi,$$

where the sum is over the h_K Hecke characters of infinity type $(2\ell, 0)$.

Proof. Let $\mathfrak{a}$ be an integral ideal. Then since the $\mathfrak{a}_j^{-1}$ are also representatives of Cl_K, one can write $\mathfrak{a} = \lambda\mathfrak{a}_j^{-1}$ for some unique j and some $\lambda \in \mathfrak{a}_j$ which is unique up to a unit. Therefore

$$\theta_\psi(q) = \frac{1}{w_K} \sum_{j=1}^{h_K} \sum_{\lambda \in \mathfrak{a}_j} \psi(\lambda\mathfrak{a}_j^{-1}) q^{N(\lambda)/N(\mathfrak{a}_j)} = \frac{1}{w_K} \sum_{j=1}^{h_K} \psi^{-1}(\mathfrak{a}_j) \sum_{\lambda \in \mathfrak{a}_j} \lambda^{2\ell} q^{N(\lambda)/N(\mathfrak{a}_j)}.$$

The second equality follows from this equality and the following orthogonality relation:

$$\sum_{\psi} \psi(\mathfrak{a}) = \begin{cases} \lambda^{2\ell} h_K & \text{if } \mathfrak{a} = \lambda \mathcal{O}_K \text{ for some } \lambda \in K \\ 0 & \text{otherwise} \end{cases}. \tag{4.8}$$

To prove this, fix a Hecke character ψ_0 of infinity type $(2\ell, 0)$ and note that

$$\sum_{\psi} \psi/\psi_0(\mathfrak{a}) = \begin{cases} h_K & \text{if } \mathfrak{a} = \lambda \mathcal{O}_K \text{ for some } \lambda \in K \\ 0 & \text{otherwise} \end{cases},$$

by the orthogonality relation for characters of the abelian group Cl_K. □

It is now possible to prove Theorem 13.

Proof. (of Theorem 13) It is clear from the modularity of the $\theta_{\mathfrak{a},\ell}$ and Proposition 9 that the θ_ψ are modular forms of the correct weight, level and character.

For $\ell \neq 0$, the cuspidality of the $\theta_{\mathfrak{a},\ell}$ also implies that of θ_ψ. Moreover, the fact that $L(\theta_\psi, s)$ has an Euler product of the right shape proves that θ_ψ is a newform (see [DS05, Thm.5.9.2], which is a partial converse to Theorem 5).

For $\ell = 0$, a finer analysis is needed to prove that θ_ψ is an Eisenstein series if and only if ψ^2 is trivial. See [Kan12, Thm.14], for example. Once this is known, the same argument as above proves that θ_ψ is a newform when it is cuspidal. □

The identities in the previous proposition can also be used to obtain the well-known decomposition of L-functions of Hecke characters in terms of Eisenstein series evaluated at CM points. To see this, first note that

$$L(\theta_{\mathfrak{a},\ell}, s) = N(\mathfrak{a})^s G_{2\ell, s-2\ell}(\bar{\mathfrak{a}}). \tag{4.9}$$

Then

$$L(\theta_\psi, s) = \frac{1}{w_K} \sum_{j=1}^{h_K} \psi^{-1}(\mathfrak{a}_j) L(\theta_{\mathfrak{a}_j,\ell}, s)$$

and so

$$L(\psi, s) = \frac{1}{w_K} \sum_{j=1}^{h_K} \psi^{-1}(\mathfrak{a}_j) N(\mathfrak{a}_j)^s G_{2\ell, s-2\ell}(\bar{\mathfrak{a}}_j). \tag{4.10}$$

The space of theta functions

By Proposition 9, the modular forms $\theta_{\mathfrak{a},\ell}$ and θ_ψ span the same space inside $M_{2\ell+1}(\Gamma_0(|D|), \chi_D)$ as $\mathfrak{a}$ ranges over all fractional ideals of K and ψ ranges over the Hecke characters of infinity type $(2\ell, 0)$. Let $\Theta_{K,\ell} \subseteq M_{2\ell+1}(\Gamma_0(|D|), \chi_D)$ denote this space.

When $\ell = 0$, relation (4.7) becomes

$$\theta_{\mathfrak{a},0} = \theta_{\bar{\mathfrak{a}},0}. \tag{4.11}$$

Letting $\mathbb{Z}/2\mathbb{Z}$ act on the set $\{\theta_{\mathfrak{a}_1,0}, \ldots, \theta_{\mathfrak{a}_{h_K},0}\}$ via ρ, where the $\mathfrak{a}_j$ are representatives of Cl_K, one sees using Burnside's Lemma that this set contains $(h_K + g_K)/2$ orbits, where $g_K = [\mathrm{Cl}_K : \mathrm{Cl}_K^2]$ is the number of genera.

This relation (4.11) and relation (4.6) are the only relations between theta functions, for any ideal and any $\ell \geq 0$. More formally:

Proposition 10. *Let $\ell \geq 0$ and let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_K}\}$ be representatives of Cl_K. Then*

- *if $\ell = 0$, the $\mathbb{C}$-vector space $\Theta_{K,\ell}$ has dimension $(h_K + g_K)/2$;*
- *if $\ell \neq 0$, the $\mathbb{C}$-vector space $\Theta_{K,\ell}$ has dimension h_K.*

Proof. This is well-known and follows from the classical theory of integral binary quadratic forms. See [Kan12, Prop.7], for example. □

One can then prove the following

Corollary 3. *Let $\psi_1, \ldots, \psi_{g_K}$ be the g_K genus characters of Cl_K. Then the Eisenstein series θ_{ψ_j} for $j = 1, \ldots, g_K$ are $\mathbb{C}$-linearly independent. Moreover, the space spanned by the $h_K - g_K$ cusp forms θ_ψ, for ψ a class character of order greater than 2, has dimension $(h_K - g_K)/2$.*

4.3 ★ Petersson inner product of theta functions attached to imaginary quadratic fields

In this section, the Rankin-Selberg method is used to find formulas for the Petersson norm of cuspidal theta functions.

Symmetric square L-function of theta functions attached to Hecke characters

Recall that the Petersson norm of a newform is related to a special value or to the residue of a twist of its symmetric square L-function. For theta functions, this L-function is related to the L-function of a Hecke character.

Proposition 11. *Let ψ be a Hecke character of infinity type $(2\ell, 0)$ for some $\ell \geq 0$ and let ω be a primitive Dirichlet character. Then*

$$L(\operatorname{Sym}^2 \theta_\psi, \omega, s) = L_D(\omega, s - 2\ell) \sum_{\mathfrak{a}} \frac{\omega(N(\mathfrak{a}))\psi^2(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where the sum is taken over all integral ideals of K.

Proof. For $\Re(s)$ large enough, all Dirichlet series involved have an Euler product expansion and the statement follows from a straightforward calculation. □

Corollary 4. *The following equalities hold:*

$$L(\operatorname{Sym}^2 \theta_\psi, 1, s) = \zeta_D(s - 2\ell) L(\psi^2, s)$$

and

$$L(\operatorname{Sym}^2 \theta_\psi, \chi_D, s) = L(\chi_D, s - 2\ell) L_D(\psi^2, s).$$

Proof. The first equation is clear and the second follows from the fact that $\chi_D(N(\mathfrak{p}))$ is 1 if $\mathfrak{p}$ divides a prime of $\mathbb{Q}$ which is unramified in K and 0 otherwise. □

Petersson norm of cuspidal theta functions attached to Hecke characters

Let ψ be a Hecke character of infinity type $(2\ell, 0)$ which is not a genus character. Then θ_ψ is a newform and $\theta_\psi^\rho = \theta_\psi$, so one can apply either the formula of Theorem 9:

$$\langle \theta_\psi, \theta_\psi \rangle = \left(\frac{\pi}{2} \frac{\phi(|D|)}{D^2} \frac{(4\pi)^{2\ell+1}}{\Gamma(2\ell+1)} \right)^{-1} L(\chi_D, 1) \operatorname{res}_{s=2\ell+1} L(\operatorname{Sym}^2 \theta_\psi, 1, s) \tag{4.12}$$

or the general formula of Theorem 8:

$$\langle \theta_\psi, \theta_\psi \rangle = \left(\frac{\pi}{2} \frac{\phi(|D|)}{D^2} \frac{(4\pi)^{2\ell+1}}{\Gamma(2\ell+1)} \right)^{-1} L(\operatorname{Sym}^2 \theta_\psi, \chi_D, 2\ell+1).$$

For no particular reason, let us use the first formula. Of course, the second gives the same result.

Proposition 12. *Let ψ be a Hecke character of infinity type $(2\ell, 0)$ which is not a genus character. Then*

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_K}{w_K} \sqrt{|D|} \frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}} L(\psi^2, 2\ell+1).$$

Proof. Using the previous Corollary, one sees that

$$\operatorname{res}_{s=2\ell+1} L(\operatorname{Sym}^2 \theta_\psi, 1, s) = \operatorname{res}_{s=1} \zeta_D(s) L(\psi^2, 2\ell+1) = \prod_{p|D} (1-p^{-1}) L(\psi^2, 2\ell+1).$$

The formula then follows from a simple calcuation, using (4.12) and the class number formula

$$L(\chi_D, 1) = \frac{2\pi h_K}{w_K \sqrt{|D|}}.$$

$\square$

Note that if $\ell = 0$ and ψ is a genus character, the above formula for $\langle \theta_\psi, \theta_\psi \rangle$ is not well-defined since in this case $L(\psi^2, s)$ is the Dedekind zeta function of K which has a pole at $s = 1$.

Using (4.10) and the previous proposition, it follows that

$$L(\psi^2, s+2\ell+1) = \frac{1}{w_K} \sum_{j=1}^{h_K} \psi^{-2}(\mathfrak{a}_j) N(\mathfrak{a}_j)^{s+2\ell+1} G_{4\ell, s-2\ell+1}(\bar{\mathfrak{a}}_j).$$

If $\ell \neq 0$, the Eisenstein series $G_{4\ell,s-2\ell+1}$ is well-defined at $s = 0$ and so

$$L(\psi^2, 2\ell+1) = \frac{1}{w_K} \sum_{j=1}^{h_K} \psi^{-2}(\mathfrak{a}_j) N(\mathfrak{a}_j)^{2\ell+1} G_{4\ell, -2\ell+1}(\bar{\mathfrak{a}}_j).$$

Using Equation (1.76) and (3.4), one can compute

$$\begin{aligned} \delta_2^{2\ell-1} E_2(\mathfrak{a}) &= 2^{-1}(2\pi i)^{-2} \delta_2^{2\ell-1} G^*_{2,0}(\mathfrak{a}) \\ &= 2^{-1}(2\pi i)^{-4\ell} G^*_{4\ell,-2\ell+1}(\mathfrak{a}) \\ &= (2\pi)^{-4\ell} \frac{1}{4}(2\ell)! \left(\frac{\sqrt{|D|} N(\mathfrak{a})}{2\pi} \right)^{-2\ell+1} G_{4\ell,-2\ell+1}(\mathfrak{a}) \\ &= (2\pi)^{-1-2\ell} \frac{1}{4}(2\ell)! (\sqrt{|D|} N(\mathfrak{a}))^{-2\ell+1} G_{4\ell,-2\ell+1}(\mathfrak{a}), \end{aligned}$$

and so

$$L(\psi^2, 2\ell+1) = \frac{4(2\pi)^{2\ell+1} \sqrt{|D|}^{2\ell-1}}{w_K (2\ell)!} \sum_{j=1}^{h_K} \psi^{-2}(\mathfrak{a}_j) N(\mathfrak{a}_j)^{4\ell} \delta_2^{2\ell-1} E_2(\bar{\mathfrak{a}}_j). \tag{4.13}$$

If $\ell = 0$, Theorem 1 says that the Eisenstein series $G^*_{0,s+1}$, and hence $G_{0,s+1}$, has a pole at $s = 0$. Using Kronecker's first limit formula, equation (3.4) and the orthogonality relation for the *non-trivial* character ψ^2, one sees that

$$L(\psi^2, 1) = -\frac{\pi}{3 w_K \sqrt{|D|}} \sum_{j=1}^{h_K} \psi^{-2}(\mathfrak{a}_j) \log(N(\mathfrak{a}_j)^6 |\Delta(\mathfrak{a}_j)|). \tag{4.14}$$

All those computations essentially prove the following

Theorem 15. *Let ψ be a Hecke character of infinity type $(2\ell, 0)$ which is not a genus character. Then if $\ell \neq 0$,*

$$\langle \theta_\psi, \theta_\psi \rangle = (|D|/4)^\ell \frac{4h_K}{w_K^2} \sum_{\mathcal{A} \in Cl_K} \psi^2(\mathcal{A}) \delta_2^{2\ell-1} E_2(\mathcal{A}),$$

while if $\ell = 0$,

$$\langle \theta_\psi, \theta_\psi \rangle = -\frac{h_K}{3w_K^2} \sum_{\mathcal{A} \in Cl_K} \psi^2(\mathcal{A}) \log(N(\mathcal{A})^6 |\Delta(\mathcal{A})|).$$

Proof. When $\ell > 0$, first note that

$$N(\mathfrak{a})^{4\ell} \delta_2^{2\ell-1} E_2(\bar{\mathfrak{a}}) = \delta_2^{2\ell-1} E_2(\mathfrak{a}^{-1})$$

and that

$$\psi(\mathfrak{a}) \delta_2^{2\ell-1} E_2(\mathfrak{a})$$

quantity depends only on the ideal class of $\mathfrak{a}$. The formula for $\langle \theta_\psi, \theta_\psi \rangle$ follows then from Proposition 12 and (4.13).

When $\ell = 0$, it suffices to use Proposition 12 and (4.14), since

$$N(\mathfrak{a}^{-1})^6 |\Delta(\mathfrak{a}^{-1})| = N(\mathfrak{a})^6 |\Delta(\mathfrak{a})|$$

and since this quantity depends only on the ideal class of $\mathfrak{a}$. □

Note that it follows from those formulas that

$$\langle \theta_\psi, \theta_\psi \rangle = \langle \theta_{\psi^{-1}}, \theta_{\psi^{-1}} \rangle$$

when $\ell = 0$, as expected (since $\theta_\psi = \theta_{\psi^{-1}}$ in this case).

Corollary 5. *Let ψ be a Hecke character of infinity type $(2\ell, 0)$ which is not a genus character and let Ω_K be the Chowla-Selberg period attached to K as in Proposition 7. Then*

$$\langle \theta_\psi, \theta_\psi \rangle \in \Omega_K^{4\ell} \bar{\mathbb{Q}}.$$

Proof. This follows from the previous theorem and the corollary to Proposition 8 applied to the nearly holomorphic modular form $\delta_2^{2\ell-1}E_2$. $\square$

Obtaining the case $\ell = 0$ from the case $\ell > 0$

The formula of Theorem 15 for $\ell > 0$ does not make sense for $\ell = 0$ since it involves the expression

$$\delta_2^{-1}E_2.$$

However, if this expression made sense, it would have to be a C^∞ modular form of weight zero such that

$$\delta_0 F = E_2.$$

We claim that

$$\delta_0 \log(\Im(\tau)^6|\Delta(\tau)|) = -12E_2(\tau),$$

where

$$\delta_0 = \frac{1}{2\pi i}\frac{d}{d\tau}.$$

This follows from the well known fact (see [RBvdG+08, Prop. 7]) that

$$\frac{d}{dq}\log\Delta(q) = \frac{1}{2\pi i}\frac{d}{d\tau}\log\Delta(\tau) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n. \tag{4.15}$$

Indeed, since

$$\log|\Delta(\tau)| = \Re(\log\Delta(\tau)),$$

it implies that

$$\frac{d}{d\tau}\log|\Delta(\tau)| = \frac{1}{2}\frac{d}{d\tau}\log\Delta(\tau)$$

(since $\frac{d\bar{F}}{d\tau} = \overline{\frac{dF}{d\bar{\tau}}} = 0$ if $F(\tau)$ is holomorphic). Letting

$$\delta_2^{-1}E_2(\mathfrak{a}) = -\frac{1}{12}\log(N(\mathfrak{a})^6|\Delta(\mathfrak{a})|) + C,$$

where C is some constant, in the formula of Theorem 15 for $\ell > 0$ gives exactly the formula for $\ell = 0$.

This relation between E_2 and the logarithmic derivative of Δ is the starting point of the calculations that will be done in the second part of this thesis.

Petersson inner product of theta functions attached to ideals

When $\ell > 0$, one can use Proposition 9 and Theorem 15 to prove the following

Proposition 13. *Let $\ell > 0$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of K. Then*

$$\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle = 4(|D|/4)^{\ell} \sum_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K} \lambda_{\mathfrak{c}}^{2\ell} \delta_2^{2\ell-1} E_2(\mathfrak{c}),$$

where the sum is over a set of representatives $\mathfrak{c}$ of ideals classes in Cl_K such that $\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \lambda_{\mathfrak{c}}\mathcal{O}_K$.

Then one has the following

Corollary 6. *Let $\ell > 0$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of K which are not in the same genus, i.e. their classes in Cl_K differ $\mod Cl_K^2$. Then $\theta_{\mathfrak{a}}$ and $\theta_{\mathfrak{b}}$ are orthogonal under the Petersson inner product.*

Proof. The sum in the above expression for $\langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle$ is empty. $\square$

This last corollary can be proved without the explicit formulas of the previous proposition. However, it is not clear how one could prove the following corollary without those formulas.

Corollary 7. *Let $\ell > 0$ and let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be fractional ideals of K. Then*

$$\langle \theta_{\mathfrak{c}\mathfrak{a},\ell}, \theta_{\mathfrak{c}\mathfrak{b},\ell} \rangle = N(\mathfrak{c})^{2\ell} \langle \theta_{\mathfrak{a},\ell}, \theta_{\mathfrak{b},\ell} \rangle.$$

Proof. Clear from the previous proposition. $\square$

CHAPTER 5
Petersson norm of weight one theta functions

In weight one, one can use the formula of Theorem 15 to obtain more algebraic information about the Petersson norm of theta series attached to class characters.

5.1 ★ Petersson norm of theta functions and Siegel units

Recall (see (3.3) and (3.5)) that the absolute value of Siegel units is given by

$$|\delta_{\mathfrak{a}}| = (N(\mathfrak{a})^6 |\Delta(\mathcal{O}_K)/\Delta(\mathfrak{a}^{-1})|)^{h_K}.$$

When ψ is a class character which is not a genus character, one can rearrange the formula in Theorem 15 to get

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{1}{3w_K^2} \sum_{j=1}^{h_K} \psi^2(\mathfrak{a}_j) \log |\delta_{\mathfrak{a}_j}|. \tag{5.1}$$

Note that w_K is always equal to 2 with these assumptions on ψ.

5.2 ★ On Stark's observation

Throughout this section, let $K = \mathbb{Q}(\sqrt{-23})$ and as before let H denote the Hilbert class field of K. Recall Stark's observation in the introduction on the relation between the Petersson norm of a theta series and the special value at $s = 1$ of an Artin L-function. He observed that the Petersson norm of the weight one theta function attached to a degree 2 Artin representation of $\mathrm{Gal}(H/\mathbb{Q}) \simeq S_3$ has Petersson norm

$$3 \log \varepsilon,$$

where ε is a real root of $x^3 - x - 1$, which generates the Hilbert class field of K (over K).

