On the p-adic variation of Heegner points

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To my parents and sisters

Abstract

In this thesis we study the so-called *big Heegner points* introduced and first studied by Ben Howard [**How07b**]. By construction these are global cohomology classes, with values in the Galois representation associated to a twisted Hida family, interpolating in weight 2 the twisted Kummer images of CM points.

In the first part, we relate the higher weight specializations of the big Heegner point of conductor one to the p-adic étale Abel–Jacobi images of Heegner cycles. This is based on a new p-adic limit formula of Gross–Zagier type obtained in the recent work [**BDP13**] of Bertolini–Darmon–Prasanna, a formula that we extend to a setting allowing arbitrary ramification at p. As a first consequence of the aforementioned relation, we deduce an interpolation of the p-adic Gross–Zagier formula of Nekovář over a Hida family.

In the second part, we extend some of these formulae in the anticyclotomic direction, showing that the p-adic L-function introduced in [**BDP13**] can be obtained as the image of a compatible sequence of big Heegner points of p-power conductor via a generalization of the Coleman power series map. By Kolyvagin's method of Euler systems, as reinvented by Kato and Perrin-Riou, we then deduce certain new cases of the Bloch–Kato conjecture for the Rankin–Selberg convolution of a cusp form with a theta series of higher weight, as well as a divisibility in the Iwasawa–Greenberg main conjecture associated with this family of motives.

Abrégé

Cette thèse est consacrée aux "points de Heegner en famille" introduits par Ben Howard dans [**How07b**]. Par définition, ce sont des classes de cohomologie globales à valeurs dans la représentation Galoisienne associée à une famille de Hida, interpolant en poids 2 les images de points CM par l'application de Kummer.

La première partie de cette thèse relie les spécialisations de la classe de Howard en poids $k \geq 2$ aux images de certains cycles de Heegner par l'application d'Abel–Jacobi p-adique. Notre démonstration de cette relation repose sur une formule de Gross–Zagier p-adique obtenue dans les travaux récents [**BDP13**] de Bertolini–Darmon–Prasanna, et que nous étendons ici à un cadre permettant de travailler avec des formes modulaires de niveau divisible par p. On déduit de nos résultats une interpolation de la formule de Gross–Zagier p-adique de Nekovář sur une famille de Hida.

La deuxième partie étend la définition de la classe de Howard "le long de la droite anticyclotomique", pour obtenir une classe de cohohomologie à deux variables. On montre que la fonction L p-adique de Hida–Rankin, telle que décrite dans [**BDP13**], est l'image de cette classe par une généralisation de l'isomorphisme de Coleman. La méthode des systèmes d'Euler de Kolyvagin, telle que réinventée par Kato et Perrin-Riou, permet d'en déduire certains nouveaux cas de la conjecture de Bloch–Kato pour la convolution de Rankin–Selberg d'une forme parabolique avec une série thêta de poids supérieur, et une divisibilité dans la conjecture principale de la théorie d'Iwasawa–Greenberg associée à cette famille de motifs.

Contents

Preface

It is a somewhat vexing fact that, to the embarrassment of many mathematicians, the most convincing theoretical evidence in support of the Birch and Swinnerton-Dyer conjecture still rests largely on the foundational works of Gross–Zagier [**GZ86**] and Kolyvagin [**Kol88**], where the classical Heegner point construction attached to the auxiliary choice of an imaginary quadratic field was stunningly exploited to establish the conjecture in the case of analytic rank at most 1 for a class of elliptic curves that now, after Wiles's breakthrough [**Wil95**] culminating in [**BCDT01**], is known to be rich enough to include all rational elliptic curves.

In this thesis we aim to further scrutinize the wealth of information accounted for by Heegner points and their p-adic variation, examining a two-variable construction by Howard [**How07b**] that extends over a Hida family and over the anticyclotomic tower.

Our new results in these directions are contained in Chapters 1 and 2, which are slightly modified versions of the papers [**Cas13a**] (to appear in Mathematische Annalen) and [**Cas13b**] (submitted for publication), and are ultimately based on the study of an anticyclotomic p-adic L-function introduced in [**BDP13**] for which the characters relevant for the Birch and Swinnerton-Dyer conjecture lie *outside* the range of classical interpolation. Because of this feature, the p-adic Gross–Zagier formulae of [**BDP13**] are certainly a less natural analogue of the result of Gross–Zagier than the p-adic formulae proven by Perrin-Riou [**PR87b**] and Nekovář [**Nek95**], but *a posteriori* they have been found to be useful for arithmetic applications.

Starting with Leopoldt's formula, similar formulae for the values of p-adic L-functions outside their range of classical interpolation have been discovered and exploited in most situations where interesting Euler systems can be shown to exist. This point of view, which is sometimes not completely apparent in the classical literature, is stressed in [**BCD$^+$13**], where the reader can see most clearly how our results fit within a broader perspective.

