Arithmetic Eisenstein theta lifts

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ABSTRACT. We study certain arithmetic group cocycles valued in differential forms arising from torus bundles over (locally) symmetric spaces, which we call Eisenstein theta lifts following the nomenclature of Bergeron-Charollois-Garcia, who constructed them using automorphic theta kernels arising from regularized Eisenstein series. By studying the Hodge theory of such torus bundles in the setting where they have the structure of an abelian family, we establish that these analytically constructed cocycles agree with the cohomology classes defined by Kings-Sprang using equivariant polylogarithm classes in coherent cohomology, showing the former have a natural integral structure and giving an analytic way to compute the latter. We then construct analogous cohomology classes (called by analogy arithmetic theta lifts) valued in Milnor K-theory using a motivic analogue of the equivariant polylogarithm, and show their de Rham regulators yield the Eisenstein theta lift.

RÉSUMÉ. Nous étudions des cocycles pour des groupes arithmétiques à valeurs dans des formes différentielles sur des tores fibrées au-dessus des espaces (localement) symmétriques, que nous appelons des relèvements thêta d'Eisenstein après Bergeron-Charollois-Garcia, qui les ont construits en utilisant des noyaux thêta automorphiques venant des séries d'Eisenstein régularisées. En étudiant la théorie de Hodge dans le cas des fibrés abéliens, nous démontrons que les classes de ces cocycles analytiques sont données par les classes abstraites définies par Kings-Sprang en utilisant le polylogarithme équivariant en cohomologie cohérente. Cela implique que les cocycles analytiques ont une structure intégrale canonique, et en même temps donne un façon de calculer les classes abstraites. Ensuite, nous construisons des classes abstraites analogues (appelées relèvements d'Eisenstein arithmétiques) à valeurs dans la K-théorie de Milnor en utilisant le polylogarithme équivariant dans le contexte motivique, puis démontrons que leurs régulateurs sont donnés par les relèvements d'Eisenstein.

Contents

1. Introduction

Theta lifts are a central tool in modern number theory. In the general framework laid out by [Howe], they offer a way to relate automorphic forms on a pair of reductive groups in ways amenable to explicit formulas involving terms of interest such as special values of L-functions. Via the Langlands philosophy, one can obtain subtle number theoretic information from these relations; some of the most well-known successes along these lines include the seminal work of Gross-Kohnen-Zagier [GKZ], Borcherds' singular theta lift [Bor], and the promising framework known as the "Kudla program" [Kud].

A key issue one encounters in arithmetic applications is that theta lifts of automorphic forms are usually constructed analytically, and so understanding how the correspondence interacts with rational or integral structures on these forms is difficult, relying on delicate period computations; see [Pra] for a survey of such computations.

Work in recent years has produced many different incarnations by many different authors of a particular theta lift which seems much more amenable to algebraic formalism. Given the large number of different related constructions, some of which are subsumed by others, we note a few of the most pertinent works to our present approach: the fullest grounding of the analytic theory was explicated and given the name of "Eisenstein theta lift" in [BCG1], then further developed by the same authors in [BCG3]. An analogous "Eisenstein-Kronecker cohomology class" valued in automorphic forms is defined in a formal algebraic way in [KS]. Finally, [SV] defined using explicit complexes a "motivic Eisenstein class" valued in Milnor K-theory in a narrower setting. In this thesis, we relate all of these cocycles; namely, we show the first two constructions give the same theta lift, while the latter also yields the same class upon taking regulators, and thus could be considered an "arithmetic Eisenstein theta lift" refining the usual one.

We also generalize, both via the abstract formalism and more explicit methods, all these constructions to new settings, and consider some basic aspects of the p-adic variation of this theta lift with an eye towards arithmetic applications.

1.1. **Main idea: Eisenstein cocycles.**

All definitions of the Eisenstein theta lift are (sometimes non-obviously) related to the construction of an *equivariant polylogarithm class*. In this article, we will only consider the theory one obtains from using the "base class", i.e. with constant coefficients rather than the whole of the logarithm sheaf (as is done in [KS] and [BKL], for example).

The starting point is a $2n$-dimensional relative commutative group parameterized over a base $\pi : E \to B$, with the fiberwise action of a group Γ. Throughout this paper, we take the convention of a *left* action of the group on the space, with a resulting pushforward left action on cohomology (i.e. pullback by the inverse)[1]

Given a Borel equivariant cohomology theory on Γ-spaces $H^\bullet_\Gamma$,[2] suppose that we have an equivariant localization sequence

$$\ldots \to H^{2n-1}_\Gamma(E - E[c]) \xrightarrow{\text{res}} H^0_\Gamma(E[c]) \to H^{2n}_\Gamma(E) \to \ldots$$

The group $H^0_\Gamma(E[c])$ is identified with the Γ-invariant formal sums of connected components of $E[c]$. The unlabeled map is a fiberwise degree map (or alternately zero if the fibers are not proper). Using certain projectors built from isogenies ("Lieberman's trick"[3]) we can find a canonical splitting of the map res, and hence produce an injective map

$$(H^0(E[c])^{\deg=0})^\Gamma \hookrightarrow H^{2n-1}_\Gamma(E - E[c]), \mathcal{C} \mapsto z_\mathcal{C}$$

producing cohomology classes with prescribed residues in the form of codimension-$2n$ cycles built from the fiberwise c-torsion. The resulting class $z_\mathcal{C}$ is known as the *equivariant polylogarithm* (base) *class* for E.

The relation to theta lifts is that this class $z_\mathcal{C}$, or rather a component of it, is like a family (parameterized by E) of "theta kernels" between the group Γ and the space B. Roughly, the idea of a theta kernel is that if we have two groups, say $\mathbf{G}$ and $\mathbf{H}$, a function $\theta(g, h)$ which has an automorphic transformation law for *both* groups can be used to map automorphic forms for $\mathbf{G}$ to $\mathbf{H}$ using θ as a kernel function:

$$\mathbf{G}\text{-automorphic forms} \ni f \mapsto \int_\mathbf{G} f(g)\theta(g, h)\, dg \in \mathbf{H}\text{-automorphic forms}$$

[1]This is in contrast to [SV], which considers right actions on spaces so that we have a left pullback action on cohomology theories. Since we do not consider monoid actions in this thesis, it does not end up really mattering to us except in superficial conventions of notation, though there are reasons for preferring this convention.
[2]See section 2.1 for the properties we want from such a theory.
[3]It is unclear to us where this name comes from, but it is common to refer to it as such in the literature; for example, it is used in one of our primary references [BCG1].

In the context of the theta correspondence [Howe], this map has representation-theoretic significance, but this will not be our primary concern in this thesis.

In our context, if Γ is an arithmetic subgroup of a reductive group and B a locally symmetric space (for some second group H) equipped with a family of tori E, then we wish to construct from $z_{\mathcal{C}}$ group cohomology classes for Γ valued in the cohomology of E: this is the step which (at least heuristically) involves "isolating a component" of $z_{\mathcal{C}}$ as we mentioned above, which we will end up calling $\Theta_{\mathcal{C}}$.

After pullback by a torsion section of E, this class becomes valued in the cohomology of B. Given the existence of Eichler-Shimura-type theorems relating cohomology to automorphic forms, the "theta kernel" label is then justified by the chain of relations

$$\Gamma\text{-automorphic forms} \xrightarrow{\text{E.-S.}} \Gamma\text{-homology} \xrightarrow{\Theta_{\mathcal{C}}} \text{differential forms on } E \to H\text{-automorphic forms}$$

where the last step, passing from differential forms on E to to sections of automorphic bundles on the base B, involves some kind of pullback.

This description is mainly motivational; we do not actually wish to translate to the setting of automorphic forms on the Γ-side in this thesis, and are content to work on the level of cohomology. More precisely, what we what we want to with this "theta kernel" $z_{\mathcal{C}}$ is:

- integrate it "in the Γ-direction," i.e. deduce from it a group cocycle for Γ, and
- optionally, pull back the resulting Γ-cocycle from E to B along a section (or else view it as a family of cocycles valued on B parameterized by the torsion sections of E).

Remark 1.1. An important issue to note is that the way we end up pulling back to the base is more subtle than it appears: we do not actually want to just take the pullback of differential forms from B to E, but rather make some intermediary transformations - namely, contraction with a vector field.[4] Indeed, suppose

$$x^* z_{\mathcal{C}} \in H^{2n-1}_{\Gamma}(B)$$

[4]Taking the naive pullback to the base, however, does end up being the approach we want to take later on in the motivic setting.

via some Γ-invariant section $x : B \to E$. Observe that the triviality of the Γ-action on B affords us a Künneth splitting and thus a class in $H^{n-1}(\Gamma, H^n(B))$.[5] This is convenient, but loses access to a good deal of the interesting information contained in the class $z_{\mathcal{C}}$: for instance, let us consider the case where Γ is trivial and $E \to B$ is the universal elliptic curve $\mathcal{E} \to \mathcal{Y}$ over the open modular curve at some suitable level. Then the class $z_{\mathcal{C}} \in H^1(\mathcal{E} - \mathcal{E}[c])$ in de Rham cohomology is represented by the 1-form which is the total logarithmic derivative of a Kato-Siegel unit as in [Kato1, Proposition 1.3]. Any pullback by a torsion section $x^* z_{\mathcal{C}}$ is the component coming from the logarithmic derivative along the base, and hence a weight-2 Eisenstein series. However, the pullback loses the information about the logarithmic derivative along the fiber, which is a weight-1 Eisenstein series once one identifies it with a section of the Hodge bundle on the total space and then pulls it back. We are interested in the analogue of this weight-1 Eisenstein series in a more general context. See [BCG1, (9.6)] for this computation.

Can we avoid this issue by producing a big class

$$\Theta_{\mathcal{C}} \in H^{n-1}(\Gamma, \Omega^n(E - E[c]))$$

out of $z_{\mathcal{C}}$, with values on the whole bundle? Something similar is indeed possible, and there are a variety of approaches in the literature. Analytically, [BCG1] outlines an approach to construct an explicit representative of the polylogarithm class in de Rham cohomology, whose projection yields an appropriate theta kernel. This kernel has good functorial properties and relates well to automorphic constructions, making it possible to explicitly calculate its integrals along homology cycles in many situations arising from automorphic theory.

On the other hand, in the presence of scheme-theoretic structure, one can attempt to force enough degeneration of the Hochschild-Serre spectral sequence for $H^\bullet_\Gamma$ to afford an edge map

$$H^{2n-1}_\Gamma(E) \to H^{n-1}(\Gamma, H^n(E)),$$

for example by shrinking E: this is the approach followed by parts of [BCG3] and [KS] for example. This has the advantage of being purely algebraic, yielding results on rationality, integrality, etc. for free if one

[5] We first learned this idea from a talk of Romyar Sharifi.

can maintain these structures on the cohomology. The disadvantage is that the layers of abstraction make it harder to understand what is happening explicitly.

Another algebraic approach, yielding more explicit results, is to define Γ-equivariant complexes computing equivariant cohomology, find a way to make them exact, and then take the resulting cocycle defined using the Γ-action on an exact complex. This approach, followed by [SV] in the motivic setting, marries many of the advantages of both other approaches, in being both explicit as well as yielding many nice properties purely formally. However, it also can involve significant technical difficulties.

We will follow all these approaches, and relate the resulting cocycles to better study their properties.

1.2. **Outline of thesis and results.**

1.2.1. *Eisenstein theta lift: the differential cocycle.* We define a "Eisenstein theta kernel" cocycle

$$\Theta_{\mathcal{C},x} \in H^{n-1}(\Gamma, \Omega_B^n)$$

valued in n-forms on the base, parameterized by some torsion section data. We begin by constructing an explicit analytic representative of the theta kernel as in the first approach, and then use an equivariant polylogarithm class in *coherent* cohomology in the second approach (using spectral sequences).

The first construction, due to [BCG1], starts by constructing analytic representatives (via the Mathai-Quillen formalism [MQ]) for the polylogarithm class in ordinary cohomology, for general *topological* torus bundles endowed with an invariant metric; these representatives are linear combinations of certain Eisenstein-regularized series E_ψ. For Γ acting on a universal torus bundle over some (locally) symmetric space, there is a topological incarnation of this Γ-bundle given by the whole family as fibered over the classifying space $B\Gamma$. In the principal cases of interest, Γ is itself an arithmetic group, so $B\Gamma$ also has a canonical Riemannian model given by a locally symmetric space, allowing us to run the Mathai-Quillen formalism. Then via explicit computations done in [BCG2], a certain component of the resulting Eisenstein series E_ψ is closed along the base $B\Gamma$, making it into a kernel function from homology classes for Γ to automorphic forms on the torus bundle.

The second construction, due to [KS], begins with the polylogarithm class in Γ-equivariant coherent cohomology of a universal abelian scheme $\mathcal{A}$ over Shimura varieties. In order to obtain group cohomology classes from this, we need a existence of a certain edge map in the resulting Hochschild-Serre spectral sequence requiring vanishing of certain terms on the second page; see (2.12). The strategy to prove this vanishing is by cutting down the fibers of $\mathcal{A}$ to be affine, thus reducing their cohomological dimension. In the case where $\mathcal{A}$ is isogenous to a sum of elliptic curves, one can find natural families of ample hyperplanes, but otherwise, we have to use a very coarse localization at certain torsion sections, then establish certain integrality properties by a more careful analysis.

In Theorem 3.27, we prove that the two constructions above coincide in a high degree of generality. Many of the ingredients of this result have appeared in various publications and preprints: in the case when the abelian family $\mathcal{A}$ splits into elliptic curves, [BCG3] showed that a class derived from the equivariant polylogarithm via formality results for hyperplane complements (*Orlik-Solomon formality*) could be computed by a variant of the construction in [BCG1]. The formality statements, together with the closedness lemma Proposition 3.8 from [BCG2], show that this variant agrees with the integral of the theta kernel E_ψ, and the class extracted from the equivariant de Rham polylogarithm can be compared with the equivariant coherent polylogarithm via Hodge theory. Thus, the comparison with the cocycle of [KS] was already implicit in the work of the authors of [BCG1] for split $\mathcal{A}$.

For more general $\mathcal{A}$, the violent localization process we use makes this type of argument infeasible. Instead, when $\mathcal{A}$ is a universal family over a Shimura variety, we use constellations of special points over which $\mathcal{A}$ *does* split; using functoriality properties relating the constructions for these specialized cocycles to the constructions over the whole base, one can prove by continuity that the comparison remains true for $\mathcal{A}$; see Theorem 3.27.

This comparison is significant because while the analytic theta kernel is very explicit and computable by automorphic means (see, for example, [BCG1, §13]), the Kings-Sprang theta kernel is definable in very general settings and has very good algebraic properties for purely formal reasons. For example, if one has a model of a Shimura variety defined integrally (over $\mathbb{Z}_p$, for example), it can furnish "automatic" integrality results for the analytic Eisenstein theta correspondence; the construction is exploited to this end in [KS].

1.2.2. *Arithmetic Eisenstein theta lift: the motivic cocycle.* In section 4, we turn to motivic refinements of the Eisenstein theta lift, following the ideas of [SV]. In contrast to the principal method of loc. cit., we use a Hochschild-Serre edge map from *equivariant motivic cohomology* to define the cocycle, much as in the differential forms setting.[6] This is made possible by the existence of suitable functorial complexes (Bloch's cubical complexes, in our case) computing motivic cohomology, which we can turn into an equivariant version by taking group cochains.

Then we may imitate the "violent" localization process we used for the differential cocycle using a motivic vanishing result for local rings, allowing us to treat a setting more general than [SV]. This affords us a group cocycle $\Theta^{M}_{\mathcal{C},x}$ valued in bidegree-(n, n) *motivic* cohomology, whose regulator we identify in section 4.2.3 with (a period multiple of) the differential theta kernel $\Theta_{\mathcal{C},x}$ via a regulator map on the level of *complexes* from the equivariant Gersten complex to an equivariant Dolbeault complex computing the appropriate coherent cohomology groups.

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2. Cohomological formalism of the polylogarithm class

Following the outline in the introduction, we give in this section a more detailed treatment of the construction of the equivariant polylogarithm class with trivial coefficients, most of which applies with very little change in any suitable Borel-equivariant cohomology theory (a concept to be explained in this section). Our aim is to clarify exactly what properties a cohomology theory needs to have for our purposes.

In this section, put ourselves in the setting of a discrete group Γ acting fiberwise on a relative commutative group

$$\pi : E \to B$$

[6] It should be noted that this idea is also presented in [SV, §1.2.3], but not expanded upon.

of relative dimension $d = 2n$. We will end up working with various manifestations of this notion in various categories of spaces, but the central case one should keep in mind for geometric intuition is that of a bundle of $2n$-dimensional topological tori over some manifold base. In this thesis, we will always be in this setting, or even the more restrictive setting of a relative dimension-n abelian scheme over a base with algebraic structure.

The formalism also applies to the closely related case of self-products of $\mathbb{G}_m$ or twists thereof, as considered in [BCG3, §8] or [SV, §3-5]. Certainly, this case is also interesting, and we anticipate pursuing it using most of the same methods in future work. However, we omit it from the current thesis in favor of focusing on abelian families.[7]

2.1. **Equivariant polylogarithm class.** Suppose we have a cohomology theory

$$H^\bullet : \mathbf{S}^{op} \to \mathbf{A}$$

a family of contravariant functors, indexed by nonnegative integers (for the placeholder $\bullet$), from some category of spaces $\mathbf{S}$ to some abelian category $\mathbf{A}$. For an arrow f of spaces, we thus have the pullback in $\mathbf{A}$, which we will denote per convention by f^* (rather than $H^\bullet(f)$ as one usually does for functors). For finite maps in $\mathbf{S}$, we also demand the structure of pushforwards f_*, covariantly functorial in the sense that $(f \circ g)_* = f_* \circ g_*$ such that such that

$$f_* f^* = \deg f \tag{2.1}$$

where the right-hand scalar denotes multiplication in the $\mathbb{Z}$-linear structure on $\mathbf{A}$. Here, we presume the existence of a class of "finite maps" with an associated integer degree $\deg$ (multiplicative under composition) in the category $\mathbf{S}$. We will not axiomatize this, but in each particular setting we will consider it is a well-trod concept (corresponding intuitively to the notion of branched cover).

We will need also an *Borel-equivariant* version of the cohomology $H^\bullet_\Gamma$ for Γ-spaces, whose construction we give a high-level overview of here, with the intention of explaining the details in each particular manifestation

[7] In fact, [BCG3] even constructs interesting group cocycles when E is simply a vector bundle, though this situation is quite different and it is unclear whether any of our methods yield results of interest.

later in the thesis. We will frequently call these cohomology theories simply "equivariant cohomology."[8] Elementary but comprehensive references are hard to find for this material; see [Bott] for an overview of the concept, as well as [NSS] for a more complete but significantly more abstract treatment in very different language from what we use in this thesis.

In each situation we will consider, the functor $H^\bullet$ will factor as

$$\mathbf{S}^{op} \xrightarrow{D} C(\mathbf{A}) \xrightarrow{R^\bullet\Gamma} \mathbf{A}.$$

Here, $C^+(\mathbf{A})$ is the bounded below complex category of $\mathbf{A}$ and $R^i\Gamma$ the corresponding ith cohomology functor; D hence should be viewed as a functor sending a space X to a complex $D^\bullet(X)$ which computes its (bare) cohomology.

The corresponding Borel-equivariant cohomology can then by formed by constructing the functorial double complex $C^\bullet(\Gamma, D^\bullet(X))$ of group cochains, i.e.

$$\begin{array}{ccccc} D^0(X) & \xrightarrow{\quad d \quad} & D^1(X) & \xrightarrow{\quad d \quad} & \dots \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\ C^1(\Gamma, D^0(X)) & \xrightarrow{d} & C^1(\Gamma, D^1(X)) & \xrightarrow{d} & \dots \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\ \vdots & & \vdots & & \dots \end{array} \tag{2.2}$$

and defining $H^i_\Gamma(X)$ to be the ith cohomology of the total complex. Here, we take the convention for double complexes that the differential is

$$d + (-1)^{i+j}\partial$$

on $C^i(\Gamma, D^j(X))$. In this thesis, we will always write $C^\bullet(\Gamma, -)$ for *inhomogenous* group cochains.

The spectral sequence of a double complex then yields a Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\Gamma, H^q(X)) \Rightarrow H^{p+q}_\Gamma(X). \tag{2.3}$$

[8] To homotopy theorists, these are distinct from "genuine" equivariant cohomology (hence the customary "Borel" in that field). We have no need of genuine equivariant cohomology in this thesis, and so find it acceptable to muddle the terminological waters.

As mentioned in the introduction, we also need the theory $H^\bullet_\Gamma$ to have a localization sequence

$$\dots \to H^{d-1}_\Gamma(E) \to H^{d-1}_\Gamma(E - C) \to H^d_{\Gamma,C}(E) \to H^d_\Gamma(E) \to \dots \tag{2.4}$$

for any Γ-invariant closed subspace $C \subset E$. This means we also need a corresponding theory of (bare) cohomology with supports for closed embeddings $Z \subset X$. This can be defined formally using the shifted mapping cone (or mapping cylinder) complex [Wei]

$$(X, Z) \mapsto D_Z(X) := \text{Cone}(D(X) \xrightarrow{\iota^*} D(X - Z))[-1]$$

where $\iota : X - Z \hookrightarrow X$ is the inclusion map. In many of our settings, the restriction map ι^* is actually surjective, allowing the simpler and canonically quasi-isomorphic choice

$$(X, Z) \mapsto D_Z(X) := \ker(D(X) \xrightarrow{\iota^*} D(X - Z)). \tag{2.5}$$

Either way, this then yields a localization sequence in cohomology

$$\dots \to H^{d-1}(X) \to H^{d-1}_\Gamma(X - Z) \to H^d_{\Gamma,Z}(X) \to H^d_\Gamma(X) \to \dots \tag{2.6}$$

for $Z \hookrightarrow X$ a Γ-fixed subspace, by the snake lemma. Being that the construction of $D(X)$ and $D_Z(X)$ are functorial, we similarly have a functorial triangle of double complexes

$$[C(\Gamma, D_Z(X))] \to [C(\Gamma, D(X))] \to [C(\Gamma, D(X - Z))] \tag{2.7}$$

which results in (2.4) upon taking $X = E$ and $Z = C$.

In the equivariant localization sequence, we will eventually choose a Γ-invariant subset $C \subset E[c]$ for the c-torsion sections of E, so that it is of pure codimension d along each fiber. We then also demand an isomorphism

$$H^d_{\Gamma,C}(E) \cong H^0_\Gamma(C) = H^0(C)^\Gamma \tag{2.8}$$

which is functorial for isogenies $[a] : E \to E$ (and their restrictions). This will be deduced by constructing a Γ-equivariant quasi-isomorphism of complexes

$$D(C)[-d] \xrightarrow{\sim} D_C(E),$$

which we will call the *Gysin* isomorphism in analogue with its usual name in algebraic geometry, and then checking that the induced map on cohomology is functorial for such isogenies. We will further call the map

$$H^d_{\Gamma,C}(E) \to H^d_\Gamma(E)$$

the *degree* map, in analogy with the usual name for the analogous map in (non-equivariant) singular cohomology. Putting these ingredients together, we have thus the identification

$$\mathrm{im}(H^{d-1}_\Gamma(E - C) \to H^d_{\Gamma,C}(E)) = \ker(H^d_{\Gamma,C}(E) \to H^d_\Gamma(E)), \tag{2.9}$$

i.e. a free module over the cohomology of a point generated by the connected components of C, where the subscript $\deg = 0$ denotes the "liftable" classes, meaning in the kernel of the degree map (so everything, if the fibers of E are not proper and the target of $\deg$ is zero).

We further demand all constructions above to be functorial also for the finite pushforward maps f_*; this amounts demanding functorial pushforwards f_* on the level of the complexes $Z(X)$ inducing f_* on the cohomology groups, and which are compatible with the localization sequence and Gysin isomorphisms on the level of complexes.

The upshot of this formalism is that for any $\mathcal{C} \in H^0(*)\{\pi_0(C)\}^\Gamma_{\deg=0}$, we can lift it to some $\tilde{\mathcal{C}} \in H^{d-1}_\Gamma(E - C)$, with an ambiguity coming from the image of $H^{d-1}_\Gamma(E)$.

The final assumption we need to make is the existence of a *Lieberman projector* e_L acting on the cohomology spaces in the localization sequence, such that e_L annihilates $H^{d-1}_\Gamma(E)$ and fixes $H^d_{\Gamma,C}(E)$. To make sure of this, we will add the stipulation that the isogeny $[a]$ maps C to itself (not necessarily surjectively).