With our notation, the theta series considered by Stark is θ_ψ, where ψ is one of the two non-trivial class characters of K. Its Petersson norm could be expressed as a linear combination of logarithms of Siegel units as in (5.1), but to obtain Stark's result one has to work directly from the formula of Theorem 15. Then

$$\langle \theta_\psi, \theta_\psi \rangle = -\frac{1}{4} \sum_{j=1}^{3} \psi^2(\mathfrak{a}_j) \log(N(\mathfrak{a}_j)^6 |\Delta(\mathfrak{a}_j)|) = -3 \sum_{j=1}^{3} \psi^2(\mathfrak{a}_j) \log(\sqrt{N(\mathfrak{a}_j)} |\eta(\mathfrak{a}_j)|^2),$$

where

$$\eta(q) = \exp(2\pi i\tau/24) \prod_{n=1} (1 - q^n)$$

is the Dedekind eta function. As is well known, the function $|\eta(\tau)|^2$ on $\mathcal{H}$ is a C^∞ modular form of weight $(0, 1)$ (this follows from [DS05, Prop.1.2.5], for example), so it has a well-defined value on lattices in $\mathcal{L}$. Define

$$\Phi(\mathfrak{a}) = \sqrt{N(\mathfrak{a})} |\eta(\mathfrak{a})|^2.$$

Then Φ depends only on the ideal class of $\mathfrak{a}$ and has the property that

$$\Phi(\mathfrak{a}) = \Phi(\overline{\mathfrak{a}}) = \Phi(\mathfrak{a}^{-1}).$$

Moreover,

$$\left(\frac{\Phi(\mathcal{O}_K)}{\Phi(\mathfrak{a}^{-1})} \right)^{12h_K} = |\delta_\mathfrak{a}|. \tag{5.2}$$

With this notation,

$$\langle \theta_\psi, \theta_\psi \rangle = -3 \sum_{j=1}^{3} \psi^2(\mathfrak{a}_j) \log \Phi(\mathfrak{a}_j^{-1}) = 3 \log \prod_{j=1}^{3} \Phi(\mathfrak{a}_j^{-1})^{-\psi^2(\mathfrak{a}_j)}.$$

Letting $\mathfrak{a}$ be any non-principal ideal of K, we see that

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \frac{\Phi(\mathfrak{a}^{-1})}{\Phi(\mathcal{O}_K)}.$$

This computation uses the fact that

$$\psi^2(\mathfrak{a}) + \psi^2(\mathfrak{a}^{-1}) = 2\Re(\zeta_3) = -1,$$

where ζ_3 is any non-trivial third root of unity. As one can verify numerically,

$$\frac{\Phi(\mathfrak{a}^{-1})}{\Phi(\mathcal{O}_K)}$$

is a real root of $x^3 - x - 1$, so we recover Stark's example.

5.3 ★ On generalisations of Stark's observation

Based on the above computation and the formula

$$\langle \theta_\psi, \theta_\psi \rangle = h_K \log \prod_{j=1}^{h_K} \Phi(\mathfrak{a}_j)^{-\psi^2(\mathfrak{a}_j)}, \tag{5.3}$$

it might seem reasonable to conjecture that

$$\kappa_\psi = \prod_{j=1}^{h_K} \Phi(\mathfrak{a}_j)^{-\psi^2(\mathfrak{a}_j)} \tag{5.4}$$

is a unit in the Hilbert class field of K. Note that the individual numbers $\Phi(\mathfrak{a})$ are usually transcendental and that $\psi^2(\mathfrak{a})$ is a root of unity. In fact, when ψ is a genus character κ_ψ can be computed explicitly in terms of the Chowla-Selberg period defined in Proposition 7 (actually, this statement is equivalent to the Chowla-Selberg formula, see [RBvdG+08, Sec.6.3]). It follows from those observations that the algebraicity of κ_ψ depends crucially on the character ψ.

Generalisation to class number 3

There are exactly 16 imaginary quadratic fields of class number 3, the one with the largest discriminant in absolute value being $\mathbb{Q}(\sqrt{-907})$ (see [Wat04, Table 4]). In this case, the same computation as above shows that

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \frac{\Phi(\mathfrak{a}^{-1})}{\Phi(\mathcal{O}_K)},$$

where ψ is one of the two non-trivial class characters of K and the following lemma shows that κ_ψ is a unit.

Lemma 2. *Let K be an imaginary quadratic field and let $\mathfrak{a}$ be an ideal of K. Then*

$$\frac{\Phi(\mathfrak{a}^{-1})}{\Phi(\mathcal{O}_K)}$$

is an algebraic integer and a unit.

Proof. It suffices to note that

$$\left(\frac{\Phi(\mathfrak{a}^{-1})}{\Phi(\mathcal{O}_K)}\right)^{12h_K} = |\delta_{\mathfrak{a}}|.$$

□

When K has class number 3, it appears numerically that the unit $\Phi(\mathfrak{a}^{-1})/\Phi(\mathcal{O}_K)$ is in fact *in the Hilbert class field* of K. This gives numerical evidence that Stark's observation generalizes to all imaginary quadratic fields of class number 3.

Generalisation to class number 5

When K is one of the 25 imaginary quadratic fields of class number 5 (see [Wat04, Table 4]), Stark's observation does not seem to generalize. In other words, the complex number κ_ψ does not seem to be a unit for any class character ψ.

The main difference between the class number 3 and class number 5 case is that the later involves non-trivial 5th roots of unity, which do not have rational real part.

The general case

The above observations lead to the following

Proposition 14. *Let ψ be a character of the class group of K as above and suppose that ψ^2 is a non-trivial character with rational real part. Then κ_ψ is an algebraic integer which is a unit. If ψ^2 is a non-trivial genus character corresponding to the factorization $D = D_1 D_2$, with $D_1 > 0$, then*

$$\kappa_\psi = \epsilon_{D_1}^{\frac{4h_{D_1}h_{D_2}}{w_K w_{D_2}}},$$

where ϵ_{D_1} is a fundamental unit of $\mathbb{Q}(\sqrt{D_1})$, h_{D_j} is the class number of $\mathbb{Q}(\sqrt{D_j})$ and w_{D_2} is the number of roots of unity in $\mathbb{Q}(\sqrt{D_2})$.

Proof. Since ψ^2 is non-trivial,

$$\kappa_\psi = \prod_{j=1}^{h_K} \Phi(\mathfrak{a}_j)^{-\psi^2(\mathfrak{a}_j)} = \prod_{j=1}^{h_K} \left(\frac{\Phi(\mathfrak{a}_j)}{\Phi(\mathcal{O}_K)}\right)^{-\psi^2(\mathfrak{a}_j)}.$$

The first claim follows from the previous lemma and the observation that

$$\left(\frac{\Phi(\mathfrak{a})}{\Phi(\mathcal{O}_K)}\right)^{-\psi^2(\mathfrak{a})} \left(\frac{\Phi(\overline{\mathfrak{a}})}{\Phi(\mathcal{O}_K)}\right)^{-\psi^2(\overline{\mathfrak{a}})} = \left(\frac{\Phi(\overline{\mathfrak{a}})}{\Phi(\mathcal{O}_K)}\right)^{-2\Re(\psi^2(\overline{\mathfrak{a}}))}$$

if $\mathfrak{a}$ is not equivalent to $\overline{\mathfrak{a}}$ in Cl_K and

$$\left(\frac{\Phi(\mathfrak{a})}{\Phi(\mathcal{O}_K)}\right)^{-\psi^2(\mathfrak{a})} = \frac{\Phi(\mathfrak{a})}{\Phi(\mathcal{O}_K)}$$

otherwise.

For the second part, first note that

$$L(\psi^2, s) = L(\chi_{D_1}, s) L(\chi_{D_2}, s),$$

where χ_{D_j} is the Kronecker symbol attached to the discriminant D_j (see [Coh07, Prop.10.5.19]). Then

$$L(\psi^2, 1) = L(\chi_{D_1}, 1) L(\chi_{D_2}, 1) = \frac{4\pi h_{D_1} h_{D_2}}{w_{D_2}\sqrt{|D|}} \log \varepsilon_{D_1}$$

by the class number formula and so by Proposition 12

$$h_K \log \kappa_\psi = \langle \theta_\psi, \theta_\psi \rangle = \frac{4h_K\sqrt{|D|}}{w_K(4\pi)} L(\psi^2, 1) = h_K \frac{4h_{D_1}h_{D_2}}{w_K w_{D_2}} \log \varepsilon_{D_1}.$$

$\square$

Note that the power of ε_{D_1} which appears in the above proposition is not always integral. For example, with $D = -39, D_1 = 13$ and $D_2 = -3$, the proposition gives $\kappa_\psi = \epsilon_{13}^{\frac{1}{3}}$. This example also proves that κ_ψ is not always in the Hilbert class field (since H/K has degree 4 in this case).

It also follows from that proposition that κ_ψ, does not always generate the Hilbert class field of K (even when it is in H).

Since ψ^2 has rational real part if and only if it has order dividing 4 of 6, it follows that κ_ψ is a unit for some ψ whenever $\gcd(h_k, 6) > 1$. We believe that the converse is true.

Conjecture 1. *If K has class number coprime to 6, then κ_ψ is not algebraic for any ψ.*

Numerical evidence for this conjecture is given in Chapter 12.

Part II

p-adic Interpolation

CHAPTER 6
p-adic modular forms

In the next few sections, test objects, weight space and Tate curves will be introduced over rings, leading to the definition of algebraic modular forms and generalized p-adic modular forms. Our treatment of the theory of algebraic and generalized p-adic modular forms follows closely Chapters II and V of [Kat76], respectively. The first section is there to motivate the passage from the classical analytic theory (formulated in terms of points in the upper half-plane and lattices) to the algebraic theory (to be formulated in terms of moduli spaces of enhanced elliptic curves). The fact that the algebraic theory of modular forms is equivalent over $\mathbb{C}$ to the theory presented in Chapter 1 will lead to an algebraic interpretation of the complex formulas of the first part. One can then use the powerful theory of p-adic modular forms to interpolate them.

6.1 Towards an algebraic theory of modular forms

Throughout this section, the notation is the same as in Chapter 1.

The idea of the algebraic theory of modular forms is to interpret modular forms as functions on moduli spaces of elliptic curves with extra structure. To see why this idea is not far fetched, let $L \in \mathcal{L}$ be a lattice. Then it is well-known that the quotient space $\mathbb{C}/L$ is a smooth Riemann surface of genus one, i.e. a torus. One can also explicitly realize this space as the set of complex points of an elliptic curve over $\mathbb{C}$ via the map

$$\begin{aligned} \phi_L : \mathbb{C}/L &\longrightarrow E_L(\mathbb{C}) \\ z &\longmapsto [\wp_L(z) : \wp_L'(z) : 1], \end{aligned}$$

where $\wp_L$ is the Weierstrass $\wp$ function attached to the lattice L and

$$E_L: \quad Y^2 = 4X^3 - g_2(L)X - g_3(L),$$

with

$$g_2 = 60G_4 \qquad \text{and} \qquad g_3 = 140G_6.$$

(see [Sil09, Coro.5.1.1]). This illustrates the fact that, loosely speaking, the set of lattices $\mathcal{L}$ can be thought of as a set of elliptic curves over $\mathbb{C}$.

To view modular forms as functions of elliptic curves, one should first turn this last statement into an actual bijection by finding a way to attach a lattice to an elliptic curve E over $\mathbb{C}$. The way one can do this depends on how this elliptic curve E is presented. If E is defined by a Weierstrass equation

$$E_{A,B}: \quad Y^2 = 4X^3 + AX + B,$$

then it is well known (see [Sil09, Thm.5.1]) that there exists a unique lattice $L(A,B)$ such that

$$g_2(L(A,B)) = -A \qquad \text{and} \qquad g_3(L(A,B)) = -B.$$

However, if E is given geometrically as a nonsingular curve over $\mathbb{C}$ with a distinguished point, there is no canonical way to attach a Weierstrass equation to it. To solve this problem, recall that one can recover L from the space $\mathbb{C}/L$ by integrating the translation invariant differential dz along the elements of the singular homology group $H_1(\mathbb{C}/L, \mathbb{Z})$:

$$L = \left\{ \int_\gamma dz : \gamma \in H_1(\mathbb{C}/L, \mathbb{Z}) \right\}$$

(see [Sil09, Prop.5.6]). Letting

$$L(E,\omega) = \left\{ \int_\gamma \omega : \gamma \in H_1(E(\mathbb{C}), \mathbb{Z}) \right\}$$

for any elliptic curve E over $\mathbb{C}$ and any invariant differential ω on E, it follows that

$$L(E_L, \omega_L) = L,$$

where

$$\omega_L = \tfrac{dX}{Y}.$$

This proves that there is a sequence of bijections of sets

$$\mathcal{L} \xrightarrow{\sim} \left\{Y^2 = 4X^3 + AX + B : A^3 - 27B^2 \neq 0\right\} \xrightarrow{\sim} \{(E, \omega)\} / \approx,$$

where $(E, \omega) \approx (E', \omega')$ if there is a isomorphism $\phi : E \xrightarrow{\sim} E'$ such that $\phi^*(\omega') = \omega$. A couple (E, ω) is called a *framed elliptic curve.*

Now let f be a modular form of weight k and level one. By definition, it satisfies

$$f(\lambda L) = \lambda^{-k} f(L)$$

for all $\lambda \in \mathbb{C}^\times$ and $L \in \mathcal{L}$. It is a simple exercise to see that if $L \in \mathcal{L}$ corresponds to the framed elliptic curve (E, ω), the lattice λL corresponds to the framed elliptic curve $(E, \lambda\omega)$. Viewing f as a function on framed elliptic curves, this is saying that

$$f(E, \lambda\omega) = \lambda^{-k} f(E, \omega).$$

At the level of Weierstrass equations, the change from L to λL corresponds to the change of variables

$$X = \lambda^2 X' \qquad \text{and} \qquad Y = \lambda^3 Y'.$$

This proves that the weight of a modular form can be recovered from the corresponding function on framed elliptic curves.

One can also recover the q-expansion from this function on framed elliptic curves. To see this, first note that for fixed $\tau_0 \in \mathcal{H}$, one has the following sequence of equalities in $\mathbb{C}$

$$\begin{aligned} f(\tau_0) &= f([\tau_0, 1]) \\ &= f\left(\mathbb{C}/[\tau_0, 1], dz\right) \\ &= f\left(Y^2 = 4X^3 - g_2(\tau_0)X - g_3(\tau_0), \frac{dX}{Y}\right) \\ &= f\left(Y^2 = 4X^3 - g_2(q_0)X - g_3(q_0), \frac{dX}{Y}\right), \end{aligned}$$

where $q_0 = e^{2\pi i \tau_0} \in \mathcal{H}$. The Weierstrass equation

$$Y^2 = 4X^3 - g_2(q)X - g_3(q)$$

defines an elliptic curve over $\mathbb{C}((q))$ (use (1.22)). Applying the change of variables

$$X = (2\pi i)^2 X' \qquad \text{and} \qquad Y = (2\pi i)^3 Y'$$

gives an isomorphism

$$\left(Y^2 = 4X^3 - g_2(q)X - g_3(q), \frac{dX}{Y}\right) \approx \left(Y'^2 = 4X'^3 - (2\pi i)^{-4} g_2(q)X' - (2\pi i)^{-6} g_3(q), \frac{1}{2\pi i}\frac{dX'}{Y'}\right)$$

and the last elliptic curve is defined over $\mathbb{Q}((q))$. In fact, one can do even better by letting

$$X' = x + \frac{1}{12} \qquad \text{and} \qquad Y' = x + 2y,$$

which gives an isomorphism

$$\left(Y'^2 = 4X'^3 - (2\pi i)^{-4} g_2(q)X' - (2\pi i)^{-6} g_3(q), \frac{dX'}{Y'}\right) \approx \left(y^2 + xy = x^3 + B(q)x + C(q), \frac{dx}{x+2y}\right),$$

where

$$\begin{aligned}
B(q) &= -5\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}\\
C(q) &= -\frac{1}{12}\sum_{n=1}^{\infty}\frac{(7n^5+5n^3)q^n}{1-q^n}.
\end{aligned}$$

Note that this last elliptic curve is now defined over $\mathbb{Z}((q))$ (in fact, it has coefficients in $\mathbb{Z}[[q]]$). Its discriminant is

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

It follows from this last sequence of change of variables that

$$f(q_0) = (2\pi i)^k f\left(y^2+xy = x^3+B(q_0)x+C(q_0), \frac{dx}{x+2y}\right).$$

One might hope that

$$f(q) = (2\pi i)^k f\left(y^2+xy = x^3+B(q)x+C(q), \frac{dx}{x+2y}\right), \tag{6.1}$$

where now q is seen as a variable. However, this equality does not make sense yet, since f is only defined on lattices and the elliptic curve

$$y^2+xy = x^3+B(q)x+C(q)$$

is not defined over $\mathbb{C}$ (so one cannot attach a lattice to it). This idea still leads to a good definition of the q-expansion and one can indeed make sense of this equality.

6.2 Test objects and trivialized elliptic curves

Let R be a ring (commutative with 1). An elliptic curve over R is defined as a smooth and proper morphism

$$\begin{tikzcd} E \arrow[d] \\ \mathrm{Spec}(R) \end{tikzcd}$$

whose geometric fibres are connected curves of genus one, together with a section

$$\begin{tikzcd} E \arrow[d] \\ \mathrm{Spec}(R) \arrow[u, bend left, "e"] \end{tikzcd} .$$

Let E be an elliptic curve over a ring R and let $N \geq 1$ be a positive integer. The kernel of multiplication by N is a finite flat commutative group scheme over R of rank N^2, which is denoted $E[N]$. It is étale if and only if N is invertible in R.

6.2.1 Level N test objects

As above, let $N \geq 1$ be a positive integer and let $E_{/R}$ be an elliptic curve over R. As in [Kat76, Sec.2.0], a $\Gamma(N)^{arith}$ *level structure* is defined as an isomorphism

$$\beta : \mu_N \times \mathbb{Z}/N\mathbb{Z} \longrightarrow E[N]$$

of group schemes over R such that the Weil pairing on $E[N]$ corresponds to the canonical pairing

$$\langle (\zeta_1, n), (\zeta_2, m) \rangle = \frac{\zeta_1^m}{\zeta_2^n}$$

on $\mu_N \times \mathbb{Z}/N\mathbb{Z}$. A $\Gamma(N)^{arith}$-*test object* over R is a triple

$$(E_{/R}, \omega, \beta),$$

where

- $E_{/R}$ is an elliptic curve over R;
- ω is a nowhere vanishing invariant differential on E;
- β is a $\Gamma(N)^{\mathrm{arith}}$ level structure

(see [Kat76, Sec.2.1]).

6.2.2 Trivialized elliptic curves

A p-adic ring B is defined as a ring which is complete and separated with respect to its p-adic topology (e.g. $\mathbb{Z}_p$ or $\mathbb{F}_p$, but not $\mathbb{Q}_p$).

Let $E_{/B}$ be an elliptic curve over B. As in [Kat76, Sec.5.0], a *trivialization* of E is defined as an isomorphism

$$\varphi : \hat{E} \longrightarrow \hat{\mathbb{G}}_m$$

of formal groups over B. Note that a trivialization φ of $E_{/B}$ is equivalent to a compatible sequence of inclusions

$$\mu_{p^r} \hookrightarrow E$$

for $r \geq 0$. In particular, trivialized elliptic curves are fibre-by-fibre ordinary (note that the residue field of every maximal ideal of B is of characteristic p). A $\Gamma(N)^{\mathrm{arith}}$ level structure β on $E_{/B}$ is said to be compatible with the trivialization φ if the composition of maps

$$\mu_{p^r} \overset{\beta}{\hookrightarrow} \hat{E} \xrightarrow{\varphi} \hat{\mathbb{G}}_m$$

is the inclusion, where $p^r || N$ and $\mathbb{G}_m$ is the multiplicative group scheme. A *trivialized* $\Gamma(N)^{arith}$ *elliptic curve* over B is a triple

$$(E_{/B}, \varphi, \beta),$$

where

- $E_{/B}$ is an elliptic curve over B;
- φ is a trivialization of E;

- β is a compatible $\Gamma(N)^{\mathrm{arith}}$ level structure

(see [Kat76, Sec.5.1]).

6.3 Weight and Tate curves

Throughout this section, let $N \geq 1$ be a positive integer, and let $(E_{/R}, \omega, \beta)$ and $(E_{/B}, \varphi, \beta)$ be as above.

6.3.1 Weight

Let f be a map from the set of $\Gamma(N)^{\mathrm{arith}}$-test objects over R to R. For $\lambda \in R^{\times}$, define the map $[\lambda]f$ as

$$[\lambda]f(E, \omega, \beta) = f(E, \lambda^{-1}\omega, \beta). \tag{6.2}$$

Then f is said to be of weight $k \in \mathbb{Z}$ if

$$[\lambda]f = \lambda^k f \qquad \text{for all } \lambda \in R^{\times}. \tag{6.3}$$

Note that in level 1 over $\mathbb{C}$ one recovers the notion of weight introduced in Section 1.2.