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CHAPTER 1

Higher weight specializations of big Heegner points

Summary

Let $\mathbf{f}$ be a p-ordinary Hida family of tame level N, and let K be an imaginary quadratic field satisfying the Heegner hypothesis relative to N. By taking a compatible sequence of twisted Kummer images of CM points over the tower of modular curves of level $\Gamma_0(N) \cap \Gamma_1(p^s)$, Howard [**How07b**] has constructed a canonical class $\mathfrak{Z}$ in the cohomology of a self-dual twist of the big Galois representation associated to $\mathbf{f}$. If a p-ordinary eigenform f on $\Gamma_0(N)$ of weight $k > 2$ is the specialization of $\mathbf{f}$ at ν, one thus obtains from $\mathfrak{Z}_\nu$ a higher weight generalization of the Kummer images of Heegner points. In this chapter we relate the classes $\mathfrak{Z}_\nu$ to the étale Abel–Jacobi images of Heegner cycles when p splits in K.

Introduction

Fix a prime $p > 3$ and an integer $N > 4$ such that $p \nmid N\phi(N)$. Let

$$f_o = \sum_{n>0} a_n q^n \in S_k(X_0(N))$$

be a p-ordinary newform of even weight $k = 2r \geq 2$ and trivial nebentypus. Thus f_o is an eigenvector for all the Hecke operators T_n with associated eigenvalues a_n, and a_p is a p-adic unit for a choice of embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ that will remain fixed throughout this paper. Also let $\mathcal{O}$ denote the ring of integers of a (sufficiently large) finite extension $L/\mathbf{Q}_p$ containing all the a_n.

For $s > 0$, let X_s be the compactified modular curve of level

$$\Gamma_s := \Gamma_0(N) \cap \Gamma_1(p^s),$$

and consider the tower

$$\cdots \longrightarrow X_s \xrightarrow{\alpha} X_{s-1} \longrightarrow \cdots$$

with respect to the degeneracy maps described on the non-cuspidal moduli by

$$(E, \alpha_E, \pi_E) \longmapsto (E, \alpha_E, p \cdot \pi_E),$$

where α_E denotes a cyclic N-isogeny on the elliptic curve E, and π_E a point of E of exact order p^s. The group $(\mathbf{Z}/p^s\mathbf{Z})^\times$ acts on X_s via the diamond operators

$$\langle d \rangle : (E, \alpha_E, \pi_E) \longmapsto (E, \alpha_E, d \cdot \pi_E)$$

compatibly with α under the reduction $(\mathbf{Z}/p^s\mathbf{Z})^\times \longrightarrow (\mathbf{Z}/p^{s-1}\mathbf{Z})^\times$. Set $\Gamma := 1 + p\mathbf{Z}_p$. Letting J_s be the Jacobian variety of X_s, the inverse limit of the system induced by Albanese functoriality,

$$\cdots \longrightarrow \mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathcal{O} \longrightarrow \mathrm{Ta}_p(J_{s-1}) \otimes_{\mathbf{Z}_p} \mathcal{O} \longrightarrow \cdots, \tag{1.0.1}$$

is equipped with an action of the Iwasawa algebras $\widetilde{\Lambda}_{\mathcal{O}} := \mathcal{O}[[\mathbf{Z}_p^\times]]$ and

$$\Lambda_{\mathcal{O}} := \mathcal{O}[[\Gamma]].$$

Let $\mathfrak{h}_s$ be the $\mathcal{O}$-algebra generated by the Hecke operators T_ℓ ($\ell \nmid Np$), $U_\ell := T_\ell$ ($\ell | Np$), and the diamond operators $\langle d \rangle$ ($d \in (\mathbf{Z}/p^s\mathbf{Z})^\times$) acting on the space $S_k(X_s)$ of cusp forms of weight k and level Γ_s. Hida's ordinary projector

$$e^{\mathrm{ord}} := \lim_{n\to\infty} U_p^{n!}$$

defines an idempotent of $\mathfrak{h}_s$, projecting to the maximal subspace of $\mathfrak{h}_s$ where U_p acts invertibly. We make each $\mathfrak{h}_s$ into a $\widetilde{\Lambda}_{\mathcal{O}}$-algebra by letting the group-like element attached to $z \in \mathbf{Z}_p^\times$ act as $z^{k-2}\langle z\rangle$.

Taking the projective limit with respect to the restriction maps induced by the natural inclusion $S_k(X_{s-1}) \hookrightarrow S_k(X_s)$ we obtain a $\widetilde{\Lambda}_{\mathcal{O}}$-algebra

$$\mathfrak{h}^{\mathrm{ord}} := \varprojlim_s e^{\mathrm{ord}}\mathfrak{h}_s \tag{1.0.2}$$

which can be seen to be *independent* of the weight $k \geq 2$ used in its construction.

After a highly influential work [**Hid86b**] of Hida, one can associate with f_o a certain local domain $\mathbb{I}$ quotient of $\mathfrak{h}^{\mathrm{ord}}$, finite flat over $\Lambda_{\mathcal{O}}$, with the following properties. For each n, let $\mathbf{a}_n \in \mathbb{I}$ be the image of T_n under the projection $\mathfrak{h}^{\mathrm{ord}} \longrightarrow \mathbb{I}$, and consider the formal q-expansion

$$\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]].$$

We say that a continuous $\mathcal{O}$-algebra homomorphism $\nu : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p$ is an *arithmetic prime* if there is an integer $k_\nu \geq 2$, called the *weight* of ν, such that the composition

$$\Gamma \longrightarrow \mathbb{I}^\times \longrightarrow \overline{\mathbf{Q}}_p^\times$$

agrees with $\gamma \longmapsto \gamma^{k_\nu - 2}$ on an open subgroup of Γ of index $p^{s_\nu - 1} \geq 1$. Denote by $\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ the set of arithmetic primes of $\mathbb{I}$, which will often be seen as sitting inside $\mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$. If $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, F_ν will denote its residue field. Then:

- for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, there exists an ordinary p-stabilized newform[1]

  $$\mathbf{f}_\nu \in S_{k_\nu}(X_{s_\nu})$$

  such that $\nu(\mathbf{f}) \in F_\nu[[q]]$ gives the q-expansion of $\mathbf{f}_\nu$;
- if $s_\nu = 1$ and $k_\nu \equiv k \pmod{2(p-1)}$, there exists a normalized newform $\mathbf{f}_\nu^\sharp \in S_{k_\nu}(X_0(N))$ such that

$$\mathbf{f}_\nu(q) = \mathbf{f}_\nu^\sharp(q) - \frac{p^{k_\nu - 1}}{\nu(\mathbf{a}_p)}\mathbf{f}_\nu^\sharp(q^p); \tag{1.0.3}$$

- there exists a unique $\nu_o \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ such that $f_o = \mathbf{f}_{\nu_o}^\sharp$.

In particular, after "p-stabilization" (1.0.3), the form f_o fits in the p-adic family $\mathbf{f}$.

[1] As defined in [**NP00**, (1.3.7)].

Similarly for the associated Galois representation V_{f_o}: the continuous $\mathfrak{h}^{\mathrm{ord}}$-linear action of the absolute Galois group $G_{\mathbf{Q}}$ on the module

$$\mathbb{T} := \mathbb{T}^{\mathrm{ord}} \otimes_{\mathfrak{h}^{\mathrm{ord}}} \mathbb{I}, \qquad \text{where} \qquad \mathbb{T}^{\mathrm{ord}} := \varprojlim_{s} e^{\mathrm{ord}}(\mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathcal{O}), \tag{1.0.4}$$

gives rise to a "big" Galois representation $\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbb{T})$ such that

$$\nu(\rho_{\mathbf{f}}) \cong \rho^*_{\mathbf{f}_\nu}$$

for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, where $\rho^*_{\mathbf{f}_\nu}$ is the contragredient of the (cohomological) p-adic Galois representation $\rho_{\mathbf{f}_\nu} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}(V_{\mathbf{f}_\nu})$ attached to $\mathbf{f}_\nu$ by Deligne; in particular, one recovers $\rho^*_{f_o}$ from $\rho_{\mathbf{f}}$ by specialization at ν_o.

Assume from now on that the residual representation $\bar{\rho}_{f_o}$ is irreducible; then $\mathbb{T}$ can be shown to be free of rank 2 over $\mathbb{I}$. (See [**MT90**, Théorème 7] for example.) Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$ containing an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with

$$\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}, \tag{1.0.5}$$

and denote by H the Hilbert class field of K. Under this *Heegner hypothesis* relative to N (but with no extra assumptions on the prime p), the work [**How07b**] of Howard produces a compatible sequence $U_p^{-s} \cdot \mathfrak{X}_s$ of cohomology classes with values in a certain twist of the ordinary part of (1.0.1), giving rise to a canonical "big" cohomology class $\mathfrak{X}$, the *big Heegner point* (of conductor 1), in the cohomology of a self-dual twist $\mathbb{T}^\dagger$ of $\mathbb{T}$. Moreover, if every prime factor of N splits in K, it follows from his results that the class

$$\mathfrak{Z} := \mathrm{Cor}_{H/K}(\mathfrak{X})$$

lies in Nekovář's extended Selmer group $\widetilde{H}^1_f(K, \mathbb{T}^\dagger)$. In particular, for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ with $s_\nu = 1$ and $k_\nu \equiv k \pmod{2(p-1)}$ as above, the specialization $\mathfrak{Z}_\nu$ belongs to the Bloch–Kato Selmer group $H^1_f(K, V_{\mathbf{f}^\sharp_\nu}(k_\nu/2))$ of the self-dual representation $\mathbb{T}^\dagger \otimes_{\mathbb{I}} F_\nu \cong V_{\mathbf{f}^\sharp_\nu}(k_\nu/2)$. The classes $\mathfrak{Z}_\nu$ may thus be regarded as a natural higher weight analogue of the Kummer images of Heegner points, on modular Abelian varieties (associated with weight 2 eigenforms).

But for any of the above $\mathbf{f}^\sharp_\nu$, one has an alternate (and completely different!) method of producing such a higher weight analogue. Briefly, if $k_\nu = 2r_\nu > 2$, associated to any elliptic curve A with CM by $\mathcal{O}_K$, there is a null-homologous cycle $\Delta^{\mathrm{heeg}}_{A,r_\nu}$, a so-called *Heegner cycle*, on the $(2r_\nu-1)$-dimensional Kuga–Sato variety W_{r_ν} giving rise to an H-rational class in the Chow group $\mathrm{CH}^{r_\nu+1}(W_{r_\nu})_0$ with $\mathbf{Q}$-coefficients. Since the representation $V_{\mathbf{f}^\sharp_\nu}(r_\nu)$ appears in the étale cohomology of W_{r_ν}:

$$H^{2r_\nu-1}_{\text{ét}}(\overline{W}_{r_\nu},\mathbf{Q}_p)(r_\nu)\xrightarrow{\pi_{\mathbf{f}^\sharp_\nu}} V_{\mathbf{f}^\sharp_\nu}(r_\nu),$$

by taking the images of the cycles $\Delta^{\text{heeg}}_{A,r_\nu}$ under the p-adic étale Abel-Jacobi map

$$\Phi^{\text{ét}}_H:\mathrm{CH}^{r_\nu+1}(W_{r_\nu})_0(H)\longrightarrow H^1(H,H^{2r_\nu-1}_{\text{ét}}(\overline{W}_{r_\nu},\mathbf{Q}_p)(r_\nu))$$

and composing with the map induced by $\pi_{\mathbf{f}^\sharp_\nu}$ on H^1's, we may consider the classes

$$\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu}):=\mathrm{Cor}_{H/K}(\pi_{\mathbf{f}^\sharp_\nu}\Phi^{\text{ét}}_H(\Delta^{\text{heeg}}_{A,r_\nu})).$$

By the work [**Nek00**] of Nekovář, these classes are known to lie in the same Selmer group as $\mathfrak{Z}_\nu$, and the question of their comparison thus naturally arises.

Main Theorem (Thm. 1.4.12). *Assume that p splits in $K=\mathbf{Q}(\sqrt{-D})$ and that the class $\mathfrak{Z}$ is not $\mathbb{I}$-torsion. Then for all but finitely many $\nu\in\mathcal{X}_{\text{arith}}(\mathbb{I})$ of weight $k_\nu=2r_\nu>2$ with $k_\nu\equiv k \pmod{2(p-1)}$ and trivial character, we have*

$$\langle\mathfrak{Z}_\nu,\mathfrak{Z}_\nu\rangle_K=\left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)^4\frac{\langle\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu}),\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu})\rangle_K}{u^2(4D)^{r_\nu-1}},$$

where $\langle,\rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K,V_{\mathbf{f}^\sharp_\nu}(r_\nu))$, and $u:=|\mathcal{O}_K^\times|/2$.

Thus assuming the non-degeneracy of the p-adic height pairing, it follows that the étale Abel–Jacobi images of Heegner cycles are p-adically interpolated by $\mathfrak{Z}$. We also note that $\mathfrak{Z}$ is conjecturally *always* not $\mathbb{I}$-torsion ([**How07b**, Conj. 3.4.1]), and that by [**How07a**, Cor. 5] this conjecture can be verified in any given case by exhibiting the non-vanishing of an appropriate L-value (a derivative, in fact). But arguably the main interest of the above result is to be found in connection with p-adic L-functions, as we indicate below.

Let $\mathscr{G}_\infty$ be the Galois group of the unique $\mathbf{Z}_p^2$-extension of K. In their recent proof [**SU13**] one divisibility in the Iwasawa Main Conjecture for $\mathbf{GL}_2$, Skinner and Urban construct an element $\mathcal{L}_p(\mathbf{f}\otimes K)\in\mathbb{I}[[\mathscr{G}_\infty]]$ which interpolates a certain two-variable p-adic L-function $\mathcal{L}_p(\mathbf{f}_\nu\otimes K)\in\mathcal{O}_\nu[[\mathscr{G}_\infty]]$ attached to the specializations $\mathbf{f}_\nu$. For any ν as in the above Main Theorem, the work [**Nek95**] of Nekovář proves a p-adic analogue of the Gross–Zagier formula for $\mathcal{L}_{\mathbf{f}_\nu,K}$. Combined with the existence of an $\mathbb{I}$-valued "height pairing" $\langle,\rangle_{K,\mathbb{T}^\dagger}$ on $\widetilde{H}^1_f(K,\mathbb{T}^\dagger)$, we can easily deduce the following.

Corollary (Thm. 1.5.1). *Let $\mathcal{L}'_{\mathbf{f},K}$ be the linear term in the expansion of $\mathcal{L}_{\mathbf{f},K}$ restricted to the cyclotomic line. Under the assumptions of the Main Theorem, we have*

$$\mathcal{L}'_{\mathbf{f},K}(\mathbb{1}_K)=\langle\mathfrak{Z},\mathfrak{Z}\rangle_{K,\mathbb{T}^\dagger}\pmod{\mathbb{I}^\times}.$$

This paper is organized as follows. Section 1.1 is aimed at proving an expression for the formal group logarithms of ordinary CM points on X_s using Coleman's theory of p-adic integration. Our methods here are drawn from [**BDP13**, §3], which we extend in weight 2 to the case of level divisible by an arbitrary power of p, but with ramification restricted to a *potentially crystalline* setting. Not quite surprisingly, this restriction turns out to make our computations essentially the same as theirs, and will suffice for our purposes.

In Section 1.2 we recall the generalised Heegner cycles and the formula for their p-adic Abel-Jacobi images from *loc.cit.*, and discuss the relation between these and the more classical Heegner cycles.

In Section 1.3 we deduce from the work [**Och03**] of Ochiai the construction of a "big" logarithm map that will allow as to move between different weights in the Hida family.