Then the situation we will commonly encounter is that the $H^{d-1}_\Gamma(E)$ decomposes under the action of the multiplication-by-a isogeny pushforward $[a]_*$ into only copies of the characters $[a]_* \mapsto a^k$ (for all $a \in \mathbb{Z}$) for

$k \in \{1, \ldots, d\}$. The same isogenies act also on

$$H^{\bullet}_{\Gamma,C}(E)$$

which can be identified through the Gysin isomorphism with its pushforward action on $\pi_0(C)$, acts on $H^{\bullet}(E - C)$ via the composite

$$H^{\bullet}(E - C) \to H^{\bullet}(E - [a]^{-1}C) \xrightarrow{[a]_*} H^{\bullet}(E - C) \tag{2.10}$$

where the first arrow is pullback by the inclusion (since $[a]$ maps C into C), and the localization sequence is equivariant for these actions.

Then if we take any $a \equiv 1 \pmod{c}$, the operator

$$e_L^{(a)} := \frac{([a] - a) \ldots ([a] - a^d)}{(1 - a) \ldots (1 - a^d)}$$

certainly annihilates $H^{d-1}_{\Gamma}(E)$ and fixes all elements in $H^0(*)\{\pi_0(C)\}^{\Gamma}$.[9] However, its action on (2.4) only is defined over some localization of $\mathbb{Z}$ thanks to the denominators.

We wish to refine this projector to obtain the smallest possible denominators. We claim that the greatest common divisor of the denominators of these projectors (as a varies) divides a power of c: indeed, for any prime p not dividing c, we can find $a \equiv 1 \pmod{c}$ with $p \mid a$, so that none of the $1 - a^i$ are divisible by p. Choose $e_L^{(a_1)}, \ldots, e_L^{(a_k)}$ with corresponding denominators $D_1, \ldots, D_k$ with greatest common denominator c^t; there thus exist integers $b_1, \ldots, b_k$ with

$$\sum_{i=1}^{k} b_i D_i = c^t.$$

Then the linear combination

$$e_L := \frac{1}{c^t} \sum_{i=1}^{k} b_i D_i e_L^{(a_i)}$$

defines a projector with $\mathbb{Z}[1/c]$ coefficients annihilating $H^{d-1}_{\Gamma}(E)$ and fixing any class in $H^d_{\Gamma,C}(E)$ (by the congruence condition on the a_i).

[9]The latter statement is true assuming $H^0(*)$ is torsion-free, which will be true in all settings we consider except coherent cohomology. (The presence of torsion can cause different weights for the isogenies to become congruent, annihilating some classes in $H^0(*)$.) However, even in the coherent setting, we always are only concerned with classes in a submodule which is torsion-free, so this subtlety is immaterial.

Thus, $z_{\mathcal{C}} := e_L\tilde{\mathcal{C}} \in H^{d-1}_\Gamma(E - C)$ is well-defined independent of the choice of lift $\tilde{\mathcal{C}}$, and it is this class we call the polylogarithm class associated to $\mathcal{C}$.

Remark 2.1. The approach we are taking to Borel equivariant cohomology, taking group cochains of functorial chain complexes computing cohomology, is slightly different from the usual way one encounters the term in topology, where the Borel construction (which we consider later in (3.2) in the topological setting) is at the center. In the context of the de Rham cohomology of topological spaces, we show that these approaches coincide in the section following (3.2).

2.2. **Hochschild-Serre edge maps and chasing spectral sequences.** To obtain a group cocycle from $z_{\mathcal{C}}$, we use the spectral sequence (2.3). In order to make this work, we must restrict to a subspace of E to ensure a certain vanishing.

Suppose there is a diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B & = & B \end{array} \tag{2.11}$$

such that U has cohomological dimension $n = d/2$ (with respect to the "bare" cohomology theory). Then this vanishing of the cohomology of U affords us an edge map

$$H^{d-1}_\Gamma(U) \to E_2^{d-1-r,r} = H^{n-1}(\Gamma, H^n(U)) \tag{2.12}$$

This map then sends $j^*z_{\mathcal{C}}$ to a class

$$\theta_{\mathcal{C}} \in H^{n-1}(\Gamma, H^n(U)).$$

As suggested in the introduction, the last step is thus relating $H^n(U)$ to a space of differential forms of degree n; more precisely, we want a map $H^n(U) \to \Omega^n_U$ splitting the map $\Omega^n_U \to H^n(U)$ coming from Hodge theory. At least in the algebraic setting, this is possible via Hodge theory (see §3.5 and §3.5.2) and so we can (in an abuse of notation) consider $\theta_{\mathcal{C}}$ to be valued in Ω^n_U. We can then take the contraction with a suitable Γ-invariant n-vector field X on U,[10] denoted as per convention in differential geometry by $\iota_X\theta_{\mathcal{C}}$. We can view this class

[10]Also called the *interior product* with X.

as lying in the space

$$H^{n-1}(\Gamma, H^0(U, \pi'^*\pi'_*\Omega^n_U)),$$

as follows: on an open $V \subset U$, sections of $\pi'_*\pi'_*\Omega^n_U$ can be identified with sections of $\mathcal{O}_V \otimes_{\mathcal{O}_{\pi'V}} \pi'_*\Omega^n$ on π'_*V. Then we define the image of $\omega \in H^0(U, \Omega^n_U)$ on the open set V to be

$$1 \otimes_{\mathcal{O}_{\pi'V}} \iota_X\omega \in \mathcal{O}_V \otimes_{\mathcal{O}_{\pi'V}} \pi'_*\Omega^n_U. \tag{2.13}$$

where we can view $\iota_X\omega$ as a section of $\pi'_*\Omega^n_U$ on π'_*V.[11]

Then if there is a Γ-invariant section $s : B \to U$, we can take the pullback

$$s^*\iota_X\theta_{\mathcal{C}} \in H^{n-1}(\Gamma, H^0(B, \pi'_*\Omega^n_U))$$

to obtain a group cohomology class valued in sections of a bundle over the base.

In the setting of de Rham or coherent cohomology, this yields a map from the homology of an arithmetic group Γ (which can then be naturally identified as a Hecke module with some space of automorphic forms) to sections of automorphic line bundles on on B a locally symmetric space. Hence, we can justifiably call it a theta lift; in particular, the family of lifts that we can construct in this way we term "Eisenstein" theta lifts to extend the terminology of [BCG1, §13].

In other settings, the pulled back class itself $s^*\theta_{\mathcal{C}} \in H^{n-1}(\Gamma, H^n(B))$ can be interesting, as we will see in the motivic/Milnor K-theory setting.

Regardless of which setting we are in, we need to obtain some explicit control of the edge map (2.12). To do this, we return to the functor D defined earlier; recall that the cohomology $H^\bullet(X)$ is computed as the cohomology of the complex

$$D^\bullet(X) := \left[D^0(X) \xrightarrow{d} D^1(X) \xrightarrow{d} D^2(X) \xrightarrow{d} \dots\right]$$

and the Γ-equivariant refinement comes from the double complex $C^\bullet(\Gamma, D^\bullet(X))$ as in (2.2). Write $C^{\bullet,\bullet}$ for this complex for the remainder of the section, for brevity of notation.

[11]Besides the step of the contraction, this is how one explicitly writes the unit of the adjunction between π'_* and π'^*.

The Hochschild-Serre spectral sequence then results from the spectral sequence of a double complex for $C^{\bullet,\bullet}$, with the vertical filtration (in the orientation drawn above), i.e. so that $F^iC^{\bullet,\bullet} := C^{\bullet,i+\bullet}$

$$E_1^{p,q} = C^p(G, H^q(X)).$$

Let us return to the setting of the previous section, taking $G = \Gamma$ and $X = U$. We have a class $z \in H_\Gamma^{d-1}(U)$; suppose we are given even an explicit representative $z = [\omega]$ for some

$$\omega \in \bigoplus_{p+q=d-1} C^p(\Gamma, D^q).$$

We will $\omega^{(i,j)}$ for the component of ω in bidegree (i,j). Above, we exploited the fact that the acyclicity of $D^\bullet(U)$ above degree n gives us an edge map

$$H_\Gamma^{d-1}(U) \to H^{n-1}(\Gamma, H^n(U)).$$

Can the image of z under this edge map be related to ω in a simple way?

The following lemma, a general diagram chasing result holding in any spectral sequence of a double complex with analogous acyclicity conditions, will show us how:

Lemma 2.2. *In the situation above, we can write*

$$\omega^{(n-1,n)} = \omega_{cl}^{(n-1,n)} + \partial\sigma$$

for $\omega_{cl}^{(n-1,n)}$ d-closed and some $\sigma \in C^{n-2}(\Gamma, D^n(U))$. Then

$$[\omega^{(n-1,n)}] = [\omega_{cl}^{(n-1,n)}] \in H^{n-1}(\Gamma, H^n(U)).$$

is the image of $[\omega]$ under the edge map considered above, and this class is independent of choices. Further,

$$[\omega_{cl}^{(n-1,n)}] \in H^{n-1}(\Gamma, D^n(U)_{cl})$$

furnishes a cocycle representative of a cohomology class valued in d-closed elements giving a representative of the above class valued in closed elements.

Proof. The choice of cohomological degree $n = d/2$ is actually immaterial; any cohomological degree $r < d-1$ for U yields the analogous result for the bidegree $(d-1-r,r)$-component and the edge map valued in

$$H^{d-r-1}(\Gamma, H^r(U)).$$

We phrase the proof in this generality since it helps to clarify and distinguish the role of the cohomological degree of U; the lemma as stated is recovered by setting $r = n$.

First, note that the statement makes sense because $\partial\omega^{(d-1-r,r)}$ is a d-coboundary, and hence $\omega^{(d-1-r,r)}$ is ∂-closed considered as a cochain valued in $H^r(U)$.

In the degenerate case $r = d-1$, the assumption abou the cohomological degree is denegerate, and $\omega^{(0,d-1)}$ must already be d-closed. Hence we take $\sigma = 0$ and recover the classical expression for the usual edge map

$$H^{d-1}(C^{\bullet,\bullet}) \to E_2^{0,d-1} = H^0(\Gamma, H^{d-1}(U)).$$

Now, we construct the decomposition in question in general. Since $\omega^{(0,d-1)}$ is d-closed and $D^\bullet(U)$ is acyclic above degree r, there hence exists α_0 in bidegree $(0, d-2)$ with $d\alpha_0 = -\omega^{(0,d-1)}$. We can compute that

$$d(\omega^{(1,d-2)} - \partial\alpha_0) = d\omega^{(1,d-2)} - \partial\omega^{(0,d-1)} = 0$$

so we $\omega^{(1,d-2)} - \partial\alpha_0$ is again d-closed, and hence d-exact if we are still to the right of r, enabling us to construct $\alpha_1 \in \omega^{(1,d-3)}$ with $d\alpha_1 = \omega^{(1,d-2)} - \partial\alpha_0$. We continue in this fashion, constructing $\alpha_2, \alpha_3, \ldots, \alpha_{d-2-r}$ with α_i in bidegree $(i, d-2-i)$ such that

$$d(-\alpha_i) = \omega^{(i,d-1-i)} - \partial\alpha_{i-1}.$$

In particular, at the end of the sequence,

$$\omega^{(d-1-r,r)} - \partial\alpha_{d-2-r}$$

is again d-closed (though we are no longer able to conclude it is d-exact), so we can take $\sigma = \alpha_{d-2-r}$ to get the representation posited in the lemma, i.e.

$$\omega_{cl}^{(d-1-r,r)} := \omega^{(d-1-r,r)} - \partial\alpha_{d-2-r}.$$

Furthermore, we have that

$$\partial\omega_{cl}^{(d-1-r,r)} = \partial\omega^{(d-1-r,r)} = -d\omega^{d-r,r+1},$$

meaning that considered as a $H^q(U)$-cohomology valued cochain, $\omega_{cl}^{(d-1-r,r)}$ is ∂-closed and hence yields a class

$$[\omega_{cl}^{(d-1-r,r)}] \in H^{d-1-r}(\Gamma, H^q(U))$$

We claim this is the image of $[\omega]$ under the E_2 edge map. As a sanity check, different choices of α in the process above can only change $\omega_{cl}^{(d-1-r,r)}$ by a ∂-exact form, so this is well-defined.

Indeed, write $\alpha := \sum \alpha_i$. Then $\omega - \partial\alpha \in F^{d-1-r}C^{\bullet,\bullet}$ is a representative for the class

$$[\omega] \in F^{d-1-r}H^{d-1}(C^{\bullet,\bullet}) \subset H^{d-1}(C^{\bullet,\bullet}).$$

The cohomology of $F^{d-1-r}H^{d-1}(C^{\bullet,\bullet})$ is computed by $F^{d-1-r}C^{\bullet,\bullet}$. Changing the labels on the bidegrees along

$$\bullet,\bullet \mapsto \bullet,\bullet - r.$$

we see this filtered part of the complex now falls into the degenerate case of the classical edge map considered above. Hence, $\omega_{cl}^{(d-1-r,r)}$ is a representative of the image of

$$[\omega] = [\omega - \partial\alpha] \in F^{d-1-r}H^{d-1}(C^{\bullet,\bullet})$$

under the edge map for $F^{d-1-r}C^{\bullet,\bullet}$

$$F^{d-1-r}H^{d-1}(C^{\bullet,\bullet}) \to H^{d-1-r}(\Gamma, H^r(U)).$$

The functoriality of edge maps under the inclusion of complexes $F^{d-1-r}C^{\bullet,\bullet} \hookrightarrow C^{\bullet,\bullet}$ affords us the conclusion. $\square$

This lemma and its proof imply in particular that if some ω is such that $\omega^{(d-1-r,r)}$ is already d-closed, then the latter gives a cocycle representative

$$[\omega] \in H^{d-1-r}(\Gamma, D^r(U)_{cl})$$

for the $H^r(U)$-coefficients image of $[\omega]$ under the edge map.

This lemma and especially the "staircase" technique in its proof will be our principal tool in comparing the spectral sequence construction with more explicit constructions.

3. Analytic and algebraic constructions

In this section, we recapitulate the theory first laid out in [BCG1] of representing the Eisenstein theta correspondence in terms of an explicit theta kernel, in the algebraic setting. We then give an algebraic approach to such a theta kernel, following in part the construction of [KS], and apply the methods and results of [BCG3] in our setting to compare the two classes.

3.1. **Some setup and notation.**

We begin by setting up some notational conventions for our constructions.

Let $\mathbf{G}/\mathbb{Q}$ and $\mathbf{H}/\mathbb{Q}$ be a pair of reductive algebraic groups equipped with embeddings ι_G, ι_H of $\mathbf{G}$ and $\mathbf{H}$ into $\mathbf{GL}_d/\mathbb{Q}$ such that $\mathbf{G} \subset \mathbf{Z}_{\mathbf{GL}_d}(\mathbf{H})$, i.e. the images commute.

In addition, the principal setting of interest in this thesis will assume the following two algebraicity conditions:

- (**Alg1**) $\mathbf{H}$ admits a Shimura datum D in the sense of [Del], i.e. a symmetric Hermitian domain D parameterizing an $\mathbf{H}(\mathbb{R})$-conjugacy class of embeddings of the Deligne torus

$$\mathbf{S}_{Del} := \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}\mathbf{G}_m \hookrightarrow \mathbf{H}_{\mathbb{R}},$$

  and such that the stabilizer of a point of D for the left conjugation action of $\mathbf{H}(\mathbb{R})$ on these embeddings can be identified with a maximal compact subgroup $K_\infty(\mathbf{H}_{\mathbb{R}}) \subset \mathbb{H}(\mathbb{R})$. In particular, this yields an identification

$$D \cong X_{\mathbf{H}} := K_\infty(\mathbf{H}_{\mathbb{R}})\backslash\mathbb{H}(\mathbb{R})$$

as $\mathbf{H}(\mathbb{R})$-homogenous spaces.

- (**Alg2**) The family of Hodge structures on $\mathbf{V}$ associated to ι_H, i.e. the one that associates for each point of D the composite embedding

$$\mathbf{S}_{Del} \hookrightarrow \mathbf{H}_{\mathbb{R}} \xrightarrow{\iota_H} (\mathbf{GL}_d)_{\mathbb{R}},$$

is a family of polarizable Hodge structures of weight -1.[12] In particular, it admits a family of positive-definite Hermitian forms and its complexification decomposes (as a representation of $\mathbf{S}_{Del}$) into copies of the characters $z \mapsto z^{-1}$ and $(z_1, z_2) \mapsto \overline{z}^{-1}$. These are precisely the weights of the Hodge structure on the first homology group uniformizing a complex torus, and the polarizability implies that this complex torus has the structure of an abelian variety.

When these conditions are satisfied, we will call $(\mathbf{G}, \mathbf{H})$ an *algebraic pair*. These assumptions are what allow us to put an algebraic structure on the corresponding family of complex tori over the locally symmetric space, a prerequisite condition to most of this thesis.

Remark 3.1. Theta correspondences usually start from the perspective of groups which are additionally mutual centralizers, i.e. $\mathbf{Z}_{\mathbf{GL}_d}(\mathbf{G}) = \mathbf{H}$ and vice versa. If we imposed this condition in the current setting, then at each place of $\mathbb{Q}$ the pair $(\mathbf{G}, \mathbf{H})$ would yield a dual pair of type II in the sense of [Howe] inside $\mathbf{Sp}_{2d}$, via the embedding

$$\mathbf{GL}_d \to \mathbf{Sp}_{2d}, A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix}.$$

Our avenue of exploration in this thesis will not use the "duality" aspects of the theta correspondence, not least because in most of our constructions, the roles of $\mathbf{G}$ and $\mathbf{H}$ are fundamentally asymmetric: we only consider algebraic/complex structures on the locally symmetric space of the latter rather than the former. Thus, we do not restrict ourselves to true dual pairs, though many of the primary examples of interest (for example $(\mathbf{GL}_n, \mathbf{GL}_2)$) do happen to be dual.

[12] Note that this is true for the whole family if and only if it is true for one point in D, since we can change the polarization by the adjoint action.

$\mathbf{GL}_d$ has a natural structure over $\mathbb{Z}$, and acts via its standard representation on a d-dimensional affine space we call $\mathbf{V}/\mathbb{Z}$; $\mathbf{G}$ and $\mathbf{H}$, as subgroups, acquire $\mathbb{Z}$-structures as well by taking their Zariski closures inside $\mathbf{GL}_d/\mathbb{Z}$. For any scheme S, $\mathbf{V}(S)$ is then a representation of $\mathbf{G}(S)$ and $\mathbf{H}(S)$ via ι_G, ι_H.

There is a locally symmetric space $Y(H)$ for any arithmetic subgroup $H \subset \mathbf{H}(\mathbb{Q})$ given by

$$K_f(H)\mathbf{H}(\mathbb{Q})\backslash(X_{\mathbf{H}} \times \mathbf{H}(\mathbb{A}_f))/\mathbf{Z}_{\mathbf{H}}(\mathbb{A}_f)$$

where $X_{\mathbf{H}} = \mathbf{Z}_{\mathbf{H}}(\mathbb{R})K_\infty(\mathbf{H}_{\mathbb{R}})\backslash\mathbf{H}(\mathbb{R})$ is the symmetric space associated to $\mathbf{H}$ given by quotienting by a maximal compact subgroup, and $K_f(H)$ is the open subgroup of $\mathbf{H}(\mathbb{A}_f)$ associated to H. Each connected component of the locally symmetric space is homeomorphic to $H\backslash X^0_{\mathbf{H}}$, where $X^0_{\mathbf{H}}$ is the neutral component of the symmetric space. $Y(H)$ has the structure of an algebraic variety in the presence of condition (**Alg1**), defined over a number field associated to the Shimura datum called the reflex field [Del].

So long as H is small enough to act without fixed points on $X_{\mathbf{H}}$, the representation associated to the inclusion $\iota_H : \mathbf{H} \hookrightarrow \mathbf{GL}_d$ gives rise to a universal family of d-dimensional tori $A(H) \to Y(H)$ over $Y(H)$, given by the uniformization

$$(\mathbf{V}(\widehat{\mathbb{Z}}) \rtimes K_f(H))(\mathbf{V} \rtimes \mathbf{H})(\mathbb{Q})\backslash(\mathbf{V} \rtimes \mathbf{H})(\mathbb{A})/\mathbf{Z}_{\mathbf{H}}(\mathbb{A}_f)K_\infty(\mathbf{H}(\mathbb{R})) \tag{3.1}$$

where the semidirect product is for the action of $\mathbf{H}$ on $\mathbf{V}$ we have defined. In the presence of (**Alg2**), $A(H) \to Y(H)$ has the structure of a family of abelian varieties of genus $\frac{d}{2}$ defined over the reflex field of the Shimura datum we associate to $\mathbf{H}$ [HS].

$A(H)$ is endowed with an action of $\mathbf{G}(\mathbb{Z})$ since this group's fiberwise action commutes with the monodromy action of H. To be precise, $\mathbf{G}(\mathbb{Z}) \subset \mathbf{G}(\mathbb{A})$ acts on $\mathbb{V}(\mathbb{A})$ in (3.1) via the representation ι_G, and this action descends through both the left and right quotients because the images of ι_G and ι_H commute by assumption.

3.2. **Equivariance and fiber bundles over** $K(\Gamma, 1)s$**.** Let us now turn to translating the equivariant formalism of 2 into the more geometric language used in [BCG1, §3]. Throughout this section, all cohomology will be the ordinary cohomology of topological spaces unless otherwise noted.[13]

[13]This is not quite precise, since there are various ways (Čech, singular, de Rham, etc.) to compute "ordinary" cohomology which can disagree in pathological situations. All the spaces we work with will be arbitrarily nice, so this will not be an issue, but for now the reader can view this statement as a generality to be made precise later.

3.2.1. *Classifying spaces and equivariance.* Write $B\Gamma$ for the simplicial model of the classifying space of the discrete group Γ, considered as a simplicial complex; for the original reference for this concept, see [Mil]. Its k-simplices are labeled by homogenous $(k+1)$-tuples $[g_0 : g_1 : \ldots : g_k]$ of elements of Γ, and it is uniformized by the contractible simplicial complex $E\Gamma$ whose k-simplices are labeled by $(k+1)$-tuples $(g_0, \ldots, g_k)$, via quotienting by the left g-action

$$g : (g_0, \ldots, g_k) \mapsto (gg_0, \ldots, gg_k).$$

There is a standard equivariant-to-geometric dictionary we will freely use taking (left, by convention) Γ-equivariant objects to objects "over" the geometric realization $|B\Gamma|$. Heuristically, the idea of this dictionary is that there is a a functor from objects with Γ-action to objects fibered over $|B\Gamma|$ given by the *Borel construction*

$$X \mapsto X \times_\Gamma |E\Gamma| \tag{3.2}$$

which is an equivalence of categories. Additional structure or auxiliary constructions on X then correspond to the same on the fiber bundle. To illustrate what this means, we give a sample of the most common translations, each of which should be interpreted as an equivalence of categories or a natural isomorphism of functors:

equivariant	geometric
Γ-representation $\rho : \Gamma \to \mathrm{GL}(V)$	local system on $\lvert B\Gamma\rvert$ with fiber V and monodromy ρ
Γ-invariants V^Γ	global sections $H^0(\lvert B\Gamma\rvert, V)$
group cohomology $H^i(\Gamma, V)$	cohomology with local coefficients $H^i(\lvert B\Gamma\rvert, V)$
Γ-space X	fibered space $X \times_\Gamma \lvert E\Gamma\rvert \to \lvert B\Gamma\rvert$
equivariant cohomology $H^i_\Gamma(X)$	cohomology of the space $H^i(X \times_\Gamma \lvert E\Gamma\rvert)$
Hochschild-Serre spectral sequence	Leray spectral sequence for the bundle $X \times_\Gamma \lvert E\Gamma\rvert \to \lvert B\Gamma\rvert$

Unfortunately, there appears to be a dearth of comprehensive references for this dictionary in the literature; one can be found in [NSS, §3.7], though it is phrased in ∞-language more abstract than necessary for our purposes. We will explicate the last two rows of this dictionary in the setting of de Rham cohomology below.

3.2.2. *The de Rham formalism, distributions, and currents.* We return now to the formalism of §2.1 in the setting of ordinary cohomology of topological spaces. While we could continue working with singular cochains, it is more practical for us to use de Rham complexes instead. We do not need the integral or rational structures in ordinary cohomology in this thesis, and all the spaces we will deal with hereon are smooth manifolds. There exists a functorial quasi-isomorphism between the Čech complex (with $\mathbb{R}$-coefficients) and the de Rham complex for such spaces by [BT], so the discussion of the previous section all applies.

For expository purposes,[14] we wish to work with not only the usual de Rham complex, but also *distributional* de Rham cohomology, defined in terms of *currents*, which intuitively are smooth differential forms with distributional coefficients. See [BT] for a textbook reference on traditional de Rham cohomology, and de Rham's original text [dR] for the distributional version; we will be brief in describing the properties we need here.

Write W^i_X, respectively $W^i_{X,c}$, for the smooth complex-valued[15] i-forms, respectively *compactly supported* i-forms, on a smooth manifold X of dimension k.[16] These are equipped with the usual exterior derivative $d : W^i_X \to W^{i+1}_X$ (and similarly for the compactly supported versions), forming the usual smooth de Rham complex.