Now let f be a map from the set of trivialized $\Gamma(N)^{\mathrm{arith}}$ elliptic curves over B to B. In this case it is not as straightforward to define the weight of f, since there is no obvious action of $B^{\times}$ on trivialized $\Gamma(N)^{\mathrm{arith}}$ elliptic curves. However, one can let $\mathbb{Z}_p^{\times}$ act on a trivialization $\varphi : \hat{E} \longrightarrow \hat{\mathbb{G}}_m$ by viewing $a \in \mathbb{Z}_p^{\times}$ as a compatible sequence of elements in $(\mathbb{Z}/p^r\mathbb{Z})^{\times}$ for $r \geq 1$. If $(E_{/B}, \varphi, \beta)$ is a trivialized $\Gamma(N)^{\mathrm{arith}}$ elliptic curves over B, then so is

$$(E_{/B}, a^{-1}\varphi, \beta \circ (a, a^{-1})).$$

More generally, following [Kat76, Sec.5.3], let

$$G(N) = \left\{(a, b) \in \mathbb{Z}_p^{\times} \times (\mathbb{Z}/N\mathbb{Z})^{\times} : a \equiv b \pmod{p^r}, \text{ where } p^r || N\right\} \tag{6.4}$$

and for $(a,b) \in G(N)$, define the map $[a,b]f$ as

$$[a,b]f(E_{/B}, \varphi, \beta) = f(E_{/B}, a^{-1}\varphi, \beta \circ (b, b^{-1})). \tag{6.5}$$

Then f is said to be of weight κ, where κ is a continuous character

$$\kappa : G(N) \longrightarrow B^{\times},$$

if

$$[a,b]f = \kappa(a,b)f \qquad \text{for all } (a,b) \in G(N). \tag{6.6}$$

If $\kappa = \kappa_k \cdot \chi$, where

$$\kappa_k(a,b) = a^k$$

for some $k \in \mathbb{Z}$ and χ is a *finite order* character of $G(N)$, then f is said to be of weight k and character (or nebentypus) χ.

6.3.2 Tate curves

In Section 6.1, it was shown that the elliptic curve

$$\mathrm{Tate}(q): \quad y^2 + xy = x^3 + B(q)x + C(q) \tag{6.7}$$

is defined over $\mathbb{Z}((q))$. This elliptic curve is called the *Tate curve* of level one. Note also that $\mathrm{Tate}(q)$ is equipped with a canonical differential

$$\omega_{\mathrm{can}} = \frac{dx}{x+2y},$$

so that

$$(\mathrm{Tate}(q), \omega_{\mathrm{can}}) \tag{6.8}$$

is a level one test object (see [Kat76, Sec.2.2]). Using the same notation as in Section 6.1, one has the following isomorphisms of test objects:

$$(\mathbb{C}/[\tau_0,1],dz) \approx (\mathbb{C}/2\pi i[\tau_0,1], 2\pi i dz) \approx \left(\mathbb{C}^\times/q_0^{\mathbb{Z}}, \frac{dt}{t}\right),$$

where the first map is $z \mapsto 2\pi i z$, the second is $z \mapsto e^{2\pi i z}$, and $t = e^{2\pi i z}$ is the parameter on $\mathbb{C}^\times$. Therefore, one can think of Tate(q) as "$\mathbb{G}_m/q^{\mathbb{Z}}$". In level N, the relevant Tate curve, denoted Tate(q^N), is given by the Weierstrass equation

$$\mathrm{Tate}(q^N): \quad y^2 + xy = x^3 + B(q^N)x + C(q^N) \tag{6.9}$$

and can be thought of as "$\mathbb{G}_m/q^{N\mathbb{Z}}$". With that in mind, it is clear that the N-torsion of Tate(q^N) should correspond to the N^2 points

$$\zeta_N^i q^j \qquad \text{for } 0 \le i,j \le N-1,$$

where ζ_N is a primitive Nth root of unity. This gives a canonical $\Gamma(N)^{\mathrm{arith}}$ level structure

$$\beta_{\mathrm{can}} : \mu_N \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathrm{Tate}(q^N)$$

and so

$$(\mathrm{Tate}(q), \omega_{\mathrm{can}}, \beta_{\mathrm{can}}) \tag{6.10}$$

is a $\Gamma(N)^{\mathrm{arith}}$-test object (see [Kat76, Sec.2.2]). Over the ring $\widehat{\mathbb{Z}_p((q))}$, the p-adic completion of $\mathbb{Z}_p((q))$, one can also trivialize the Tate curve Tate(q^N) in a canonical way (the formal completion of $\mathbb{G}_m/q^{N\mathbb{Z}}$ is $\hat{\mathbb{G}}_m$). Since this trivialization, denoted φ_{can}, is compatible with β_{can}, one obtains trivialized $\Gamma(N)^{\mathrm{arith}}$ elliptic curves

$$(\mathrm{Tate}(q^N), \varphi_{\mathrm{can}}, \beta_{\mathrm{can}})$$

(see [Kat76, Sec.5.2.0]).

6.4 Algebraic and generalized p-adic modular forms

As in the previous section, let $N \geq 1$ be an integer.

6.4.1 Algebraic modular forms

Let R be a ring and let A be an R-algebra. Following [Kat76, Sec.2.1], an *algebraic modular form defined over R of weight $k \in \mathbb{Z}$ and level $\Gamma(N)^{arith}$* is a rule f which assigns to every $\Gamma(N)^{\mathrm{arith}}$-test object

$$(E_{/A}, \omega, \beta)$$

over A an element

$$f(E_{/A}, \omega, \beta) \in A$$

and which satisfies the following properties:

- The value $f(E_{/A}, \omega, \beta)$ depends only on the isomorphism class of $(E_{/A}, \omega, \beta)$;
- If R' is a R-algebra, the formation of f commutes with base change from R to R'. Symbolically,

  $$f((E_{/A}, \omega, \beta) \otimes_R R') = f(E_{/A}, \omega, \beta) \otimes_R 1$$

  inside $A \otimes_R R'$, where $(E_{/A}, \omega, \beta) \otimes_R R'$ is the test object obtained from $(E_{/A}, \omega, \beta)$ by base change from R to R';
- For every $\lambda \in A^\times$,

  $$f(E_{/A}, \lambda^{-1}\omega, \beta) = \lambda^k f(E_{/A}, \omega, \beta).$$

The R-module of algebraic modular forms is denoted

$$V_k^{\mathrm{alg}}(R, \Gamma(N)^{\mathrm{arith}}).$$

As in [Kat76, Sec.2.2], the q-expansion of an algebraic modular form of level $\Gamma(N)^{\mathrm{arith}}$ is defined as

$$f(\mathrm{Tate}(q^N), \omega_{\mathrm{can}}, \beta_{\mathrm{can}}) \in R \otimes \mathbb{Z}((q)) \subseteq R((q)).$$

The very important *q-expansion principle* says that the map

$$V_k^{\mathrm{alg}}(R, \Gamma(N)^{\mathrm{arith}}) \longrightarrow R((q)) \tag{6.11}$$

which sends algebraic modular forms to their q-expansion is *injective* (see [Kat76, Sec.2.2.6]). Moreover, if $R \subseteq R'$, then

$$V_k^{\mathrm{alg}}(R, \Gamma(N)^{\mathrm{arith}}) \hookrightarrow V_k^{\mathrm{alg}}(R', \Gamma(N)^{\mathrm{arith}})$$

and an element $f \in V_k^{\mathrm{alg}}(R', \Gamma(N)^{\mathrm{arith}})$ lies in $V_k^{\mathrm{alg}}(R, \Gamma(N)^{\mathrm{arith}})$ if and only if its q-expansion lies in $R((q))$ (see [Kat76, Sec.2.2.6]).

Finally, note that there is an action of *diamond operators* on $V_k^{\mathrm{alg}}(R, \Gamma(N)^{\mathrm{arith}})$ as follows: for $b \in (\mathbb{Z}/N\mathbb{Z})^\times$ define

$$\langle b \rangle f(E_{/A}, \omega, \beta) = f(E_{/A}, \omega, \beta \circ (b, b^{-1})) \tag{6.12}$$

for $f \in V_k^{\mathrm{alg}}(R, \Gamma(N)^{\mathrm{arith}})$ (see [Kat76, Sec.5.4.8]). If

$$\langle b \rangle f = \chi(b) f$$

for some homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow R^\times$, then f is said to be of character (or nebentypus) χ.

6.4.2 Generalized p-adic modular forms

Let B be a p-adic ring, let A be a p-adic ring which is also a B-algebra. Following [Kat76, Sec.5.1], a *generalized p-adic modular form defined over B of level $\Gamma(N)^{arith}$* is defined as a rule f which assigns to every trivialized $\Gamma(N)^{\mathrm{arith}}$ elliptic curve

$$(E_{/A}, \varphi, \beta)$$

over A an element

$$f(E_{/A}, \varphi, \beta) \in A$$

and which satisfies the following properties:

- The value $f(E_{/A}, \omega, \beta)$ depends only on the isomorphism class of $(E_{/A}, \omega, \beta)$;
- If B' is p-adic ring which is also a B-algebra, the formation of f commutes with base change from B to B'.

The B-module of such objects is denoted

$$V^{\mathrm{gen}}(B, \Gamma(N)^{\mathrm{arith}}).$$

Let

$$\kappa : \mathbb{Z}_p^\times \longrightarrow B^\times$$

be a continuous homomorphism. Then, as in [Kat76, Sec.5.3], a generalized p-adic modular form is said to be of weight κ if

$$[a, b]f(E_{/A}, \omega, \beta) = \kappa(a, b)f(E_{/A}, \omega, \beta)$$

for every $(a, b) \in G(N)$. The B-module of generalized p-adic modular forms of weight κ is denoted

$$V_\kappa^{\mathrm{gen}}(B, \Gamma(N)^{\mathrm{arith}}).$$

As in [Kat76, Sec.5.1], the q-expansion of a generalized p-adic modular form is defined as

$$f(\mathrm{Tate}(q^N), \varphi_{\mathrm{can}}, \beta_{\mathrm{can}}) \in \widehat{B((q))}.$$

The *q-expansion principle* says that the map

$$V_\kappa^{\mathrm{gen}}(B, \Gamma(N)^{\mathrm{arith}}) \longrightarrow \widehat{B((q))} \tag{6.13}$$

which sends generalized p-adic modular forms to their q-expansion is *injective* (see [Kat76, Sec.5.2.1]). Moreover, if $B \subseteq B'$, then

$$V_\kappa^{\mathrm{gen}}(B, \Gamma(N)^{\mathrm{arith}}) \hookrightarrow V_\kappa^{\mathrm{gen}}(B', \Gamma(N)^{\mathrm{arith}})$$

and an element $f \in V_\kappa^{\mathrm{gen}}(B', \Gamma(N)^{\mathrm{arith}})$ lies in $V_\kappa^{\mathrm{gen}}(B, \Gamma(N)^{\mathrm{arith}})$ if and only if its q-expansion lies in $\widehat{B((q))}$ (see [Kat76, Sec.5.2.2]).

6.5 Comparison between modular forms

6.5.1 Classical and algebraic modular forms over $\mathbb{C}$

The correspondence between elliptic curves over $\mathbb{C}$ and lattices in $\mathbb{C}$, presented in Section 6.1, gives a map

$$V_k^{\mathrm{alg}}(\mathbb{C}, \Gamma(1)^{\mathrm{arith}}) \longrightarrow M_k^!(\Gamma(1)),$$

where $M_k^!(\Gamma(1))$ denotes the space of weakly holomorphic modular forms of level 1 (i.e. the space of modular forms which are only required to have *meromorphic* q-expansion at the cusps (see Section 1.3)). *A priori*, there is no map in the other direction, since one cannot attach a lattice in $\mathbb{C}$ to every elliptic curve over a $\mathbb{C}$-algebra. As it turns out, this map is bijective and generalizes to higher level (see [Kat76, Sec.2.4]). However, this natural map does not quite preserve q-expansions[1] . For this reason, consider instead the normalized map which sends

$$f \in V_k^{\mathrm{alg}}(\mathbb{C}, \Gamma(N)^{\mathrm{arith}})$$

---

[1] In [Kat76, Sec.2.4], Katz mentions that this map is a q-expansion preserving bijection. This is because his definition of the q-expansion of a classical modular form differs from ours. We prefer the definition given here since it is the most widely used. For example, according to Katz's definition, the Eisenstein series $G_4(\tau) = \sum_{m,n}(m\tau + n)^{-4}$ has q-expansion with rational coefficients.

to the function

$$f^{\mathrm{an}} : \mathrm{GL}^{+} \longrightarrow \mathbb{C}$$

defined as

$$f^{\mathrm{an}}(\omega_1;\omega_2) = (2\pi i)^{-k} f\left(\mathbb{C}/[\omega_1,\omega_2], dz, \beta(e^{2\pi i a/N}, b) = \frac{a\omega_1 + b\omega_2}{N} \mod L\right) \tag{6.14}$$

(note the power of $2\pi i$). Then f^{an} is $\Gamma(N)$ invariant and homogeneous of weight k. Moreover, one has the following

Proposition 15. *Let R be a subring of $\mathbb{C}$. Then the above map $f \mapsto f^{an}$ induces a q-expansion preserving bijection between $V_k^{alg}(R, \Gamma(N)^{arith})$ and the set of weakly holomorphic modular forms in $M_k^!(\Gamma(N))$ whose q-expansion coefficients are in R.*

In particular, if the q-expansion of a classical modular form f is defined over R and the elliptic curve $\mathbb{C}/[\omega_1;\omega_2]$ is defined over R (i.e. $g_2(\omega_1;\omega_2), g_3(\omega_1;\omega_2) \in R$), then

$$(2\pi i)^k f(\omega_1;\omega_2) \in R.$$

Proof. The first part of the proposition is a restatement of [Kat76, Sec.2.4]. To prove the second statement, note that if $f \in M_k(\Gamma(N))$ corresponds to $f^{\mathrm{alg}} \in V_k^{\mathrm{alg}}(R, \Gamma(N)^{\mathrm{arith}})$, then

$$(2\pi i)^k f(\omega_1;\omega_2) = f^{\mathrm{alg}}\left(\mathbb{C}/[\omega_1;\omega_2], dz, \beta\right) \in R,$$

by the definition of algebraic modular forms. □

By defining

$$M_k^!(R, \Gamma(N)) = \{f \in M_k^!(\Gamma(N)) : f(q) \in R((q))\}, \tag{6.15}$$

one can interpret this proposition as saying that the map

$$V_k^{\mathrm{alg}}(R, \Gamma(N)^{\mathrm{arith}}) \longrightarrow M_k^!(R, \Gamma(N))$$

defined above is a q-expansion preserving bijection.

6.5.2 Algebraic and generalized p-adic modular forms

The following proposition says that there is a q-expansion preserving map

$$V_k^{\mathrm{alg}}(B, \Gamma(N)^{\mathrm{arith}}) \longrightarrow V^{\mathrm{gen}}(B, \Gamma(N)^{\mathrm{arith}})$$

which also preserves the nebentypus (see [Kat76, Sec.5.4] for a definition of the map).

Proposition 16. *Let B be a p-adic ring and let $f \in V_k^{alg}(B, \Gamma(N)^{arith})$ be an algebraic modular form of nebentypus $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow B^\times$. Then the corresponding generalized p-adic modular form has weight k and nebentypus χ, i.e. it has weight $\kappa = \kappa_k \cdot \chi_0$, where $\chi_0 : G(N) \longrightarrow B^\times$ sends (a, b) to $\chi(b)$ and $\kappa_k(a, b) = a^k$.*

Proof. See [Kat76, Lem.5.4.10]. □

6.6 p-adic Eisenstein series

Recall (see (1.24)) that the Eisenstein series E_k has q-expansion

$$E_k(q) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \qquad \text{for } k \geq 4,$$

and so it follows directly from Proposition 15 that there exist algebraic modular forms with those q-expansions. In weight 2, the Eisenstein series

$$E_2(\tau) = \frac{1}{8\pi \Im(\tau)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

of Section 1.4 is not holomorphic on $\mathcal{H}$ and so one cannot apply the same reasoning. However, one has the following

Proposition 17. *Let $k \geq 2$. Then there exists a p-adic modular form*

$$E_k \in V_k^{gen}(\mathbb{Z}_p, \Gamma(1)) \otimes \mathbb{Q}_p$$

such that

$$E_k(\mathit{Tate}(q), \varphi_{can}, \beta_{can}) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Proof. For $k > 2$, this follows from the above discussion and Proposition 16.

For $k = 2$, the proof is more involved and the reader is referred to [Kat76, Lem.5.7.8] for an algerbo-geometric definition of E_2. $\square$

Note in particular that the p-adic modular form $E_2 \in V_2^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(1)) \otimes \mathbb{Q}_p$ has q-expansion

$$E_2(q) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

This abuse of notation (using the symbol E_k to denote classical and p-adic Eisenstein series) should not cause any confusion; the Eisenstein series considered will be clear from the context and the expression $E_2(q)$ will always be equal to the last equation.

6.7 Changing the level at p

It follows from the previous results that most of the modular forms considered so far can be viewed as generalized p-adic modular forms. From now on, generalized p-adic modular forms will be simply called p-adic modular forms.

The following proposition is characteristic of the p-adic theory of modular forms.

Proposition 18. *Let B be a p-adic ring, let $N_0 \geq 1$ be an integer coprime to p and let $r \geq 0$. Then there is a $G(p^r N_0) \cong G(N_0)$ equivariant, q-expansion preserving isomorphism*

$$V^{gen}(B, \Gamma(p^r N_0)^{arith}) \cong V^{gen}(B, \Gamma(N_0)^{arith}).$$

Proof. See the discussion below and [Kat76, Sec.5.6] for a proof. $\square$

To see why such an isomorphism should exist, consider the following example. First, it follows from the results of Section 1.5 that the classical modular form $E_k^{(p)}$ defined as

$$E_k^{(p)}(q) = E_k(q) - p^{k-1}V_pE_k(q) = (1-p^{k-1})\frac{-B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}^{(p)}(n)q^n,$$

where $\sigma_{k-1}^{(p)}(n) = \sum_{p\nmid d|n} d^{k-1}$, has weight k for $\Gamma_0(p)$, hence also for $\Gamma(p)$. For most primes p, there exists a p-adic modular form in

$$V_k^{\text{gen}}(\mathbb{Z}_p, \Gamma(p)^{\text{arith}})$$

with q-expansion $E_k^{(p)}(q)$. By the previous proposition, it follows that $E_k^{(p)}$ can be seen as a p-adic modular form of weight k and level $\Gamma(1)^{\text{arith}}$. In this case, one can prove this directly. To do so, let k_i be a sequence of even integers which tend to ∞ in $\mathbb{R}$ and to k in $G(1) = \mathbb{Z}_p^\times$. Then for any $n \geq 1$, one sees that

$$\lim_{i\to\infty}\sigma_{k_i-1}(n) = \sigma_{k-1}^{(p)}(n)$$

in $\mathbb{Q}_p$. Moreover, it is also true that

$$\lim_{i\to\infty}(1-p^{k_i-1})\frac{-B_{k_i}}{k_i} = \lim_{i\to\infty}\zeta_p(1-k_i) = \zeta_p(1-k) = (1-p^{k-1})\zeta(1-k),$$

by the continuity of the Kubota-Leopoldt p-adic L-function. It follows that

$$\lim_{i\to\infty}E_{k_i}(q) = E_k^{(p)}(q)$$

in $\mathbb{Z}_p[[q]]\otimes\mathbb{Q}_p$. Viewing each $E_{k_i}(q)$ as an element of

$$V^{\text{gen}}(\mathbb{Z}_p, \Gamma(1)^{\text{arith}})\otimes\mathbb{Q}_p$$

and using the q-expansion principle, it follows that $E_k^{(p)}$ can also be viewed as a p-adic modular form of level 1.

6.8 Operators on generalized p-adic modular forms

In the classical case, the operator V_p sends a modular form

$$f(q) \in M_k(\Gamma_0(N))$$

to the modular form

$$V_p f(q) = f(q^p) \in M_k(\Gamma_0(pN)).$$

In the p-adic theory of modular forms,

$$V_\kappa^{\text{gen}}(B, \Gamma(N)^{\text{arith}}) \cong V_\kappa^{\text{gen}}(B, \Gamma(pN)^{\text{arith}}),$$

and so it seems like a p-adic analogue of the V_p operator, if it exists, should preserve the level p-adically. This is indeed the case.

Proposition 19. *Let $N \geq 1$ be an integer and let B be a p-adic ring. There exists an endomorphism V_p of*

$$V_\kappa^{gen}(B, \Gamma(N)^{arith})$$

which acts on q-expansions as $V_p f(Tate(q^N), \omega_{can}, \beta_{can}) = f(Tate(q^{pN}), \omega_{can}, \beta_{can})$ (i.e. $V_p f(q) = f(q^p)$).