Finally, in Section 1.4 we prove our results on the arithmetic specializations of the big Heegner point $\mathfrak{Z}$. The key observation is that, when p splits in K, the combination of CM points on X_s taken in Howard's construction appears naturally in the evaluation of the critical twist of a p-adic modular form at a canonical trivialized elliptic curve. The expression from Section 1 thus yields, for infinitely many ν of weight 2, a formula for the p-adic logarithm of the localization of $\mathfrak{Z}_\nu$ in terms of certain values of a p-adic modular form of weight 0 associated with $\mathbf{f}_\nu$ (Theorem 1.4.9). When extended by p-adic continuity to an arithmetic prime ν of higher even weight, this expression is seen to agree with the formula from Section 1.2, and by the interpolation properties of the big logarithm map it corresponds to the p-adic logarithm of the localization of $\mathfrak{Z}_\nu$. The above Main Theorem then follows easily from this.

We also note that an extension "in the anticyclotomic direction" of some of the results in this paper leads to a number of arithmetic applications arising from the connection between Howard's big Heegner points and a certain p-adic L-function introduced in [**BDP13**]. This connection appears implicitly here and is developed in [**Cas13b**].

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1.1. Preliminaries

1.1.1. p-adic modular forms. To avoid some issues related to the representability of certain moduli problems, in this section we change notations from the Introduction, letting X_s be the compactified modular curve of level $\Gamma_s := \Gamma_1(Np^s)$, viewed as a scheme over $\mathrm{Spec}(\mathbf{Q}_p)$. Let $\pi : \mathcal{E}_s \longrightarrow \tilde{X}_s$ be the universal elliptic curve over the complement $\tilde{X}_s \subset X_s$ of the cuspidal subscheme $Z_s \subset X_s$, and let $\underline{\omega}_{X_s}$ be the invertible sheaf X_s given by the extension of $\pi_* \Omega_{\mathcal{E}_s/\tilde{X}_s}$ to the cusps Z_s as described in [**Gro90**, §1], for example.

Algebraically, $H^0(X_s, \underline{\omega}_{X_s}^{\otimes 2})$ gives the space of modular forms of weight 2 and level Γ_s (defined over $\mathbf{Q}_p$). Consider the complex

$$\Omega^\bullet_{X_s/\mathbf{Q}_p}(\log Z_s) : 0 \longrightarrow \mathcal{O}_{X_s} \xrightarrow{d} \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s) \longrightarrow 0 \tag{1.1.1}$$

of sheaves on X_s. The algebraic *de Rham cohomology* of X_s

$$H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p) := \mathbb{H}^1(X_s, \Omega^\bullet_{X_s/\mathbf{Q}_p}(\log Z_s))$$

is a finite-dimensional $\mathbf{Q}_p$-vector space equipped with a *Hodge filtration*

$$0 \subset H^0(X_s, \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s)) \subset H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p),$$

and by the Kodaira-Spencer isomorphism $\underline{\omega}_{X_s}^{\otimes 2} \cong \Omega^1_{X_s/\mathbf{Q}_p}(\log Z_s)$, every cusp form $f \in S_2(X_s)$ (in particular) defines a cohomology class $\omega_f \in H^1_{\mathrm{dR}}(X_s/\mathbf{Q}_p)$.

Let X be the complete modular curve of level $\Gamma_1(N)$, also viewed over $\mathrm{Spec}(\mathbf{Q}_p)$, and consider the subspaces of the associated rigid analytic space X^{an}:

$$X^{\mathrm{ord}} \subset X_{<1/(p+1)} \subset X_{<p/(p+1)} \subset X^{\mathrm{an}}.$$

To define these, let $\mathcal{X}_{/\mathbf{Z}_p}$ be the canonical integral model of X over $\mathrm{Spec}(\mathbf{Z}_p)$, and let $X_{\mathbf{F}_p} := \mathcal{X} \times_{\mathbf{Z}_p} \mathbf{F}_p$ denote its special fiber. The *supersingular points* $SS \subset X_{\mathbf{F}_p}(\overline{\mathbf{F}}_p)$ is the finite set of points corresponding to the moduli of supersingular elliptic curves (with $\Gamma_1(N)$-level structure) in characteristic p.

Let E_{p-1} be the Eisenstein series of weight $p-1$ and level 1, seen as a global section of the sheaf $\underline{\omega}_X^{\otimes(p-1)}$. (Recall that we are assuming $p \geq 5$.) The reduction of E_{p-1} to $X_{\mathbf{F}_p}$ is the *Hasse invariant*, which defines a section of the reduction of $\underline{\omega}_X^{\otimes(p-1)}$ with SS as its locus of (simple) zeroes. If $x \in X(\overline{\mathbf{Q}}_p)$, let $\bar{x} \in X_{\mathbf{F}_p}(\overline{\mathbf{F}}_p)$ denote its reduction. Each point

$\bar{x} \in SS$ is smooth in $X_{\mathbf{F}_p}$, and the *ordinary locus* of X

$$X^{\mathrm{ord}} := X^{\mathrm{an}} \smallsetminus \bigcup_{\bar{x} \in SS} D_{\bar{x}}$$

is defined to be the complement of their residue discs $D_{\bar{x}} \subset X^{\mathrm{an}}$. The function $|E_{p-1}(x)|_p$ defines a local parameter on $D_{\bar{x}}$, and with the normalization $|p|_p = p^{-1}$, $X_{<1/(p+1)}$ (resp. $X_{<p/(p+1)}$) is defined to be complement in X^{an} of the subdiscs of $D_{\bar{x}}$ where $|E_{p-1}(x)|_p \leq p^{-1/(p+1)}$ (resp. $|E_{p-1}(x)|_p \leq p^{-p/(p+1)}$), for all $\bar{x} \in SS$.