The smooth i-currents are defined by the dual

$$\mathcal{D}^i_X := \hom(W^{k-i}_{X,c}, \mathbb{C}).$$

The exterior derivative $d : \mathcal{D}^i_X \to \mathcal{D}^{i+1}_X$ is defined by adjunction to the exterior derivative on forms, i.e. by $(d\alpha)(\omega) := \alpha(d\omega)$. While currents do not have a product structure like forms do, they do have a right module structure under the wedge product for forms, via the product

$$\mathcal{D}^j_X \otimes W^i_X \to \mathcal{D}^{i+j}_X, (\alpha \wedge \eta)(\omega) := \alpha(\eta \wedge \omega).$$

[14] It is possible to develop the theory in this section without distributions, but the distributional de Rham complex offers certain technical advantages, and we will need it later in this thesis anyway.

[15] Everything in this subsection is true with real coefficients as well, but just as with the integral/rational structures in singular cohomology, there is no real advantage to this since we will need complex coefficients for the main results in the next chapter. Thus for notational ease, we define everything with complex coefficients from the outset.

[16] We use W for differential forms instead of the customary Ω, which we reserve for *holomorphic* forms.

Given a smooth submersion $f : X \to Y$, we have a pullback

$$f^* : \mathcal{D}_Y^i \to \mathcal{D}_X^i, (f^*\alpha)(\omega) := \alpha(f_*\omega)$$

where $f_*\omega$ means integration along the fiber. For a finite map $f : X \to Y$, we can also define the pushforward

$$f_* : \mathcal{D}_X^i \to \mathcal{D}_Y^i, (f_*\alpha)(\omega) := \alpha(f^*\omega).$$

Both of these can be checked to commute with the exterior derivative.

For any closed submanifold $Z \subset X$ of codimension s, we have a closed *current of integration*

$$\delta_Z \in \mathcal{D}_X^s$$

defined by

$$\delta_Z(\omega) := \int_Z \omega.$$

Finally, there is a natural map

$$\upsilon : W_X^i \to \mathcal{D}_X^i, \omega \mapsto \left(\eta \mapsto \int_X \eta \wedge \omega\right)$$

which is compatible with pullback/pushforward, intertwines the respective exterior derivatives and the wedge product module structure, and yields a map of de Rham complexes turns out to be a quasi-equivalence. Via this map, we can and will implicitly view differential forms as currents.[17]

If X is a Γ-space, we can take the functor D interchangeably to be either the de Rham or distributional de Rham complexes; i.e., the equivariant ordinary cohomology $H_\Gamma^i(X)$ can then be defined in our formalism as the total cohomology of either complex $C^\bullet(\Gamma, W_X^\bullet)$ or $C^\bullet(\Gamma, \mathcal{D}_X^\bullet)$, since υ induces a quasi-isomorphism between the double complexes.

[17]In fact, in the presence of complex structure, both the definition of υ and the wedge module structure make sense not only for smooth global forms, but even those with log singularities, i.e. locally of the form

$$\log f_0 \cdot d\log f_1 \wedge \ldots \wedge d\log f_k,$$

for f_i meromorphic in each local chart. This will be important in §4.2.3.

Both approaches offer their own advantages. From the classical de Rham complex, we obtain contravariant functoriality for all smooth maps (not just submersions), though both approaches give covariant functoriality for finite maps. The localization sequence in the classical theory is the content of [BT, p. 6.49]; note that in this setting, ι^* is not surjective, cf. §2.1, so the "relative de Rham complex" constructed there is the mapping cylinder.

It is more convenient to formulate the localization sequence in the distributional setting: since $\iota : X - Z \to X$ is certainly a submersion, it is legitimate to consider the pullback

$$\iota^* : \mathcal{D}^i_X \to \mathcal{D}^i_{X-Z},$$

and this map *is* a surjection, because it is the dual of the pushforward

$$\iota_* : W^i_{X-Z,c} \to W^i_{X,c}$$

which is certainly an injection. Notice that in the resulting localization sequence, the snake lemma implies that the map

$$H^i(X - Z) \to H^{i+1}_Z(X)$$

is induced by the exterior derivative sending a closed i-current on $X - Z$ (viewed as a current on X closed along $X - Z$) to a $(i + 1)$-current in $\ker \iota^*$.

The Gysin isomorphism is made very convenient by currents: we have the natural isomorphism of complexes

$$\mathcal{D}^{i-d}_Z \xrightarrow{\sim} \ker(\mathcal{D}^i_X \to \mathcal{D}^i_{X-Z}), \eta \mapsto \delta_Z \wedge \eta \tag{3.3}$$

for a closed subspace $j : Z \subset X$, defined as the dual of the pullback of forms

$$\Omega^{k-i}_{X,c} \to \Omega^{k-i}_{Z,c}$$

from X to Z; it is formal to check that this is functorial for finite pushforwards. One can check that restricted to $(i - d)$-forms ω, this map is

$$\omega \mapsto \delta_Z \wedge \omega.$$

We wish now to translate everything on the geometric side of the dictionary. Since the geometric realizations $|B\Gamma|$ and $|E\Gamma|$ generally do not come equipped with a smooth manifold structure, the de Rham approach taken in this subsection cannot be directly applied to the Borel construction. We wish, then, to find homotopy equivalent smooth manifold models for these spaces.

Before constructing the specific de Rham model for our situation, we formulate the translation abstractly; this will reduce the notational baggage. Suppose that E is a smooth manifold with Γ-action, with a Γ-invariant subspace C. Suppose further that we have a Γ-covering $X \to Y$ of smooth manifolds such that X is contractible, along with a diagram of Γ-maps

$$\begin{array}{ccc} |E\Gamma| & \xrightarrow{\tilde{R}} & X \\ \downarrow & & \downarrow \\ |B\Gamma| & \xrightarrow{R} & Y \end{array} \tag{3.4}$$

where both horizontal maps are homotopy equivalences. We say that this "realizes Y as a classifying space for Γ."

We recall that the equivariant (de Rham) cohomology $H^\bullet_\Gamma(E)$ is computed by the double complex $C^\bullet(\Gamma, W^\bullet_E)$. On the other hand, the equivariant-geometric dictionary tells us that the same groups are computed by the "de Rham version" of the Borel construction $H^\bullet(E \times_\Gamma X)$, computed by the de Rham complex $W^\bullet_{E\times_\Gamma X}$.

Rather than constructing a direct quasi-isomorphism between the two complexes (which seems difficult), we outline why they coincide in the spirit of the dictionary, i.e. more conceptually. The idea is that we have an exact functor

$$B : \Gamma{-}\mathbf{Sh}/E \to \Gamma\mathbf{Sh}/E\times_\Gamma$$

between Γ-sheaves (of $\mathbb{R}$-vector spaces) on E and sheaves (of $\mathbb{R}$-vector spaces) on the Borel construction of E. On objects, this functor sends a sheaf $\mathcal{F}$ to the pullback by projection onto the first coordinate $\pi_1^*\mathcal{F}$ on $E \times X$; via the Γ-action on $\mathcal{F}$, this is a Γ-sheaf on the product space. It hence descends by the Γ-cover a sheaf $E \times_\Gamma X$. It is defined on arrows similarly, and since exactness of sequences of sheaves can be checked on stalks, it is formal that B is exact.

Now, the constant sheaf $\underline{\mathbb{C}}$ on E with trivial Γ-action has a acyclic resolution given by the total complex $C^\bullet(\Gamma, W_E^\bullet)$. Thus, the equivariant cohomology groups $H_\Gamma^\bullet(E)$ can be identified with the image of $\underline{\mathbb{R}}$ derived functor $R^\bullet\Gamma_E^\Gamma$ of the functor of Γ-invariant global sections[18]

$$\Gamma_E^\Gamma : \Gamma{-}\mathbf{Sh}/E \to \mathbb{C}{-}\mathbf{Vector}.$$

Meanwhile, $B(\underline{\mathbb{C}})$ is the constant sheaf on $E \times_\Gamma X$, and $W^\bullet_{E\times_\Gamma X}$ is an acyclic resolution of this object. Hence we can identify it as a derived functor as well:

$$H^\bullet(E \times_\Gamma X) \cong (R^\bullet\Gamma_{E\times_\Gamma X})(B(\underline{\mathbb{C}}))$$

But one can also check that we have the identification of abelian functors $\Gamma_E^\Gamma = \Gamma_{E\times_\Gamma X} \circ B$. Since B is exact, one finds from the the degeneracy of the corresponding Grothendieck spectral sequence [Tohoku] the identification

$$H^\bullet(E \times_\Gamma X) \cong (R^\bullet\Gamma_{E\times_\Gamma X})(B(\underline{\mathbb{C}})) \cong R^\bullet\Gamma_E^\Gamma(\underline{\mathbb{C}}) \cong H_\Gamma^\bullet(E),$$

as desired.

We abbreviate the de Rham Borel construction $E \times_\Gamma X$ to Bor(E), etc. for the rest of this section for convenience.

In the geometric translation, the localization sequence for Bor$(C) \subset$ Bor(E) is then the long exact sequence in ordinary cohomology

$$\ldots \to H^i(\text{Bor}(E)) \to H^i(\text{Bor}(E) - \text{Bor}(C)) \to H^{i+1}_{\text{Bor}(C)}(\text{Bor}(E)) \to \ldots$$

and the Gysin isomorphism

$$H^i_{\text{Bor}(C)}(\text{Bor}(E)) \cong \tilde{H}^{i-d}(\text{Bor}(C))$$

is given by the excision axiom for cohomology: the pair of spaces $(\text{Bor}(E), \text{Bor}(E) - \text{Bor}(C))$, by excising everything but a small tubular neighborhood of Bor(C) as a closed submanifold of Bor(E), is homology equivalent to $(\text{Bor}(C) \times B^d, \text{Bor}(C) \times S^{d-1})$ where B^d the closed unit d-ball (in each fiber of the normal

[18]We use the usual notation for global sections Γ, which due to the choice of the name of our group, gives an unfortunate overloading of notation.

bundle of $\mathrm{Bor}(C)$ in $\mathrm{Bor}(E)$ embedded as the tubular neighborhood), and S^{d-1} its boundary in each fiber. This last pair of spaces is the d-fold suspension $\Sigma^d\mathrm{Bor}(C)$, and so its homology can be identified as

$$H^i(\Sigma^d\mathrm{Bor}(C)) \cong H^{i-d}(\mathrm{Bor}(C)).$$

In the context of $\mathrm{Bor}(C)$ a set of torsion sections inside $\mathrm{Bor}(E)$ a torus bundle, [BCG1, §3] calls this a "Thom isomorphism" (terminology which is traditionally is used only for the zero section).

In some ways all of this this is just a formal change of language, but the upside is that we will be able to equip the space playing the role of $\mathrm{Bor}(E)$ with a natural Riemannian structure. This structure we will leverage, following [BCG1, §5-8], to obtain canonical de Rham representatives for the cohomology classes we are interested in.

3.2.3. *A de Rham model for the classifying space.* The uniformization map from the connected $\mathbf{G}(\mathbb{R})$-symmetric space

$$\mathbf{G}(\mathbb{R})/K_\infty(\mathbf{G}(\mathbb{R}))\mathbf{Z}_{\mathbf{G}}(\mathbb{R}) \cong X_{\mathbf{G}(\mathbb{R})} \to Y(\Gamma)$$

is the quotient of a contractible space by a Γ-action by holomorphic isometries. The stabilizer of a point is compact [Hel], and therefore Γ acts with finite stabilizers of points modulo the kernel of the action.

The kernel of the Γ-action is contained in $K_\infty(\mathbf{G}(\mathbb{R}))\mathbf{Z}_{\mathbf{G}}(\mathbb{R})$. For $\mathbf{G}$ with no central anisotropic torus splitting over a totally real extension, the Dirichlet unit theorem implies there is no infinite arithmetic subgroup contained in this image. In this case, we conclude that if $\Gamma \subset \mathbf{G}(\mathbb{Z})$ is a torsion-free arithmetic subgroup, then the Γ-action is free.

If $\mathbf{G}$ does have such a torus, we can instead replace $X_{\mathbf{G}(\mathbb{R})}$ by

$$X'_{\mathbf{G}(\mathbb{R})} := \mathbf{G}(\mathbb{R})/K_\infty(\mathbf{G}(\mathbb{R}))\mathbf{Z}^s_{\mathbf{G}}(\mathbb{R}) \tag{3.5}$$

where $\mathbf{Z}^s_{\mathbf{G}}(\mathbb{R})$ is just the split part of the central torus; this now again has finite stabilizers since the $\mathbb{Z}$-points of a split torus have no infinite discrete subgroup. For the rest of this section, the quotient of $X'_{\mathbf{G}(\mathbb{R})}$ by Γ is what we will mean by the notation $Y(\Gamma)$ in this specific case. Note we mean this only for the group playing the role of Γ and not for the group playing the role of H, which we need to have a natural algebraic structure (which (3.5) never carries).

Example 3.2. If $\mathbf{G} = \operatorname{Res}_{\mathbb{Z}}^{\mathcal{O}_F} \mathbf{G}_m$ for $F/\mathbb{Q}$ a real quadratic extension, we have that

$$X_{\mathbf{G}(\mathbb{R})} = \mathcal{H} \times \mathcal{H},$$

i.e. two copies of the upper half-plane. The action of $\mathrm{GL}_2(\mathcal{O}_F)$ then has kernel $\mathcal{O}_F^\times$; the replacement $X'_{\mathbf{G}(\mathbb{R})}$ is a (trivializable) $\mathbb{R}$-bundle over $\mathcal{H} \times \mathcal{H}$ on which $\mathcal{O}_F^\times$ acts nontrivially.

In any case, we can construct the de Rham model as in (3.4):

$$\begin{tikzcd} {|E\Gamma|} \arrow[r, "\tilde{\rho}"] \arrow[d] & X_{\mathbf{H}} \arrow[d] \\ {|B\Gamma|} \arrow[r, "\rho"] & Y(\Gamma) \end{tikzcd} \tag{3.6}$$

Here, $\tilde{\rho}$ is defined as follows: given a choice of basepoint τ_0 in $X_{\mathbf{H}}$, we linearly map the geometric realization of the k-simplex labelled $[g_0 : g_1 : \ldots : g_k]$ to the geodesic simplex $\Delta_{\tau_0}(g_0, g_1, \ldots, g_k)$.[19]

For torsion-free Γ, ρ is a homotopy equivalence onto the connected component containing τ_0, realizing $Y(\Gamma)$ as a classifying space for the group Γ.[20]

In the remainder of the analytic treatment, for simplicity we will only work with the single connected component $Y(\Gamma)^0$ of $Y(\Gamma)$ (and $Y(H)$, etc.) containing some given basepoint τ_0, which we will choose to be the class of the identity under the analytic uniformization by $\mathbf{G}(\mathbb{R})/K_\infty$, for simplicity. Consequently, we will drop the superscript 0 indicating the neutral connected component from the notation. We do not expect the general case does not introduce any new real complications to the theory as presented here; this choice is only for convenience of formulating the results.

Example 3.3. Choose an arithmetic subgroup $\Gamma \subset \mathrm{GL}_2(\mathbb{Z})$ acting freely on the upper half-plane $\mathcal{H}$ via Möbius transformations. Then $\mathcal{H} \to \mathcal{H}/\Gamma$ is a quotient by a free action, and can be identified (in the homotopy

[19]When $k = 2$, this is simply the geodesic from $g_0\tau_0$ to $g_1\tau_0$. Otherwise, it is defined inductively as the geodesic join $g_0\tau_0 * \Delta_{\tau_0}(g_1, \ldots, g_k)$, meaning the union of all geodesics connecting the former point to the latter simplex.

[20]More generally, the map ρ is a homology equivalence away from the primes for which Γ has torsion.

category) with $|E\Gamma| \to |B\Gamma|$ via (3.6): for example, if we choose τ to be $i \in \mathcal{H}$, then the 1-simplex $[g_0 : g_1]$ is sent to the geodesic sending $g_0 i$ to $g_1 i$. If V is the standard 2-dimensional real representation of Γ, the corresponding local system on $Y(\Gamma) = \mathcal{H}/\Gamma$ can be identified with flat sections of the Hodge bundle ω if we forget the complex structure, i.e. the relative Lie algebra of the universal elliptic curve $\mathcal{E}(\Gamma) \to Y(\Gamma)$.

3.2.4. *More torus bundles.* Recall from Section 3.1 the Γ-space $A(H)$, which is a torus bundle over $Y(H)$. To this Γ-space there corresponds, from the equivariant-geometric dictionary, a bundle $A(\Gamma, H) \to Y(\Gamma)$ whose fibers are copies of $A(H)$, which is the bundle over $Y(\Gamma)$ given by

$$A(\Gamma, H) \cong A(H) \times_\Gamma X_{\mathbf{G}(\mathbb{R})}$$

Notice that $A(\Gamma, H)$ can equally be viewed as a bundle of tori over $Y(\Gamma) \times Y(H)$ since the Γ-action fixes the base $Y(H)$.

Even assuming the fibers of this bundle carry the structure of abelian varieties when considered over the algebraic structure on $Y(H)$ alone, the total bundle usually has no canonical algebraic or even complex analytic structure even if $Y(\Gamma)$ is also a Shimura variety: the complex/algebraic structure one might hope for coming from considering the bundle over $Y(\Gamma)$ is incompatible with that coming from considering it over $Y(H)$.

Example 3.4. The example considered in [BCG1] is that of $\mathbf{G} = \mathbf{GL}_n$, $\mathbf{H} = \mathbf{GL}_2$ with $\mathbf{H}$ endowed with the usual Shimura datum parameterized by the upper half-plane. We take $d = 2n$, and $\iota_G : \mathbf{GL}_n \hookrightarrow \mathbf{GL}_{2n}$, $\iota_H : \mathbf{GL}_2 \hookrightarrow \mathbf{GL}_{2n}$ to be given by the external tensor product of the natural representations of $\mathbf{G}$ and $\mathbf{H}$. In this case, $A(H)$ is given by the self-product of n copies of the universal elliptic curve over $Y(H)$, with the natural action of $\Gamma \subset \mathrm{GL}_n(\mathbb{Z}) = \mathbf{G}(\mathbb{Z})$. The bundle $A(\Gamma, H)$ is kind of a "tensor product torus bundle" over the product $Y(\Gamma) \times Y(H)$, in the sense that the fiber over a point of $Y(\Gamma)$ is the bundle over $Y(H)$ corresponding to the n-fold self-product of the standard representation of H, and conversely, the fiber over a point of $Y(H)$ is the bundle over $Y(\Gamma)$ corresponding to the squared standard representation of Γ.

Even if $n = 2$, so that $Y(\Gamma)$ is also an open modular curve with a natural algebraic structure, the bundle $A(\Gamma, H)$ still has no natural algebraic structure over the base $Y(\Gamma) \times Y(H)$; the complex structures implied by the $\mathbf{G}$- and $\mathbf{H}$-structures are incompatible.

The space $A(\Gamma, H)$ is uniformized by the Γ-cover

$$A(X_{\mathbf{G}}, H) := A(H) \times X_{\mathbf{G}(\mathbb{R})}.$$

Over the diagram (3.6), we have the analogous diagram of bundles

$$\begin{tikzcd} A(H) \times |E\Gamma| \arrow[r] \arrow[d] & A(X_{\mathbf{G}}, H) \arrow[d] \\ A(H) \times_{\Gamma} |E\Gamma| \arrow[r] & A(\Gamma, H) \end{tikzcd} \tag{3.7}$$

where the horizontal arrows are defined in the same way, i.e. the simplex $\{a\} \times [g_0 : \ldots : g_0]$ is sent linearly to the geodesic simplex $\Delta_{(a,\tau_0)}(g_0, \ldots, g_k)$ in the top row, and this descends along the pair of Γ-covers.

$A(\Gamma, H)$ hence plays the role of the space $\mathrm{Bor}(E)$ we discussed in the more abstract treatment earlier. Per the general discussion in that subsection, we have the identification

$$H^{\bullet}_{\Gamma}(A(H)) \cong H^{\bullet}(A(\Gamma, H)).$$

The formalism of §2.1 now gives rise to the same cohomological manipulations as found in [BCG1, §3]: for any $\mathcal{C} \in H^0(C)^{\deg=0}$, we get a class in $H^{d-1}(A(\Gamma, H) - C)$, with ambiguity of $H^{d-1}(A(\Gamma, H))$. Writing $[a]$ for the fiberwise multiplication-by-a isogeny; in a slight abuse, we will also use the same notation for restrictions thereof. If $(a, c) = 1$, $[a]$ gives a map

$$[a] : A(\Gamma, H) - A(\Gamma, H)[ac] \to A(\Gamma, H) - A(\Gamma, H)[c].$$

The composition in cohomology with the reverse inclusion is then an endomorphism of $\mathcal{A}(\Gamma, H) - \mathcal{A}(\Gamma, H)[c]$, and we will abusively also refer to it as $[a]$.

Then as defined previously, for $a \equiv 1 \pmod{c}$ the projectors

$$e_L^{(a)} := \frac{([a]_* - a)([a]_* - a^2)\dots([a]_* - a^d)}{(1-a)(1-a^2)\dots(1-a^d)}$$

can combine linearly to give a map

$$e_L : H^0(C)^{\deg=0} \to H^{d-1}(A(\Gamma, H) - C), \mathcal{C} \mapsto z_{\mathcal{C}}^{top}$$

with $\mathbb{Z}[1/c]$-coefficients.

3.3. **Transgressing the Mathai-Quillen form.** Now that we are done with the set up, the goal is to construct a canonical differential form representing $z_{\mathcal{C}}^{top} \in H^\bullet(A(\Gamma, H) - A(\Gamma, H)[c])$. This subsection is a rephrasing of the contents of [BCG1, §5-8] in our language.

We work in a superficially more general context than loc. cit., in the sense that they only consider bundles over the symmetric spaces for general linear groups. However, all these constructions are functorial for maps of Riemannian bundles of metrized tori, and so can be pulled back from the analogous ones for some general linear group. In particular, all our constructions are for the representation $\mathbf{V}$ of $\mathbf{G} \times \mathbf{H}$, whose Riemannian structure will be defined (see (3.8) below) by restriction of structure from considering $\mathbf{V}$ as a representation $\mathbf{GL}_d$, which is precisely the setting considered in [BCG1].

We hence do not need any original arguments, and do not pretend to give any in this subsection. We will abbreviate calculations and technical details inessential to presenting the main ideas, since they are identical in form to those of loc. cit.. The only notational differences of note are that we will use a tuple (g, h) to represent group elements instead of just g (because we want to emphasize that we are parameterizing elements of a *product* of groups), and we will use η for the Mellin transform of ψ, notation which does not appear in the original work [BCG1] (having been introduced in the follow-ups [BCG2] and [BCG3]).

The construction proceeds as follows: there is a Hodge line bundle $\omega(\Gamma, H) = \mathrm{Lie}\, A(\Gamma, H)$; it is the geometric counterpart to the representation $\mathbf{V}(\mathbb{R})$ of $\Gamma \times H$ in the equivariant-geometric dictionary. The sublattice $L := \mathbf{V}(\mathbb{Z})$ is the uniformizing lattice for $A(\Gamma, H)$; i.e. the Borel construction for the $(\Gamma \times H)$-torus $\mathbf{V}(\mathbb{Q})/\mathbf{V}(\mathbb{Z})$ yields the torus bundle $A(\Gamma, H)$.

Associated to the pair $(\mathbf{G}, \mathbf{H})$, we have an embedding

$$(\mathbf{G} \times \mathbf{H} \ltimes \mathbf{V})(\mathbb{R})/K_\infty(\mathbf{G} \times \mathbf{H} \ltimes \mathbf{V}) \hookrightarrow (\mathbf{GL}_d \ltimes \mathbf{V})(\mathbb{R})/O(d). \tag{3.8}$$

The latter bundle has a universal metric as given in [BCG1, §4], which is a symmetric positive definite bilinear form along each fiber of the vector bundle, agrees with the metric induced by the Killing form along the base (viewed as the zero section) $\mathbf{GL}_d/O(d) \cong X_{\mathbf{GL}_d}$, and is $\mathrm{GL}_d(\mathbb{R})$-invariant. Its pullback via this embedding thus descends to a Riemannian metric on $A(\Gamma, H)$ which is a flat metric invariant by translation by the group structure on each toroidal fiber.

A construction of Mathai and Quillen [MQ] yields, given such a Riemannian structure, a canonical rapidly decreasing d-form φ on the total space of $\omega(\Gamma, H)$, such that its class in $H^d(\omega(\Gamma, H), \omega(\Gamma, H) - \{0\})$ is the Thom class. This construction is for any Riemannian bundle, and is functorial for (isometric) maps of such bundles.

They further construct a $(d-1)$-form ψ on the same space such that

$$d([t]^*\psi) = t\frac{d}{dt}[t]^*\varphi$$

where $[t]$ is the multiplication-by-t isogeny on the fibers for $t > 0$. The key is that the Mellin transform

$$\eta := \int_0^\infty [t]^*\psi \frac{dt}{t}$$

satisfies [BCG1, Proposition 15]

$$d\eta = \delta_0 - \pi^* x_0^* \varphi$$

where $\pi : \omega(\Gamma, H) \to Y(\Gamma) \times Y(H)$ is the projection and

$$x_0 : Y(\Gamma) \times Y(H) \to \omega(\Gamma, H)$$

is the zero section.