Proof. See [Kat76, Sec.5.5]. $\square$

The operator V_p is also called the Frobenius operator, since in characteristic p (e.g. if $B = \mathbb{F}_p$) it satisfies the equation

$$V_p f(q) = f(q^p) = f(q)^p.$$

6.9 Theta operator on generalized p-adic modular forms

In the classical case, recall that the Shimura-Maass operator was defined in (1.71) as

$$\delta_k f(\tau) = \frac{1}{2\pi i}\left(\frac{\partial f(\tau)}{\partial z} + \frac{kf(\tau)}{\tau - \bar{\tau}}\right)$$

on weight k modular forms. Recall also that it increases the weight by 2 and does not preserve holomorphicity. However, it preserves the ring of nearly holomorphic modular forms, which in level one is simply

$$\mathbb{C}[E_2, E_4, E_6].$$

The analogue of this differential operator in the p-adic theory is Serre's *theta operator*, which will be denoted D_N.

Proposition 20. *Let $N \geq 1$ be an integer. There exists an endomorphism D_N of*

$$V^{gen}(\mathbb{Z}_p, \Gamma(N)^{arith})$$

which acts on q-expansions as $q\frac{d}{dq}$, i.e. it makes the following diagram commutative:

$$\begin{tikzcd} V^{gen}(\mathbb{Z}_p, \Gamma(N)^{arith}) \arrow[r, "D_N"] \arrow[d] & V^{gen}(\mathbb{Z}_p, \Gamma(N)^{arith}) \arrow[d] \\ \widehat{\mathbb{Z}_p((q))} \arrow[r, "q\frac{d}{dq}"] & \widehat{\mathbb{Z}_p((q))} \end{tikzcd},$$

where $\widehat{\mathbb{Z}_p((q))}$ is the p-adic completion of $\mathbb{Z}_p((q))$. Moreover, it is of weight 2, in the sense that

$$[a, b] \circ D_N = a^2 D_N \circ [a, b]$$

for all $(a, b) \in G(N)$.

Proof. The proof involves an analysis of the Gauss-Manin connection on the DeRham cohomology of the universal elliptic curve over $V^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(N)^{\mathrm{arith}})$ and will not be given here. See [Kat76, Sec.5.8]. $\square$

CHAPTER 7
Complex multiplication from an algebraic point of view

Recall that in this text, K denotes an imaginary quadratic field and $\mathcal{O}_K$ denotes its ring of integers.

In Chapter 3, the analytic theory of complex multiplication was introduced. In particular, CM points in $\mathcal{L}$ were defined as the set of lattices of the form

$$\Omega\mathfrak{a},$$

where $\Omega \in \mathbb{C}^\times$ and $\mathfrak{a}$ is a fractional ideal in K. One may wonder if the associated elliptic curve

$$\mathbb{C}/\Omega\mathfrak{a}$$

has special properties. To answer this, first note that if

$$\mathbb{C}/L \xrightarrow{\sim} E(\mathbb{C}) \text{ and } \mathbb{C}/L' \xrightarrow{\sim} E'(\mathbb{C}),$$

then any isogeny $\phi : E \longrightarrow E'$ over $\mathbb{C}$ corresponds to a unique $\lambda \in \mathbb{C}^\times$ such that

$$\lambda L \subseteq L'$$

(see [Sil09, Thm.4.1, Ch.VI]). In particular,

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}/L) = \{\lambda \in \mathbb{C} : \lambda L \subseteq L\}$$

and so

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}/\Omega\mathfrak{a}) \supseteq \mathcal{O}_K.$$

In other words, the elliptic curve $\mathbb{C}/\Omega\mathfrak{a}$ has *extra endomorphisms* (other than the trivial ones).

Conversely, one can show that if $L \in \mathcal{L}$ and $\lambda \in \mathbb{C}^\times$ are such that

$$\lambda L \subseteq L,$$

then λ is an element in an order in a quadratic imaginary field. Therefore the endomorphism ring of elliptic curves over $\mathbb{C}$ is either $\mathbb{Z}$ or an order in an imaginary quadratic field. In the later case, the elliptic curve is called a *CM elliptic curve* or is said to have complex multiplication.

It follows from the above discussion that the set of elliptic curves over $\mathbb{C}$ with CM by the maximal order $\mathcal{O}_K$ of an imaginary quadratic field K is in bijection with the CM points of $\mathcal{L}$ corresponding to K (as defined in Chapter 3). It also follows that there are exactly h_K isomorphism classes (over $\mathbb{C}$) of elliptic curves with CM by $\mathcal{O}_K$. See [Sil94, Ch.II] for a detailed exposition of the theory of CM elliptic curves.

7.1 ★ CM values of algebraic modular forms

The first step in passing from the classical theory of complex multiplication to the algebraic one was to go from lattices in $\mathbb{C}$ to framed elliptic curves. The next step is to analyse the field of definition of CM elliptic curves.

Proposition 21. *Let E be an elliptic curve over $\mathbb{C}$ with CM by the maximal order $\mathcal{O}_K$. Then*

1. *E is isomorphic over $\mathbb{C}$ to an elliptic curve defined over H (the Hilbert class field of K);*
2. *Suppose that E is defined over a number field F. Then every endomorphism of E is defined over FK, the compositum of F and K.*

Proof. The first point follows from the fact that the j-invariant of CM elliptic curves is an algebraic integer in the Hilbert class field of K (see Proposition 10).

Then second point is proved in [Sil94, Ch.II, Thm.2.2], for example. □

Now let $(E_{/F}, \omega, \beta)$ be a $\Gamma(N)^{\mathrm{arith}}$-test object over a number field F and suppose that E has complex multiplication by $\mathcal{O}_K$. Suppose also that the complex multiplication endomorphisms are defined over F. Then one has the following

Proposition 22. *Let $f \in M_k(F, \Gamma(N))$ be a classical modular form with Fourier coefficients in F and let $(E_{/F}, \omega, \beta)$ be a $\Gamma(N)^{arith}$-test object as above. Then*

$$(2\pi i)^{2a+2b+k} \delta_{2b+k}^{a} E_2^b f(\omega_1; \omega_2) \in F$$

for any $a, b \geq 0$, where $(\omega_1; \omega_2)$ is the point of $\mathcal{L}_N = \Gamma(N) \setminus GL^+$ corresponding to the test object $(E_{/F}, \omega, \beta)$ base changed to $\mathbb{C}$.

Note that this Proposition could equivalently have been stated in terms of algebraic modular forms.

Proof. If $a = b = 0$, this follows directly from the correspondence between algebraic and classical modular forms and the definition of algebraic modular forms.

The main difficulty, and the only place where the hypothesis of complex multiplication is used, is in proving that

$$(2\pi i)^2 E_2(\omega_1; \omega_2)$$

lies in F. This is proved in [Kat76, Thm.4.0.4] (note that Katz's function S corresponds to the function $24(2\pi i)^2 E_2$ in our notation). □

In particular, one has the following

Corollary 8. *Let $f \in F[E_2, E_4, E_6]$ be a nearly holomorphic modular form of weight k with Fourier coefficients in a number field F containing the Hilbert class field H of K, and let*

$\mathfrak{a}$ be a fractional ideal of K. Then there exists a complex number $\Omega_{\mathfrak{a}}$ depending only on $\mathfrak{a}$ such that

$$f(\mathfrak{a}) \in (2\pi i \Omega_{\mathfrak{a}})^{-k} F.$$

Proof. The elliptic curve

$$\mathbb{C}/\mathfrak{a}$$

is isomorphic over $\mathbb{C}$ to an elliptic curve $(E_{/F}, \omega)$ defined over F (since F contains the Hilbert class field of K). Let $L \in \mathcal{L}$ be the lattice corresponding to the complexification of $E_{/F}$. Then there exists $\Omega_{\mathfrak{a}} \in \mathbb{C}^{\times}$ such that

$$\mathfrak{a} = \Omega_{\mathfrak{a}} L.$$

The claim follows from the equality

$$(2\pi i \Omega_{\mathfrak{a}})^k f(\mathfrak{a}) = (2\pi i)^k f(L)$$

and the previous proposition. □

This Corollary may look stronger than Proposition 6, since the number field in which f takes values at CM points is known. However, this is not really the case, since the extra ambiguity in Proposition 6 (coming from the fact that the values were only required to be algebraic) allowed us to choose the period Ω_K *explicitly* (using the closed formula in the Chowla-Selberg formula) and *independently* of the fractional ideal $\mathfrak{a}$.

7.2 CM values of p-adic modular forms

As above, let $(E_{/F}, \omega, \beta)$ be $\Gamma(N)^{\text{arith}}$-test object with CM. In this section, fix a prime $\mathfrak{P}$ of F and let p be the rational prime which it divides in $\mathcal{O}_F$. In order to talk about "CM values" of p-adic modular forms, one must make a few assumptions about the test objects. First, suppose that the test object $(E_{/F}, \omega, \beta)$ has good reduction at $\mathfrak{P}$, in the sense that there exists a $\Gamma(N)^{\text{arith}}$-test object over the ring $\mathcal{O}_{F,\mathfrak{P}}$ of $\mathfrak{P}$-integers of $\mathcal{O}_F$ which gives $(E_{/F}, \omega, \beta)$ after extension of scalars from $\mathcal{O}_{F,\mathfrak{P}}$ to F. Second, suppose that E has ordinary reduction at $\mathfrak{P}$ (note that this hypothesis is equivalent to p being split in K).

Given such a test object $(E_{/F}, \omega, \beta)$ and an embedding of F in $\mathbb{C}$, one can attach an element $(\omega_1; \omega_2) \in \mathcal{L}_N$ to its complexification. On the other hand, one can consider it as a test object $(E_{/F_{\mathfrak{P}}}, \omega, \beta)$ over the $\mathfrak{P}$-adic completion $F_{\mathfrak{P}}$ of F at $\mathfrak{P}$. On the modular forms side, any algebraic modular form $f \in M_k(F, \Gamma(N)^{\text{arith}})$ can be considered as a classical modular form with Fourier coefficients in F or as p-adic modular form in $V_k^{\text{gen}}(\mathbb{Z}_p, \Gamma(N)^{\text{arith}}) \otimes F_{\mathfrak{P}}$. Then one has the following

Theorem 16. *With the hypotheses and notations as above, the quantity*

$$D_N^a E_2^b f(E_{/F_{\mathfrak{P}}}, \omega, \beta) \in F_{\mathfrak{P}}$$

actually lies in F and is equal to the quantity

$$(2\pi i)^{2a+2b+k} \delta_{2b+k}^a E_2^b f(\omega_1; \omega_2) \in F.$$

Proof. If $a = b = 0$, this is clear.

The main difficulty is to compare the complex value

$$(2\pi i)^2 E_2(\omega_1; \omega_2)$$

with the $\mathfrak{P}$-adic value

$$E_2(E_{/F_{\mathfrak{P}}}, \omega, \beta),$$

where now E_2 is the p-adic modular form defined in Section 6.6. For a proof of this, see [Kat76, Thm.8.0.9]. $\square$

CHAPTER 8
p-adic interpolation of Petersson inner product of theta series

In this chapter, a few p-adic measures are constructed, including one with values in the p-adic ring of generalized p-adic modular forms and one with values in a subring of $\mathbb{C}_p$. The second measure naturally gives rise to a $\mathbb{C}_p$-valued analytic function on the p-adic weight space

$$\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times).$$

By evaluating this function at a point outside the range of interpolation, namely at $\kappa = -1$, a p-adic analogue of the Petersson inner product of two weight one theta series attached to ideals is found. This result can be interpreted as a p-adic analogue of Kronecker's first limit formula.

8.1 Review of p-adic integration theory

Part of the material of this section is contained in [Kat76, Sec.6.0]. Other, more detailed, references are [Col04, Sec.1.4] and [MSD74, Ch.7].

As before, let B be a p-adic ring and let

$$G = \varprojlim_n G/G_n,$$

where

$$G_1 \supseteq \cdots \supseteq G_n \cdots \supseteq \cap G_n = \{1\},$$

be a profinite group (e.g. $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$).

Define a B-*valued measure on* G as a B-linear map

$$\mu : \mathcal{C}^0(G, B) \longrightarrow B,$$

where $\mathcal{C}^0(G, B)$ is the ring of B-valued continuous functions on G. Note that μ is automatically continuous ($\mathcal{C}^0(G, B)$ is equipped with the p-adic topology). The value of μ on $f \in \mathcal{C}^0(G, B)$ is denoted

$$\int_G f d\mu \qquad \text{or} \qquad \int_G f(g) d\mu(g).$$

Equivalently, one could define a B-valued measure as a finitely additive map μ on compact open subsets of G, i.e. as a B-*valued distribution on* G. Indeed, any such distribution extends uniquely to a B-linear map

$$\mu : LC(G, B) \longrightarrow B,$$

where $LC(G, B)$ is the ring of locally constant B-valued functions on G. Using the density of $LC(G, B)$ in $\mathcal{C}^0(G, B)$, one can then show that μ extends uniquely to a B-valued measure on G.[1]

8.1.1 Integration over $\mathbb{Z}_p$

When $G = \mathbb{Z}_p$, one has a lot more information about the set of B-valued measure on $\mathbb{Z}_p$, thanks to the structure theorem for $\mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p)$.

---

[1] Note that B is simply assumed to be a p-adic ring and so it is not necessarily profinite or equipped with an absolute value. In particular, one cannot define p-adic measures as bounded p-adic distributions, as in the classical theory. However, the density of $LC(G, B)$ in $\mathcal{C}^0(G, B)$ and the extension property can still be proved using only the fact that B is complete and separated with respect to its p-adic topology.

Theorem 17 (Mahler's Theorem). *Any $f(x) \in \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p)$ can be uniquely written as*

$$f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n},$$

where

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!} \qquad \text{for all } n \geq 0$$

and

$$a_n(f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(n-i) \underset{n \longrightarrow \infty}{\longrightarrow} 0 \qquad \text{in } \mathbb{Z}_p.$$

Conversely, for any sequence $\{a_n\}$ of p-adic integers tending to 0 p-adically, the series

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

converges for $x \in \mathbb{Z}_p$ and defines an element $f(x) \in \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p)$ which is such that

$$a_n(f) = a_n \qquad \text{for all } n \geq 0,$$

where $a_n(f)$ is defined as above.

Proof. This theorem is contained in many places. See [Col04, Thm.1.3.2], for example. □

By tensoring with B over $\mathbb{Z}_p$ and using the fact that

$$\mathcal{C}^0(\mathbb{Z}_p, B) = \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p) \widehat{\otimes}_{\mathbb{Z}_p} B = \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B,$$

one has the following corollary (this is [Kat76, 6.0.3]).

Corollary 9. *Suppose that B is flat over $\mathbb{Z}_p$ (equivalently, B is of characteristic 0). Then any $f(x) \in \mathcal{C}^0(\mathbb{Z}_p, B)$ can be uniquely written as*

$$f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n},$$

where
$$a_n(f) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}f(n-i) \underset{n\longrightarrow\infty}{\longrightarrow} 0 \qquad \textit{in } B,$$
and conversely (as above).

This structure theorem for $\mathcal{C}^0(\mathbb{Z}_p, B)$ allows one to formally attach the power series
$$\mathcal{A}_\mu(T) = \int_{\mathbb{Z}_p}(1+T)^x d\mu(x) = \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}\binom{x}{n}d\mu(x)\right)T^n \in B[[T]]$$
to any B-valued measure μ on $\mathbb{Z}_p$. This map is sometimes called the *Amice transform*.

In fact, the Amice transform is more than a formal link between measures and power series. To illustrate that, let $z \in 1+pB$ and consider the series
$$f_z(x) = \sum_{n=0}^{\infty}(z-1)^n\binom{x}{n}.$$
By Mahler's theorem, this series defines a B-valued continuous function on $\mathbb{Z}_p$, which is such that
$$f_z(k) = z^k \qquad \text{for all } k \in \mathbb{Z}_{\geq 0}.$$
Because f_z satisfies the above interpolation property, $f_z(x)$ is denoted z^x. Using this notation, one has the following

Lemma 3. *Let B be a p-adic ring which is flat over $\mathbb{Z}_p$, let $z \in 1+pB$ and let $\mu \in \mathcal{D}^0(\mathbb{Z}_p, B)$. Then*
$$\mathcal{A}_\mu(z-1) = \int_{\mathbb{Z}_p} z^x d\mu(x).$$
Proof. This is a generalization of [Col04, Lem.1.4.3]. By the normal convergence of the series defining the function z^x (i.e. for any $M \in \mathbb{Z}_{\geq 0}$, there exists an index n_M such that $\sum_{n=n_M}^{\infty}(z-1)^n\binom{x}{n} \in p^M B$), one can interchange the sum and the integral in the expression
$$\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}(z-1)^n\binom{x}{n}d\mu(x).$$

□

Now suppose that B is flat over $\mathbb{Z}_p$ and for every integer $n \geq 0$ define $c_k(n) \in \mathbb{Z}_p$ for $1 \leq k \leq n$ in such a way that

$$\binom{x}{n} = \sum_{k=0}^{n} c_k(n) x^k.$$

It is then clear that a measure μ on $\mathbb{Z}_p$ is determined by its *moments*

$$m_k(\mu) = \int_{\mathbb{Z}_p} x^k d\mu, \qquad \text{for } k \in \mathbb{Z}_{\geq 0}.$$

Conversely, one has the following

Lemma 4. *Let B be a p-adic ring which is flat over $\mathbb{Z}_p$ and let $\{m_k\}_{k\geq 0}$ be a sequence of elements of B. Then there exists a measure $\mu \in \mathcal{D}^0(\mathbb{Z}_p, B)$ with moments $m_0, m_1, \ldots$ if and only if*

$$\sum_{k=0}^{n} c_k(n) m_k \in B$$

for every $n \geq 0$. If it exists, the measure is unique.

Proof. This is [Kat76, Lem.6.0.9]. □

8.1.2 Integration over $\mathbb{Z}_p^\times$

When p is an odd prime, the natural exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \mathbb{Z}_p^\times \longrightarrow \mathbb{F}_p^\times = \mu_{p-1} \longrightarrow 1,$$

where μ_{p-1} is the group of $p-1$ roots of unity and $\Gamma = 1+p\mathbb{Z}_p$, gives a canonical isomorphism

$$\mathbb{Z}_p^\times \simeq \mu_{p-1} \times \Gamma. \tag{8.1}$$

If $x \in \mathbb{Z}_p^\times$, let $\omega(x)$ and $\langle x \rangle$ denote the images of x under the projection to the first and second factors of the isomorphism (8.1), respectively.

Recall (Subsection 6.3.1) that the weight of a generalized p-adic modular form of level one with coefficients in B is by definition a continuous character

$$\kappa : \mathbb{Z}_p^\times \longrightarrow B^\times.$$

The space of all such weights

$$\mathcal{W}(B) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, B^\times)$$

is called the *weight space* (with values in B). In the case where $B = \mathbb{Z}_p$, this space is simply denoted $\mathcal{W}$.

Using the decomposition (8.1), one sees that any element of $\mathcal{W}$ is of the form

$$\kappa_{i,s}(x) = \omega^i(x)\langle x\rangle^s$$

for some $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $s \in \mathbb{Z}_p$. It follows that $\mathcal{W}$ decomposes as a direct product of $p-1$ copies of $\mathbb{Z}_p$, indexed by the characters of μ_{p-1}:

$$\kappa_{i,s} \mapsto (i,s) : \mathcal{W} \xrightarrow{\sim} (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p. \tag{8.2}$$

In fact, this decomposition also respects the group operations on both sides. The natural injection from $\mathbb{Z}$ into $\mathcal{W}$, which sends the integer k to the character

$$\kappa_k(x) = \kappa_{k,k}(x) = x^k,$$

then corresponds to the natural map from $\mathbb{Z}$ into the product decomposition (8.2). This injection induces a topology on $\mathbb{Z}$ which makes it dense in $\mathcal{W}$.[2] The weights in the image of $\mathbb{Z}$ are called *classical weights*.