Using the *canonical subgroup* H_E (of order p) attached to every elliptic curve E corresponding to a closed point in $X_{<p/(p+1)}$, the *Deligne-Tate map*

$$\phi_0 : X_{<1/(p+1)} \longrightarrow X_{<p/(p+1)}$$

is defined by sending $E \longmapsto E/H_E$ (with the induced action on the level structure) under the moduli interpretation. This map is a finite morphism which by definition lifts to characteristic zero the absolute Frobenius on $X_{\mathbf{F}_p}$. (See [**Kat73**, Thm. 3.1].)

For every $s > 0$, the Deline-Tate map ϕ_0 can be iterated $s-1$ times on the open rigid subspace $X_{<p^{2-s}/(p+1)}$ of X^{an} where $|E_{p-1}(x)|_p > p^{-p^{2-s}/(p+1)}$. Letting $\alpha_s : X_s \longrightarrow X$ be the map forgetting the "$\Gamma_1(p^s)$-part" of the level structure, define

$$\mathcal{W}_1(p^s) \subset X_s^{\mathrm{an}}$$

to be the open rigid subspace of X_s whose closed points correspond to triples (E, α_E, π_E) whose image under α_s lands inside $X_{<p^{2-s}(p+1)}$ and are such that π_E generates the canonical subgroup of E *of order* p^s (as in [**Buz03**, Defn. 3.4]).

Define $\mathcal{W}_2(p^s) \subset X_s^{\mathrm{an}}$ is the same manner, replacing $p^{2-s}/(p+1)$ by $p^{1-s}/(p+1)$ in the definition of $\mathcal{W}_1(p^s)$. Then we obtain a lifting of Frobenius $\phi = \phi_s$ on X_s making the diagram

$$\begin{tikzcd}
\mathcal{W}_2(p^s) \arrow[r, "\phi"] \arrow[d, "\alpha_s"] & \mathcal{W}_1(p^s) \arrow[d, "\alpha_s"] \\
X_{<p^{1-s}(p+1)} \arrow[r, "\phi_0"] & X_{<p^{2-s}(p+1)}.
\end{tikzcd}$$

commutative by sending a point $x = (E, \alpha_E, \imath_E) \in \mathcal{W}_2(p^s)$, where $\imath_E : \boldsymbol{\mu}_{p^s} \hookrightarrow E[p^s]$ is an embedding giving the $\Gamma_1(p^s)$-level structure on E, to $x' = (\phi_0 E, \phi_0 \alpha_E, \imath'_E)$, where $\imath'_E$ is determined by requiring that $\alpha_s(x')$ lands in $X_{<p^{2-s}/(p+1)}$ and for each $\zeta \in \boldsymbol{\mu}_{p^s} - \{1\}$, $\imath'_E(\zeta) = \phi_0 Q$ if $\imath_E(\zeta) = pQ$. (Cf. [**Col97b**, §B.2].)

Let $I_s := \{v \in \mathbf{Q} \ : \ 0 \leq v < p^{2-s}/(p+1)\}$, and for $v \in I_s$ define the affinoid subdomain $X_s(v)$ of X_s^{an} inside $\mathcal{W}_1(p^s)$ whose closed points x satisfy $|E_{p-1}(x)|_p \geq p^{-v}$. Then $X_s(0)$ is the connected component of the ordinary locus of X_s containing the cusp ∞.

Denote by $\underline{\omega}_{X_s^{\mathrm{an}}}$ the rigid analytic sheaf on X_s^{an} deduced from $\underline{\omega}_{X_s}$ and fix $k \in \mathbf{Z}$. The space of *p-adic modular forms* of weight k and level Γ_s (defined over $\mathbf{Q}_p$) is the p-adic Banach space

$$M_k^{\mathrm{ord}}(X_s) := H^0(X_s(0), \underline{\omega}_{X_s^{\mathrm{an}}}^{\otimes k}),$$

and the space of *overconvergent p-adic modular forms* of weight k and level Γ_s is the p-adic Fréchet space

$$M_k^{\mathrm{rig}}(X_s) := \varinjlim_v H^0(X_s(v), \underline{\omega}_{X_s^{\mathrm{an}}}^{\otimes k}),$$

where the limit is with respect to the natural restriction maps as $v \in I_s$ increasingly approaches $p^{2-s}/(p+1)$. By restriction, a classical modular form in $H^0(X_s, \underline{\omega}_{X_s}^{\otimes k})$ defines an (obviously) overconvergent p-adic modular form of the same weight an level. Moreover, the action of the diamond operators on X_s gives rise to an action of $(\mathbf{Z}/p^s\mathbf{Z})^\times$ on the spaces of p-adic modular forms which agrees with the action on $H^0(X_s, \underline{\omega}_{X_s}^{\otimes k})$ under restriction.