Remark 3.5. [BCG1] describes η as a "transgression of the Euler class": since φ represents the Thom class, its pullback $x_0^*\varphi$ represents the Euler class by definition (as pullback of the Thom class by the zero section).

Recall from our discussion of the de Rham presentation of the Gysin isomorphism that the class corresponding to a torsion cycle

$$\mathcal{C} = \sum_{x \in C} a_x x$$

in $H_C^{2n-1}(A(\Gamma, H))$ is represented by a sum

$$\sum_{x \in C} a_x \delta_x$$

of currents of integration corresponding to x. Morally, then, it looks like to find the form $z_{\mathcal{C}}$ lifting $\mathcal{C}$, we should take some kind of sum of suitable translates of η.

The form η, however, lives on the vector bundle $\omega(\Gamma, H)$ and not the torus bundle $A(\Gamma, H)$. Thus, we would like to to take some kind of weighted sum (with weights corresponding $\mathcal{C}$) of η over a lattice commensurate L to descend η to a form on the torus bundle $A(\Gamma, H)$ representing $z_{\mathcal{C}}$. Because η is not rapidly decreasing, we cannot actually sum it over a lattice in this way. Instead, the idea is that since η is the Mellin transform of ψ, we instead take the corresponding sum for the form ψ, and only then take its Mellin transform. This leads to the expression

$$E_\psi(s, \varphi_f; g, h, z) := \int_0^\infty t^{ds} \theta_\psi(\varphi_f; g, h, z) \frac{dt}{t}.$$

which we hence can we can view as a "regularized sum" of η over the lattice. Here, φ_f is a function in the adelic Schwartz space[21] $\mathcal{S}(\mathbf{V}(\mathbb{A}_{\mathbb{Q}}^f))$ encoding the cycle $\mathcal{C}$, i.e. if $\mathcal{C}$ correponds to the formal sum $\sum_i c_i[\lambda_i]$ for λ_i a set of coset representatives in $V(\mathbb{Q})/L$ encoding the suitable torsion sections via the uniformization of the torus bundle, then

$$\varphi_f := \sum_i c_i \mathbf{1}_{\lambda_i + V(\mathbb{A}_{\mathbb{Z}})}$$

The theta series in the integral is then defined by

$$\theta_\psi(\varphi_f; g, h, z) := \sum_{\lambda \in V(\mathbb{Q})} \varphi_f((g_f, h_f)^{-1}(\lambda - z_f))\psi(\lambda) \tag{3.9}$$

where $\psi(\lambda)$ denotes pullback by translation by $-\lambda$ of the form ψ, viewed as a form on $(\mathbf{G} \times \mathbf{H})(\mathbb{A}) \ltimes \mathbf{V}(\mathbb{A})$ via the uniformization (3.1) for the representation $\mathbf{V}$ of $\mathbf{G} \times \mathbf{H}$. Here, s is a complex variable, g and h are elements in the adelic groups $\mathbf{G}(\mathbb{A})$ and $\mathbf{H}(\mathbb{A})$ respectively, and z is an element of $\mathbf{V}(\mathbb{A})$ (and λ is viewed as

[21]Meaning the space of locally constant functions with compact support on the finite adelic points of $\mathbf{V}$.

such via the diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}$). We write g_f, respectively, g_∞, for the finite adelic, respectively archimedean, part of the element g, and similarly for h and z. The action of (g, h) on $\lambda - z$ in the expression is then just the action of $(\mathbf{G} \times \mathbf{H})(\mathbb{A})$, broken into its separate parts.

For s in a right half-plane, the regularization integral converges absolutely, and it admits analytic continuation to $s = 0$. A priori, then, $E_\psi(s)$ is a differential form on the underlying space of the group

$$\mathbf{X_G} \times \mathbf{X_H} \times (\mathbf{G} \times \mathbf{H})(\mathbb{A}_f) \ltimes \mathbf{V}(\mathbb{A}), \tag{3.10}$$

but [BCG3, Chapitre 7, §3] shows that it descends to $A(\Gamma, H)$ - or rather, they show it for the group $\mathbf{GL}_d$, which implies it also for $\mathbf{G} \times \mathbf{H}$ by pullback/restriction of structure.

Remark 3.6. In fact, the regularization integral already converges at $s = 0$, making analytic continuation unnecessary for simply defining our desired form $E_\psi(0)$, but it is still useful for the following reason: if we set

$$\eta(s) := \int_0^\infty t^{ds}[t]^*\psi \frac{dt}{t}$$

then for Re $s > 0$, the theta series does converge absolutely and we can exchange sum and integral to make valid the naive definition

$$E_\psi(s) = \sum_{\lambda \in \mathbf{V}(\mathbb{Q})} \varphi_f((g, h)^{-1}(\lambda - z_f))\eta(\lambda)$$

Thus, $E_\psi(0)$ is an analytically continued sum of η over the lattice, in a way reminiscent of the Hecke regularization of classical Eisenstein series [Hecke]. In fact, if one works this out for the universal elliptic curve over the upper half-plane, one gets precisely this classical regularization for the weight-1 and weight-2 Eisenstein series; see [BCG1, §11.2].

In any case, our desired $(d-1)$-form $E_\psi(0, \varphi_f; g, h, \mathbf{z})$ then descends to the torus bundle $A(\Gamma, H)$ via the uniformization

$$A(\Gamma, H) = \mathbf{V}(\widehat{\mathbb{Z}}) \rtimes K_f(\Gamma \times H)(\mathbf{G} \times \mathbf{H} \ltimes \mathbf{V})(\mathbb{Q})\backslash(\mathbf{G} \times \mathbf{H} \ltimes \mathbf{V})(\mathbb{A}_\mathbb{Q})/K_\infty(\mathbf{G}_\mathbb{R} \times \mathbf{H}_\mathbb{R})\mathbf{Z}_{\mathbf{G}\times\mathbf{H}}(\mathbb{A}_\mathbb{Q})$$

(with a slight alteration to the central term when $\mathbf{G}$ has a central torus of the form described at the end of section 3.2.3).

For reasons that will become clear in section 3.5, we also will wish to consider the variant

$$A(\Gamma, X_{\mathbf{H}}) = \mathbf{V}(\widehat{\mathbb{Z}})K_f(\Gamma)(\mathbf{G} \ltimes \mathbf{V})(\mathbb{Q})\backslash(\mathbf{G} \ltimes \mathbf{V})(\mathbb{A}_{\mathbb{Q}}) \times X_{\mathbf{H}}/K_\infty(\mathbf{G}_{\mathbb{R}})\mathbf{Z}_{\mathbf{G}\times\mathbf{H}}(\mathbb{A}_{\mathbb{Q}})$$

fibered over the symmetric space $X_{\mathbf{H}}$, which uniformizes $A(\Gamma, H)$ via an H-covering map.[22]

Conceptually, the role of the Schwartz function φ_f should be understood as corresponding to the torsion cycle $\mathcal{C}$. In particular, a torsion cycle is a linear combination

$$\mathcal{C} = \sum_{x \in C} a_x x$$

where each x is a torsion section, which can be viewed as an element of $\mathbf{V}(\mathbb{Q})/L$ via the uniformization of $A(\Gamma, H)$. To this, we associate the Schwartz function

$$\varphi_f(\mathcal{C}) := \sum a_x \chi(x + L)$$

where $\chi(x + L)$ denotes the compactly supported indicator function of the closure of the coset $x + L$ inside $\mathbf{V}(\mathbb{A})$ (i.e. the function which is 1 on this compact open, and 0 elsewhere). Then, as in [BCG1, Theorem 19], we have that

$$dE_\psi(0, \varphi_f(\mathcal{C}); g, z) = \delta_{\mathcal{C}} := \deg \mathcal{C} \cdot \mathrm{vol} + \sum_{x \in C} a_x \delta_x.$$

where vol is the volume form on each fiber of the torus bundle $A(\Gamma, H)$. In loc. cit., they consider this derivative only in the case $\varphi_f = \varphi_f((0))$, but the expression (3.9) for $\varphi_f(\mathcal{C})$ is just a sum of translates of the theta series for $\varphi_f((0))$ - at least, over the base $(\mathbf{G} \times \mathbf{H})(\mathbb{A})$, which is a valid way to calculate as exterior derivatives are computed locally. In particular, for $\mathcal{C}$ of degree zero, $E_\psi(0, \varphi_f(\mathcal{C}); g, z)$ is closed when considered as a differential form on $A(\Gamma, H) - C$.

As in [BCG1, Proposition 20], the form E_ψ satisfies the isogeny invariance property

$$[a]_* E_\psi(0, \varphi_f; \tau, z) = E_\psi(0, [a]_* \varphi_f; \tau, z)$$

[22] The reason for this is technical: to obtain the edge map (2.12) in the argument sketched in section 2.2, we need the base (whose role is played by $Y(H)$ or $X_{\mathbf{H}}$ in this case) to be acyclic, which the latter is and the former is not (necessarily). We thus will run the argument over the latter base, then descend back to results over $Y(H)$ using H-invariance of the constructions.

for any integer a. In particular, $[a]_* E_\psi(0, \varphi_f; \tau, z) = E_\psi(0, \varphi_f; \tau, z)$ for any a with $[a]_*\varphi_f$ - in particular, for $\varphi_f = \varphi_f(\mathcal{C})$ for $\mathcal{C}$ annihilated by an integer c, this is true for all a with $(a, c) = 1$.

Putting the above together, we have the following culmination of our desires, which is [BCG1, Theorem 21]:

Proposition 3.7. *Writing $z_{\mathcal{C}}^{top}$ for the Eisenstein class defined earlier, we have*

$$[E_\psi(0, \varphi_f; g, z)] = z_{\mathcal{C}}$$

for any torsion cycle $\mathcal{C}$ (over $A(\Gamma, H)$) of degree zero.

Proof. As in the proof of [BCG1, Theorem 21], this follows from the fact that isogeny-invariance and the exterior derivative together characterize the class $z_{\mathcal{C}}^{top}$ by construction. □

Naturally, the same holds for the form and the cohomology class viewed (via pullback) as H-invariant constructions on $A(\Gamma, X_{\mathbf{H}})$.

3.4. **Analytic Eisenstein theta correspondence in the algebraic setting.** From here on, we always assume (**Alg1**) and (**Alg2**) in the rest of the thesis, except where otherwise specifically noted.

Recall that we write $W^i(\bullet)$ for the smooth complex-valued i-forms on a space. From the uniformization by (3.10), any component of $A(\Gamma, H)$ is a discrete quotient of

$$X_{\mathbf{G}} \times X_{\mathbf{H}} \times \mathbf{V}(\mathbb{R})/L.$$

The contangent bundle (whose sections are 1-forms) on this product space then splits according to the product structure. Moreover, the quotient in (3.10) preserves this decomposition, meaning we have a corresponding decomposition of the cotangent bundle of $A(\Gamma, H)$ into corresponding parts which we denote as

$$W^1_{Y(\Gamma)} \oplus W^1_{X_{\mathbf{H}}} \oplus W^1_A$$

which in turn induces a splitting on all differential forms by taking exterior powers, along with a splitting on the differential $d = d_\Gamma + d_H + d_A$. Further, since the fibers are naturally complex tori,[23] we have the further

[23]Using the complex structure coming from the $\mathbf{H}(\mathbb{R})$-structure.

splitting

$$W_A^1 = W_A^{(1,0)} \oplus W_A^{(0,1)}$$

into holomorphic and anti-holomorphic parts, and a corresponding splitting $d_A = \partial + \overline{\partial}$. Thus, in total, we have the decomposition

$$W^1_{Y(\Gamma)} \oplus W^1_{X_{\mathbf{H}}} \oplus W_A^{(1,0)} \oplus W_A^{(0,1)} \tag{3.11}$$

and we can project any smooth differential k-form on $A(\Gamma, H)$ to its $(a, b, c_\partial, c_{\overline{\partial}})$-component for any four-tuple of nonnegative integers such that $a + b + c_\partial + c_{\overline{\partial}} = k$.

Naturally, all of the preceding discussion applies also to the cover $A(\Gamma, X_{\mathbf{H}})$. Considered on either space, we wish to consider the component

$$\theta_{\varphi_f} := E_\psi(\varphi_f)^{(d/2-1,0,d/2,0)}.$$

We have the following important closedness result:

Proposition 3.8. *For any $\varphi_f = \varphi_f(\mathcal{C})$, θ_{φ_f} is closed along $Y(\Gamma)$, i.e. $d_\Gamma \theta_{\varphi_f} = 0$.*

Proof. A weaker version of this proposition (after pulling back by a torsion section) for the globally split case $(\mathbf{GL}_N, \mathbf{GL}_2)$ is claimed without proof in [BCG1, Lemma 28]. The proof we give below follows from a computation by the same authors in [BCG2], and so is surely known to them.

We will prove the result in fact for the (non-closed) form

$$\theta := E_\psi^{(d/2-1,0,d/2,0)}(\varphi((0))),$$

since, as we mentioned previously, every other case is a linear combination of translates of this one.

Choose some point $p \in X_{\mathbf{H}}$; it suffices to prove the result for the fiber $p^*\theta$ for all p; each of these fibers is a punctured d-torus bundle over $Y(\Gamma)$.

We can consider of $p^*\theta$ as a form on (an open subset of) the uniformizing space

$$X_{\mathbf{G}(\mathbb{R})} \times \mathbf{V}(\mathbb{R}) \cong \mathbf{G}^{ad}(\mathbb{R}) \times_{K_\infty(\mathbf{G}^{ad}(\mathbb{R}))} \mathbf{V}(\mathbb{R})$$

where the isomorphism is given by $([g], v) \mapsto [g, g^{-1}v]$. Since closedness is a local property, it is equivalent to check it on this space.

The representation $\iota_{\mathbf{G}}$ gives rise to an embedding

$$\mathbf{G}(\mathbb{R}) \hookrightarrow \mathbf{GL}_d(\mathbb{R}).$$

Since $\mathbf{G}$ centralizes $\mathbf{H}$, it also preserves the $\mathbb{C}$-linear structure on $\mathbf{V}(\mathbb{R})$ given by any Hodge structure parameterized by some point of $X_{\mathbf{H}(\mathbb{R})}$, and in particular for the point p. It hence gives rise to a more refined embedding

$$\mathbf{G}(\mathbb{R}) \hookrightarrow \mathbf{GL}_{d/2}(\mathbb{C})$$

corresponding to ι_G and the complex structure associated to p, which induces an embedding of bundles

$$\Phi_p : \mathbf{G}^{ad}(\mathbb{R}) \times_{K_\infty(\mathbf{G}^{ad}(\mathbb{R}))} \mathbf{V}(\mathbb{R}) \hookrightarrow \mathrm{SL}_{d/2}(\mathbb{C}) \times_{\mathrm{SU}_{d/2}} \mathbb{C}^{d/2}$$

which fiberwise is an isomorphism of complex vector spaces.

The Riemannian structure on the latter bundle considered in [BCG2] agrees with the one pulled back from restriction of structure from the $\mathrm{GL}_d(\mathbb{R})$-structure on $\mathbb{C}^{d/2}$, as we described in the beginning of §3.3. Hence, if we run the Mathai-Quillen formalism using this metric to obtain a degree-d Thom form $\varphi_{\mathbb{C}^{d/2}}$ and a degree-$(d-1)$ transgression form $\eta_{\mathbb{C}^{d/2}}$ (and its s-deformations), then we have the functorial relationship $\eta = \Phi_p^* \eta_{\mathbb{C}^{d/2}}$, and we have similar functorial agreements for the analogously defined $(E_\psi)_{\mathbb{C}^{d/2}}$, $\theta_{\mathbb{C}^{d/2}}$, etc. In [BCG2, §3.27], it is shown that the component $\theta_{\mathbb{C}^{d/2}}$ is closed along $X_{SL_{d/2}(\mathbb{C}}$, whence

$$d_\Gamma \theta = d_\Gamma \Phi^* \theta_{\mathbb{C}^{d/2}} = \Phi^* d_{X_{SL_{d/2}(\mathbb{C}}} \theta_{\mathbb{C}^{d/2}} = 0.$$

$\square$

As a result of this computation, we may obtain a cohomological invariant by integrating θ (or its stabilizations θ_{φ_f} for various Schwartz functions φ_f) along closed $(d/2-1)$-cycles in $Y(\Gamma)$. This proceeds as follows: consider θ as a form in

$$W_{cl}^{d/2-1}(X_{\mathbf{G}(\mathbb{R})}, W^{(0,d/2,0)}(A(X_{\mathbf{H}}) - 0)).$$

Here, the notation should be interpreted as closed $(d/2-1)$-forms on $X_{\mathbf{G}(\mathbb{R})}$ valued in multidegree-$(0, d/2, 0)$ forms on the fiber $A(X_{\mathbf{H}})-0$ (where we drop the first coordinate corresponding to $Y(\Gamma)$ in the decomposition (3.11)). Since θ is Γ-invariant, this can equally be viewed as a closed differential form on $Y(\Gamma)$ with values in the local system defined by $W^{(0,d/2,0)}(A(X_{\mathbf{H}}))$, representing hence a cohomology class

$$\Theta^{an} := [\theta] \in H^{d/2-1}(Y(\Gamma), W^{(0,d/2,0)}(A(X_{\mathbf{H}}) - 0)).$$

We note now that by the H-invariance of the form θ, we can equally do all of this on the level of the quotient by H, giving a cocycle representing a cohomology class in

$$\widetilde{\Theta}^{an} := [\theta] \in H^{d/2-1}(Y(\Gamma), W^{(0,d/2,0)}(A(H) - 0)),$$

i.e. valued in smooth $(0, d/2, 0)$-forms on the fibers over $Y(\Gamma)$. This is the cocycle and class we are interested in, though it turns out the target can be given a more refined structure, as we shall see.

Example 3.9. The case $\mathbf{G} = \mathbf{GL}_n/\mathbb{Q}$ and $\mathbf{H} = \mathbf{GL}_2/\mathbb{Q}$ is the one considered in [BCG1, §13]. In particular, the evaluation on the $(d-1)$-homology class associated to the embedding of a nonsplit torus

$$T_F \subset \mathrm{GL}_n(\mathbb{R})$$

coming from a totally real field F of degree d in [BCG1, p. 13.3] yields, upon pullback by a $(\Gamma \times H)$-invariant torsion section, the diagonal restriction of a Hilbert-Eisenstein series associated to F.

3.5. **Hodge theory and the de Rham polylogarithm.** Having constructed a Γ-cocycle valued in differential forms analytically, we turn now to working with the class $z_{\mathcal{C}}^{top}$. In order to obtain an appropriate edge map in the formalism from §2, we need to decrease the cohomological dimension of $A(\Gamma, X_{\mathbf{H}})$ to $d/2$.[24]

Under the geometric-equivariant dictionary, $z_{\mathcal{C}}^{top}$ corresponds to a class

$$z_{\mathcal{C}}^{eq} \in H_{\Gamma}^{d/2-1}(A(H) - C).$$

[24] It is here that it becomes necessary to work with Θ^{an} rather than $\widetilde{\Theta}^{an}$, as it is not clear to us how to do this when the fibers are copies of $A(H)$ rather than $A(X_{\mathbf{H}})$: the base $Y(H)$ (unlike $X_{\mathbf{H}}$) is not necessarily contractible, and can contribute extra cohomology.

In the previous section, we were computing the total cohomology of the bundle $A(\Gamma, X_{\mathbf{H}})$ via the de Rham complex

$$W^{\bullet}(A(\Gamma, X_{\mathbf{H}})) = W^{(\bullet,\bullet,\bullet,\bullet)}(A(\Gamma, X_{\mathbf{H}}))$$

to which we assigned a quadruple grading (3.11).

Similarly, the Γ-equivariant cohomology groups of $A(X_{\mathbf{H}})$ and related spaces with Γ-action can be computed with a de Rham model which looks like

$$C^{\bullet}(\Gamma, W^{(\bullet,\bullet,\bullet)}(A(X_{\mathbf{H}})))$$

where the triple grading is as in the previous subsection, dropping the first coordinate of (3.11).

For our chosen basepoint $\tau_0 \in Y(\Gamma)$, the natural map of $\mathbb{Z}^4$-graded complexes

$$W^{(\bullet,\bullet,\bullet,\bullet)}(A(\Gamma, X_{\mathbf{H}})) \to C^{\bullet}(\Gamma, A^{(\bullet,\bullet,\bullet)}(A(X_{\mathbf{H}})))$$

given by

$$\omega \mapsto \left((\gamma_1, \gamma_2, \ldots, \gamma_k) \mapsto \int_{\Delta_{\tau_0}(1,\gamma_1,\ldots,\gamma_k)} \omega \right) \tag{3.12}$$

is a quasi-isomorphism inducing the equivalence between the cohomology theories from the equivariant-geometric dictionary; further, the map on cohomology groups is independent of the basepoint τ_0.

In particular, (3.12) associates to $E_\psi(\varphi_f)$ a Γ-cochain

$$\mathcal{A}_\psi \in C^{\bullet}(\Gamma, W^{\bullet}(A(X_{\mathbf{H}}) - 0))$$

of total degree $d-1$ (but in various Γ-degrees).[25] For $\varphi_f = \varphi_f(\mathcal{C})$, this yields an element with total differential zero representing the class $z_{\mathcal{C}}^{eq}$. Further, Proposition 3.8 shows that its $(d/2-1, 0, d/2, 0)$-component is a Γ-cochain.

[25]This notation is intended to be reminiscent of the simplicial differential form considered in [BCG3], since they are essentially the same thing.

It now remains to finesse the fibers of the Γ-space $A(X_{\mathbf{H}})$ to cut down their cohomological dimension. However, we will first only be doing this for certain pairs $(\mathbf{G}, \mathbf{H})$.

3.5.1. *Formality and comparison for split global pairs.* In this subsection, we consider the case where the family $A(X_{\mathbf{H}})$ (or equivalently $A(H)$) is isogenous to a power of a (relative) elliptic curve. Equivalently, we ask that $\mathbf{V}_{\mathbb{Q}}$ decomposes into a power of a rank-two representation of $\mathbf{H}_{\mathbb{Q}}$, since this implies a splitting of the corresponding complex torus uniformized by (3.1) up to isogeny.

We will call this the *rationally-split setting*. The subclass of examples in which $A(H)$ genuinely splits, not just up to isogeny, we will call the *split setting*.

Remark 3.10. In the case where $(\mathbf{G}, \mathbf{H})$ form a type II dual pair, it is equivalent to ask that $\mathbf{V}_{\mathbb{Q}}$ decomposes into rank-two summands (alternately, that the complex torus bundle is isogenous to a sum of elliptic curves). Indeed, one can prove using the classification of endomorphism rings of elliptic curves over $\mathbb{C}$ that all the type II dual pairs satisfying (**Alg1**) and (**Alg2**) with a representation $\mathbf{V}$ of this form are either globally split of the form $(\mathbf{GL}_n, \mathbf{GL}_2)$, or "CM-split", of the form

$$(\mathrm{Res}_{K/\mathbb{Q}}\mathbf{GL}_n, \mathrm{Res}_{K/\mathbb{Q}}\mathbf{GL}_1)$$

for some imaginary quadratic field K.

In a slightly different but essentially equivalent setting, [BCG3] have constructed an edge map class in

$$H^{d/2-1}(\Gamma, H^{d/2}(\text{a pro-subspace of } A(X_{\mathbf{H}})))$$

by excising an *elliptic hyperplane arrangements*. By this we mean an effective divisor, flat over the base $Y(H)$, such that in each fiber it is a union of images of abelian subvarieties embedded affinely (i.e. by a translation of a group homomorphism).

Done correctly, this excision makes the fibers affine as we discussed in §2.2, implying they have cohomological dimension at most n [Hamm]. As the base $X_{\mathbf{H}}$ is contractible, this implies the same for the total space, yielding the existence of the Hochschild-Serre edge map (2.12). They hence deduce an explicit class valued in meromorphic differential forms by mapping (relative) cohomology classes to differential forms via

Orlik-Solomon formality of hypersurface complements in families, as treated in [Dup]. We roughly follow their approach in this subsection, though with the important caveat that they do not work up to isogeny.

Remark 3.11. We are also mostly interested in the split setting, and not the rationally-split one, as all of the most naturally occurring rationally-split examples are actually split. In fact, to a large extent, results about the rationally-split setting can simply be deduced from the split setting if one analyzes how the various cohomology classes and cocycles we construct transform under isogenies. We choose in this thesis to directly consider the isogeny-split setting on equal terms mostly because it is useful as a technical tool in the proof of Theorem 3.27.