Similarly to the case of $\mathbb{Z}_p$, the space $\mathcal{W}(B)$ decomposes as a product of copies of $B^\times$ indexed by

$$\mu^*_{p-1}(B) = \mathrm{Hom}(\mu_{p-1}, B^\times).$$

Now let

$$\mu \in \mathcal{D}^0(\mathbb{Z}_p^\times, B)$$

be a measure on $\mathbb{Z}_p^\times$. By mapping $\mathcal{W}$ to $\mathcal{C}^0(\mathbb{Z}_p^\times, B)$, one obtains a continuous function on weight space

$$L_\mu : \mathcal{W} \longrightarrow B$$

whose values at integers $k \geq 0$ are the moments of μ:

$$L_\mu(k) = L_\mu(\kappa_k) = \int_{\mathbb{Z}_p^\times} x^k d\mu.$$

Then one has the following

Proposition 23. *Let $\mu \in \mathcal{D}^0(\mathbb{Z}_p^\times, B)$ be a p-adic measure, let $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ and let $u = 1 + p$. Then there exists a unique measure $\Gamma_\mu^{(i)} \in \mathcal{D}^0(\mathbb{Z}_p, B)$ on $\mathbb{Z}_p$ such that*

$$\int_{\mathbb{Z}_p^\times} \omega^i(x)\langle x\rangle^s d\mu(x) = \int_{\mathbb{Z}_p} u^{sy} d\Gamma_\mu^{(i)}(y)$$

for all $s \in \mathbb{Z}_p$.

[2] This induced topology is the one where two integers are close if they are p-adically close *and* congruent $\mod p-1$.

Proof. This is a straightforward generalization of the proof of [Col04][Prop.1.5.9] to the case of measures with values in p-adic rings. $\square$

The measure $\Gamma_\mu^{(i)}$ is sometimes called Leopoldt's gamma transform. In the notation of the previous proposition, one sees using Lemma 3 that

$$\int_{\mathbb{Z}_p} u^{sy} d\Gamma_\mu^{(i)}(y) = \mathcal{A}_{\Gamma_\mu^{(i)}}(u^s - 1), \qquad \text{for all } s \in \mathbb{Z}_p.$$

It follows that the function F_μ has a power series expansion in s on each of the $p-1$ components of $\mathcal{W}$. Indeed, for each fixed $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ one has

$$L_\mu(\kappa_{i,s}) = \int_{\mathbb{Z}_p^\times} \omega^i(x)\langle x\rangle^s d\mu(x) = \mathcal{A}_{\Gamma_\mu^{(i)}}(u^s - 1),$$

which can be expressed as a power series in s (since $u^s - 1 = sv + s^2v^2/2 + \dots$, where $v = \log u$). Such functions on $\mathbb{Z}_p$ (or $\mathcal{W}$) are called p-adic analytic.

To summarize, p-adic measures on $\mathbb{Z}_p^\times$ naturally give rise to continuous functions on weight space, which in turn can be viewed as a finite collection of p-adic analytic functions on $\mathbb{Z}_p$ which interpolate the moments of the measure.

8.2 Construction of measures with values in the ring of generalized p-adic modular forms

From now on, suppose that p is different from 2 and 3 and let V denote the p-adic ring of generalized p-adic modular forms of level one:

$$V = V^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(1)^{\mathrm{arith}}).$$

8.2.1 The measure μ_E on $\mathbb{Z}_p$ with values in V

One has the following

Lemma 5. *There exists a unique measure μ_E on $\mathbb{Z}_p$ with values in V whose moments are given by*

$$\int_{\mathbb{Z}_p} x^k d\mu_E = D_1^k E_2 \qquad \text{for all } k \geq 0.$$

Proof. By Lemma 4, it suffices to show that

$$\sum_{k=0}^{n} c_k(n) D_1^k E_2 \in V$$

for all $n \geq 0$. But an easy computation using the q-expansion of $D_1^k E_2$ shows those p-adic modular forms have q-expansions

$$\sum_{m=1}^{\infty} \binom{m}{n} \sigma(m) q^m \qquad \text{if } n > 0$$

and

$$E_2(q) = -\frac{1}{24} + \sum_{m=1}^{\infty} \sigma_1(m) q^m \qquad \text{if } n = 0,$$

which are in V (i.e. their q-expansions have coefficients in $\mathbb{Z}_p$). Note that the assumption $p \neq 2, 3$ is needed when $n = 0$. □

The calculation done in the proof of this lemma can be used to prove the following

Proposition 24. *Let μ_E be the measure of Lemma 5 and let $f \in \mathcal{C}^0(\mathbb{Z}_p, V)$. Then*

$$\int_{\mathbb{Z}_p} f d\mu_E = -\frac{f(0)}{24} + \sum_{n=1}^{\infty} f(n)\sigma(n) q^n.$$

Proof. The proof of Lemma 5 shows that the statement is true for the functions $\binom{x}{n} \in \mathcal{C}^0(\mathbb{Z}_p, V)$. The result then follows from Corollary 9, the fact that

$$a_0(f) = f(0)$$

and the continuity of μ_E. □

8.2.2 Restriction of μ_E to $\mathbb{Z}_p^\times$

Since integration over $\mathbb{Z}_p^\times$ gives rise to analytic functions on weight space, it is desirable to restrict μ_E to $\mathbb{Z}_p^\times$. Define

$$\mu_E^{[p]} \in \mathcal{D}^0(\mathbb{Z}_p^\times, V)$$

as

$$\int_{\mathbb{Z}_p^\times} g(x) d\mu_E^{[p]}(x) = \int_{\mathbb{Z}_p} g_0(x) d\mu_E(x) = \int_{\mathbb{Z}_p} g_0(x)\chi_{\mathbb{Z}_p^\times}(x) d\mu_E(x),$$

where $g \in \mathcal{C}^0(\mathbb{Z}_p^\times, V)$ and $g_0 \in \mathcal{C}^0(\mathbb{Z}_p, V)$ is g extended by 0 on $p\mathbb{Z}_p$. Then one has the following

Proposition 25. *Let $\mu_E^{[p]}$ be the measure defined above. Then*

$$\int_{\mathbb{Z}_p^\times} x^k d\mu_E^{[p]} = D_1^k E_2^{[p]},$$

where the p-adic modular form $E_2^{[p]}$ with q-expansion

$$E_2^{[p]}(q) = \sum_{(n,p)=1} \sigma(n) q^n$$

is the p-depletion of $E_2(q)$.

Proof. This follows directly from the definition of $\mu_E^{[p]}$ and Proposition 24. □

That $E_2^{[p]}$ is a p-adic modular form of level $\Gamma(1)^{\text{arith}}$ follows formally from the previous proposition, since $\mu_E^{[p]}$ is a p-adic measure with values in V. One can also prove directly the following

Lemma 6. *As in the previous proposition, let*

$$E_2^{[p]}(q) = \sum_{(n,p)=1} \sigma(n) q^n.$$

Then if

- $$E_2(\tau) = \frac{1}{8\pi\Im(\tau)} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n$$

  is the nearly holomorphic modular form of Chapter 1, one has the equality

  $$E_2^{[p]}(q) = E_2(\tau) - (p+1)V_pE_2(\tau) + pV_p^2E_2(\tau),$$

  where $q = e^{2\pi i\tau}$. Moreover, $E_2^{[p]}(q)$ is a holomorphic modular form in $M_2(\Gamma_0(p^2))$.

- $$E_2(q) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)q^n$$

  is the p-adic modular form of Chapter 6, one has the equality

  $$E_2^{[p]}(q) = E_2(q) - (p+1)V_pE_2(q) + pV_p^2E_2(q).$$

  Moreover, $E_2^{[p]}(q)$ is a p-adic modular form in $V_2^{gen}(\mathbb{Z}_p, \Gamma(1)^{arith})$.

Proof. For the first point, note that the non-holomorphic parts in the right hand side of the equation cancel. The claim then follows from a computation on q-expansions and the general fact that the V_p operator sends weight k functions on $\mathrm{SL}_2(\mathbb{Z})$ to weight k functions on $\Gamma_0(p^2)$.

The second point follows from the same computation on q-expansions and the fact that the V_p operator preserves the level p-adically. $\square$

More generally, one sees that

$$\int_{\mathbb{Z}_p^\times} x^\kappa d\mu_E^{[p]} = D_1^\kappa E_2^{[p]},$$

where D_1^κ is defined as

$$D_1^\kappa E_2^{[p]}(q) = \sum_{(n,p)=1} n^\kappa \sigma(n) q^n$$

and $x^\kappa = \kappa(x)$ denotes the image of $x \in \mathbb{Z}_p^\times$ under $\kappa \in \mathcal{W}$. It then follows from the theory exposed above that the measure $\mu_E^{[p]}$on $\mathbb{Z}_p^\times$ induces a p-adic analytic function

$$L_{\mu_E^{[p]}} : \mathcal{W} \longrightarrow V,$$

which is defined as

$$L_{\mu_E^{[p]}}(\kappa) = D_1^\kappa E_2^{[p]}. \tag{8.3}$$

8.3 Construction of a measure with values in $\mathbb{C}_p$

Our goal in this section is to evaluate the p-adic modular forms $L_{\mu_E^{[p]}}(\kappa)$ at CM points and to relate those values to the CM values of the Shimura-Maass derivatives of E_2 which, as was shown in the first part of this thesis, intervene in the formulas for the Petersson inner product of theta series. In order to do so, one has to attach trivialized $\Gamma(1)^{\mathrm{arith}}$ elliptic curves to ideals in quadratic fields.

8.3.1 Trivialized elliptic curves attached to CM elliptic curves

This is well explained and done in more generality in Section 8.3 of [Kat76], so we only recall the main idea here.

As usual, let K be an imaginary quadratic field and let H be its Hilbert class field. From now on, let $\mathfrak{a}$ be a fractional ideal of K and suppose the p splits as

$$p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$$

in K. [3]

[3] Asking for p to split in K is a big restriction, since only half of the prime in $\mathbb{Q}$ do so. However, this restriction is necessary since the CM elliptic curves attached to K are not ordinary at the inert primes of K, which means that they cannot be trivialized. Since the differential operator D_1 does not preserve overconvergence in general, it is impossible to evaluate all the p-adic modular forms $D_1^k E_2^{[p]}$ (for $k \in \mathbb{Z}_{\geq 0}$) at non-ordinary elliptic curves.

Fix an isomorphism

$$\widetilde{\varphi} : \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \bigcup_{n\geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{a}/\mathfrak{a} \tag{8.4}$$

and let

$$H(\bar{\mathfrak{p}}^{\infty}) = \bigcup_{n\geq 1} H(\bar{\mathfrak{p}}^n),$$

where $H(\bar{\mathfrak{p}}^n)$ is the ray class field $\mod \bar{\mathfrak{p}}^n$ over K.

By CM theory, the couple $(\mathfrak{a}, \widetilde{\varphi})$ determines a complex embedding of $H(\bar{\mathfrak{p}}^{\infty})$. [4] Fix a place $\mathfrak{p}_\infty$ of $H(\bar{\mathfrak{p}}^{\infty})$ which divides $\mathfrak{p}$ and let $\mathcal{O}_{\mathfrak{p}_\infty}$ be its valuation ring. Then the elliptic curve over $\mathbb{C}$ determined by $\mathfrak{a}$ has a model $E_{\mathfrak{a}}$ over $H \cap \mathcal{O}_{\mathfrak{p}_\infty}$. Suppose that the action of $\mathcal{O}_K$ on $H^0(\Omega^1)$ is compatible with the inclusion $\mathcal{O}_K \hookrightarrow H \hookrightarrow \mathbb{C}$.

Over $H \cap \mathcal{O}_{\mathfrak{p}_\infty}$, there is a canonical splitting

$$\bigcup_{n\geq 1} \ker(p^n) = \bigcup_{n\geq 1} \ker(\mathfrak{p}^n) \times \bigcup_{n\geq 1} \ker(\bar{\mathfrak{p}}^n).$$

Taking the Cartier dual of $\widetilde{\varphi}$ and using the Weil pairing on p^n torsion, one obtains an isomorphism

$$\varphi : \bigcup_{n\geq 1} \ker(\mathfrak{p}^n) = \bigcup_{n\geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{a}/\mathfrak{a} \xrightarrow{\sim} \mu_{p^\infty} \tag{8.5}$$

Out of the above data, one can extract at least two things. First, the isomorphism

$$\varphi^{-1} \times \widetilde{\varphi} : \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \bigcup_{n\geq 1} \ker(p^n) \tag{8.6}$$

[4] The part of CM theory used to prove such a statement was not covered in this thesis. The missing result is that, roughly speaking, the $\bar{\mathfrak{p}}^n$ torsion points of elliptic curves with CM by $\mathcal{O}_K$ generate the extension $H(\bar{\mathfrak{p}}^n)$.

gives rise to $\Gamma(p^n)^{\text{arith}}$-level structures on $E_{\mathfrak{a}}$ for every $n \geq 0$. Second, passing over to $\widehat{\mathcal{O}}_{\mathfrak{p}_\infty}$, the isomorphism φ becomes equivalent to a trivialization

$$\varphi : \hat{E}_{\mathfrak{a}} \xrightarrow{\sim} \hat{\mathbb{G}}_m.$$

In this way, to any couple $(\mathfrak{a}, \widetilde{\varphi})$ one can attached trivialized $\Gamma(p^n)^{\text{arith}}$ elliptic curves

$$(E_{\mathfrak{a}}, \varphi, \beta_n)_{/\widehat{\mathcal{O}}_{\mathfrak{p}_\infty}}$$

for all $n \geq 0$.

8.3.2 The measure on $\mathbb{Z}_p^\times$ with values in $\mathbb{C}_p$

Let $(\mathfrak{a}, \widetilde{\varphi})$ be as above. By evaluating generalized p-adic modular forms at the trivialized elliptic curve attached to $(\mathfrak{a}, \widetilde{\varphi})$, one obtains a measure valued in the p-adic ring $\widehat{\mathcal{O}}_{\mathfrak{p}_\infty}$. For reasons that will become clear later, it is preferable to view our measures as valued in p-adic modular forms of level p^2 rather than 1. This leads to the definition of

$$L_{\mu_E^{[p]}}((\mathfrak{a}, \widetilde{\varphi}), \kappa) : \mathcal{W} \longrightarrow \widehat{\mathcal{O}}_{\mathfrak{p}_\infty}$$

as the composition

$$\mathcal{W} \longrightarrow \mathcal{C}^0(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \xrightarrow{\mu_E^{[p]}} V^{\text{gen}}(\mathbb{Z}_p, \Gamma(1)^{\text{arith}}) \xrightarrow{\sim} V^{\text{gen}}(\mathbb{Z}_p, \Gamma(p^2)^{\text{arith}}) \xrightarrow{\text{eval}} \widehat{\mathcal{O}}_{\mathfrak{p}_\infty}.$$

This function can be seen as a $\mathbb{C}_p$-valued p-adic analytic function on $\mathcal{W}$.

Our next goal is to relate the values of this function at classical weights to the CM values of some classical modular form, using Theorem 16. In order to do so, it is necessary to go from trivialized elliptic curves to elliptic curves equipped with differentials. As in the previous subsection, we follow [Kat76, Sec.8.3]. Let

$$(E_{\mathfrak{a}}, \varphi, \beta_n)_{/\widehat{\mathcal{O}}_{\mathfrak{p}_\infty}}$$

be a trivialized elliptic curve over $\widehat{\mathcal{O}}_{\mathfrak{p}_\infty}$. Then the trivialization

$$\varphi : \hat{E}_{\mathfrak{a}} \longrightarrow \hat{\mathbb{G}}_m$$

gives rise to the differential $\varphi^*(dT/1+T)$ on $E_{\mathfrak{a}}$ defined over $\widehat{\mathcal{O}}_{\mathfrak{p}_\infty}$ (see [Kat76], right before paragraph 8.3.16, for more details). Since $E_{\mathfrak{a}}$ is defined over $H \cap \mathcal{O}_{\mathfrak{p}_\infty}$, there exists $\Omega_p(\mathfrak{a}) \in \widehat{\mathcal{O}}_{\mathfrak{p}_\infty}^\times$ such that

$$\Omega_p(\mathfrak{a})\varphi^*(dT/1+T)$$

is defined over $\mathcal{O}_{\mathfrak{p}_\infty}$. Using the complex embedding of $\mathcal{O}_{\mathfrak{p}_\infty}$, one can also find $\Omega_{\mathbb{C}}(\mathfrak{a}) \in \mathbb{C}^\times$ such that

$$\Omega_p(\mathfrak{a})\varphi^*(dT/1+T) = \Omega_{\mathbb{C}}(\mathfrak{a})dz.$$

The period lattice of $E_{\mathfrak{a}}$ is then $\Omega_{\mathbb{C}}(\mathfrak{a})\mathfrak{a}$.

Theorem 18. *Let $k \in \mathcal{W}$ be a classical weight. Then using the notation introduced above, one has the following equality in H*

$$\frac{L_{\mu_E^{[p]}}((\mathfrak{a}, \widetilde{\varphi}), k)}{\Omega_p(\mathfrak{a})^{2k+2}} = \frac{\delta^k E_2^{[p]}(\mathfrak{a})}{((2\pi i)^{-1}\Omega_{\mathbb{C}}(\mathfrak{a}))^{2k+2}}. \tag{8.7}$$

Proof. After all this work, the proof is a relatively simple computation using the previous calculations and Theorem 16:

$$\begin{aligned}
L_{\mu_E^{[p]}}((\mathfrak{a}, \widetilde{\varphi}), k) &= D_1^k E_2^{[p]}(E_{\mathfrak{a}}, \varphi, \beta_2)_{/\widehat{\mathcal{O}}_{\mathfrak{p}_\infty}} \\
&= \Omega_p(\mathfrak{a})^{2k+2} D_1^k E_2^{[p]}(E_{\mathfrak{a}}, \Omega_p(\mathfrak{a})\varphi^*(dT/1+T), \beta_2)_{/H\cap\mathcal{O}_{\mathfrak{p}_\infty}} \\
&= \Omega_p(\mathfrak{a})^{2k+2}(2\pi i)^{2k+2}\delta_2^k E_2^{[p]}(\Omega_{\mathbb{C}}(\mathfrak{a})\mathfrak{a}) \\
&= \Omega_p(\mathfrak{a})^{2k+2}(2\pi i)^{2k+2}\Omega_{\mathbb{C}}(\mathfrak{a})^{-(2k+2)}\delta_2^k E_2^{[p]}(\mathfrak{a}).
\end{aligned}$$

Note that the fact that $E_2^{[p]}$ is a *holomorphic* modular form of weight 2 and level $\Gamma_0(p^2)$ is used when Theorem 16 is applied. $\square$

8.4 p-adic interpolation of Petersson inner product of theta series

Our goal in this section is to see how one could p-adically interpolate the quantities

$$\delta_2^k E_2(\mathfrak{a}),$$

where $\mathfrak{a}$ is some fractional ideal of K, which are directly related via Proposition 13 to the Petersson inner product of theta series. To do so, it will be necessary to use the following technical

Proposition 26. *Let $f \in V^{gen}(\mathbb{Z}_p, \Gamma(p^n)^{arith})$ and let $(E_{\mathfrak{a}}, \varphi, \beta_n)$ be the trivialized $\Gamma(p^n)^{arith}$ elliptic curve defined over $\widehat{\mathcal{O}}_{\mathfrak{p}_\infty}$ which was attached to the couple $(\mathfrak{a}, \widetilde{\varphi})$ in the previous section. Then, letting $(E_{\mathfrak{a}}, \varphi, \beta_0)$ denote the trivialized $\Gamma(1)^{arith}$ elliptic curve attached to $(\mathfrak{a}, \widetilde{\varphi})$ and letting $f^{(0)}$ be the level $\Gamma(1)^{arith}$ p-adic modular form corresponding to f under the canonical isomorphism of Proposition 18, one has the following equalities:*

$$f^{(0)}(E_{\mathfrak{a}}, \varphi, \beta_0) = V_p^n f(E_{\mathfrak{a}}, \varphi, \beta_n) = \mathrm{Frob}_{\mathfrak{p}}^n(f(E_{\mathfrak{a}}, \varphi, \beta_n)),$$

where $\mathrm{Frob}_{\mathfrak{p}} = \left(\frac{H(\bar{\mathfrak{p}}^\infty)/K}{\mathfrak{p}}\right)$ *is the Artin symbol.*

Proof. This is [Kat76, Lem.8.3.25]. □

It follows from Lemma 6 and the fact that

$$D_N V_p = p V_p D_N,$$

as operators on $V^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(N)^{\mathrm{arith}})$, that

$$D_1^k E_2^{[p]} = (1 - p^k(1+p)V_p + p^{2k+1}V_p^2) D_1^k E_2 = (1 - p^k V_p)(1 - p^{k+1}V_p) D_1^k E_2.$$

Using the definition of

$$L_{\mu_E^{[p]}}((\mathfrak{a}, \widetilde{\varphi}), k),$$

and the previous proposition, it then follows as in the proof of Theorem 18 that

$$\frac{\mathrm{Frob}_{\mathfrak{p}}^2(L_{\mu_E^{[p]}}((\mathfrak{a},\widetilde{\varphi}),k))}{\Omega_p(\mathfrak{a})^{2k+2}} = (1-p^k\,\mathrm{Frob}_{\mathfrak{p}})(1-p^{k+1}\,\mathrm{Frob}_{\mathfrak{p}})\left(\frac{\delta_2^k E_2(\mathfrak{a})}{((2\pi i)^{-1}\Omega_{\mathbb{C}}(\mathfrak{a}))^{2k+2}}\right)$$

or equivalently

$$\frac{L_{\mu_E^{[p]}}((\mathfrak{a},\widetilde{\varphi}),k)}{\Omega_p(\mathfrak{a})^{2k+2}} = (\mathrm{Frob}_{\mathfrak{p}}^{-1}-p^k)(\mathrm{Frob}_{\mathfrak{p}}^{-1}-p^{k+1})\left(\frac{\delta^k E_2(\mathfrak{a})}{((2\pi i)^{-1}\Omega_{\mathbb{C}}(\mathfrak{a}))^{2k+2}}\right) \tag{8.8}$$

Finally, one can prove that the Petersson inner product of theta series of weight greater than one can be p-adically interpolated, at least when K has genus one.