We say that a ring R is a *p-adic ring* if the natural map $R \longrightarrow \varprojlim R/p^nR$ is an isomorphism. For varying s, the data of a compatible sequence of embeddings $\boldsymbol{\mu}_{p^s} \hookrightarrow E$ as R-group schemes, amounts to the data of an embedding $\boldsymbol{\mu}_{p^\infty} \hookrightarrow E[p^\infty]$ of p-divisible groups, and also to the given of a *trivialization* of E over R, i.e. an isomorphism

$$\imath_E : \hat{E} \longrightarrow \hat{\mathbf{G}}_m$$

of the associated formal groups. The space $\mathbf{M}(N)$ of *Katz p-adic modular functions* of tame level N (over $\mathbf{Z}_p$) is the space of functions f on trivialized elliptic curves with $\Gamma_1(N)$-level structure over arbitrary p-adic rings, assigning to the isomorphism class of a triple $(E, \alpha_E, \imath_E)$ over R a value $f(E, \alpha_E, \imath_E) \in R$ whose formation is compatible under base change. If R is a fixed p-adic ring, by only considering p-adic rings which are R-algebras, we obtain the notion of Katz p-adic modular functions defined over R, forming the space $\mathbf{M}(N)\widehat{\otimes}_{\mathbf{Z}_p} R$ which will also be denoted by $\mathbf{M}(N)$ by an abuse of notation.

The action of $z \in \mathbf{Z}_p^\times$ on a trivialization gives rise to an action of $\mathbf{Z}_p^\times$ on $\mathbf{M}(N)$:

$$\langle z \rangle f(E, \alpha_E, \imath_E) := f(E, \alpha_E, z^{-1} \cdot \imath_E),$$

and given a character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, R^\times)$, we say that $f \in \mathbf{M}(N)$ has *weight-nebentypus* χ if $\langle z \rangle f = \chi(z) f$ for all $z \in \mathbf{Z}_p^\times$. If k is an integer, denoting by z^k the k-th power character on $\mathbf{Z}_p^\times$, the subspace $M_k^{\mathrm{ord}}(Np^s, \varepsilon)$ of $M_k^{\mathrm{ord}}(X_s)$ consisting of p-adic modular forms with nebentypus $\varepsilon : (\mathbf{Z}/p^s\mathbf{Z})^\times \longrightarrow R^\times$ can be recovered as

$$M_k^{\mathrm{ord}}(Np^s, \varepsilon) \cong \{f \in \mathbf{M}(N) : \langle z \rangle f = z^k \varepsilon(z) f, \quad \text{for all } z \in \mathbf{Z}_p^\times\}. \tag{1.1.2}$$

Since it will play an important role later, we next recall from [**Gou88**, §III.6.2] the definition in terms of moduli of the twist of p-adic modular forms by characters of not necessarily finite order. Let R be a p-adic ring, and let $(E, \alpha_E, \imath_E)$ be a trivialized elliptic curve with $\Gamma_1(N)$-level structure over R. For each s, consider the quotient $E_0 := E/\imath_E^{-1}(\boldsymbol{\mu}_{p^s})$, and let $\varphi_0 : E \twoheadrightarrow E_0$ denote the projection. Since $p \nmid N$, φ_0 induces a $\Gamma_1(N)$-level structure α_{E_0} on E_0, and since $\ker(\varphi_0) \cong \boldsymbol{\mu}_{p^s}$, the dual $\check{\varphi}_0 : E_0 \longrightarrow E$ is étale, inducing an isomorphism of the associated formal groups. Thus (with a slight abuse of notation) $\imath_{E_0} := \imath_E \circ \check{\varphi}_0 : \hat{E}_0 \xrightarrow{\sim} \hat{\mathbf{G}}_m$ is a trivialization of E_0, and since we have an embedding $\mathbf{Z}/p^s\mathbf{Z} \cong \ker(\check{\varphi}_0) \hookrightarrow E_0[p^s]$, we deduce an isomorphism

$$E_0[p^s] \cong \boldsymbol{\mu}_{p^s} \oplus \mathbf{Z}/p^s\mathbf{Z}$$

which we use to bijectively attach a p^s-th root of unity ζ_C to every étale subgroup $C \subset E_0[p^s]$ of order p^s, in such a way that 1 is attached to $\ker(\check{\varphi}_0)$.

Now for $f \in \mathbf{M}(N)$ and $a \in \mathbf{Z}_p$, define $f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}$ to be the rule on trivialized elliptic curves given by

$$f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}(E, \alpha_E, \imath_E) = \frac{1}{p^s} \sum_C \zeta_C^{-a} \cdot f(E_0/C, \alpha_C, \imath_C) \tag{1.1.3}$$

where the sum is over the étale subgroups $C \subset E_0[p^s]$ of order p^s, and where α_C (resp. $\imath_C$) denotes the $\Gamma_1(N)$-level structure (resp. trivialization) on the quotient E_0/C naturally induced by α_{E_0} (resp. $\imath_{E_0}$).