We define our family of hyperplanes as follows: pick $\mathbf{T}$ an $\mathbf{H}$-representation such that

$$\mathbf{V}_{\mathbb{Q}} = \mathbf{T}_{\mathbb{Q}}^{\oplus d/2}$$

and we have an index-N inclusion of lattices

$$L \subset \mathbf{T}^{\oplus d/2} \tag{3.13}$$

as $\mathbf{H}$-representations, with N minimal. Fix a torsion cycle $\mathcal{C}$ annihilated by an integer $c > 1$ such that $(c, N) = 1$.[26]

For any $p \in \mathbb{P}^{d/2-1}(\mathbb{Q})^\vee$, let L_p be the locus in 3.10 defined by the kernel of the composition

$$\mathbf{V}(\mathbb{R})/\mathbf{V}(\mathbb{Z}) \twoheadrightarrow (\mathbf{T}(\mathbb{R})/\mathbf{T}(\mathbb{Z}))^{\oplus d/2} \xrightarrow{\text{projection}} \mathbf{T}(\mathbb{R})/\mathbf{T}(\mathbb{Z}) \xrightarrow{[c]} \mathbf{T}(\mathbb{R})/\mathbf{T}(\mathbb{Z}) \tag{3.14}$$

where the first arrow is the N-isogeny dual to the inclusion of lattices (3.13), and the second arrow is given by the quotient corresponding to the hyperplane x. In particular, the family of hypersurfaces corresponding to L_p (which we abusively denote by L_p as well) contains the c-torsion in $A(X_{\mathbf{H}})$, and is fixed by the parabolic subgroup of $\mathbf{G}(\mathbb{Z})$ fixing the kernel of projection to the last coordinate.

[26]This is perhaps unnecessarily restrictive, but it suffices for us, as we are not overly concerned with optimizing the rationally-split setting per se.

For any subset $S \subset \Gamma$, consider the open hypersurface complement

$$U_S := A(X_{\mathbf{H}}) - \bigcup_{\gamma \in S} \gamma L_p; \tag{3.15}$$

these are partially ordered by inclusion in a system $U_\bullet$. This inverse system is finally affine fiberwise (in the categorical sense of finality) and, by construction, Γ-stable.[27]

Since colimits are exact, we may take the Γ-pro-bundle $\varprojlim U_\bullet$ as the target of our restriction of $z_{\mathcal{C}}^{eq}$. The pro-affine family $U_\bullet$ over the contractible base $X_{\mathbf{H}}$ thus has cohomological dimension at most $d/2$ [Hamm].

We deduce from (2.12) a class

$$H^{d/2-1}(\Gamma, \varinjlim H^{d/2}(U_\bullet)). \tag{3.16}$$

Fix some integer a which is $1 \pmod{c}$. By construction, $z_{\mathcal{C}}^{eq}$ is in the submodule on which $[a]_*$ acts by the identity. We also have an action of $[a]_*$ on the cohomology of $U_\bullet$ just as in (2.10), since the kernel of (3.14) maps to itself under multiplication-by-a.

Then by functoriality, the image of $z_{\mathcal{C}}^{eq}$ in (3.16) is in the $([a]_* = 1)$-part as well. In fact, we claim that it even can be refined to a class in

$$H^{d/2-1}(\Gamma, \varinjlim H^{d/2}(U_\bullet)^{[a]_*=1}).$$

This follows from the following result:

Lemma 3.12. *Under the action on the cohomology of the pro-space $U_\bullet$ by any isogeny $[a] : U_\bullet \to U_\bullet$ with $a \equiv 1 \pmod{c}$, it transforms under said isogenies by the weights $[a] \mapsto a^k$ for $k \in \{0, \ldots, d\}$.*

Proof. It suffices to prove this for U_S for any S, since the limit of modules satisfying this condition also satisfies the condition. Recalling the definition (3.15), there is a Leray spectral sequence converging to the cohomology of U_S whose terms are the cohomology of the various pure strata of the arrangement, i.e. intersections of γL_p [W].

[27] If one prefers not to think about pro-spaces, one can equivalently consider the Leray spectral sequence for the corresponding bundle over $E\Gamma$ as in the proof of [BCG3, Théorème 8].

Each of these strata is topologically a union of torus bundles of dimension $\leqslant d$, which is a subgroup of the bundle $A(X_{\mathbf{H}})$ translated by a c-torsion point. Thus, they are acted on by $[a]$ via their own a-isogeny for any $a \equiv 1 \pmod{c}$; hence, each of their cohomologies is a module for such $[a]_*$ satisfying the condition of the lemma by the same argument we already used for the total bundle (from [BCG1, §3]). By functoriality of the Leray spectral sequence under the pushforwards $[a]_*$, the result follows. □

Thus, the Lieberman projector e_L makes any cocycle representative of (3.16) actually isogeny-invariant by the isogeny a, and not just up to a Γ coboundary.[28]

By the edge map in the Leray spectral sequence for the pro-bundle $\pi' : U_\bullet \to X_{\mathbf{H}}$, we deduce an element in

$$H^{d/2-1}(\Gamma, H^0(X_{\mathbf{H}}, R^{d/2}\pi'_*\mathbb{Z}[1/c])^{[a]*=1}).$$

With $\mathbb{C}$-coefficients, [BCG3, §3] proves using the formality[29] results of [Dup] that the map

$$\pi'_*(\Omega^{d/2}_{U_\bullet/Y(H)})^{[a]*=1} \to \mathcal{O}_{X_{\mathbf{H}}} \otimes (R^{d/2}\pi'_*\mathbb{C})^{[a]*=1} \tag{3.17}$$

induced by the Hodge-de Rham spectral sequence is finally an isomorphism. We hence deduce a class

$$\Theta_{\mathcal{C}} \in H^{d/2-1}(\Gamma, \varinjlim H^0(X_{\mathbf{H}}, \pi'_*\Omega^{d/2}_{U_\bullet/X_{\mathbf{H}}})^{[a]*=1}) = H^{d/2-1}(\Gamma, \varinjlim H^0(U_\bullet, \Omega^{d/2}_{U_\bullet/X_{\mathbf{H}}})^{[a]*=1}),$$

the equality by the definition of the pushforward sheaf.

Recall that integrating $\theta_{\mathcal{C}}$ yielded in the previous subsection a class

$$\Theta^{an}_{\mathcal{C}} := [\theta(\varphi_f)] \in H^{d/2-1}(\Gamma, W^{(0,d/2,0)}(A(X_{\mathbf{H}}) - 0)^{[a]*=1}).$$

where $\varphi_f = \varphi_f(\mathcal{C})$.

Proposition 3.13. *Under the inclusion*

$$H^0(U_\bullet, \Omega^{d/2}_{U_\bullet/X_{\mathbf{H}}})^{[a]*=1} \hookrightarrow W^{(0,d/2,0)}(U_\bullet)^{[a]*=1},$$

[28]Note that this also follows from [BCG3, Proposition 9.5], cited below.
[29]Here, the term “formality” is in the sense it is used in the field of rational homotopy theory, wherein cohomology classes are assigned distinguished differential form representatives.

Θ^{an} *and* $\Theta_{\mathcal{C}}$ *are cohomologous.*

Proof. Write $\mathbf{g} := (g_0, \dots, g_{d/2-1})$. [BCG3, Proposition 9.5] tells us that

$$\int_{\Delta_{\tau_0}(\mathbf{g})} E_\psi(\varphi_f) \tag{3.18}$$

is equal up to a Γ-coboundary to

$$\mathbf{S}^*_{\mathbf{ell}}[\varphi_f](\mathbf{g}) + (d_H + d_{\mathcal{A}})\mathcal{H}_{d/2-1}(\varphi_f)(\mathbf{g})$$

where $\mathbf{S}^*_{\mathbf{ell}}[\varphi_f]$ is an explicit $[a]_*$-invariant Γ-cocycle defined in [BCG3, §7])[30] and $\mathcal{H}_{d/2-1}(\varphi_f)$ is the integral of $E_\psi(\varphi_f)$ over a certain $d/2$-cycle (indexed by g) in the Tits compactification of $E\Gamma$.

By the closedness of $E_\psi(\varphi_f)$ and Proposition 3.8, the contribution of $(d_H + d_A)\mathcal{H}_{d/2-1}(\varphi_f)(\mathbf{g})$ to the $(0, d/2, 0)$-component

$$\int_{\Delta_{\tau_0}(\mathbf{g})} \theta(\varphi_f)$$

of (3.18) is then necessarily zero. The corresponding component $\mathbf{S}^*_{\mathbf{ell}}(\mathbf{g})^{(0,d/2,0)}$ is valued in isogeny-invariant holomorphic (or rather meromorphic) forms, and $\mathbf{S}^*_{\mathbf{ell}}(\mathbf{g})$ differs from this component by a $(d_H + d_A)$-exact form by [BCG3, Lemme 9.6]. Hence, it represents the class $\Theta_{\mathcal{C}}$ by Proposition 2.2 and the formality result (3.17). This class then is cohomologous to $\Theta^{an}_{\mathcal{C}} = [\theta(\varphi_f)]$. □

Remark 3.14. $\mathbf{S}^*_{\mathbf{ell}}(\mathbf{g})^{(0,d/2,0)}$ actually furnishes a cocycle valued in $H^0(U_\bullet, \Omega^{d/2}_{U_\bullet/Y(H)})$, i.e. over $Y(H)$ and not just over $X_{\mathbf{H}}$. Notice that since $E_\psi(\varphi_f)$ is Γ-invariant, $\Theta^{an}_{\mathcal{C}}$ also descends to a cocycle valued in

$$H^0(U_\bullet, A^{(0,d/2,0)}_{U_\bullet/Y(H)}),$$

i.e. also over $Y(H)$. However, the Hochschild-Serre edge map can only be defined over $X_{\mathbf{H}}$, as it is not clear how to cut down the cohomological dimension of $A(H)$ to $d/2$ due to the possibly nontrivial contribution of the base $Y(H)$. Thus, we have the curious state of affairs that these two cocycles, both of which descend to the bundle over $Y(H)$, can only be proven cohomologous over $X_{\mathbf{H}}$. This tension disappears upon pullback by a torsion section, as we will see later.

[30]It is also given by an integral of $\theta(\varphi_f)$, in fact, by moving it to infinity and integrating it "along the Tits boundary".

Remark 3.15. It is worth remarking that the method from [BCG3, §7-8] of integrating along the Tits boundary to produce $\mathbf{S}^*_{\mathbf{ell}}(\mathbf{g})$ thus is the only method thus far that produces a cocycle valued in meromorphic forms over the base $Y(H)$. We will see another way (due to [KS]) using coherent cohomology in the next section, though that produces only rather a cohomology class rather than a cocycle.

Write ω for the weight-1 Hodge bundle on $Y(H)$. For any $(\Gamma \times H)$-invariant section $x : Y(H) \to U_\bullet$, consider the composite

$$e(x) : A^{(d/2,0)}(U_\bullet) \to H^0(U_\bullet, \pi'^*\omega^{\otimes d/2} \otimes C^\infty(U_\bullet)) \to H^0(Y(H), \omega^{\otimes d/2} \otimes C^\infty(Y(H)))$$

given by contraction with the polyvector field

$$\partial_{z_1} \otimes \partial_{z_2} \otimes \ldots \otimes \partial_{z_{d/2}}$$

and then pullback by x, to *smooth* sections of the bundle $\omega^{\otimes d/2}$ on the base $Y(H)$. Here, we are identifying

$$A^{(d/2,0)}(U_\bullet) \cong H^0(U_\bullet, \Omega^{d/2}_{U_\bullet} \otimes_{\mathcal{O}_{U_\bullet}} C^\infty(U_\bullet))$$

to define the first map, as every $(d/2, 0)$-smooth differential is locally a holomorphic differential times a smooth function, then using the same argument to in (2.13) to pass to the pullback of the Hodge bundle.

With this setup, the preceding proposition implies:

Corollary 3.16. *The cocycle given by $e(x)^*\theta(\varphi_f)$ is cohomologous to $e(x)^*\mathbf{S}^*_{\mathbf{ell}}[\varphi_f]$ (from the proof of 3.13), both representing the class*

$$\Theta_{\mathcal{C},x} := e(x)^*\Theta_{\mathcal{C}}.$$

In particular,

$$e(x)^*\theta(\varphi_f) \in H^{d/2-1}(\Gamma, \omega^{\otimes d/2} \otimes C^\infty(Y(H)))$$

considered via the universal coefficients theorem as a homomorphism

$$H_{d/2-1}(\Gamma) \to H^0(Y(H), \omega^{\otimes d/2} \otimes C^\infty(Y(H))),$$

is actually valued in $H^0(Y(H), \omega^{\otimes d/2})$.

In review, we have finally exhibited $\theta_{\mathcal{C}}$ as an explicit representative of our Eisenstein theta class after pullback.

Remark 3.17. When $d = 4$ and we are looking at degree-1 cohomology, the 1-simplices are themselves closed cycles, as the standard model of $B\Gamma$ has a unique 0-simplex. Thus, due to the lack of 1-coboundaries, cocycle representatives are unique and $e(x)^*\theta(\varphi_f)$ is actually identical to $e(x)^*\mathbf{S}^*_{\mathbf{ell}}[\varphi_f]$ as a *cocycle*. This is the case, for example, for the split pair $(\mathbf{GL}_2, \mathbf{GL}_2)$.

Remark 3.18. Since we are removing hyperplanes to obtain $U_\bullet$, the choice of the pullback section x must be considered carefully: for example, if Γ is too large, we may end up with no torsion sections at all.

Remark 3.19. Note that the cocycle given by integrating $\theta_{\mathcal{C}}$ over geodesic simplices does not necessarily take value in meromorphic forms; it is only cohomologous to $\Theta_{\mathcal{C}}$ after extending coefficients to smooth forms. After the aforementioned pullback to the base, the preceding corollary says that there is the nice interpretation that the integrals over closed cycles on $Y(\Gamma)$ are in fact holomorphic since now the Γ-action is trivial, but *a priori* it is still unknown whether the integrals over $\Delta_{\mathbf{g}}$ (for simplices of dimension > 1) are holomorphic or only smooth.

If the values of $\theta_{\mathcal{C}}$ (as a Γ-cocycle) are known to be in meromorphic forms on the bundle $A(H)$, then we can say more. These values are given by regularized sums

$$\sum_{v \in V(\mathbb{Q})} \int_{\Delta(\mathbf{g})} v^*\eta(s)\varphi_f(v) \tag{3.19}$$

analytically continued to $s = 0$. In some cases, this sum is computable to be holomorphic; e.g. on real-split anisotropic tori, it reduces to the definition of Hecke regularized Hilbert-Eisenstein series [BCG1, §13]. For Γ corresponding to modular or Shimura curves, for example, the classes of such tori exhaust the 1-homology of $B\Gamma$. It is not clear to us what can be said more generally as of writing this. Assuming we can compute the sum (3.19) and check that it is holomorphic, Lemma 2.2 and the formality result of [BCG3, §3] show that $\theta_{\mathcal{C}}$ coincides with (a stabilization of) the cocycle denoted $\mathbf{S}_{\mathbf{ell}}$ in loc. cit. Additionally, the φ_f-stabilization of the cocycle denoted $\mathbf{S}^*_{\mathbf{ell}}$ in loc. cit. (which we referenced above) differs from it by a Γ-coboundary.

3.5.2. *Coherent cohomology and the Kings-Sprang construction.* We now turn to a different perspective on the construction of these cocycles, due to [KS], which is more abstract and algebraic in flavor. Roughly, it is the top Hodge-filtered layer of the de Rham construction, using coherent cohomology of schemes in place of Betti or de Rham cohomology. This gives a more flexible approach which gives us the ability to work with arithmetic coefficients and avoid the formality result (3.17), at the cost of being harder to compare to explicit formulas (more on which later).

In this section, we consider an *arbitrary* discrete group Γ acting on an *arbitrary* abelian scheme $\pi : \mathcal{A} \to \mathcal{Y}$ over an integral base on which a fixed integer c is invertible.

The formalism of the polylogarithm in coherent cohomology is slightly different from, but very analogous to that presented in §2.1. Following [KS, Appendix A], we can define the Borel equivariant coherent cohomology of a sheaf $\mathscr{F}$ on a scheme $\mathcal{S}$ as the derived functors of the abelian functor of Γ-invariant global sections on Γ-quasi-coherent sheaves

$$\Gamma_{\mathcal{S}}^{\Gamma} : \Gamma\textbf{-QCohSh}(\mathcal{S}) \to \mathcal{O}_{\mathcal{S}}\textbf{-Mod}$$

without reference for complexes. This makes functoriality and the localization sequence formal, and the Hochschild-Serre spectral sequence is the Grothendieck spectral sequence for the identification of the functor $\Gamma_{\mathcal{S}}^{\Gamma}$ as the composition of the functor of Γ-invariants with the functor of global sections on $\mathcal{S}$. The Gysin isomorphism is proven in [KS, Corollary 2.14].

It will still be useful to us, however, to compare this definition with the formalism we gave in §2.1, for the purposes of comparison maps with other constructions we are interested in. Rather than follow that formalism precisely, however, we will establish a link with functorial complexes solely in the case where the sheaf $\mathscr{F}$ is the sheaf of algebraic i-differentials Ω^i, and the base scheme is defined over over $\mathcal{C}$, which will suffice for our purposes.[31]

To be precise, we employ the distributional Dolbeault resolution

$$\Omega_{\mathcal{S}}^{i} \hookrightarrow \mathcal{D}_{\mathcal{S}}^{i,0} \xrightarrow{\overline{\partial}} \mathcal{D}_{\mathcal{S}}^{i,1} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathcal{D}_{\mathcal{S}}^{i,2\dim \mathcal{S}-i} \tag{3.20}$$

[31]To be clear, we will pursue the Kings-Sprang construction over arbitrary integral bases, but the comparison with the topological/analytic construction of the previous sections is necessarily defined only over $\mathbb{C}$, as is the archimedean regulator we will consider in the motivic case later in §4.2.3 - actually, the latter can be defined over $\mathbb{R}$ also, but this is not of huge concern to us.

where $\mathcal{D}^{p,q}$ are the smooth (p,q)-currents: these are the subspace of the smooth $(p+q)$-currents defined in §3.2.2 supported on $(\dim \mathcal{S} - p, \dim \mathcal{S} - q)$-forms. The Dolbeault operator (or anti-holomorphic derivative) $\overline{\partial}$ is, as in the earlier setting, defined as the adjoint of the analogous operator on forms.

Remark 3.20. This can be thought of as a refinement of the distributional de Rham complex by the Hodge filtration afforded from the complex structure on $\mathcal{S}$: concretely, the natural inclusion map from (3.20) to the distributional de Rham complex (as i varies) realizes the Hodge decomposition of de Rham cohomology [Voi, §2.3].

This construction gives an acyclic resolution of $\Omega^i_{\mathcal{S}}$. As per our formalism, the corresponding double complex of Γ-cochains

$$C^\bullet(\Gamma, \mathcal{D}^{i,\bullet}_{\mathcal{S}}) \tag{3.21}$$

can be used to define $H^\bullet_\Gamma(\mathcal{S}, \Omega^i)$. Since (3.21) is an acyclic resolution of $\mathscr{F}$ as a Γ-sheaf on $\mathcal{S}$, this definition coincides with the derived functor definition of [KS], by the general formalism laid out in [Tohoku].

Our desired functoriality properties, the localization sequence, and the Hochschild-Serre spectral sequence are realized at the level of complexes identically to those in the setting of §3.2.2 considering currents without the Hodge refinement. Even the Gysin isomorphism is defined by the same map: the map (3.3) respects the Hodge decomposition in the sense that it restricts to

$$\mathcal{D}^{i-k,j-k}_Z \to \ker(\mathcal{D}^{i,j}_X \to \mathcal{D}^{i,j}_{X-Z})$$

for $Z \subset X$ of complex codimension k (so real codimension $2k$), leading to a Gysin isomorphism of the form

$$H^{j-k}_\Gamma(Z, \Omega^{i-k}) \cong H^j_{\Gamma,Z}(X, \Omega^i).$$

Remark 3.21. It actually is possible to adapt the formalism of §2.1 over an arbitrary base with arbitrary quasi-coherent sheaves, not just in this special setting over $\mathbb{C}$. In this case, the category of spaces **S** should be thought of as pairs $(S, \mathscr{F})$ of a scheme equipped with a quasicoherent sheaf, and we can take $D(S, \mathscr{F})$ to be the *Godemont resolution* of $\mathscr{F}$ [God]. This is a suitably functorial complex giving an acyclic resolution of an arbitrary sheaf, meaning its complex of global sections can be used to define the equivariant cohomology. It is well-suited to defining the localization sequence (2.4), but otherwise it is a quite formal and unwieldy object not very well-suited to explicit manipulations. It is possible to define a natural transformation from the Godemont complex to the distributional Dolbeault complex when $\mathscr{F} = \Omega^i$ for schemes over $\mathbb{C}$, but we have no need for this comparison.

Having concluded these preliminaries, we briefly reproduce now the proof of [KS, Theorem 2.18] in our language, specialized to the base class with trivial coefficients.

As before, suppose we have a degree-zero Γ-invariant cycle $\mathcal{C}$ made of torsion sections; we write C for the support of $\mathcal{C}$, and suppose that $c > 0$ is the positive integer annihilating C. $\mathcal{C}$ then naturally can be identified with a Γ-invariant element in the coherent cohomology group $H^0(C, \mathcal{O}_C)$. Moreover, the fact that it is degree zero means that it further lives in

$$H^0(C, \mathcal{O}_C)^0 := \ker(H^0(C, \mathcal{O}_C) \xrightarrow{\pi_*} H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})).$$

The main point is that the replacement for the cycle class encoding residues in $H^d_\Gamma(A - C)$ in the coherent setting is the group $H^{d/2}_\Gamma(\mathcal{A} - C, \Omega^{d/2})$ (where we implicitly identify the sheaf with its restriction to the complement of C in a slight abuse).

For brevity, we will denote $\Omega^{d/2}_{\mathcal{A}/\mathcal{Y}}$ by $\Omega^{d/2}$ in what follows; we also write $i : C \to \mathcal{A}$ denotes the inclusion map.

Concretely, we have a map

$$H^0(C, \mathcal{O}_C)^\Gamma \to \mathrm{Ext}^{d/2}_{\mathcal{O}_\mathcal{A}}(i_*\mathcal{O}_C, \Omega^{d/2})^\Gamma$$

To see this, we can identify

$$H^0(C, \mathcal{O}_C) \cong \mathrm{Ext}^0_{\mathcal{O}_\mathcal{A}}(i_*\mathcal{O}_C, \mathcal{O}_\mathcal{A}) \cong \mathrm{Ext}^{d/2}_{\mathcal{O}_\mathcal{A}}(i_*\mathcal{O}_C, \Omega^{d/2})$$

as Γ-modules: the first isomorphism is the adjunction between i_* and i^*, and the second is Serre duality on $\mathcal{A}$ with the dualizing sheaf $\Omega^{d/2}$.. By the argument in [KS, §2.6], there exists a canonical inclusion

$$\mathrm{Ext}^{d/2}_{\mathcal{O}_{\mathcal{A}}}(i_*\mathcal{O}_C, \Omega^{d/2}) \hookrightarrow H^{d/2}_C(\mathcal{A}, \Omega^{d/2}).$$

The edge map in the Hochschild-Serre spectral sequence yields

$$H^{d/2}_C(\mathcal{A}, \Omega^{d/2})^{\Gamma} \to H^{d/2}_{\Gamma,C}(\mathcal{A}, \Omega^{d/2}).$$

Finally, we have the equivariant localization sequence [KS, §2.3.2]

$$H^{d/2-1}_{\Gamma}(\mathcal{A}, \Omega^{d/2}) \to H^{d/2-1}_{\Gamma}(\mathcal{A} - C, \Omega^{d/2}) \to H^{d/2}_{\Gamma,C}(\mathcal{A}, \Omega^{d/2}) \xrightarrow{\deg} H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$$

and therefore by Lieberman's trick (recalling that c is invertible on the base) we obtain a map

$$H^0(C, \mathcal{O}_C)^{0,\Gamma} \to H^{d/2-1}_{\Gamma}(\mathcal{A} - C, \Omega^{d/2}) = H^{d/2-1}_{\Gamma}(\mathcal{A} - C, \pi^*\omega^{\otimes d/2}),$$

where $\omega := \pi_*\Omega^1$ is the Hodge bundle in this context.

We define $z^{KS}_{\mathcal{C}}$ to be the image of $\mathcal{C}$, considered as a section of $\mathcal{O}_C$ constant on connected components, under this map.[32]

We wish to obtain a group cocycle valued in sections of $\pi^*\omega^{\otimes d/2}$ on something approximating the torus bundle $\mathcal{A}$. Indeed, if it were possible pass to an affine subspace $\mathcal{U} \subset \mathcal{A}$ still rich in torsion sections, vanishing of coherent cohomology on affines in the Hochschild-Serre spectral sequence would buy us a cocycle

$$\Theta^{KS}_{\mathcal{C}} \in H^{d/2-1}(\Gamma, H^0(\mathcal{U}, \Omega^{d/2})) = H^{d/2-1}(\Gamma, H^0(\mathcal{U}, \pi'^*\omega^{d/2})).$$

If $\mathcal{A}$ is rationally split, for example if it is isogenous to a power $\mathcal{E}^n$ of the universal elliptic curve over some open modular curve $\mathcal{Y} = \mathcal{Y}(H)$ over a characteristic zero base, we can take advantage of the fact that $\mathcal{Y}$ itself is affine: as in the previous section, we can take $\mathcal{U}_{\mathcal{Y}} = (\mathcal{U}_\bullet)_{\mathcal{Y}}$ to be the pro-bundle given by excising hyperplanes in the same manner as before, as these hyperplanes were all already defined scheme-theoretically.