Theorem 19. *Let K be an imaginary quadratic field of genus one with Hilbert class field H and let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals which are such that*

$$\mathfrak{a}\overline{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K.$$

Let also $\theta_{\mathfrak{a},\ell}$ and $\theta_{\mathfrak{b},\ell}$ be the theta series of weight $2\ell+1$ attached to $\mathfrak{a}$ and $\mathfrak{b}$ for some $\ell > 0$. Then for any prime $p > 3$ which splits in K, say $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, and any isomorphism

$$\widetilde{\varphi} : \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigcup_{n\geq 1} \bar{\mathfrak{p}}^{-n}\mathfrak{c}/\mathfrak{c},$$

there exists a p-adic analytic function

$$F : \mathcal{W} \longrightarrow \mathbb{C}_p$$

with the property that

$$\frac{F(\ell)}{\Omega_p(\mathfrak{c})^{4\ell}} = (\mathrm{Frob}_{\mathfrak{p}}^{-1}-p^{2\ell-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1}-p^{2\ell})\left(\frac{\langle\theta_{\mathfrak{a},\ell},\theta_{\mathfrak{b},\ell}\rangle}{((2\pi i)^{-1}\Omega_{\mathbb{C}}(\mathfrak{c}))^{4\ell}}\right) \qquad \text{for all } \ell > 0,$$

where $\mathrm{Frob}_{\mathfrak{p}} = \left(\frac{H/K}{\mathfrak{p}}\right)$ *is the Artin symbol.*

Proof. Note that it follows from the theory of complex multiplication and the explicit formulas of Proposition 13 that

$$\left(\frac{\langle\theta_{\mathfrak{a},\ell},\theta_{\mathfrak{b},\ell}\rangle}{((2\pi i)^{-1}\Omega_{\mathbb{C}}(\mathfrak{c}))^{4\ell}}\right)\in H,$$

so it makes sense to apply $\mathrm{Frob}_{\mathfrak{p}}$ to it. In the light of the above computations and the explicit formulas of Proposition 13, it suffices to take

$$F(\kappa)=4(|D_K|/4)^{\kappa}L_{\mu_E^{[p]}}((\mathfrak{c},\widetilde{\varphi}),2\kappa-1).$$

Note that

$$(|D_K|/4)\in\mathbb{Z}_p^{\times},$$

so $(|D_K|/4)^{\kappa}$ is analytic on $\mathcal{W}$. $\square$

8.5 p-adic analogue of Kronecker's First Limit Formula

Recall that when $\ell=0$, the theta series attached to ideals in imaginary quadratic fields are not cuspidal and so it makes no sense *a priori* to consider the quantity

$$\langle\theta_{\mathfrak{a},0},\theta_{\mathfrak{b},0}\rangle.$$

However, one can makes sense of $F(0)$, since F is analytic on $\mathcal{W}$. Using the definition of F given in the proof of Theorem 19, one sees that

$$F(0)=4L_{\mu_E^{[p]}}((\mathfrak{c},\widetilde{\varphi}),-1)=4\int_{\mathbb{Z}_p^{\times}}x^{-1}d\mu_E^{[p]}(E_{\mathfrak{c}},\varphi,\beta_2).$$

Using the ideas of Propositions 24 and 25, one sees that

$$\int_{\mathbb{Z}_p^{\times}}x^{-1}d\mu_E^{[p]}=D_1^{-1}E_2^{[p]}=\sum_{(n,p)=1}n^{-1}\sigma(n)q^n.$$

Our goal in this section is to find an expression for the p-adic modular form given by the q-expansion above in terms of known objects. It will then appear that our formula can be seen as a p-adic analogue of Kronecker's First Limit formula.

Let g_0 be the modular unit given by

$$g_0 = \frac{\Delta}{V_p\Delta}, \tag{8.9}$$

where $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is the classical modular form defined in Chapter 1, whose q-expansion is given by

$$\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}. \tag{8.10}$$

Then g_0 is a weakly holomorphic modular form of level $\Gamma_0(p)$ and weight 0. Define its p-depletion as

$$g_0^{(p)} = \frac{V_p g_0}{g_0^p} = \frac{(V_p\Delta)^{p+1}}{\Delta^p V_p^2\Delta}. \tag{8.11}$$

Then one has the following

Lemma 7. *Let $g_0^{(p)}$ be as above. Then*

$$g_0^{(p)} \in 1 + pV_0^{gen}(\mathbb{Z}_p, \Gamma(p^2)^{arith}).$$

Proof. To see that

$$g_0^{(p)} \in V_0^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(p^2)^{\mathrm{arith}}),$$

it suffices to note that $g_0^{(p)} \in M_0(\Gamma_0(p^2))$ and that it has q-expansion given by

$$g_0^{(p)}(q) = \prod_{n\geq 1}\left(\frac{(1-q^{pn})^{p+1}}{(1-q^n)^p(1-q^{p^2n})}\right)^{24},$$

which is an element of $\mathbb{Z}_p[[q]]$. To see that $g_0^{(p)} \in 1 + pV_0^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(p^2)^{\mathrm{arith}})$, it suffices to show that

$$g_0^{(p)} \equiv 1 \pmod{p\mathbb{Z}_p[[q]]},$$

which follows directly from the fact that

$$(1-q^{pn})^{p+1} \equiv (1-q^{pn})^p(1-q^{pn}) \equiv (1-q^{p^2n})(1-q^n)^p \pmod{p\mathbb{Z}_p[[q]]}.$$

□

Since the power series

$$\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n+1}x^n}{n}$$

converges for any $x \in pV_0^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(p^2)^{\mathrm{arith}})$, one can take the logarithm of $g_0^{(p)}$ and obtain a p-adic modular form

$$\log_p g_0^{(p)} \in V_0^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(p^2)^{\mathrm{arith}}).$$

Here, $\log_p$ denotes the function defined by the above power series for $x \in pV_0^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(p^2)^{\mathrm{arith}})$. Then one can prove the following p-adic analogue of Kronecker's First Limit formula.

Proposition 27. *In the above notation, one has that*

$$D_1^{-1}E_2^{[p]} = \int_{\mathbb{Z}_p^\times} x^{-1} d\mu_E^{[p]} = \frac{-1}{24p} \log_p g_0^{(p)}.$$

Proof. First note that the following equality holds in $\mathbb{Z}_p[[q]]$:

$$\log_p \prod_{n\geq 1} (1-q^{p^r n})^{24} = 24 \sum_{n\geq 1} \sigma_{-1}(n) q^{p^r n}$$

for any integer $r \geq 0$, where $\sigma_{-1}(n) = \sum_{d|n} d^{-1}$. It then follows that

$$\begin{aligned}
\log_p g_0^{(p)} &= -24\left(p\sum_{n\geq 1}\sigma_{-1}(n)q^n - (p+1)\sum_{n\geq 1}\sigma_{-1}(n)q^{pn} + \sum_{n\geq 1}\sigma_{-1}(n)q^{p^2n}\right) \\
&= -24p\left(\sum_{n\geq 1}\sigma_{-1}(n)q^n - (1+p^{-1})\sum_{n\geq 1}\sigma_{-1}(n)q^{pn} + p^{-1}\sum_{n\geq 1}\sigma_{-1}(n)q^{p^2n}\right) \\
&= -24p\sum_{(n,p)=1}\sigma_{-1}(n)q^n \\
&= -24pD_1^{-1}E_2^{[p]}.
\end{aligned}$$

□

A slightly more conceptual way of proving the above result is as follows. When $k \geq 4$, first note that the q-expansion of E_k, which is given by (1.24), can be written as

$$E_k(q) = \frac{\zeta(1-k)}{2} + \sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

When $k = 0$, the above formula does not make sense because the Riemann zeta function has a pole at $s = 1$. However, forgetting the constant term, one can define formally

$$E_0(q) = \sum_{n\geq 1}\sigma_{-1}(n)q^n.$$

Then one can show that

$$T_pE_0 = (U_p + p^{-1}V_p)E_0 = (1+p^{-1})E_0$$

in the same way as one shows that

$$T_pE_k = (1+p^{k-1})E_k$$

when $k \geq 4$. It then follows formally that

$$E_0 - (1+p^{-1})V_pE_0 + p^{-1}V_p^2E_0 = E_0 - V_p(U_p + p^{-1}V_p)E_0 + p^{-1}V_p^2E_0 = (1 - V_pU_p)E_0 = E_0^{[p]}.$$

Since

$$E_0^{[p]}(q) = D_1^{-1}E_2^{[p]}(q)$$

and

$$\log_p \prod_{n\geq 1}(1 - q^{p^r n}) = V_p^r E_0(q),$$

this gives another proof of the above Proposition.

The above proposition implies that

$$\frac{-1}{24p}D_1 \log_p g_0^{(p)} = E_2^{[p]}$$

(compare with the computations done at the end of Section 4.3) and so

$$F(0) = 4D_1^{-1}E_2^{[p]}(E_{\mathfrak{a}}, \varphi, \beta_2) = \frac{-1}{6p}\log_p g_0^{(p)}(E_{\mathfrak{a}}, \varphi, \beta_2). \tag{8.12}$$

Generally speaking, one usually hopes that evaluating a p-adic L-function outside its range of interpolation gives an expression which is analogous in some sense to the corresponding complex value. Think of Kubota's formula for that value at 1 of the p-adic zeta function, for example. In the case of (8.12), the corresponding complex analogue would be the Petersson inner product of weight one theta series attached to ideals, which is of course not defined. However, one could still *formally* use the formula of Theorem 15 for $\ell = 0$ and obtain

$$\langle \theta_{\mathfrak{a},0}, \theta_{\mathfrak{b},0}\rangle = \frac{-1}{3}\log(N(\mathfrak{c})^6|\Delta(\mathfrak{c})|)$$

(we still suppose that K has genus one and that $\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{c}^2 = \mathcal{O}_K$). Using the same kind of reasoning that lead to (8.8), one can then see that

$$F(0) = \frac{-1}{6}(\mathrm{Frob}_{\mathfrak{p}}^{-1} - p^{-1})(\mathrm{Frob}_{\mathfrak{p}}^{-1} - 1)\log_p \Delta(\mathfrak{c})$$

formally (since $\Delta \notin 1 + pV^{\mathrm{gen}}(\mathbb{Z}_p, \Gamma(1)^{\mathrm{arith}})$). Notice how the last equation looks like the p-stabilized version of the one before it.

Part III

Computations

In the third and final Part of this thesis, many computations and numerical experiments are presented. Almost all computations were done using the PARI/GP computer algebra system (see [PAR16]).

Using the scripts

A big effort was made to make the code written for this project easy to use for anyone who is a little bit familiar with PARI/GP. Some functions in PARI/GP 2.9 are required, so it is necessary to have a version of PARI/GP which is ≥ 2.9.

To download the code, one can access the url [Sim] and download manually the whole `ENT` repository or simply run the command

```
git clone https://github.com/NicolasSimard/ENT.git
```

on the terminal of any machine which has the git version control system installed (see [Git]). Once the code is downloaded, navigate to the `ENT/` folder and start the PARI/GP calculator there.

PARI/GP Session 1.

```
git clone https://github.com/NicolasSimard/ENT.git
cd ENT
gp
```

About the PARI/GPSessions

It is assumed that every PARI/GP session is run directly from the `ENT/` folder. Most PARI/GP sessions start with the `read("init.gp")` command, which loads the main scripts that were written for this project. The functions that where written by the author are highlighted in green and in slanted form. The output of the commands are sometimes truncated to save some space. This is indicated by the symbol `[...]`.

CHAPTER 9
Petersson inner product

This chapter contains examples of computations of Petersson norm of newforms.

9.1 Petersson norm of Δ

To compute the Petersson norm of the modular form Δ, one could use directly the definition of the Petersson inner product as a double integral:

PARI/GP Session 2.

```
gp > default(realprecision, 50);
gp > delta(x) = eta(x,1)^24;
gp > intnum(x = -1/2, 1/2, intnum(y = (1-x^2)^(1/2),[[1],4*Pi], \
    norm(delta(x+y*I))*y^10))

%1 = 1.0353620568043209223478168122251645932249079609504 E-6
```

The last command runs in 4 seconds. A much more efficient method consists in defining the symmetric square L-function of Δ and Formula (2.12).

PARI/GP Session 3.

```
gp > default(realprecision, 50);
gp > N = 1; k = 12;
gp > a = (n -> vector(n,i,sumdiv(i, d, \
    (-1)^bigomega(d)*d^(k-1)*ramanujantau(i/d)^2)));
gp > L = lfuncreate([a,1,[0,1,2-k],2*k-1,N^2,1]);
gp > (Pi/2*(4*Pi)^k/(k-1)!/N)^-1*lfun(lfuncreate(Ldata),k)

%1 = 1.0353620568043209223478168122251645932249079609373 E-6
```

The formula for the coefficients of the Dirichlet series are taken from [Coh13, Thm.2.1]. The last command runs in a few milliseconds, nearly 100 times faster than the previous one. Note that the time it takes to initialize the L-function may not always be negligible. It depends, in particular, on how efficiently one can compute the coefficients of the Dirichlet series.

There are many more ways of computing the Petersson norm of Δ, like Poincaré series, Haberland's formula, periods or other formulas involving special values of Dirichlet series. Some methods, like Haberland's formula, are even faster than the symmetric square L-function method. For more on those methods, see [Coh13].

9.2 Petersson norm of $\Delta_5(\tau) = (\eta(\tau)\eta(5\tau))^4$

The modular form $\Delta_5(\tau) = (\eta(\tau)\eta(5\tau))^4$ is a newform of weight 2 and level $\Gamma_0(5)$. To compute its Petersson norm, one could again use the definition. To do so, one needs to find a fundamental domain for $\Gamma_0(5) \setminus \mathcal{H}$. Taking the set

$$\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} : 0 \leq j \leq 4 \right\}$$

as a set of representatives for the quotient $\mathrm{SL}_2(\mathbb{Z})/\Gamma_0(5)$, one can write the double integral over $\Gamma_0(5) \setminus \mathcal{H}$ as a sum of integrals over translates of the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ by these representatives. Another method is to use (2.12) again.

PARI/GP Session 4.

```
gp > default(realprecision, 50);
gp > k=4; N=5; anf(n) = lfunan(lfunetaquo([1,4;5,4]),n);
gp > avec(n) = {
    my(an=anf(n), ap2);
    direuler(p=2,n,
        if(N%p==0,
            1/(1 - an[p]^2*X), /*factor at the bad primes*/
```

```
            ap2 = an[p]^2-p^(k-1);
            1/(1 - ap2*X + p^(k-1)*ap2*X^2 - p^(3*(k-1))*X^3)))};
gp > L = lfuncreate([n -> avec(n),1,[0,1,2-k],2*k-1,N^2,1]);
gp > (Pi/2*(4*Pi)^k/(k-1)!/N)^-1*lfun(L,k)

%1 = 0.00087080010497869125684556081325455557786399599585989
```

This last computation is more than 1000 times faster than the previous one. The expression for the Euler factors at p of the symmetric square L-function in terms of the Fourier coefficients are easily obtained directly from the definition (2.6).

9.3 Petersson norm of $\Delta_7(\tau) = (\eta(\tau)\eta(7\tau))^3$

The modular form $\Delta_7(q) = (\eta(q)\eta(q^7))^3$ is newform of weight 3, level $\Gamma_0(7)$ and character χ_{-7}, the character attached to the quadratic field of discriminant -7 (given by the Kronecker symbol).

To compute the Petersson norm of Δ_7, one can use Theorem 8 or Theorem 9. For technical reasons, we use the first one.[1]

PARI/GP Session 5.

```
gp > default(realprecision, 50);
gp > k=3; N=7; chi(n) = kronecker(-7,n);
gp > anf(n) = lfunan(lfunetaquo([1,3;7,3]),n);
gp > avec(n) = {
    my(an=anf(n), ap2);
    direuler(p=2,n,
        if(N%p == 0,
            1/(1-p^(k-1)*X), /*factor at the bad primes*/
```

[1] The second formula involves the computation of the residue of an L-function. In general, this data is required to define the L-function in PARI/GP. In some cases, the function `lfunrootres()` can be used to find this data numerically, but we have not been able to compute it in this case.

```
            ap2 = an[p]^2-chi(p)*p^(k-1);
            1/(1 - chi(p)*ap2*X + chi(p)*ap2*p^(k-1)*X^2 - p^(3*(k-1))*X^3)))};
gp > L = lfuncreate([n -> avec(n),1,[0,1,2-k],2*k-1,N^2,1]);
gp > (Pi/2*eulerphi(N)*(4*Pi)^k/N^2/(k-1)!)^-1*(lfun(L,k)*6/7) /*Note the 6/7*/

%1 = 0.0052288338547890255263565535732901581103627066353333
```

Note that the one has to multiply $L(\mathrm{Sym}^2\,\Delta_7, \chi_{-7}, s)$ by the Euler factor

$$(1 - p^{2-s})^{-1}$$

at the bad prime $p = 7$ to obtain a nice functional equation (this explains the presence of the factor $6/7$ in the above computation). In general, finding the bad Euler factors can be difficult. In this case, it turns out the Δ_7 is also a theta series and so one can use Proposition 4 to guess the missing Euler factor.

CHAPTER 10
Complex multiplication

10.1 Singular moduli

In this section we illustrate the statements of Theorem 10, which is in some sense the starting point of the theory of complex multiplication. The following calculation illustrates the first three points of this Theorem.

PARI/GP Session 6.

```
gp > read("init.gp"); default(realprecision, 500);
gp > K = bnfinit(x^2+47); reps = redrepshnf(K); hK = #reps;
gp > f = algdep(ellj(idatouhp(K,reps[1])),hK) /*Verify point 1*/

%1 = x^5 + 2257834125*x^4 - 9987963828125*x^3 + 5115161850595703125*x^2 -
    14982472850828613281250*x + 16042929600623870849609375

gp > nfisincl(f,polcompositum(K.pol,quadhilbert(K.disc))[1]) != []

%2 = 1

gp > poldegree(polcompositum(K.pol,f)[1]) == 2*hK /*Verify point 2*/

%3 = 1

gp > f == round(prod(i=1,hK,x-ellj(idatouhp(K,reps[i])))) /*Verify point 3*/

%4 = 1
```

The last point of Theorem 10 could also be verified, but we didn't try to. Note that the fact that the polynomial f is monic proves numerically that the singular values are integral.

10.2 Siegel units

In this section we illustrate the statements of Theorem 11.

PARI/GP Session 7.

```
gp > read("init.gp"); default(realprecision , 500);
gp > phi(K,ida) = delta(idatolat(K,1))/delta(idatolat(K,idealinv(K,ida)));
gp > K = bnfinit(x^2+71); reps = redrepshnf(K); hK = #reps;
gp >  f = algdep(phi(K,reps[2]),2*hK) /*Verify point 1*/

%1 = 68719476736*x^6 + 2785017856*x^5 + 14351421440*x^4 + 412493295*x^3 +
    3503765*x^2 + 166*x + 1

gp > nfisincl(f,polcompositum(K.pol,quadhilbert(K.disc))[1]) != [] /* phi(K,ida) \
    is in H*/

%2 = 1

gp > factor(polcoeff(f,6)) /*Verify point 2*/

%3 =
[2 36]

gp > idealfactor(K,reps[2]) /*Verify point 2*/

%4 =
[[2, [0, 1]~, 1, 1, [1, -6; 1, 0]] 1]
```

It should also be possible to verify point 3 of Theorem 11, but we haven't tried.