[32] The fact that we choose only such functions here, instead of more general functions along the base, is fundamentally the same restriction as our choosing only flat sections in (3.17).

Then the total space $\mathcal{U}_{\mathcal{Y}}$ is itself again (pro-)affine by Serre's criterion for affineness [EGA, Vol. II 5.2.2], and the definition of $\Theta_{\mathcal{C}}^{KS}$ makes sense:

Definition 3.22. *If in the case where $\mathcal{A}$ is rationally split, we define the big Kings-Sprang Eisenstein theta kernel*

$$\Theta_{\mathcal{C}}^{KS} \in H^{d/2-1}(\Gamma, H^0(\mathcal{U}_{\mathcal{Y}}, \pi'^*\Omega^{d/2}))$$

as the image of $z_{\mathcal{C}}^{KS}$ under the edge map (2.12) in the coherent cohomology of $\mathcal{U}_{\mathcal{Y}}$, per the preceding discussion.

We have the following comparison result between this definition and the earlier analytic one:

Proposition 3.23. *Upon base change to $\mathbb{C}$ and analytification, $\Theta_{\mathcal{C}}^{KS}$ coincides with the class $\Theta_{\mathcal{C}}$ defined earlier, as elements in $H^{d/2-1}(\Gamma, H^0(\mathcal{U}_{\mathcal{Y}}, \pi'^*\Omega^{d/2}))$.*

Proof. We work entirely in the complex analytic category, since this is where the comparison is being made; write A and Y for the complex-analytic spaces associated to the schemes $\mathcal{A}$ and $\mathcal{Y}$. First, we claim that the image of $\Theta_{\mathcal{C}}^{KS}$ and $\Theta_{\mathcal{C}}$ coincide under the natural pullback map

$$H^{d/2-1}(\Gamma, H^0(U_Y, \Omega^{d/2}_{U_Y/Y})) \to H^{d/2-1}(\Gamma, H^0(U_{X_{\mathbf{H}}}, \Omega^{d/2}_{U/X_{\mathbf{H}}}))^H. \tag{3.22}$$

Indeed, recall that the image of $\Theta_{\mathcal{C}}$ under the composite

$$H^{d/2-1}(\Gamma, H^0(U_Y, \Omega^{d/2}_{U_Y/Y})) \to H^{d/2-1}(\Gamma, H^0(U_{X_{\mathbf{H}}}, \Omega^{d/2}_{U/X_{\mathbf{H}}})) \to H^{d/2-1}(\Gamma, H^{d/2}(U_{X_{\mathbf{H}}})) \tag{3.23}$$

is the image of $z_{\mathcal{C}}$ under the edge map for the affine bundle $U_{X_{\mathbf{H}}}$ over the contractible space $X_{\mathbf{H}}$. Noting that $z_{\mathcal{C}}$ is the image of $z_{\mathcal{C}}^{KS}$ under the equivariant Hodge-de Rham edge map

$$H^{d/2-1}_{\Gamma}(A - C, \Omega^{d/2}_{U/Y}) \to H^{d-1}_{\Gamma}(A - C)$$

by Remark 3.20 (compare [KS, Remark 2.20]), we see that functoriality of the corresponding Hochschild-Serre spectral sequences implies that the image of $\Theta_{\mathcal{C}}^{KS}$ under the same Hodge-de Rham edge map is the same class.

Since Proposition 3.13 tells us that $\Theta_{\mathcal{C}}$ in fact gives a class in $H^{d/2-1}(\Gamma, H^{d/2}(U_{X_{\mathbf{H}}})^{[a]_*=1})$, which by formality (3.17) can be identified with a submodule of $H^{d/2-1}(\Gamma, H^0(U_{X_{\mathbf{H}}}, \Omega^{d/2}_{U/X_{\mathbf{H}}})^{[a]_*=1})$, we see that the two images under (3.22) indeed coincide.

We claim now that (3.22) is injective, from which the result would follow. Indeed, consider the composite

(3.24)

$$H^{d/2-1}(\Gamma, H^0(U_Y, \Omega^{d/2}_{U_Y/Y})) \to H^{d/2-1}(\Gamma, H^0(U_{X_{\mathbf{H}}}, \Omega^{d/2}_{U/X_{\mathbf{H}}}))^H \to H^0(Y, \underline{H^{d/2-1}(\Gamma, H^0(U_Y, \Omega^{d/2}_{U_Y/Y}))})$$

where the notation $\underline{H^{d/2-1}(\Gamma, H^0(U_Y, \pi'^*\Omega^{d/2})}$ means the sheaf on the complex-analytic space Y which is the sheafification of the presheaf with sections on an open $V \subset Y$ given by $H^{d/2-1}(\Gamma, H^0(U_V, \Omega^{d/2}_{U_V/V}))$. The last map in the composite is defined as follows: given

$$\varphi \in H^{d/2-1}(\Gamma, H^0(U_{X_{\mathbf{H}}}, \Omega^{d/2}_{U/X_{\mathbf{H}}}))^H$$

we send φ to a section s_φ defined as follows: on any sufficiently small analytic open $V \subset Y$ such that there exists $V' \subset X_{\mathbf{H}}$ so that $V' \to V$ is an isomorphism under the restriction of the H-uniformizing map $X_{\mathbf{H}} \to Y$, s_φ is defined to be the restriction of φ to $H^{d/2-1}(\Gamma, H^0(U_{V'}, \Omega^{d/2}_{U_{V'}/V'}))$. The various choices of V' are permuted by H, so this gives a well-defined map as φ is H-invariant.

But in fact, $\underline{H^{d/2-1}(\Gamma, H^0(U_Y, \Omega^{d/2}_{U_Y/Y}))}$ is a coherent sheaf on the analytic space Y, because these group cohomology sheaves can be computed via the complex of Γ-injective sheaves on Y

$$\pi'_*\Omega^{d/2}_{U_Y/Y} \to \underline{C^1(\Gamma, \pi'_*\Omega^{d/2}_{U_Y/Y})} \to \underline{C^2(\Gamma, \pi'_*\Omega^{d/2}_{U_Y/Y})} \to \dots$$

each of which is visibly coherent. $H^0(Y, -)$ is an exact functor on coherent sheaves by Cartan's Theorem B [Ca] since Y is Stein (i.e. $\mathcal{Y}$ is affine), and thus the composite (3.24) is actually an isomorphism. Injectivity of (3.22) follows. $\square$

3.5.3. *Bootstrapping to the non-split case.* When $\mathcal{A}$ is not rationally split, it is unclear whether there exists a suitable family of ample divisors whose Γ-orbits avoid prescribed torsion sections, so we do not know what the analogue of the hyperplanes should be. Thus, we do not have an easy analogue of the formality result (3.17), nor of [BCG3]'s technique of "moving to the boundary." The Kings-Sprang construction in coherent cohomology is therefore our only recourse. We further note the extra complication that the base $\mathcal{Y}$ is not necessarily affine in general.

In the absence of a natural family of ample divisors to excise from the fibers, we instead resort to violence: write D for a finite Γ-invariant set of torsion sections disjoint from C, and η_D for the union of the generic points of the corresponding closed subscheme. We set $\mathcal{U}_D$ to be the localization of $\mathcal{A}$ at η_D.

Writing $\eta_{\mathcal{Y}}$ for the generic point of $\mathcal{Y}$, we have the restricted projection map $\mathcal{U}_D \to \eta_{\mathcal{Y}}$. With this structure map, $\mathcal{U}$ is the spectrum of a semi-local ring over the function field of $\mathcal{Y}$, and in particular is affine.

The discussion of the previous section then affords us a cocycle

$$\Theta_{\mathcal{C}}^{KS} \in H^{d/2-1}(\Gamma, H^0(\mathcal{U}_D, \pi'^* \omega^{\otimes d/2}))$$

Remark 3.24. There is a natural inclusion $\mathcal{U}_D \subset \mathcal{U}_S$ for any subset $S \subset \Gamma$ such that there exists L_p with

$$\left(\bigcup_{\gamma \in S} \gamma L_x \right) \cup D = \varnothing.$$

Under pullback by this inclusion, we see that this notation is then consistent with the definition of $\Theta_{\mathcal{C}}^{KS}$ by excising hyperplanes from the previous section, by functoriality of Hochschild-Serre spectral sequences.

For any Γ-invariant section $x \in D$, $\Theta_{\mathcal{C}}^{KS}$ can be pulled back to

$$\Theta_{\mathcal{C},x}^{KS} \in H^{d/2-1}(\Gamma, H^0(\eta_{\mathcal{Y}}, \omega^{\otimes d/2})).$$

It may seem like a major problem that this class is defined only on the generic point, but in fact we can bootstrap ourselves to a class defined over the whole base:

Proposition 3.25. *$\Theta_{\mathcal{C},x}^{KS}$ takes values in the submodule $H^0(\mathcal{Y}, \omega^{\otimes d/2}) \hookrightarrow H^0(\eta_{\mathcal{Y}}, \omega^{\otimes d/2})$.*

Proof. For any point $p \in \mathcal{Y}$, write D_p for the fiber of D over p, and $\mathcal{U}_{D_p}$ for the corresponding localization. This is again a semi-local scheme, and following the same construction preceding this proposition affords us a class valued in sections over the localization of $\mathcal{Y}$ at p

$$H^{d/2-1}(\Gamma, H^0(\mathcal{Y}_p, \omega^{\otimes d/2}))$$

whose restriction to $\eta_{\mathcal{Y}}$ agrees with $\Theta^{KS}_{\mathcal{C},x}$ by functoriality. This is true for every point $p \in \mathcal{Y}$, so the proposition follows. □

Thus we have successfully produced a cocycle valued in holomorphic forms on the base, but without the argument of §3.5 in the split case, it is unclear how to find an explicit representative for this.

Let us now respecialize to the case $\mathcal{A} = \mathcal{A}(H)$ and $\mathcal{Y} = \mathcal{Y}(H)$. For the remainder of the section, we may as well take the latter as defined over $\mathbb{C}$, since our interest is in comparing the Kings-Sprang construction with analytic constructions in non-rationally split cases.

Using different embeddings of Shimura varieties and the functoriality of these two constructions, we will bootstrap from the split comparison to the non-split comparison, for the pulled back cocycles Θ^{an}_x and $\Theta^{KS}_{\mathcal{C},x}$.[33]

In particular, considering the following statement, depending on any arithmetic group Γ acting fiberwise on any abelian family $\mathcal{A}$ (and thus the corresponding topological bundle over $\Gamma\backslash X_{\mathbf{G}(\mathbb{R})}$) fixing the point x, and invariant torsion cycle $\mathcal{C}$ of order relatively prime to x:

- (**Comp**) $\Theta^{an}_{\mathcal{C},x}$ is equal to $\Theta^{KS}_{\mathcal{C},x}$.

Proposition 3.26. *(**Comp**) is true for any Γ, $\mathcal{C}$, x in any rationally split case.*

Proof. Proposition 3.23 (or rather Corollary 3.16) shows that this is true so long as we can pull back by the section x after excising the Γ-orbit of the hyperplane collection L_p. But since the torsion orders of x and $\mathcal{C}$ are relatively prime, we certainly can pick p so that L_p avoids x, and then its Γ-orbit must do so as well, since Γ fixes x. isogenies. □

We now prove a "bootstrapping" lemma which says that if we can find a single fiber of the family over $Y(H)$ on which Γ acts which splits as a power of an elliptic curve, then we can extend by using the "constellation" given by the rational orbit of that point.

[33]There is little hope of a comparison up on the bundle, considering how violent the localization process is.

Theorem 3.27. *If any fiber of $A(H)$ is rationally split, then **(Comp)** is true for $(A(H), \Gamma, \mathcal{C}, x)$, for* any Γ*, x, and $\mathcal{C}$ with the torsion section and cycle fixed by Γ, and such that the torsion orders of x and $\mathcal{C}$ are relatively prime.*

Proof. Let τ_0 be a point in $X_{\mathbf{H}(\mathbb{R})}$ mapping to the point of $Y(H)$ (under the uniformization (3.1)) whose fiber is rationally split. Then every fiber of the family $A(H)$ over a point of $\mathbf{H}(\mathbb{Q})\tau_0$ is then also rationally split, since the matrix $M \subset \mathbf{H}(\mathbb{Q})$ furnishes an isogeny between the fibers over τ_0 and $M\tau_0$. Thus, the family $A(H)$ in fact has *infinitely* many fibers which are isogenous to an elliptic self-product.

For each such point $p \in Y(H)$, consider the automorphism group $\mathrm{Aut}(A(H)_p)$. Certainly it contains Γ; further, by construction, it is contained in

$$\mathrm{Aut}^0(A(H)_p) = \mathbf{GL}_{d/2}(K)$$

as an arithmetic subgroup, where K is the rational endomorphism ring of the associated elliptic curve (either $\mathbb{Q}$ or imaginary quadratic). Hence we can apply Proposition 3.26 deduce that $\Theta^{KS}_{\mathcal{C}_p,x_p}$ and $\Theta^{an}_{\mathcal{C}_p,x_p}$ coincide as $\mathrm{Aut}(A(H)_p)$-cocycles valued in $H^0(k(p), \omega^{\otimes d/2})$, hence by restriction as Γ-cocycles.

By the universal coefficients theorem, and the fact that all the cocycles we are considering are valued in modules with trivial Γ-action, we can think of them as homomorphisms from the group homology groups $H_{d/2-1}(\Gamma, \mathbb{C})$. Then by functoriality, for any closed $(d/2-1)$-cycle $\mathbf{\Delta}$ in this group homology, we have the restriction

$$(\Theta^{KS}_{\mathcal{C},x}(\mathbf{\Delta}))_p = \Theta^{KS}_{\mathcal{C}_p,x_\tau}(\mathbf{\Delta})$$

and

$$(\Theta^{an}_{\mathcal{C},x}(\mathbf{\Delta}))_p = \Theta^{an}_{\mathcal{C}_p,x_\tau}(\mathbf{\Delta}).$$

Thus, for all $\mathbf{\Delta}$, the differential forms $= \Theta^{KS}_{\mathcal{C},x}(\mathbf{\Delta})$ and $\Theta^{an}_{\mathcal{C},x}(\mathbf{\Delta})$ agree on a analytically dense subset of $\mathcal{Y}(H)$, whence they coincide as homomorphisms from $H_{d/2-1}(\Gamma, \mathbb{C})$, i.e. as cohomology classes. $\square$

This result allows us to prove that the Kings-Sprang cocycle coincides with the analytic one in a wide range of settings; we give some examples:

Example 3.28. Let $\mathbf{B}$ be the units of an indefinite quaternion algebra over $\mathbb{Q}$; then the dual pair $(\mathbf{B}, \mathbf{B})$ corresponds to the action of $\mathbf{B}(\mathbb{Z})$ on the universal abelian variety with quaternionic multiplication (by precisely that group). Each CM point in the upper half-plane corresponding to an imaginary quadratic field K yields a seesaw

$$\begin{tikzcd}[column sep=small] \mathbf{GL}_2(K) \arrow[d, dash] \arrow[drr, dash] & & \mathbf{B} \arrow[d, dash] \arrow[dll, dash] \\ \mathbf{B} & & \mathbf{GL}_1(K) \end{tikzcd} \tag{3.25}$$

Since the CM dual pairs are split and thus satisfy (**Comp**), so does the pair $(\mathbf{B}, \mathbf{B})$. The first homology of a quaternionic Shimura curve is spanned by real quadratic geodesics, each of whose corresponding quaternionic modular form under $\Theta_{\mathcal{C},x}$ we can explicitly compute as a certain restriction of a Hilbert-Eisenstein series following the example of 3.9 (in the modular curve case).

Example 3.29. For the dual pair $(F^\times, \mathrm{GL}_2(F))$ for a totally real field F of degree n over $\mathbb{Q}$, the corresponding Hilbert modular scheme parameterizes abelian schemes with F-multiplication. Any sum of n identical CM elliptic curves, together with a distinguished F-torus in its automorphism group, has a corresponding dense orbit in the corresponding symmetric space $\mathcal{H}^n$. We obtain that the class

$$\Theta^{KS}_{\mathcal{C},x}(\gamma),$$

for γ a generator of the free rank-1 group $\Gamma \subset \mathcal{O}_F^\times$, is given by a Hilbert-Eisenstein series of parallel weight $(1, \ldots, 1)$, as computed in [BCG1, Theorem 30].

Example 3.9 is the diagonal restriction of this example along the seesaw

$$\begin{tikzcd}[column sep=small] \mathrm{GL}_2(\mathbb{Q}) \arrow[d, dash] \arrow[drr, dash] & & \mathrm{GL}_2(F) \arrow[d, dash] \arrow[dll, dash] \\ F^\times & & \mathrm{GL}_2(\mathbb{Q}) \end{tikzcd} \tag{3.26}$$

written down in [BCG1, §13].

We henceforth feel justified in denoting this common cocycle simply by $\Theta_{\mathcal{C},x}$ for any arithmetic group acting on a family of principally polarized abelian varieties, without any qualifying superscript.

Remark 3.30. What can we say if we drop the assumptions (**Alg1**) and (**Alg2**) on the dual pair $(\mathbf{G}, \mathbf{H})$? Is there anything to be done in the non-algebraic setting? Consider just the split case $(\mathbf{GL}_a, \mathbf{GL}_b)$ for $b > 2$: we have something like $\mathrm{GL}_a(\mathbb{Z})$ acting on a copies of a b-torus bundle over some locally symmetric space for $\mathbf{GL}_b$. By removing codimension-b tori, we can cut the cohomological dimension of this torus bundle down to $a(b-1)$; this suggests the possibility of obtaining an $(a-1)$-cocycle valued in weight $(ab-a)$-forms. Indeed, 2.2 implies that integration of E_ψ gives a theta kernel representing, at the level of cohomology on either side.

It seems likely there is some formality statement generalizing Orlik-Solomon, but in the absence of Proposition 3.8 and the close relationship to complex structures, it is not clear how to formulate it; probably one needs some representation-theoretic generalization.

4. Arithmetic and motivic refinements

Notational convenience: in the rest of this thesis, the symbols $K_\bullet$ denote *Milnor* K-theory, which is usually denoted $K_\bullet^M$. We will not make use of (Quillen's) algebraic K-theory at any point in this thesis, so this will cause no ambiguity.

In this section, based on the idea in [SV], we produce an analogue of the Eisenstein theta kernel $\Theta_\mathcal{C}$ valued in motivic cohomology or Milnor K-theory (the latter being just parallel-degree motivic cohomology for a semi-local scheme). Over a characteristic zero field, this is possible unconditionally without too much trouble, using a similar "violent localization" approach as in the previous section. We further show in subsection 4.2.3 that the differential cocycle $\Theta_\mathcal{C}$ (using the Kings-Sprang definition) is the regulator of the motivic cocycle up to a simple period by exhibiting a map of complexes realizing the regulator map. Finally, we consider some extra structure we can put on the motivic cocycles we construct, with respect to their distribution and norm-compatibility properties.

To begin, we will again work in the context of a *arbitrary* discrete group Γ acting on an abelian scheme $\mathcal{A}$ over a general base $\mathcal{Y}$ defined over a characteristic zero field. Further additional conditions will be specified as needed.

4.1. **Technical preliminaries.**

4.1.1. *Motivic cohomology.* For a smooth equidimensional scheme X over a general base, we define the motivic cohomology

$$H^i(X, \mathbb{Z}(n))$$

as the Zariski hypercohomology

$$\mathbb{H}^i(X, \underline{\mathbb{Z}(n)_X})$$

of Bloch's weight-n *cubical* complex of sheaves $\underline{\mathbb{Z}(n)_X}$; this is the approach followed in [Tot] and [GL], for example. It is more standard to use simplicial language instead of cubical, but the two are equivalent for formal reasons, as proven in loc. cit.

Remark 4.1. This construction should more properly be called *Borel-Moore motivic homology* (or, historically, "higher Chow groups"), but we elide this technical point in this thesis, since for smooth schemes over a perfect field, Borel-Moore motivic cohomology agrees with the "standard" motivic cohomology, defined via Voevodsky-style motivic complexes, thanks to the results in [V] and [FS].

The Bloch complex of sheaves is defined as follows: let

$$\tilde{z}^n(U, i) := Z^n(U \times \square^i)$$

be the group of codimension-n cycles on $U \times \square^i$ meeting all faces properly. Here, $\square^i$ is the algebraic i-cube which is simply the affine space

$$\text{Spec } \mathbb{Z}[t_1, \ldots, t_i]$$

and the jth face map is given by the difference of the pullbacks to the subvarieties cut out by $t_j = 0$, respectively $t_j = 1$; the alternating sum of face maps gives, as usual, a differential from $\tilde{z}^n(U, i)$ to $\tilde{z}^n(U, i-1)$. It turns out that the resulting complex splits into a direct sum

$$\tilde{z}^n(U, i) = d^n(U, i) \oplus z^n(U, i)$$

where the former summand consists of *degenerate* cycles which can be pulled back from one of the faces of $\square^i$ given by $t_j = 0$, and the latter summand consists of the *reduced cycles* which are in the kernel of of the restriction to each face $t_j = 0$. We define

$$(\mathbb{Z}(n)_X)^i(U) := z^n(U, 2n - i).$$

This complex is suitably functorial for flat pullbacks and proper pushforwards.

By Zariski descent for the cohomology groups of this cycle complex (see [FS, Corollary 12.2]) the natural map

$$H^i(\mathbb{Z}(n)_X) \to \mathbb{H}^i(X, \mathbb{Z}(n)_X)$$

is an isomorphism, where $\mathbb{Z}(n)_X := \Gamma_X(\underline{\mathbb{Z}(n)_X})$ is the global sections on X of the Bloch complex of sheaves; we will refer to this complex as the *Bloch cycle complex*.

Therefore, if Γ is a discrete group acting on X, we may define the *equivariant motivic cohomology* using the global sections of $\mathbb{Z}(n)_X$ for $D(X)$, in our usual style of defining Borel-equivariant analogues from §2.1. For the same reason as the non-equivariant theory, the cohomology of the corresponding double complex also computes the Γ-hypercohomology

$$\mathbb{H}^i_\Gamma(X, \underline{\mathbb{Z}(n)}_X).$$

Functoriality and the existence of the Hochschild-Serre spectral sequence are, as usual, fully formal from the generalities in §2.1, but we need to check the existence of the localization sequence and Gysin isomorphism. The two together follow from the distinguished triangle of complexes

$$\underline{\mathbb{Z}(n-d)_Z}[-d] \to \underline{\mathbb{Z}(n)_X} \to \underline{\mathbb{Z}(n)_{X-Z}} \tag{4.1}$$

whose existence is proven in [Bl2, §3], which gives the localization/Gysin sequence in the form

$$\ldots \to H^{i-d}_\Gamma(Z, \mathbb{Z}(n-d)) \to H^i_\Gamma(X, \mathbb{Z}(n)) \to H^i_\Gamma(X - Z, \mathbb{Z}(n)) \to \ldots \tag{4.2}$$

It will also be useful for us to consider the *coniveau spectral sequence* in bare motivic cohomology given by the filtration on cycles by codimension [Geis, §4], which is of the form

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H^{q-p}(k(x), \mathbb{Z}(n-p)) \Rightarrow H^{p+q}(X, \mathbb{Z}(n)) \tag{4.3}$$

where $X^{(p)}$ denotes the codimension-p points.

4.1.2. *The Gersten conjecture and Milnor K-theory.* A key tool for us is the *Gersten complex*, a complex associated to various cohomology theories with Zariski descent; one form of it is constructed in [Bl2]. The Gersten complex for Milnor K-theory, the case of interest to us, is given by

(4.4)
$$Z_{Ger(K(n))}(X) := K_n(X) \to \bigoplus_{x \in X^{(0)}} K_n(k(x)) \to \bigoplus_{x \in X^{(1)}} K_{n-1}(k(x)) \to \ldots \to \bigoplus_{x \in X^{(d)}} K_{n-d}(k(x)) \to 0$$

for a semi-local scheme X over a characteristic zero field,[34] where the first map is the natural pullback and the successive maps are the tame residue symbols.

The exactness of this complex is known as the *Gersten conjecture* for Milnor K-theory, which is known for regular schemes over a field by [Kerz]. Motivic cohomology (in our sense) also admits a Gersten complex; we will focus only on the degree (n, n)-case. In the same setting as above, it is given by

$$H^n(X, \mathbb{Z}(n)) \to \bigoplus_{x \in X^{(0)}} H^n(k(x), \mathbb{Z}(n)) \to \ldots \to \bigoplus_{x \in X^{(d)}} H^{n-d}(k(x), \mathbb{Z}(n-d)) \to 0 \tag{4.5}$$

for which we write $Z_{Ger(H(n))}(X)$. For any semi-local X, there is a canonical morphism

$$K_1(X) \to H^1(X, \mathbb{Z}(1))$$

which is an isomorphism if X is the spectrum of a field; this induces

$$\mu : K_n(X) \to H^n(X, \mathbb{Z}(n)) \tag{4.6}$$

by sending the symbol $f_1 \otimes \ldots \otimes f_n$ to $[f_1] \smile \ldots \smile [f_n]$; this factors through the Steinberg relation by [MVW, Proposition 5.9]. For a field, this is an isomorphism.

We also have a natural map of complexes $Z_{Ger(K(n))}(X) \to Z_{Ger(H(n))}(X)$ induced by μ, which we also denote by μ in a slight abuse of notation; this is a degree-wise isomorpism, since all the terms involved are motivic cohomology/Milnor K-theory of fields, meaning that the Gersten conjecture is also known in the motivic setting.

[34]This hypothesis is more restrictive than necessary, but we choose it to avoid technicalities.

A different phrasing of the Gersten conjecture is that we have the exact sequence of sheaves

$$\mathcal{H}^n_X(\mathbb{Z}(n)) \to \bigoplus_{x\in X^{(0)}} H^n(k(x),\mathbb{Z}(n)) \to \ldots \to \bigoplus_{x\in X^{(d)}} H^{n-d}(k(x),\mathbb{Z}(n-d)) \to 0.$$

Here $\mathcal{H}^n_X(\mathbb{Z}(n))$ is the Zariski sheaf associated to motivic cohomology, which is the Zariski sheafification of the presheaf

$$U \mapsto H^n(U,\mathbb{Z}(n)),$$

whose stalk at a point x is the motivic cohomology of the localization of X at x. The other terms are interpreted as sums of skyscraper sheaves associated to the corresponding points x.

Consider now the hypercohomology spectral sequence

$$E_2^{p,q} = H^p(X,\mathcal{H}^q_X(\mathbb{Z}(n))) \Rightarrow H^{p+q}(X,\mathbb{Z}(n)). \tag{4.7}$$

In the presence of the Gersten conjecture, this $E_2^{p,q}$ term can be then be interpreted as the hypercohomology of the complex of sheaves $Z^\bullet_{Ger(H(n))}(X)$, which is just the cohomology of the complex of the corresponding global sections

$$\bigoplus_{x\in X^{(0)}} H^n(k(x),\mathbb{Z}(n)) \to \ldots \to \bigoplus_{x\in X^{(d)}} H^{n-d}(k(x),\mathbb{Z}(n-d)) \to 0.$$

(now interpreted as just literal groups, not sheaves) since skyscraper sheaves are flasque. We will equally write $Z_{Ger(H(n))}$ for this complex of global sections as in (4.5), even for non-local schemes X, though in general exactness is of course lost. We will do the same for $Z_{Ger(K(n))}$ as in (4.4). From this discussion, we see that descent spectral sequence can be identified with the coniveau spectral sequence (4.3) from the E_2-term on.

Proposition 4.2. *There is a map of complexes of Zariski sheaves*

$$\psi^\bullet_X : (\mathbb{Z}(n)^\bullet_X) \to Z^\bullet_{Ger(K^n)}(X)[-n]$$

defined as follows: for the class of an irreducible closed subvariety $[Z] \in z^n(U,2n-i)$, the projection of Z to U is at most codimension-$(i-n)$. If it is strictly higher codimension, we set $\psi^i_X([Z]) = 0$. Otherwise, Z is dominant over a codimension-$(i-n)$ integral closed subscheme $\tilde{Z}$ of X; write $p : Z \to \tilde{Z}$ for the projection *map. We set*

$$\psi^i_X(z) := \mathbf{N}_p([\sigma t_1 \smile \ldots \smile \sigma t_{2n-i}]) \in K_{2n-i}(k(\tilde{Z})) \tag{4.8}$$

where

$$\sigma x := \frac{x}{x-1}$$

and $\mathbf{N}$ *is the norm*[35] *in Milnor* K*-theory (as defined in [BaTa, §5]) for the finite map* p.[36] *The map* ψ_X *is functorial for quasi-finite flat pullbacks and proper pushforwards.*

Proof. This map was first constructed and its properties proven in [La]. □

Remark 4.3. One may wonder about the reason for the transformation σ in the above statement. It should be thought of as the Mobiüs transformation for the 2-torsion matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

which fixes 0 and interchanges 1 and ∞ in $\mathbb{P}^1_{\mathbb{Z}}$. The reason for this is that the tame symbols in the Gersten complex consider zeroes and poles, i.e. restrict to behavior at 0 and ∞, but the cubical face maps in the complex $\mathbb{Z}(n)$ restrict to behavior at the faces $t_i = 0$ and $t_i = 1$, so 1 and ∞ need to be swapped to compare the two. Perhaps the more canonical approach, followed in [Tot], would be to give a more natural definition of the cubical complex by identifying $\square^i$ with $(\mathbb{P}^1 - \{1\})^i$ and setting the cubical face maps to correspond to 0 and ∞. We opted instead to use the more standard definition of the cubical complex, which also is more obviously geometrically "cubical" at a glance.

Remark 4.4. The isomorphism $K_k(X) \to H^{2k}(X, \mathbb{Z}(k))$ is given by sending

$$(k(Z), [f_1 \otimes \ldots \otimes f_k]) \mapsto [\Gamma_{(\sigma f_1, \ldots, \sigma f_k)}]$$

[35] Also called transfer map or pushforward.

[36] Recall from the definition of the cubical Bloch complex that t_i are the coordinate functions on the algebraic cube.

where $(\sigma f_1, \ldots, \sigma f_k) : Z \dashrightarrow \square^k$ is a (rational) map from the cycle Z to the algebraic k-cube, and $\Gamma_\bullet$ is its graph. Note, however, that these maps cannot be lifted to the level of cubical cycles, and so do not furnish any kind of inverse to $\psi_X^\bullet$.

Proposition 4.5. *If X is a semi-local scheme defined over a characteristic zero field, the Γ-equivariant motivic cohomology of X can be computed as*

$$C^\bullet(\Gamma, Z^\bullet_{Ger(H(n))}(X)).$$

Proof. We claim that ψ_X is a quasi-isomorphism in this setting; then by functoriality, it is a map of Γ-modules and the result is formal.

In [Zhong, §2], it is proven that the maps $\mu \circ \psi_X^\bullet$ are the edge maps in the coniveau spectral sequence (4.3) for X. As we identified the hypercohomology/coniveau spectral sequences earlier, this map hence induces the hypercohomology spectral sequence edge map

$$H^i(X, \mathbb{Z}(n)) \to H^{i-n}(X, \mathcal{H}^n(\mathbb{Z}(n)))$$

for each $i \geqslant n$. In particular, when X is semi-local, the Gersten conjecture implies that the only non-zero term is for $i = n$, where the edge map is thus an isomorphism; see [Geis, Corollary 4.4]. $\square$

4.2. **Constructing cocycles.**

4.2.1. *The equivariant motivic polylogarithm.* We now briefly recapitulate the formalism of section 2 in the motivic context. Recall that $\pi : \mathcal{A} \to \mathcal{Y}$ is our relative abelian scheme over an integral base defined over a characteristic zero field, with Γ-action trivial on the base $\mathcal{Y}$. Recall also that C is a closed Γ-invariant subscheme of $\mathcal{A}$ consisting of a union of c-torsion sections on each fiber.

We take the Γ-equivariant theory

$$H^\bullet = H^\bullet(-, \mathbb{Z}(d/2))$$

where $d/2$ is the relative dimension of $\mathcal{A}$; we then have the equivariant Gysin sequence (4.2) in the form

$$\ldots \to H^\bullet_\Gamma(\mathcal{A}, \mathbb{Z}(d/2)) \to H^\bullet_\Gamma(\mathcal{A} - C, \mathbb{Z}(d/2)) \to H_\Gamma^{\bullet+1-d/2}(C, \mathbb{Z}) \to H_\Gamma^{\bullet+1}(\mathcal{A}, \mathbb{Z}(d/2)) \to \ldots$$

In addition, by the construction of the maps in [Bl2, §3] we can identify the map in the above sequence

$$\mathbb{Z}\{\pi^0(C)\}^\Gamma = H^0(C, \mathbb{Z}(0))^\Gamma \to H^d_\Gamma(\mathcal{A}, \mathbb{Z}(d/2))$$

with the cycle class map. Thus, after inverting c, any degree-0 cycle C lies in the kernel of this map, since the cycle class map preserves torsion order on abelian schemes.[37]

The same Lieberman projector e_L used in section 3 annihilates $H^{d-1}_\Gamma(\mathcal{A}, \mathbb{Z}(d/2))[1/(d+1)!]$ over a characteristic zero field; this follows from the argument in [SV, §6]. We thus write

$$\mathbb{Z}' := \mathbb{Z}\left[\frac{1}{c(d+1)!}\right]$$

for the finest coefficients we are allowed. Following §2.1, we hence deduce a cocycle

$$z^M_{\mathcal{C}} \in H^{d-1}_\Gamma(\mathcal{A} - C, \mathbb{Z}'(d/2)).$$

4.2.2. *The motivic cocycle over a field.* As in section 3.5.3, we localize $\mathcal{A}$ at η_D the union of the generic points of a torsion cycle D of order relatively prime to c, and call the result $\mathcal{U}$; this is a semi-local scheme, so

$$H^i(\mathcal{U}, \mathbb{Z}(d/2)) = 0$$

for $i > d/2$. As before, we have the restricted projection map $\mathcal{U} \to \eta_{\mathcal{Y}}$.

Thus, we obtain as in 2.12 a Hochschild-Serre edge map

$$H^{d-1}_\Gamma(\mathcal{U}, \mathbb{Z}'(d/2)) \to H^{d/2-1}(\Gamma, K_{d/2}(\mathcal{U}, \mathbb{Z}'))$$

and we define $\Theta^M_{\mathcal{C}}$ as the image of the restriction of $z^M_{\mathcal{C}}$ under this map. For any torsion section $x : \mathcal{Y} \to \mathcal{A}$ whose image is contained in D, we can also define the pullback

$$\Theta^M_{\mathcal{C},x} = x^*\Theta^M_{\mathcal{C}} \in H^{d/2-1}(\Gamma, K_{d/2}(\eta_{\mathcal{Y}}, \mathbb{Z}'))$$

valued in Milnor K-theory of the base.

[37] This is due to the push-pull identity $[c]_*[c]^* = c^d$ in motivic cohomology, which is visible already on the level of the Bloch complex $\mathbb{Z}(n)_X$.

Analogously to section 3.5.3, we wish for this class to be defined over the whole base, not just the generic point η. To do this, we will follow a similar approach as in that section.

For any codimension-1 point $y \in \mathcal{Y}$, let $\mathcal{A}_{D_y}$ be the localization of $\mathcal{A}$ at the finite set of points D_y (and $\mathcal{Y}_y$ the analogous localization of the base); we have the natural projection map $\mathcal{A}_{D_y} \to \mathcal{Y}_y$. Running the above formalism in the setting of these semi-local schemes, we obtain a class in

$$H^{d/2-1}(\Gamma, K_{d/2}(\mathcal{Y}_y, \mathbb{Z}'))$$

which restricts to $\Theta^M_{\mathcal{C},x}$ by functoriality. This implies that the pushforward of $\Theta^M_{\mathcal{C},x}$ by the tame symbol

$$K_{d/2}(\mathcal{Y}_y, \mathbb{Z}') \to K_{d/2-1}(k(y), \mathbb{Z}')$$

vanishes. Using the existence of these restrictions for as y varies, the best we are able to prove is:

Proposition 4.6. *There exists a class in*

$$H^{d/2-1}(\Gamma, H^0(\mathcal{Y}, \mathcal{H}^{d/2}(\mathbb{Z}'(d/2))))$$

which restricts to the class in of $\Theta^M_{\mathcal{C},x}$ over $\eta_{\mathcal{Y}}$. If we invert $h = |H_{d/2-2}(\Gamma, \mathbb{Z})^{tor}|$, then there is a unique such lift.

Proof. By the Gersten conjecture and the left exactness of the global sections functor, we have a left exact sequence

$$H^0(\mathcal{Y}, \mathcal{H}^{d/2}(\mathbb{Z}'(d/2))) \to H^{d/2}(\mathcal{Y}, \mathbb{Z}'(d/2)) \to H^{d/2}(\eta_{\mathcal{Y}}, \mathbb{Z}'(d/2)) \to \bigoplus_{y \in \mathcal{Y}^{(1)}} H^{d/2}(k(y), \mathbb{Z}'(d/2)) \tag{4.9}$$

where the second arrow is the sum over tame symbols of all codimension-1 points. From the preceding discussion, the pushforward of $\Theta^M_{\mathcal{C},x}$ by the last map is zero. Then by the long exact sequence in cohomology associated to (4.9), there certainly exists a lift of $\Theta^M_{\mathcal{C},x}$ to $H^{d/2-1}(\Gamma, H^{d/2}(\mathcal{Y}, \mathbb{Z}'(d/2)))$, though not canonically determined.

If we invert h, by the universal coefficients theorem Γ-cohomology classes in degree $d/2-1$ valued in $\mathbb{Z}[1/h]$-modules with trivial action can be identified with homomorphisms from the Γ-homology in the same degree.

$\Theta^M_{\mathcal{C},x}$ is identified then with a homomorphism

$$\hom(H_{d/2-1}(\Gamma), K_{d/2}(\eta_{\mathcal{Y}}, \mathbb{Z}'))[1/h]$$

such that its composition with every tame symbol is trivial; hence again by (4.9) it is actually valued in $H^0(\mathcal{Y}, \mathcal{H}^{d/2}(\mathbb{Z}'(d/2)))$. □

When Γ is an arithmetic group (so in particular, has finitely generated cohomology), we thus recover unconditionally a motivic theta cocycle valued in $H^0(\mathcal{Y}, \mathcal{H}^{d/2}(\mathbb{Z}'(d/2)))$ with integral coefficients outside a finite number of primes.

It would be nicer to obtain a class in $H^{d/2}(\mathcal{Y}, \mathbb{Z}'(d/2))$. In fact, we can do this if we are willing to invert more primes:

Proposition 4.7. *If we invert the set S of all primes for which the integral cohomology of Γ has torsion, there exists a unique class in*

$$H^{d/2-1}(\Gamma, H^{d/2}(\mathcal{Y}, \mathbb{Z}'(d/2)))[S^{-1}]$$

which restricts to the class of $\Theta^M_{\mathcal{C},x}$ over $\eta_{\mathcal{Y}}$.

Proof. We begin with the class

$$x^* z^M_{\mathcal{C}} \in H^{d-1}_{\Gamma}(\mathcal{Y}, \mathbb{Z}'(d/2))[S^{-1}],$$

noticing that the Γ-action on the space $\mathcal{Y}$ is trivial. As such, the double complex computing this equivariant cohomology

$$C^\bullet(\Gamma, \mathbb{Z}'[S^{-1}](d/2)_Y)$$

is actually a tensor product

$$C^\bullet(\Gamma, \mathbb{Z}[S^{-1}]) \otimes \mathbb{Z}'[S^{-1}](d/2)_Y$$

and hence by the Künneth theorem (see [Hatcher, Theorem 3B.5] for a statement in this generality) we have a natural decomposition

$$H^{d-1}_{\Gamma}(\mathcal{Y}, \mathbb{Z}'(d/2))[S^{-1}] \cong \bigoplus_{p+q=d-1} H^p(\Gamma, H^q(\mathcal{Y}, \mathbb{Z}'(d/2)))[S^{-1}]. \tag{4.10}$$

where we have used the inversion of S to ensure the vanishing of the Tor term in [Hatcher, Theorem 3B.5] due to the freeness of the cohomology of Γ. Taking the projection to the term with $(p,q) = (d/2-1, d/2)$ affords us a class in the desired group $H^{d/2-1}(\Gamma, H^{d/2}(\mathcal{Y}, \mathbb{Z}'(d/2)))[S^{-1}]$.

Upon restriction to $\eta_{\mathcal{Y}}$, we claim that the edge map (2.12) for the Γ-scheme $\mathcal{Y}$

$$H^{d-1}_{\Gamma}(\eta_{\mathcal{Y}}, \mathbb{Z}'(d/2)))[S^{-1}] \to H^{d/2-1}(\Gamma, H^{d/2}(\eta_{\mathcal{Y}}, \mathbb{Z}'(d/2)))[S^{-1}]$$

coincides with the Künneth projection (4.10). Indeed, the Künneth decomposition implies that we can take a representative of the equivariant class in the double complex considered in Proposition 2.2 each of whose components is already both $d-$ and ∂-closed, whereupon the explicit construction of the edge map in that proposition is reduced to simply taking the corresponding component.

Then the pullback by x maps the Hochschild-Serre spectral sequence (and hence the edge map (2.12)) for $\mathcal{U}$ (the subbundle of $\mathcal{A}$, recall) to that for $\mathcal{Y}$, showing that the Künneth-projected class coincides with $\Theta^M_{\mathcal{C},x}$. $\square$

4.2.3. *Computing regulators in equivariant Dolbeault complexes.* We proceed to comparing the motivic cocycle $\Theta^M_{\mathcal{C}}$ to the differential cocycle $\Theta_{\mathcal{C}}$ over $\mathbb{C}$.

There is a Betti/de Rham regulator of motivic cohomology given in bidegree-(n,n) by[38]

$$H^n(X, \mathbb{Z}(n)) \to H^n(X, \mathbb{C}) \tag{4.11}$$

$$[f_1 \smile \ldots \smile f_n] \mapsto [d\log f_1 \wedge \ldots \wedge d\log f_n], f_i \in k(X)^\times. \tag{4.12}$$

This is explained, for example, in [LW, §2.1.5]. In fact, this regulator is well-defined even in the top-filtered part of Hodge cohomology [G1], in that it factors through

$$H^n(X, \mathbb{Z}(n)) \to H^0(X, \Omega^n) \to H^n(X, \mathbb{C}); \tag{4.13}$$

we will use this to compare $\Theta^M_{\mathcal{C}}$, valued in Milnor K-theory, with $\Theta_{\mathcal{C}}$, valued in coherent cohomology.

[38]Note that not every cup product of elements in $k(X)^\times$ will give an element in the motivic cohomology group (because of poles and zeroes), but by the exact sequence (4.9), every element in the motivic cohomology can be written as a sum of such cup products.

Remark 4.8. The map in the form (4.13) is sometimes called the *Bloch map*, first defined in [Bl3] (and attributed to "secret papers" of Gersten). Note that despite the fact that we will only define it over $\mathbb{C}$ using the Dolbeault complex (see below), it actually is algebraically defined and thus makes sense over any base.

Let $\mathcal{U}$ be as in the previous section. Recall from (3.20) the Dolbeault resolution of $\Omega_{\mathcal{U}}^{d/2}$

$$\Omega_{\mathcal{U}}^{d/2} \hookrightarrow \mathcal{D}_{\mathcal{U}}^{d/2,0} \xrightarrow{\bar{\partial}} \mathcal{D}_{\mathcal{U}}^{d/2,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{D}_{\mathcal{U}}^{d/2,d/2} \tag{4.14}$$

where we write $\mathcal{D}_{\mathcal{U}}^{p,q}$ for the direct limit under pullbacks by the inclusion maps

$$\varinjlim_{\mathcal{U}\subset\mathcal{V}\subset\mathcal{A}} \mathcal{D}_{\mathcal{V}}^{p,q}$$

of Zariski opens $\mathcal{V}$ containing $\mathcal{U}$, and $\mathcal{D}_{\mathcal{V}}^{p,q}$ denotes the smooth (p,q)-currents on the analytification of $\mathcal{V}$ over $\mathbb{C}$.

We have a map of complexes [Kerr, Lemma 2.2]

$$\begin{tikzcd} \mathcal{K}_{d/2}(\mathcal{U}) \arrow[r] \arrow[d, "d\log^{\otimes d/2}"] & \bigoplus_{x\in\mathcal{U}^{(0)}} K_{d/2}(k(x)) \arrow[r] \arrow[d, "\rho_0(d/2)(\mathcal{U})"] & \dots \arrow[r] & \bigoplus_{x\in\mathcal{U}^{(d/2)}} K_0(k(x)) \arrow[d, "\rho_{d/2}(d/2)(\mathcal{U})"] \\ \Omega_{\mathcal{U}}^{d/2} \arrow[r] & \mathcal{D}_{\mathcal{U}}^{d/2,0} \arrow[r] & \dots \arrow[r] & \mathcal{D}_{\mathcal{U}}^{d/2,d/2} \end{tikzcd} \tag{4.15}$$

where $\mathcal{K}_{d/2}$ is the Zariski Milnor K-theory sheaf (in the same manner as $\mathcal{H}^\bullet(\mathbb{Z}(n))$ for motivic cohomology earlier), and where $\rho_k(d/2)(\mathcal{U})$ is the regulator map defined by sending

$$(k(x), [f_1 \otimes \ldots \otimes f_{n-k}]) \mapsto (2\pi i)^k \delta_x \wedge d\log f_1 \wedge \ldots \wedge d\log f_{n-k}$$

where δ_x is the current of integration along the closed cycle x. This map of complexes induces the *de Rham regulator* on the Milnor K-group $K_{d/2}$ as defined in [LW, §2.1.5]. Further, the definition of the map makes it clear it is Γ-equivariant, so we can upgrade the collection of $\rho_i(d/2)$ to a map of equivariant complexes

$$\rho_\Gamma^{\bullet,\bullet}(d/2)(\mathcal{U}) : C^\bullet(\Gamma, Z_{Ger}^\bullet(\mathcal{U})) \to C^\bullet(\Gamma, \mathcal{D}^{d/2,\bullet}(\mathcal{U})) \tag{4.16}$$

inducing on cohomology maps

$$H^{\bullet}_{\Gamma}(\mathcal{U},\mathbb{Z}(d/2)) \to H^{\bullet-d/2}_{\Gamma}(\mathcal{U},\Omega^{d/2}_{\mathcal{U}/\mathbb{C}}). \tag{4.17}$$

which can justly be called equivariant de Rham regulators.

We similarly can define maps of complexes $\rho^{\bullet,\bullet}(n)(X)$, $\rho^{\bullet,\bullet}_{\Gamma}(n)(X)$ for any smooth finite type scheme, respectively Γ-scheme, $X/\mathbb{C}$ and any integer n. These maps have the following properties.

Proposition 4.9. (1) *The maps $\rho^{\bullet,\bullet}(n)(X)$ and $\rho^{\bullet,\bullet}_{\Gamma}(n)(X)$ are functorial for flat pullback of schemes, respectively Γ-schemes, X, as well as for pushforwards by finite maps.*

(2) *The map on cohomology induced by $\rho^{\bullet,\bullet}(n)(X)$ refines the usual de Rham regulator of motivic cohomology (under the Hodge filtration).*

(3) *The maps $\rho^{\bullet,\bullet}_{\Gamma}(n)(X)$ induce maps of Hochschild-Serre spectral sequences, from that of equivariant motivic cohomology to that of equivariant coherent cohomology.*

(4) *Given a closed subset $Z \subset X$, the maps $\rho^{\bullet,\bullet}(n)(X)$, respectively $\rho^{\bullet,\bullet}_{\Gamma}(n)(X)$, induce maps from the localization sequence for motivic cohomology to the localization sequence in coherent cohomology, respectively localization sequence for equivariant motivic cohomology to the localization sequence in equivariant coherent cohomology; unfortunately, we were unable to locate one in the literature.*

Proof. (1) is proven in [LW, §2]. (2) is a consequence of the fact, also from [LW], that the $d\log$ map induces the usual de Rham regulator for motivic cohomology. (3) follows because $\rho^{\bullet,\bullet}_{\Gamma}(n)(X)$ are defined via the maps of double complexes (4.16), and the respective Hochschild-Serre spectral sequences are simply the spectral sequence of these respective double complexes. (4) follows because $\rho^{\bullet,\bullet}_{\Gamma}(n)(X)$ induces maps of the corresponding distinguished triangles used to construct these localization sequences. □

Theorem 4.10. *The map* (4.17) *sends $z^M_{\mathcal{C}}$ to $(2\pi i)^{d/2} z^{KS}_{\mathcal{C}}$.*

Proof. $z^{KS}_{\mathcal{C}}$ is characterized as the unique e_L-fixed class whose residue is $\delta_{\mathcal{C}}$ in its localization sequence, for the pair $C \subset \mathcal{A}$. By Proposition 4.9(4), (4.17) maps the motivic localization sequence to the coherent localization sequence, equivariantly for the action of isogenies by Proposition 4.9(1). Since $z_{\mathcal{C}}^M$ has residue $\mathcal{C}$ and is also e_L-fixed, its regulatir has residue $(2\pi i)^{d/2}\delta_{\mathcal{C}}$ and the result follows. $\square$

From the above theorem and (3) from proposition 4.9, we finally deduce the comparison theorem:

Corollary 4.11. *The de Rham regulator of the big motivic theta cocycle is the big differential theta cocycle; i.e.*

$$(d\log^{\otimes d/2})_*\Theta_{\mathcal{C}}^M = (2\pi i)^{d/2}\Theta_{\mathcal{C}}.$$

Notice that it is *not* the case that the regulator of $\Theta_{\mathcal{C},x}^M$ is $\Theta_{\mathcal{C},x}$, since the latter pulled-back cocycle was constructed by contracting with a polyvector field in (2.13) before pulling back by a section x. To illustrate in a toy example with one-dimensional fibers, we have

$$d\log\theta(\tau,x) = \frac{\theta_1(\tau,x)}{\theta(\tau,x)}d\theta \tag{4.18}$$

which is different from

$$[\iota_{\partial z} d\log\theta(\tau,z)]_{z=x} = \frac{\theta_2(\tau,x)}{\theta(\tau,x)} \tag{4.19}$$

Here θ is a function on the bundle $\mathcal{A}$ with partial derivatives θ_1 and θ_2 with respect to the coordinates τ and z on the base and fiber respectively, and $z = x$ defines the locus of the torsion section. In the most classical setting where $\mathcal{A}$ is the universal elliptic curve over a modular curve, (4.18) yields weight-two Eisenstein series while (4.19) yields weight-one Eisenstein series; see [BCG1, (9.6)].