In the following PARI/GPsession, we experiment with Siegel units.

PARI/GP Session 8.

```
gp > read("init.gp"); default(realprecision , 500);
gp > phi(K,ida) = delta(idatolat(K,1))/delta(idatolat(K,idealinv(K,ida)));
gp > siegelunit(K,ida) = complexgen(K,idealpow(K,ida,hK))^12*phi(K,ida)^hK;
gp > K = bnfinit(x^2+31); hK = K.clgp.no; reps = redrepshnf(K); /* hK = 3 */
gp > f = algdep(siegelunit(K,reps[2]),2*hK) /*Siegel units are... units!*/
```

```
%1 = x^6 - 1590927*x^5 + [...] + 896395574769*x^2 - 1590927*x + 1

gp > nfisincl(f,polcompositum(K.pol,quadhilbert(K.disc))[1]) != [] /*Siegel units \
    belong to H*/

%2 = 1
```

Note that the fact that the leading and constant terms of f are both equal to 1 proves numerically that this Siegel unit is indeed a unit.

10.3 CM values of modular forms: classically and algebraically

Recall the the Chowla-Selberg period attached to K, defined in Proposition 7 as

$$\Omega_K = \frac{1}{\sqrt{4\pi|D|}} \left(\prod_{n=1}^{|D|-1} \Gamma\left(\frac{n}{|D|}\right)^{\chi_D(n)} \right)^{w_K/(4h_K)},$$

could be used to algebraize the values of level one modular forms at lattices corresponding to fractional ideals in K.

PARI/GP Session 9.

```
gp > read("init.gp"); default(realprecision, 500);
gp > K = bnfinit(x^2+23); Om_K = CSperiod(K.disc); reps = redrepshnf(K);
gp > factor(algdep(delta(idatolat(K,reps[2]))/Om_K^12,10))

%1 =
[x + 1 1]
[262144*x^6 - 1617920*x^5 + 5304960*x^4 - 15701487*x^3 + 25590080*x^2 - 3235840*x +
    262144 1]

gp > factor(algdep(delta(idatolat(K,reps[2]))/Om_K^12,20,100)) /*Retry, see below.*/

%2 =
[x 2]
[262144*x^6 - 1617920*x^5 + 5304960*x^4 - 15701487*x^3 + 25590080*x^2 - 3235840*x +
    262144 1]
```

```
gp > factor(algdep(E(2,idatolat(K,1))/Om_K^2,20)) /*Suspicious output*/

%3 =
[14270068201587690836857*x^20 + [...] - 116255875884396643732694 1]

gp > default(realprecision, 1000); /* Need to increase precision. */
gp > K = bnfinit(x^2+23); Om_K = CSperiod(K.disc);
gp > factor(algdep(E(2,idatolat(K,1))/Om_K^2,20))

%4 =
[x 1]
[47952687595882920802373496471552*x^18 - 280961949079190786543261319168*x^12 -
    22089242062943952187392*x^6 - 5411082280083481 1]
```

The degree of the polynomial output by `algdep()` was chosen at random, since the only information we have about the quotients $\Delta(\mathfrak{a})/\Omega_K^{12}$ and $E_2(\mathfrak{a})/\Omega_K^2$ is that they are algebraic. To convince ourselves that the polynomial for $\Delta(\mathfrak{a})/\Omega_k^{12}$ was good, we increased the degree and used only the first 100 digits of the quotient in the second call to `algdep()`. Since both polynomials have a common factor, it seems plausible that the quotient is a root of this common factor. For E_2, the first call to `algdep()` output a suspicious polynomial (it has degree equal to 20, large height and is a sum of monomials of every degree less than 20). After doubling the precision, the polynomial becomes much simpler and one is lead to believe that the quotient attached to E_2 is a root of it. One could again double the precision and see that the polynomial remains the same.

It was shown in Chapter 7 that one could attach a period $\Omega_{\mathfrak{a}}$ to any fractional ideal $\mathfrak{a}$ of K in such a way that

$$f(\mathfrak{a}) \cdot (2\pi i \Omega_{\mathfrak{a}})^k \in H,$$

where f is any nearly holomorphic modular form of weight k and level 1. Note that here there is control over the algebraic field in which the numbers land.

PARI/GP Session 10.

```
gp > read("init.gp"); default(realprecision, 500);
gp > K = bnfinit(x^2+31); hK = K.clgp.no; /*hK = 3*/
gp > reps = redrepshnf(K); Om_ida = canperiod(K,reps[2]);
gp > ida = idatolat(K,reps[2]); /*Lattice attached to ideal reps[2]*/
gp > f = factor(algdep(delta(ida)*(2*Pi*I*Om_ida)^12,2*hK))

%1 =
[x 1]
[10230338448241477052305617123765768999*x^3 + [...] + 6330188685371917375801 1]

gp > f = f[2,1];
gp > nfisincl(f,polcompositum(K.pol,quadhilbert(K.disc))[1]) != [] /*Lands in H*/

%2 = 1

gp > pol = shimuramaass(E2,3) /*d^3E_2*/

%3 = -48*E2^4 + 120*E4*E2^2 - 14*E6*E2 + 25*E4^2

gp > z = substvec(pol,[E2,E4,E6],[E(2,ida),E(4,ida),E(6,ida)]); /*d^3E_2(ida)*/
gp > f = algdep(z*(2*Pi*I*Om_ida)^8,2*hK)

%4 = 41824442823309987855304220413776681*x^3 - [...] - 2480023220943452566129

gp > nfisincl(f,polcompositum(K.pol,quadhilbert(K.disc))[1]) != [] /*Lands in H*/

%5 = 1
```

The period $\Omega_{\mathfrak{a}}$, returned by the function `canperiod()`, is computed by first finding an elliptic curve defined over H isomorphic to $\mathbb{C}/\mathfrak{a}$ over $\mathbb{C}$ (this can be done explicitly by cooking up a Weierstrass equation out of the value $j(\mathfrak{a}) \in H$) and then calling the built-in function `ellperiods` on this elliptic curve.

As one can see, by using the period $\Omega_{\mathfrak{a}}$, one trades the explicit Chowla-Selberg formula for more information about the algebraicity of the values.

CHAPTER 11
Petersson inner product of theta series

11.1 Hecke characters

One advantage of Hecke characters of type A_0 (as opposed to general Hecke characters) is that they are easy to compute with. To see this, let K be an imaginary quadratic field, let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_d\}$ be a set of generators of Cl_K, let o_i be the order of $\mathfrak{a}_i$ in Cl_K, let $\mathfrak{a}$ be any fractional ideal of K and let ψ be a Hecke character of infinity type $T = (k_1, k_2)$. Then

$$\psi(\mathfrak{a}_i)^{o_i} = \psi(\mathfrak{a}_i^{o_i}) = \psi(\alpha_i) = \alpha_i^{k_1}\bar{\alpha_i}^{k_2},$$

where $\mathfrak{a}_i^{o_i} = (\alpha_i)\mathcal{O}_K$, so

$$\psi(\mathfrak{a}_i) = \zeta_{o_i}\alpha_i^{k_1/o_i}\bar{\alpha_i}^{k_2/o_i}$$

for some o_i^{th} root of unity ζ_{o_i} and so

$$\psi(\mathfrak{a}_i) = \exp(2\pi i c_i/o_i)\alpha_i^{k_1/o_i}\bar{\alpha_i}^{k_2/o_i}$$

for some $0 \le c_i < o_i$. Writing

$$\mathfrak{a} = \mu \prod_{i=1}^{d} \mathfrak{a}_i^{e_i},$$

for some $0 \le e_i < o_i$ and $\mu \in K^\times$, it follows that

$$\psi(\mathfrak{a}) = \mu^{k_1}\bar{\mu}^{k_2}\exp\left(2\pi i \sum_{i=1} e_i c_i/o_i\right)\prod_{i=1}^{d}\alpha_i^{k_1 e_i/o_i}\bar{\alpha_i}^{k_2 e_i/o_i}.$$

Therefore, ψ is completely determined by its infinity type and the tuple $c = (c_1, \ldots, c_d)$. Conversely, it is clear that any such sequence and infinity type (satisfying a certain parity condition) defines a Hecke character. For this reason, such a Hecke character is represented by the tuple

$$[c, T]$$

in the `ENT/` repository.

PARI/GP Session 11.

```
gp > read("init.gp");
gp > K = bnfinit(x^2+17*23*47); qhcdata = qhcinit(K);
gp > K.clgp.cyc /*order of cyclic components of Cl_K*/

%1 = [24, 2, 2]

gp > qhc = [[15,1,0],[8,0]]; /*some Hecke character of oo-type (8,0)*/
gp > p11 = idealprimedec(K,11)[1]; /*a prime of K above 11*/
gp > qhceval(qhcdata,qhc,p11)

%2 = 12495.16962[...] - 7630.83331[...]*I

gp > factor(round(norm(%))) /*The above number has norm 11^8, as expected*/

%3 = [11 8]
```

11.2 Theta functions

To compute the q-expansion of the theta function attached to a Hecke character, one could use the built-in PARI/GPfunction `ideallist`, which returns a list of all ideals of norm less than a bound. However, it turns out to be more efficient to compute $\theta_{\mathfrak{a},\ell}$ for each $\mathfrak{a}$ in a set of representatives for Cl_K and then use the identity in Proposition 9.

PARI/GP Session 12.

```
gp > read("init.gp"); default(realprecision, 5000);  default(seriesprecision,1000);
gp > K = bnfinit(x^2+13*31); qhcdata = qhcinit(K);
gp > K.clgp.cyc /*order of cyclic components of Cl_K*/

%1 = [2]
```

```
gp > qhc = [[0],[4,0]]; /*some Hecke character*/
gp > L = ideallist(K,default(seriesprecision));
gp > F(qhc)=q*Ser(apply((L->sum(i=1,#L,qhceval(qhcdata,qhc,L[i]))),L),q);
gp > F(qhc); /*few seconds*/
gp > bintheta(K,qhc); /*few milliseconds*/
gp > vecmax(abs(Vec(F(qhc)-bintheta(K,qhc)))) /*the two q-exp are equal*/

%2 = 1.5223063546968858611 E-5003
```

11.3 Petersson inner product of theta functions

Of course, the formulas used in Chapter 9 could be used to compute the Petersson norm of theta series. However, it is often more efficient to use the formulas

$$\langle \theta_\psi, \theta_\psi \rangle = \frac{4h_K}{w_K}\sqrt{|D|}\frac{\Gamma(2\ell+1)}{(4\pi)^{2\ell+1}}L(\psi^2, 2\ell+1) \tag{11.1}$$

and

$$\langle \theta_\psi, \theta_\psi \rangle = (|D|/4)^\ell \frac{4h_K}{w_K^2}\sum_{j=1}^{h_K}\psi^2(\mathfrak{a}_j)\delta_2^{2\ell-1}E_2(\mathfrak{a}_j) \tag{11.2}$$

of Proposition 12 and Theorem 15, respectively. To see this, take again the field $K = \mathbb{Q}(\sqrt{-7})$. Then the subspace of theta functions in $S_3(\Gamma_0(7), \chi_{-7})$ is one-dimensional and

$$\theta_{\mathcal{O}_K} = 2\theta_\psi,$$

where ψ is the unique Hecke character of K of infinity type $(2, 0)$. A trivial bound for the dimension of $S_3(\Gamma_0(7), \chi_{-7})$ as a $\mathbb{C}$-vector space is 3 (see [RBvdG+08, Prop.3]) and so it follows from a q-expansion calculation that

$$\Delta_7 = \theta_\psi.$$

PARI/GP Session 13.

```
gp > read("init.gp"); default(realprecision, 100); default(timer,1);
```

```
gp > K = bnfinit(x^2+7); hK = K.clgp.no; ell = 1; qhc = [[],[2*ell,0]];
gp > Lsym2 = lfuncreate(lsym2data(K,qhc));
gp > Lpsi2 = lfuncreate(qhcLdata(K,2*qhc));

time = 16 ms.

gp > (2*ell)!*abs(K.disc)/(4*Pi)^(2*ell+1)*2/Pi*lfun(Lsym2,2*ell+1)

time = 1,607 ms.
%5 = 0.0052288338547890255263565535732901581103627066353332689519853455514[...]

gp > 4*hK/K.tu[1]*sqrt(abs(K.disc))*(2*ell)!/(4*Pi)^(2*ell+1)*lfun(Lpsi2,2*ell+1)

time = 15 ms.
%6 = 0.0052288338547890255263565535732901581103627066353332689519853455514[...]

gp > pnorm(pipinit(K),qhc)

%7 = 0.0052288338547890255263565535732901581103627066353332689519853455514[...]
```

Compare the results of these computations with those of the PARI/GPSession 5.

Note that the formula using the Hecke L-function is about 100 times faster that the one using the symmetric square L-function. The reason for this is that the symmetric square L-function has degree 3, while the Hecke L-function has degree 2. Note also that the function `pnorm`, which uses (11.2), runs in a negligible amount of time.

The main advantage of (11.2) is that the derivatives of E_2 can be expressed recursively in terms of E_2, E_4 and E_6 (see (1.73), (1.74) and (1.75)), and so the evaluation of $\delta_2^{2\ell-1}E_2(\mathfrak{a})$ boils down to the evaluation of a 3 variable polynomial (which depend only on ℓ) at the point $(E_2(\mathfrak{a}), E_4(\mathfrak{a}), E_6(\mathfrak{a})) \in \mathbb{C}^3$. The only problem is that the degree and the height of that polynomial grows with ℓ and it becomes expensive to compute. However, since it depends *only* on ℓ it can be computed in advance and stored. The large use of computation power is then transferred to a large use in memory.[1] This leads to the search for yet another algorithm which would scale well with ℓ. An idea is to compute explicitly the "q-expansion" of $\delta_2^n E_2$ using the formula

$$\delta_2^n E_2(\tau) = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(r+2)_{n-r}}{(4\pi\Im(\tau))^{n-r}} D^r E_2(\tau),$$

where $(a)_d = a(a+1)\dots(a+d-1)$ is the Pochhammer symbol and

$$D = \frac{1}{2\pi i}\frac{\partial}{\partial z}$$

(see [RBvdG+08, Eqn.56]). A calculation then shows that

$$\begin{aligned}\delta_2^n E_2(\tau) =& (-1)^n \left(\frac{1}{8\pi\Im(\tau)} - \frac{n+1}{24}\right)\frac{n!}{(4\pi\Im(\tau))^n} \\ &+ \sum_{m\geq 1}\sigma(m)\left(\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\frac{(r+2)_{n-r}}{(4\pi\Im(\tau))^{n-r}} m^r\right) q^m.\end{aligned} \tag{11.3}$$

The running time of all those formulas can be compared for varying ℓ and precision. Here is a small comparison of the timing of the computation of the Petersson norm of θ_ψ for some Hecke character ψ of $K = \mathbb{Q}(\sqrt{-47})$ of infinity type $(2\ell, 0)$ using 4 different algorithms at 1000 digits of precision. The timings are in seconds.

		ℓ			
		1	25	50	100
Algorithms	Eq. (11.1)	5.3	9.2	14.9	31.0
	Eq. (11.2)	0	0.2	1.7	25.5
	Eq. (11.3)	0.4	2.5	6.3	18.8
	Eq. (11.3)’	0.4	0.6	0.7	1.2

[1] For example, the first 220 polynomials $\delta_2^1 E_2, \dots, \delta_2^{437} E_2$ take about 900 Mb of space.

Here, equation (11.3)' is a vectorized implementation of the formula (11.3), i.e. it uses a different algorithm to evaluate the q-expansion. Here is the same comparison, but with 2000 digits of precision.

		ℓ			
		1	25	50	100
Algorithms	Eq. (11.1)	23.3	35.3	50.7	84.1
	Eq. (11.2)	0	0.3	2.6	31.2
	Eq. (11.3)	1.9	8.4	18.5	47.4
	Eq. (11.3)'	1.8	2.4	2.8	4.1

As expected, equation (11.2) behaves well when the precision increases, but behaves badly when ℓ increases. One also sees that the vectorized implementation of (11.3) does consistently well.

One advantage of having more than one formula to compute a quantity numerically is that one can have an idea of the number of correct digits of the results by comparing the output of two different algorithms. Here is an example.

PARI/GP Session 14.

```
gp > default(realprecision,1000); read("init.gp");
gp > K = bnfinit(x^2+53); pipdata = pipinit(K); K.clgp

%4 = [6, [6], [[3, 2; 0, 1]]]

gp > pnorm(pipdata,[[1],[10,0]])-pnorm(pipdata,[[1],[10,0]],"qexpv")

%5 = 1.8923323355184722792 E-991 + 8.480963825248301816 E-1006*I
```

The first call to `pnorm` uses (11.2), while the second uses the vectorized implementation of (11.3). The difference between the two quantities suggest that around 990 out of the 1000 digits of precision are correct.

Finally, in the next PARI/GPsession, we verify (4.14), which gives an expression for the value at $s = 1$ of Hecke L-functions of non-trivial class characters.

PARI/GP Session 15.

```
gp > read("init.gp");
gp > K = bnfinit(x^2+47); wK = K.tu[1]; reps = redrepshnf(K);
gp > qhc = [[2],[0,0]]; /*some class character*/
gp > Psi(ida) = qhceval(qhcinit(K),qhc,ida);
gp > Lpsi2 = lfuncreate(qhcLdata(K,2*qhc));
gp > F(ida) = idealnorm(K,ida)^6*abs(delta(idatolat(K,ida)));
gp > lfun(Lpsi2,1)

%1 = 0.64666083128645259893546663118873725573 - 7.4[...] E-60*I

gp > -Pi/3/wK/sqrt(abs(K.disc))*sum(i=1,#reps,Psi(reps[i])^-2*log(F(reps[i])))

%2 = 0.64666083128645259893546663118873725572 + 0.E-38*I
```

CHAPTER 12
Stark's observation on weight one theta functions

12.1 Stark's original observation in the field $\mathbb{Q}(\sqrt{-23})$

Stark noticed that if θ_ψ is the theta function attached to a non-trivial class character of $K = \mathbb{Q}(\sqrt{-23})$, then

$$\langle \theta_\psi, \theta_\psi \rangle = 3 \log \epsilon,$$

where ϵ is a real root of $x^3 - x - 1$. This is easy to verify numerically.

PARI/GP Session 16.

```
gp > read("init.gp"); default(realprecision, 500);
gp > Phi(K,ida) = my(L=idatolat(K,ida)); \
    sqrt(idealnorm(K,ida))*abs(L[1]^-1*eta(L[2]/L[1],1)^2);
gp > K = bnfinit(x^2+23); data = pipinit(K);
gp > qhc = [[1],[0,0]]; /*A non-trivial class character*/
gp > algdep(exp(pnorm(data,qhc)/3),3)

%1 = x^3 - x - 1

gp > p31 = idealprimedec(K,31)[1]; /*31 splits in K*/
gp > abs(pnorm(data,qhc)-3*log(Phi(K,idealinv(K,p31))/Phi(K,1)))

%2 = 2.583511420128310153 E-500
```

12.2 Generalizing Stark's observation to class number 3 quadratic fields

One may wonder if the Petersson norm of theta functions is always the logarithm of a unit in the Hilbert class field. This was proven to be the case for all class number 3 number fields and can be seen numerically.