Appendix A. Work of Goncharov and the weight-2 archimedean regulator

[G1, §3.5] constructs the Beilinson regulator map to Deligne cohomology directly on the level of complexes in weight 2.[39] We here put ourselves in the split $(\mathrm{GL}_2, \mathrm{GL}_2)$ setting for simplicity.

[39] He also has a construction for weights 1 and 3, but these are, respectively, not very interesting and much messier.

By identifying his weight-2 polylogarithmic motivic complex with the Gersten complex earlier, obtain the following version of his construction:

(A.1)
$$\begin{tikzcd} K_2(\mathbb{Q}(\mathcal{A}(H))) \arrow[r, "\partial"] \arrow[d, "r_2(2)"] & \bigoplus_{Z\in\mathcal{A}(H)^{(1)}} K_1(\mathbb{Q}(Z)) \arrow[r, "\partial"] \arrow[d, "r_2(3)"] & \bigoplus_{Z\in\mathcal{A}(H)^{(2)}} K_0(\mathbb{Q}(Z)) \arrow[d, "r_2(4)"] \\ \frac{(\mathcal{D}^{1,0}\oplus\mathcal{D}^{0,1})_{\mathbb{R}}(1)}{\mathcal{D}^{0,0}_{\mathbb{R}}(1)} \arrow[r, "D_{1,1}"] & \mathcal{D}^{1,1}_{\mathbb{R}}(1) \arrow[r, "2\bar\partial\partial"] & \mathcal{D}^{2,2}_{cl,\mathbb{R}}(2) \end{tikzcd}$$

For an explanation of all the notation, see [G1, §2, 4, 5]; we will not fully explain it here, only give a brief description and indicate any differences.

- $\mathcal{D}^{p,q}_{\mathbb{R}}(k)$ denotes the space of smooth (p,q)-currents on $\mathcal{A}(H)$, with values in $\mathbb{R}(k) := (2\pi i)^k\mathbb{R}$. If $p \neq q$, this real structure only makes sense if we sum $\mathcal{D}^{p,q}$ and $\mathcal{D}^{q,p}$, since one needs complex coefficients to separate these.
- D denotes the de Rham differential on currents.[40]
- We, unlike Goncharov, have denoted the second occurrence of D with a subscript $D_{1,1}$, to make clear that we are considering not the full differential, but only the projection to the $(1,1)$-distributions (i.e. throwing out the part in $(2,0)+(0,2)$).
- The subscript cl in the bottom right term indicates we only consider closed distributions.

The top complex, as we have seen, computes motivic cohomology $H^i(\mathcal{A}(H),\mathbb{Z}(2))$ at the ith place, while the bottom computes Deligne cohomology $H^i(\mathcal{A}(H)_{\mathbb{R}},\mathbb{R}(2))$. The maps $r_2(i)$ induce the Beilinson regulator map on these cohomology groups. For us, the relevant formulas are:

(A.2)
$$r_2(2) : f\wedge g \mapsto -\log|f|di\arg g + \log|g|di\arg f$$

(A.3)
$$r_2(3) : (Y,f) \mapsto 2\pi i\log|f|\delta_Y$$

(A.4)
$$r_2(4) : Y \mapsto (2\pi i)^2\delta_Y$$

[40]Note that it does not necessarily coincide with the de Rham differential on forms $d = \partial+\overline{\partial}$ embedded inside distributions; in particular, the famous Poincaré-Lelong formula gives the example $d^2 i\arg f = 0$, but $Ddi\arg f = \delta_{\text{div } f}$, the current associated to the divisor of f. That is, the two differentials differ on residue behavior. On globally holomorphic forms, the two coincide.

(Here, we use concatenation instead of wedge product for the module structure on currents by forms defined in §3.2.2, to follow Goncharov's notation.)

To obtain the differential form-valued cocycles we focused on in this thesis, we extend the diagram (A.1) to

(A.5)
$$
\begin{tikzcd}
K_2(K(\mathcal{A})) \arrow[r, "\partial"] \arrow[d, "r_2(2)"] & \bigoplus_{Z\in\mathcal{A}_{(1)}} K_1(K(Z)) \arrow[r, "\partial"] \arrow[d, "r_2(3)"] & \bigoplus_{Z\in\mathcal{A}^{(2)}} K_0(K(Z)) \arrow[d, "r_2(4)"] \\
\frac{(\mathcal{D}^{1,0}\oplus\mathcal{D}^{0,1})_{\mathbb{R}}(1)}{\mathcal{D}^{0,0}_{\mathbb{R}}(1)} \arrow[r, "D_{1,1}"] \arrow[d, "\pi_2^{-1}\circ d_{2,0+0,2}"] & \mathcal{D}^{1,1}_{\mathbb{R}}(1) \arrow[r, "2\bar{\partial}\partial"] \arrow[d, "\pi_2^{-1}\circ d"] & \mathcal{D}^{2,2}_{cl,\mathbb{R}}(2) \arrow[d, "\pi_3^{-1}"] \\
F^2\mathcal{D}^2 \arrow[r, "D"] & F^2\mathcal{D}^3 \arrow[r, "D"] & F^2\mathcal{D}^4
\end{tikzcd}
$$

where the last row now computes the second Hodge filtered piece of de Rham cohomology. The lower map of complexes realizes the natural map from Deligne to de Rham cohomology

$$H^2(\mathcal{A}_{\mathbb{C}}, \mathbb{R}(2)_{\mathcal{D}}) \to F^2H^2(\mathcal{A}, \mathbb{C})$$

coming from the Deligne complex's defining triangle

$$\Omega^{<2}_{\mathcal{A}_{\mathbb{C}}}[-1] \to \mathbb{R}(2)_{\mathcal{D}} \to \mathbb{R}(2).$$

This computation is implicit in the proof of [G3, Proposition 2.1].

Let us call the bottom vertical maps of (A.5) $s_2(i)$ for $i = 2, 3, 4$ by analogy. We have the formula for the composition

$$(s_2(2) \circ r_2(2))([g_1 \wedge g_2]) = d\log g_1 \wedge d\log g_2$$

which is the regulator map sending the motivic cocycle to the differential one. We thus see that Goncharov's regulator map is a strict refinement of the one we consider.

Remark A.1. In [BCG1, §13.5], it is computed (though details are not given) that the Beilinson regulator $r_2(2)(\Theta^M_{\mathcal{C}})$ is represented by a "doubly transgressed" theta kernel defined using the "further transgressed"

Mathai-Quillen forms due to [BGS]. In more general settings, we ewxpect these forms can be used to construct theta kernels valued in these refined regulators in Deligne cohomology; this could be an interesting subject of future work.

APPENDIX B. COMPARISON WITH SHARIFI-VENKATESH AND EXPLICITIZATION

In the case when $\mathcal{A} = \mathcal{E}^2$ with the action of $\Gamma \subset \mathrm{GL}_2(\mathbb{Z})$ (i.e. the case of the type II dual pair $(\mathrm{GL}_2, \mathrm{GL}_2)$), [SV, §6] constructs a cohomology class (indeed even a cocycle) for Γ valued in a subbundle of $\mathcal{E}^2$, as well as pulled back versions, with greater explicitness than our methods.

In this appendix, we show that their construction, when restricted to an appropriate subbundle of $\mathcal{A}$ for the purposes of comparison, coincides with our motivic cocycle $\Theta_{\mathcal{C}}^M$. This will furnish a an example (in, arguably, the simplest nontrivial case) of how one can leverage our approach to equivariant cohomology with double complexes to explicitize our constructions, a subject we will broach in greater generality in future work.

To begin, we briefly recapitulate the construction of [SV]:

Proposition B.1. *[SV, Proposition 6.2.1] We have an exact sequence of Γ-modules*

$$0 \to (K_2(k(\mathcal{A})))^{(0)} \to \left(\bigoplus_{x \in \mathcal{A}^{(1)}} K_1(k(x))\right)^{(0)} \to \left(\bigoplus_{x \in \mathcal{A}^{(2)}} K_0(k(x))\right)^{(0)} \tag{B.1}$$

where the superscript (0) denotes the fixed parts under the isogenies $[a]_*$ for all but finitely many integers a.

Thus, for any Γ-fixed class

$$\mathcal{C} \in \left(\bigoplus_{x \in \mathcal{A}^{(2)}} K_0(k(x))\right)^{(0)} = Z^2(\mathcal{A})$$

that lifts to a class

$$\eta \in \left(\bigoplus_{x \in \mathcal{A}^{(1)}} K_1(k(x))\right)^{(0)},$$

we can define a Γ-cocycle by sending $\gamma \in \Gamma$ to the unique lift of $(\gamma - 1)\eta$ in $(K_2(k(\mathcal{A})))^{(0)}$, as is standard in group cohomology from a short exact sequence.

The problem is finding the lifting class η. Sharifi and Venkatesh's solution is very reminiscent of the hypersurfaces idea discussed in the previous cohomological constructions: in [SV, (6.3.2)], they find a particularly natural choice of $\mathcal{C}$ built out of c-torsion sections which they call $e_{(c)}$; namely,

$$e_{(c)} = V_{(c)}(0)$$

where $V_{(c)}$ is a certain degee-zero operator built from Hecke operators. This $e_{(c)}$ lifts to an η which is fixed by a parabolic subgroup of $\mathrm{GL}_2(\mathbb{Z})$, resulting in a *parabolic* cocycle representing a class we will call

$$[\theta^{SV}] \in H^1_{par}(\mathrm{SL}_2(\mathbb{Z}), K_2^M(U)^{(0)})$$

where U is a certain complement of hyperplanes, and the par subscript means that the class vanishes upon restriction to any parabolic subgroup. Thanks to the parabolicity, this cocycle turns out to be very computable in terms of explicit cup products of theta functions.

Remark B.2. The method of [SV] in fact applies equally well to any case with $n = 2$, i.e. where the abelian scheme is a relative surface, in the sense that we may always construct the exact sequence (B.1). However, when the dual pair is nonsplit (e.g. the Hilbert modular case, or a quaternionic abelian surface over a Shimura curve), the desired lift η does not exist for any nontrivial torsion cycle, as there are essentially no nontrivial fiberwise divisors in the (0)-part of the K_1 term. Thus, the exact sequence is useless for producing a Γ-cocycle by the same means in these cases. Note, however, that η does exist for the action of $\mathrm{GL}_2(\mathcal{O}_K)$ on the square of an elliptic curve with CM by an imaginary quadratic field K; this is the subject of ongoing work by the first author of [SV].

We now turn to comparing their construction with $\Theta^M_{V_c(0)}$. This requires introducing quite a bit of new-looking yet familiar formalism, in the form of yet another instantiation of §2.1: we define the equivariant theory for Γ-schemes X

$$H^\bullet_\Gamma(X, \mathcal{H}^2(\mathbb{Z}(2)))$$

as the cohomology of the double complex

$$C^\bullet(\Gamma, Z_{Ger(K(2)}(X))$$

with the localization sequence and Gysin isomorphisms for the pair $(\mathcal{A}, \mathcal{A} - C)$ realized by the distinguished triangle

$$Z_{Ger(K(0)}(C))[-2] \to Z_{Ger(K(2)}(\mathcal{A})) \to Z_{Ger(K(2)}(\mathcal{A} - C))$$

where the latter map simply forgets all terms in the direct sums corresponding to points contained in the closed subvariety $C \subset \mathcal{A}$, and the former map is concentrated in the single degree

$$\bigoplus_{x \in C^{(0)}} K_0(k(x)) \to \bigoplus_{x \in \mathcal{A}^{(2)}} K_0(k(x))$$

given by considering codimension-0 points of C (i.e. generic points of its connected components) as codimension-2 points of $\mathcal{A}$.

Now the key to linking this to our constructions in motivic cohomology is the following result:

Proposition B.3. *$\mu \circ \psi^{\bullet}_{\mathcal{A}}$ induces a Γ-equvariant quasi-isomorphism from $\tau_{\geqslant 2}\mathbb{Z}(2)_{\mathcal{A}}$ to the Gersten complex in degrees* $[2, 4]$

(B.2) $$Z_{Ger(K(2))}(\mathcal{A}) = K_2(k(\mathcal{A})) \to \bigoplus_{x \in \mathcal{A}^{(1)}} K_1(k(x)) \to \bigoplus_{x \in \mathcal{A}^{(2)}} K_0(k(x)).$$

Proof. From the analysis of the coniveau spectral sequence in [SV, Example 2.2.2], it converges already on the second page at the terms corresponding to (B.2). As in Proposition 4.5, the result then follows from the fact that $\mu \circ \psi^{\bullet}_{\mathcal{A}}$ gives the edge maps also in the hypercohomology spectral sequence. Hence the maps they induce on cohomology

$$H^i(\mathcal{A}, \mathbb{Z}(2)) \to H^{i-2}(\mathcal{A}, \mathcal{H}^n(\mathbb{Z}(n)))$$

are isomorphisms, implying the proposition. □

This proposition implies that in the localization sequence we constructed

(B.3) $$\ldots \to H^1_{\Gamma}(\mathcal{A}, \mathcal{H}^n(\mathbb{Z}(2))) \to H^1_{\Gamma}(\mathcal{A} - C, \mathcal{H}^n(\mathbb{Z}(2))) \to H^0_{\Gamma}(C, \mathcal{H}^n(\mathbb{Z}(2))) \to \ldots$$

that $\mu \circ \psi_{\mathcal{A}-C}$ induces an isomorphism from the equivariant motivic cohomology $H^3_{\Gamma}(\mathcal{A}, \mathbb{Z}(2))$ to the term $H^1_{\Gamma}(\mathcal{A}, \mathcal{H}^n(\mathbb{Z}(2)))$, hence it is killed by the Lieberman projector e_L. We thus can construct, still following

§2.1, a class we call

$$\tilde{z}^M_{V_c(0)} \in H^1_\Gamma(\mathcal{A} - C, \mathcal{H}^n(\mathbb{Z}(2)))$$

which, as always, is characterized by having residue $V_c(0) \in H^0_\Gamma(C, \mathcal{H}^n(\mathbb{Z}(2)))$ and being fixed by almost all isogenies $[a]_*$. Further, the preceding proposition implies that image of $z^M_{V_c(0)}$ under $\mu \circ \psi_{\mathcal{A}-C}$ is $\tilde{z}^M_C$. We conclude:

Proposition B.4. *The Sharifi-Venkatesh cocycle defined above by*

$$\gamma \mapsto \theta^{SV}(\gamma)$$

represents the restriction of $\Theta^M_{V_c(0)}$ *to* $K_2(k(\mathcal{A}))$.

Proof. We claim that the class $\tilde{z}^M_{V_c(0)}$ is represented by the total-degree-1 element

$$\theta^{SV} + \eta \in C^\bullet(\Gamma, Z_{Ger(K(2))}(\mathcal{A})).$$

Indeed, this element is closed when considered as an element of

$$C^\bullet(\Gamma, Z_{Ger(K(2))}(\mathcal{A} - C)^\bullet)$$

but has total differential $V_c(0)$ when considered in the obvious way as an element of

$$C^\bullet(\Gamma, Z_{Ger(K(2))}(\mathcal{A})^\bullet),$$

hence by the usual snake lemma construction of the boundary map in the localization sequence, it has residue $V_c(0)$ in (B.3). Further, by construction it is fixed by almost all isogenies $[a]_*$. Since these two properties characterize the class $\tilde{z}^M_{V_c(0)}$, we conclude this is a representative.

Upon restriction to $k(\mathcal{A})$, Proposition 2.2 implies the result. □

Remark B.5. The argument above applies not just to the restriction to the generic point $k(\mathcal{A})$, as its only property we used is the fact that its weight-n motivic cohomology vanishes above degree n. We only used the generic point for convenience and to avoid introducing technicalities of particular semi-localizations, hyperplanes avoiding particular torsion sections, etc. since these are secondary to the main idea we wished to illustrate.

REFERENCES

[BaTa] Hyman Bass and John Tate. *The Milnor ring of a global field.* 1972.

[BCG1] Nicolas Bergeron, Pierre Charollois, and Luis Garcia. "Transgressions of the Euler class and Eisenstein cohomology of $\mathrm{GL}_N(\mathbb{Z})$." In: *Japanese Journal of Mathematics* 15 (2020), pp. 311–379.

[BCG2] Nicolas Bergeron, Pierre Charollois, and Luis Garcia. *Eisenstein cohomology classes for GL_N over imaginary quadratic fields.* 2021. URL: https://drive.google.com/file/d/18_36dIOfsmoCmd3Q-6GT4OgPrTYCBYlL/view?usp=sharing.

[BCG3] Nicolas Bergeron, Pierre Charollois, and Luis Garcia. *Cocycles de Sczech et arrangements d'hyperplans dans les cas additif, multiplicatif et elliptique.* In preparation.

[BGS] Jean-Michel Bismut, Henri Gillet, and Christophe Soulé. "Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion." In: *Communications in Mathematical Physics.* 115.1 (1988), pp. 49–78.

[BKL] Alexander Beilinson, Guido Kings, and Andrey Levin. "Topological polylogarithms and p-adic interpolation of L-values of totally real fields." In: *Mathematische Annalen* 371 (2018), pp. 1449–1495.

[Bl2] Spencer Bloch. "Algebraic cycles and higher K-theory." In: *Advances in Mathematics.* 61 (1986), pp. 267–304.

[Bl3] Spencer Bloch. "On the tangent space to Quillen K-theory." In: *Lecture Notes in Mathematics.* 341 (1973), pp. 205–210.

[Bor] Richard Borcherds. "The Gross-Kohnen-Zagier theorem in higher dimensions." In: *Duke Mathematical Journal* 97 (1999), pp. 219–233.

[Bott] Raoul Bott. "An introduction to equivariant cohomology." In: (1998), pp. 35–58.

[BT] Raoul Bott and Loring Tu. *Differential Forms in Algebraic Topology.* Springer Verlag, 1924.

[Ca] Henri Cartan. "Variétés analytiques complexes et cohomologie." In: *Colloque tenu à Bruxelles* (1953), pp. 41–55.

[Del] Pierre Deligne. "Travaux de Shimura." In: *Séminaire Bourbaki* 13 (1971), pp. 123–165.

[dR] Georges de Rham. *Variétés différentiables: formes, courants, formes harmoniques.* Hermann, 1955.

[Dup] Clément Dupont. "Périodes des arrangements d'hyperplans et coproduit motivique." PhD thesis. Université Pierre et Marie Curie, Sept. 2014.

[EGA] Jean Dieudonné and Alexander Grothendieck. "Éléments de géométrie algébrique II. Étude globale élémentaire de quelques classes de morphismes". In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques.* 8 (1961).

[FS] Eric Friedlander and Andre Suslin. "The spectral sequence relating algebraic K-theory to motivic cohomology." In: *Annales Scientifiques de l'École Normale Supérieure.* 4 (2002), pp. 773–875.

[G1] Alexander Goncharov. "Regulators." In: *Algebraic K-theory Handbook*. 2005.

[G3] Alexander Goncharov. "Polylogarithms, regulators, and Arakelov motivic complexes." In: *Journal of the American Mathematical Society* 18 (2005), pp. 1–60.

[Geis] Thomas Geisser. "Motivic cohomology over Dedekind rings." In: *Mathematische Zeitschrift.* 248 (2004), pp. 773–794.

[GKZ] Benedict Gross, Winfried Kohnen, and Don Zagier. "Heegner points and derivative of L-series, II." In: *Mathematische Annalen* 278 (1987), pp. 497–562.

[GL] Thomas Geisser and Marc Levine. "The K-theory of fields in characteristic p." In: *Inventiones mathematicae.* 139 (2000), pp. 459–493.

[God] Roger Godemont. "Topologie algébrique et théorie des faisceaux." In: *Actualités Scientifiques et Industrielles.* 1252 (1958).

[Hamm] Helmut Hamm. "Zum Homotopietyp Steinischer Raüme." In: *Journal fur die reine und angewandte Mathematik.* 338 (1983), pp. 121–135.

[Hatcher] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[Hecke] Erich Hecke. "Theorie der Eisensteinschen Reihen höheren Stufe und ihre Anwendung auf Funktiontheorie und Arithmetik." In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 5 (1927), pp. 199–224.

[Hel] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces.* Vol. 80. Academic Press Inc, 1986.

[Howe] Roger Howe. "θ-series and invariant theory." In: *Automorphic forms, representations, and L-functions (Proceedings of Symposia in Pure Mathematics).* XXXIII.1 (1977), pp. 275–285.

[HS] Klaus Hulek and Gregory Sankaran. "The geometry of Siegel modular varieties." In: *Advanced Studies in Pure Mathematics* 35 (2002), pp. 89–156.

[Kato1] Kazuya Kato. "p-adic Hodge theory and values of zeta functions of modular forms." In: *Astérisque.* 295 (2004), pp. 117–290.

[Kerr] Matt Kerr. "A regulator formula for Milnor K-groups." In: *K-Theory.* 29 (2003), pp. 175–210.

[Kerz] Moritz Kerz. "Milnor K-theory of local rings." PhD thesis. Universität Regensburg, Apr. 2008.

[KS] Guido Kings and Johannes Sprang. *Eisenstein-Kronecker classes, integrality of critical values of Hecke L-functions and p-adic interpolation.* Preprint. URL: https://arxiv.org/pdf/1912.03657.pdf.

[Kud] Stephen Kudla. "Special cycles and derivatives of Eisenstein series." In: *Special Values of Rankin L series* (MSRI). 2001.

[La] Steven Landsburg. "Relative Chow groups." In: *Illinois Journal of Mathematics.* 35.4 (1991).

[LW] Florence Lecomte and Nathalie Wach. "Réalisations de Hodge des motifs de Voevodsky". In: *Manuscripta Mathematica.* 141 (2013), pp. 663–697.

[Mil] John Milnor. "Construction of Universal Bundles, II". In: *Annals of Mathematics.* 63 (3 1956), pp. 430–436.

[MQ] Varghese Mathai and Daniel Quillen. "Superconnections, Thom classes, and equivariant differential forms". In: *Topology.* 25 (1 1986), pp. 85–110.

[MVW] Carlo Mazza, Charles Weibel, and Vladimir Voevodsky. *Lecture notes on motivic cohomology.* Clay Mathematics Monographs. 2006.

[NSS] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson. "Principal -bundles – Presentations." In: *Journal of Homotopy and Related Structures.* 10 (3 2015), pp. 565–622.

[Pra] Kartik Prasanna. *Arithmetic aspects of the theta correspondence and periods of modular forms.* 2009.

[SV] Romyar Sharifi and Akshay Venkatesh. *Eisenstein cocycles in motivic cohomology.* Preprint. 2020. URL: https://arxiv.org/pdf/2011.07241.

[Tohoku] Alexander Grothendieck. "Sur quelques points d'algèbre homologique." In: *Tôhoku Mathematical Journal.* 9.2 (1957), pp. 119–221.

[Tot] Burt Totaro. "Milnor K-theory is the simplest part of algebraic K-theory". In: *K-Theory.* 6 (1992), pp. 177–189.

[V] Vladimir Voevodsky. "Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic." In: *International Mathematical Research Notices.* 7 (2002).

[Voi] Claire Voisin. *Hodge theory and Complex algebraic geometry II.* Vol. 77. Cambridge Studies in Advanced Mathematics, 2003.

[W] Andrzej Weber. *Leray spectral sequence for complements of certain arrangements of smooth submanifolds.* 2016.

[Wei] Charles Weibel. *An Introduction to Homological Algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[Zhong] Changlong Zhong. "Comparison of dualizing complexes." In: *Journal für die reine und angewandte Mathematik.* 695 (2014), pp. 1–39.