PARI/GP Session 17.

```
gp > read("init.gp"); default(realprecision, 1000);
gp > L = discofclassno(3); /*There are 16 of them*/
gp > test(K) = polredbest(algdep(exp(pnorm(pipinit(K),[[1],[0,0]])/3),3));
gp > for(i=1,#L,print("D=",L[i],": ",test(bnfinit(x^2-L[i]))))

D=-907: x^3 - x^2 - 7*x + 12
D=-883: x^3 - 2*x^2 - 12*x - 11
D=-643: x^3 - 2*x - 5
D=-547: x^3 - x^2 - 3*x - 4
D=-499: x^3 + 4*x - 3
D=-379: x^3 - x^2 + x - 4
D=-331: x^3 - x^2 + 3*x - 4
D=-307: x^3 - x^2 + 3*x + 2
D=-283: x^3 + 4*x - 1
D=-211: x^3 - 2*x - 3
D=-139: x^3 - 8*x - 9
D=-107: x^3 - x^2 + 3*x - 2
D=-83: x^3 - x^2 + x - 2
D=-59: x^3 + 2*x - 1
D=-31: x^3 + x - 1
D=-23: x^3 - x - 1

gp > for(i=1,#L,print("D=",L[i],": ",polredbest(quadhilbert(L[i]))))

D=-907: x^3 - x^2 - 7*x + 12
D=-883: x^3 - 2*x^2 - 12*x - 11
D=-643: x^3 - 2*x - 5
D=-547: x^3 - x^2 - 3*x - 4
D=-499: x^3 + 4*x - 3
D=-379: x^3 - x^2 + x - 4
D=-331: x^3 - x^2 + 3*x - 4
D=-307: x^3 - x^2 + 3*x + 2
D=-283: x^3 + 4*x - 1
D=-211: x^3 - 2*x - 3
D=-139: x^3 - 8*x - 9
D=-107: x^3 - x^2 + 3*x - 2
```

```
D=-83: x^3 - x^2 + x - 2
D=-59: x^3 + 2*x + 1
D=-31: x^3 + x - 1
D=-23: x^3 - x - 1
```

12.3 Generalizing Stark's observation to other quadratic fields

In this section, we give examples of computations which support the claims and conjectures made in Section 5.3. Recall the notation

$$\langle \theta_\psi, \theta_\psi \rangle = h_K \log \kappa_\psi,$$

where

$$\kappa_\psi = \prod_{\mathfrak{a} \in \mathrm{Cl}_K} \Phi(\mathfrak{a}^{-1})^{-\psi(\mathfrak{a})^2}.$$

PARI/GP Session 18.

```
gp > read("init.gp"); default(realprecision, 1000);
gp > Phi(K,ida) = my(L=idatolat(K,ida)); \
    sqrt(idealnorm(K,ida))*abs(L[1]^-1*eta(L[2]/L[1],1)^2);
gp > k_psi(K,qhc) = {
        my(reps=redrepshnf(K));
        prod(i=1,K.clgp.no,Phi(K,reps[i])^(-qhceval(qhcinit(K),qhc,reps[i])))};
gp > Q(D) = bnfinit(x^2-D); /*for convenience*/
gp > algdep(k_psi(Q(-47),[[1],[0,0]]),50) /*Class number 5: hummm*/

%1 = 388601289028*x^50 - [...] - 6975618148752847

gp > factor(algdep(k_psi(Q(-87),[[1],[0,0]]),50)) /*Class number 6: it is a unit!*/

%2 =
[                                      x - 1 1]
[x^6 - x^5 - 6*x^4 - 17*x^3 - 6*x^2 - x + 1 1]

gp > nfisincl(%[2,1],polcompositum(Q(-87).pol,quadhilbert(-87))[1]) != [] /*Is in \
    H*/
```

```
%3 = 1

gp > D10 = discofclassno(10)[-1..-1][1]; /*Class number 10*/
gp > algdep(k_psi(Q(D10),[[1],[0,0]]),50) /*Garbage for this Hecke char*/

%4 = 387398835037687*x^50 + [...] + 4477373333556254

gp > algdep(k_psi(Q(D10),[[5],[0,0]]),50) /*Works with this Hecke char*/

%5 = x^31 - 8*x^30 - x^29
```

Many other experiments like these where run and they all support the conjectures made in Chapter 5.

CHAPTER 13
Computational experiments and conjectures

In this last chapter of the thesis, we present some computational experiments which lead to some conjectures. For most of these conjectures, we have only numerical evidence.

13.1 Gram matrix of the Petersson inner product on the space of theta series

If $f_1, \ldots, f_n \in S_k(\Gamma_0(N), \chi)$ are cusp forms, define

$$\mathrm{Gram}(f_1, \ldots, f_n) = \det(\langle f_i, f_j \rangle)_{1 \leq i,j \leq n}.$$

This is the determinant of the Gram matrix of the space spanned by $f_1, \ldots, f_n$ and equipped with the Petersson inner product.

When $\ell > 0$, recall that by Proposition 10 the space

$$\Theta_{K,\ell} \subseteq M_{2\ell+1}(\Gamma_0(|D|), \chi_D)$$

of theta series has dimension h_K. It is then natural to consider

$$\mathrm{Gram}(\theta_{\psi_1}, \ldots, \theta_{\psi_{h_K}}),$$

where the θ_{ψ_i} are the theta series attached to Hecke characters of infinity type $(2\ell, 0)$. This quantity depends only on K and ℓ and by Corollary 5, it is an algebraic multiple of $\Omega_K^{4\ell h_K}$, where Ω_K is the Chowla-Selberg period attached to K. In fact, numerically we find that this algebraic number, denoted

$$A(K, \ell) = \frac{\mathrm{Gram}(\theta_{\psi_1}, \ldots, \theta_{\psi_{h_K}})}{\Omega_K^{4\ell h_K}} = \frac{\prod_{i=1}^{h_K} \langle \theta_{\psi_i}, \theta_{\psi_i} \rangle}{\Omega_K^{4\ell h_K}}, \tag{13.1}$$

is almost always an integer (some small powers of 2 and 3 appear in the denominator sometimes when $\ell = 1$).

In the following PARI/GP session, the function `invA` returns A.

PARI/GP Session 19.

```
gp > read("init.gp"); default(realprecision, 500);
gp > ell = 1; for(D = 5, 50, if(isfundamental(-D),print(-D,": \
    ",algdep(invA(-D,ell),5))));

-7: 3*x^5 - x^4
-8: 2*x - 1
-11: x - 1
-15: x - 4
-19: 3*x^5 - 13*x^4
-20: x - 64
-23: x - 621
-24: x - 196
-31: x - 7254
-35: x - 324
-39: x - 4076800
-40: x - 3364
-43: 3*x - 214
-47: x - 538443750

gp > ell = 2; for(D = 5, 50, if(isfundamental(-D),print(-D,": \
    ",algdep(invA(-D,ell),5))));

-7: x - 1
-8: x - 5
-11: x - 10
-15: x - 6084
-19: x - 142
-20: x - 21904
-23: x - 7303581
-24: x - 318096
-31: x - 404717958
```

```
-35: x - 5702544
-39: x - 16446807606528
-40: x - 36820624
-43: x - 22588
-47: x - 480609496085043750
```

This observation about the rationality of the normalized product of special values of Hecke L-functions in (13.1) is similar in nature to what was observed by Gross and Zagier in [GZ80] or by Villegas and Zagier in [VZ92]. In those papers, they observe (and prove in certain cases), that the central critical value of Hecke characters attached to CM elliptic curves is an integral multiple of powers of the corresponding Chowla-Selberg period, up to some explicit factors.

One strategy to prove this rationality property would be to introduce a Galois action and to show that the product in (13.1) is Galois invariant. The following experimentation explores this idea.

PARI/GP Session 20.

```
gp > read("init.gp"); default(realprecision , 2000);
gp > K = bnfinit('x^2+23); data = pipinit(K); Om_K = CSperiod(K.disc);
gp > ell = 1; qhcs = qhchars(K,[2*ell,0]);
gp > for(i = 1, K.clgp.no , print(algdep(pnorm(data,qhcs[i])/Om_K^(4*ell),10)))

x^9 - 6966*x^6 + 11569230*x^3 - 239483061
x^9 - 6966*x^6 + 11569230*x^3 - 239483061
x^9 - 6966*x^6 + 11569230*x^3 - 239483061

gp > ell = K.clgp.no; qhcs = qhchars(K,[2*ell,0]);
gp > for(i = 1, K.clgp.no , print(algdep(pnorm(data,qhcs[i])/Om_K^(4*ell),10)))

x - 5055
x^6 - 16287872873193*x^3 + 30021979248651078296845875
x^6 - 16287872873193*x^3 + 30021979248651078296845875
```

Many observations can be made after those computations. First, one sees that the quantities

$$N(\psi,\ell)=\frac{\langle\theta_\psi,\theta_\psi\rangle}{\Omega_K^{4\ell}}$$

are Galois conjugate over $\mathbb{Q}$ when ℓ is not a multiple of h_K (this phenomenon repeats not only when $\ell=1$). Moreover, the $N(\psi,\ell)$ are h_Kth roots of an algebraic integer. Also, there is this interesting phenomenon that when ℓ is a multiple of h_K, one of the $N(\psi,\ell)$ is rational and the other two are Galois conjugate over $\mathbb{Q}$. Similar observations were made for all the number fields tested (including some number fields with more than one genera).

The last observation suggests that one Hecke character is singled out when ℓ is a multiple of the class number, which is surprising at first. However, one sees that when $h_K|\ell$, the map

$$\psi_{0,\ell}(\mathfrak{a})=\alpha^{2\ell/h_K},$$

where α is a generator of $\mathfrak{a}^{h_K}$ is a well-defined Hecke character of infinity type $(2\ell,0)$. This $\psi_{0,\ell}$ is the singled out Hecke character. In the notation of Section 11.1, it corresponds to the tuple $c=(0,\dots,0)$. Note that when $\ell=0$, this is just the trivial characters. It would be interesting to see what can be said about the special values of the Hecke L-function corresponding to this special Hecke character.

13.2 Invariant theta series attached to imaginary quadratic fields

When $\ell>0$, one can also compute the determinant of the Gram matrix of the Petersson inner product on the space $\Theta_{K,\ell}$ with respect to a basis consisting of theta series attached to fractional ideals of K. If $\{\mathfrak{a}_1,\dots,\mathfrak{a}_{h_K}\}$ is a choice of representatives of Cl_K, one could choose the basis of $\Theta_{K,\ell}$ to be

$$\{\theta_{\mathfrak{a}_1,\ell},\dots,\theta_{\mathfrak{a}_{h_K},\ell}\}.$$

The problem is that

$$\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell},\ldots,\theta_{\mathfrak{a}_{h_K},\ell})$$

depends on this choice. Indeed,

$$\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell},\ldots,\theta_{\mu\mathfrak{a}_i,\ell},\ldots,\theta_{\mathfrak{a}_{h_K},\ell}) = N(\mu)^{2\ell}\,\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell},\ldots,\theta_{\mathfrak{a}_i,\ell},\ldots,\theta_{\mathfrak{a}_{h_K},\ell}) \qquad \text{forall } \mu \in K^{\times}.$$

The obvious way to solve this problem is to normalize the Gram determinant using the norm function. This leads to the quantity

$$B(K,\ell) = \frac{\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell},\ldots,\theta_{\mathfrak{a}_{h_K},\ell})}{\Omega_K^{4\ell h_K}\prod_{i=1}^{h_K} N(\mathfrak{a}_i)^{2\ell}},$$

an algebraic number which depends only on K and ℓ. Numerically, one notices that $B(K,\ell)$ is rational. In fact, this follows from the observation that $A(K,\ell)$ is rational, because of the following

Proposition 28. *Let $A(K,\ell)$ and $B(K,\ell)$ be the quantities defined above. Then*

$$B(K,\ell) = \left(\frac{w_K^2}{h_K}\right)^{h_K} A(K,\ell).$$

Note that the proportionality factor does not depend on ℓ.

Proof. Let

$$\mathcal{B}_1 = \{\theta_{\mathfrak{a}_1,\ell},\ldots,\theta_{\mathfrak{a}_{h_K},\ell}\}$$

and

$$\mathcal{B}_2 = \{\theta_{\psi_1},\ldots,\theta_{\psi_{h_K}}\}$$

be bases for the space $\Theta_{K,\ell}$. Using Proposition 9, one computes that

$$\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell},\ldots,\theta_{\mathfrak{a}_{h_K},\ell}) = \det(T^t)\,\mathrm{Gram}(\theta_{\psi_1},\ldots,\theta_{\psi_{h_K}})\,\det(\overline{T}), \tag{13.2}$$

where T is the $h_k \times h_k$ matrix whose ij entry is $\frac{w_k}{h_K}\psi_i(\mathfrak{a}_j)$, the bar denotes complex conjugation and the t denotes transposition.

Now

$$\det(T) = \left(\frac{w_K}{h_K}\right)^{h_K} \left(\prod_{j=1}^{h_k} \psi_1(\mathfrak{a}_j)\right) \det\left(\frac{\psi_i}{\psi_1}(\mathfrak{a}_j)\right).$$

As i goes from 1 to h_K, the class character $\chi_i = \psi_i/\psi_1$ goes through all the class characters exactly once. Letting M be the matrix with ij entry $\chi_i(\mathfrak{a}_j)$, one then sees that

$$\det(T^t\overline{T}) = \left(\frac{w_K}{h_K}\right)^{2h_K} \left(\prod_{i=1}^{h_k} |\psi_1(\mathfrak{a}_i)|^2\right) \det(M^t\overline{M}) = \left(\frac{w_K}{h_K}\right)^{2h_K} \left(\prod_{i=1}^{h_k} N(\mathfrak{a}_i)\right) \det(M^t\overline{M}).$$

Now the ij entry of $M^t\overline{M}$ is

$$\sum_{k=1}^{h_K} \chi_k(\mathfrak{a}_i)\overline{\chi_k(\mathfrak{a}_j)} = h_K\delta_{i,j},$$

so that

$$\det(M^t\overline{M}) = h_K^{h_K}.$$

Putting everything together, one has

$$\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell}, \ldots, \theta_{\mathfrak{a}_{h_K},\ell}) = \left(\frac{w_K^2}{h_K}\right)^{h_K} \left(\prod_{i=1}^{h_k} N(\mathfrak{a}_i)\right) \mathrm{Gram}(\theta_{\psi_1}, \ldots, \theta_{\psi_{h_K}}), \tag{13.3}$$

which proves the claim. □

Note also that by Corollary 7, one has the equality

$$\frac{\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell}, \ldots, \theta_{\mathfrak{a}_{h_K},\ell})}{\prod_{i=1}^{h_K} N(\mathfrak{a}_i)^{2\ell}} = \det\left(\langle \theta_{\mathcal{O}_K,\ell}, \theta_{\mathfrak{a}_i^{-1}\mathfrak{a}_j,\ell}\rangle\right)_{1\leq i,j\leq h_K}.$$

Although the computations with the invariant $A(K,\ell)$ seem to suggest that the Chowla-Selberg is a good choice of period to normalize the Petersson inner product of theta series, it would be nice to find a way to get rid of this choice of period. In what follows, we do this when $D_K < -4$.

First, define

$$\theta_{\mathcal{A},\ell} = \frac{\theta_{\mathfrak{a},\ell}}{E_2(\mathfrak{a}^{-1})^{\ell}},$$

where $\mathcal{A} \in \mathrm{Cl}_K$ and $\mathfrak{a}$ is a representative of $\mathcal{A}$.[1] Then the set $\{\theta_{\mathcal{A}_1,\ell}, \ldots, \theta_{\mathcal{A}_{h_K},\ell}\}$, where $\mathrm{Cl}_K = \{\mathcal{A}_1, \ldots, \mathcal{A}_{h_K}\}$ is a basis of $\Theta_{K,\ell}$ with the property that

$$\langle \theta_{\mathcal{A}_i,\ell}, \theta_{\mathcal{A}_j,\ell} \rangle \in H$$

for any $1 \leq i, j \leq h_K$ by the theory of complex multiplication. Note that no choice of period is involved in the previous statement. Moreover, note that when $\ell = 0$, we recover the usual theta series $\theta_{\mathcal{A},0}$.

Define

$$C(K,\ell) = \mathrm{Gram}(\theta_{\mathcal{A}_1,\ell}, \ldots, \theta_{\mathcal{A}_{h_K},\ell}).$$

Then

$$C(K,\ell) = \frac{\mathrm{Gram}(\theta_{\mathfrak{a}_1,\ell}, \ldots, \theta_{\mathfrak{a}_{h_K},\ell})}{\prod_{i=1}^{h_K} |E_2(\mathfrak{a}_i^{-1})|^{2\ell}},$$

where $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_K}\}$ is any choice of class representative for Cl_K. Numerically, one notices that $C(K,\ell)$ is rational whose denominator is a 2ℓth power of a fixed number (as usual, up to powers of 2 or 3). Here is a table of the denominators that appear for class number $1, 2, 3$ and 4 and small values of ℓ

[1] The reason for the exclusion of the imaginary quadratic fields of discriminant -3 and -4 is that E_2 vanishes at the CM point corresponding to them.

		K			
		$\mathbb{Q}(\sqrt{-163})$	$\mathbb{Q}(\sqrt{-187})$	$\mathbb{Q}(\sqrt{-23})$	$\mathbb{Q}(\sqrt{-203})$
ℓ	1	$2 \cdot 3 \cdot 181^2$	$2^4 13^2$	419^2	$2^4 57259^2$
	2	$2^4 181^4$	$2^{10} 13^4$	419^4	$2^{12} 57259^4$
	3	$2^{10} 181^6$	$2^{18} 13^6$	419^6	$2^{18} 57259^6$
	4	$2^8 181^8$	$2^{10} 13^8$	419^8	57259^8
	5	$2^{12} 181^{10}$	$2^{26} 13^{10}$	419^{10}	$2^{32} 57259^{10}$
	6	$2^{19} 181^{12}$	$2^{36} 13^{12}$	419^{12}	$2^{40} 57259^{12}$
	7	$2^{19} 181^{14}$	$2^{40} 13^{14}$	419^{14}	$2^{52} 57259^{14}$
	8	$2^{20} 181^{16}$	$2^{42} 13^{16}$	419^{16}	$2^{52} 57259^{16}$
	9	$2^{23} 181^{18}$	$2^{44} 13^{18}$	419^{18}	$2^{46} 57259^{18}$
	10	$2^{25} 181^{20}$	$2^{52} 13^{20}$	419^{20}	$2^{64} 57259^{20}$

Those numbers also appear in a relation between the invariants B and C. Indeed, it seems numerically that

$$B(K,\ell) = \frac{n(K)^{2\ell}}{(12^2|D|)^{h_K \ell}} C(K,\ell),$$

where $n(K)$ is the number such that $n(K)^{2\ell}$ is the denominator of $C(K,\ell)$. Of course, this conjecture is not very precise, but hopefully the following computations might convince the reader that it makes sense.

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```
gp > read("init.gp"); default(realprecision, 1000);
gp > D = -23; hK = qfbclassno(D)

%1 = 3

gp > for(ell = 1, 10, \
     print(ell,":",factor(invB(D,ell)/invC(D,ell)*(12^2*abs(D))^(hK*ell))))

1:Mat([419, 2])
```

```
2:Mat([419, 4])
3:Mat([419, 6])
4:Mat([419, 8])
5:Mat([419, 10])
6:Mat([419, 12])
7:Mat([419, 14])
8:Mat([419, 16])
9:Mat([419, 18])
10:Mat([419, 20])

gp > default(realprecision, 2000); D = -95; hK = qfbclassno(D)

%1 = 8

gp > for(ell = 1, 10, \
     print(ell,":",factor(invB(D,ell)/invC(D,ell)*(12^2*abs(D))^(hK*ell))))

1:[1531, 2; 242798651, 2]
2:[1531, 4; 242798651, 4]
3:[1531, 6; 242798651, 6]
4:[1531, 8; 242798651, 8]
5:[1531, 10; 242798651, 10]
6:[1531, 12; 242798651, 12]
7:[1531, 14; 242798651, 14]
8:[1531, 16; 242798651, 16]
9:[1531, 18; 242798651, 18]
10:[1531, 20; 242798651, 20]
```

It would be interesting to see if the numbers $n(K)$ can be related to K in some other way.

Conclusion

As was seen in this thesis, Stark's observation on the relation between the Petersson norm of the weight one theta series attached to a non-trivial character of $K = \mathbb{Q}(\sqrt{-23})$ and the logarithm of a unit can sometimes be generalized to other number fields. It would be interesting to see if Conjecture 1 holds. In any case, it was shown using the explicit formulas of Theorem 15 that the Petersson norm of weight one theta series attached to class characters of imaginary quadratic fields is a linear combination of logarithms of Siegel units.

In the second part, it was shown that the Petersson inner product of higher weight theta series can be p-adically interpolated. At the point corresponding to weight one, which was outside the range of interpolation, it was shown that one obtains a p-adic analogue of the Petersson inner product of weight one theta series (if it could be defined).

The possibility of efficiently computing the Petersson inner product of theta series allows one to experiment with them and observe some interesting patterns. Some experiments were presented in the last chapter. It would be interesting to further investigate the many open questions that were made there. In particular, the relation between the invariant $C(K, \ell)$ and the Chowla-Selberg period seems worthy of interest.

Another direction for further research would be to study the Petersson inner product of theta series attached to type A_0 Hecke characters with non-trivial conductor. Those theta series share many common properties with the ones considered in this thesis and all of the steps in finding the explicit formulas of Theorem 15 should generalize.

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