Lemma 1.1.1. *The assignment*

$$a + p^s\mathbf{Z}_p \quad \rightsquigarrow \quad \big(f \longmapsto f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}\big)$$

gives rise to an $\mathrm{End}_R\mathbf{M}(N)$*-valued measure* μ_{Gou} *on* $\mathbf{Z}_p$*.*

Proof. Let $\sum_n a_n q^n$ be the q-expansion of f, i.e. the value that it takes at the triple $(\mathrm{Tate}(q), \alpha_{\mathrm{can}}, \imath_{\mathrm{can}}) = (\mathbf{G}_m/q^{\mathbf{Z}}, \zeta_N, \boldsymbol{\mu}_{p^\infty} \hookrightarrow \mathbf{G}_m/q^{\mathbf{Z}})$ over the p-adic completion of $R((q))$. By the q-expansion principle, the claim follows immediately from the equality

$$f \otimes \mathbb{1}_{a+p^s\mathbf{Z}_p}(q) = \sum_{n \equiv a \bmod p^s} a_n q^n,$$

which is shown by adapting the arguments in [**Gou88**, p. 102]. □

DEFINITION 1.1.2 (Gouvêa). Let $f \in \mathbf{M}(N)$ and $\chi : \mathbf{Z}_p \longrightarrow R$ be any continuous multiplicative function. The *twist* of f by χ is

$$f \otimes \chi := \left(\int_{\mathbf{Z}_p} \chi(x) d\mu_{\mathrm{Gou}}(x) \right)(f) \in \mathbf{M}(N).$$

This operation is compatible with the usual character twist of Hecke eigenforms:

LEMMA 1.1.3. *Let $\chi : \mathbf{Z}_p^\times \longrightarrow R^\times$ be a continuous character extended by zero on $p\mathbf{Z}_p$. If $f \in \mathbf{M}(N)$ has q-expansion $\sum_n a_n q^n$, then $f \otimes \chi$ has q-expansion $\sum_n \chi(n) a_n q^n$, and if f has weight-nebentypus $\kappa \in \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, R^\times)$, then $f \otimes \chi$ has weight-nebentypus $\chi^2\kappa$.*

PROOF. See [**Gou88**, Cor. III.6.8.i] and [**Gou88**, Cor. III.6.9]). $\square$

In particular, twisting by the identity function of $\mathbf{Z}_p$ we obtain an operator

$$d : \mathbf{M}(N) \longrightarrow \mathbf{M}(N)$$

whose effect on q-expansions is $q\frac{d}{dq}$. For every $k \in \mathbf{Z}$, we see from (1.1.2) and Lemma 1.1.3, that this restricts to a map

$$d : M_k^{\mathrm{ord}}(X_s) \longrightarrow M_{k+2}^{\mathrm{ord}}(X_s)$$

which increases the weight by 2 and preserves the nebentypus. Moreover, for $k = 0$, the arguments in [**Col96**, Prop. 4.3] can be adapted to show that d gives rise to a linear map $M_0^{\mathrm{rig}}(X_s) \longrightarrow M_2^{\mathrm{rig}}(X_s)$, viewing $M_k^{\mathrm{rig}}(X_s)$ as a subspace of $M_k^{\mathrm{ord}}(X_s)$ via the natural restriction map.

1.1.2. Comparison isomorphisms. Let ζ_s be a primitive p^s-th root of unity, and let F be a finite extension of $\mathbf{Q}_p(\zeta_s)$ over which X_s acquires stable reduction, i.e. such that the base extension $X_s \times_{\mathbf{Q}_p} F$ admits a stable model over the ring of integers $\mathcal{O}_F$ of F. For the ease of notation, from now on we will denote $X_s \times_{\mathbf{Q}_p} F$ (as well as the associated rigid analytic space) simply by X_s.

Let $\mathscr{X}_s$ be the minimal regular model of X_s over $\mathcal{O}_F$, and denote by F_0 the maximal unramified subfield of F. The work [**HK94**] of Hyodo-Kato endows the F-vector space $H^1_{\mathrm{dR}}(X_s/F)$ with a canonical F_0-structure

$$H^1_{\mathrm{log-cris}}(\mathscr{X}_s) \hookrightarrow H^1_{\mathrm{dR}}(X_s/F) \tag{1.1.4}$$

equipped with a semi-linear Frobenius operator φ.

After the proof [**Tsu99**] of the so-called semistable conjecture of Fontaine–Jannsen, these structures are known to agree with those attached by Fontaine's theory to the p-adic

G_F-representation

$$V_s := H^1_{\text{ét}}(\overline{X}_s, \mathbf{Q}_p). \tag{1.1.5}$$

More precisely, since X_s has semistable reduction, V_s is semistable in the sense of Fontaine, and there is a canonical isomorphism $D_{\rm st}(V_s) \xrightarrow{\sim} H^1_{\log-{\rm cris}}(\mathscr{X}_s)$, inducing an isomorphism

$$D_{\rm dR}(V_s) \xrightarrow{\sim} H^1_{\rm dR}(X_s/F) \tag{1.1.6}$$

as filtered φ-modules after extending scalars to F.

Consider the étale Abel–Jacobi map $\mathrm{CH}^1(X_s)_0(F) \longrightarrow H^1(F, V_s(1))$ constructed in [**Nek00**], which in this case agrees with the usual Kummer map

$$\delta_F : J_s(F) \longrightarrow H^1(F, \mathbf{Q}_p \otimes \mathrm{Ta}_p(J_s)),$$

where $J_s = \mathrm{Pic}^0(X_s)$ is the connected Picard variety of X_s. (See [*loc.cit.*, Example(2.3)].)

Let $g \in S_2(X_s)$ be a newform with primitive nebentypus, denote by V_g the p-adic Galois representation associated to g, which is equipped with a Galois-equivariant projection $V_s \longrightarrow V_g$, and let V_g^* be the representation contragredient to V_g, so that $V_g(1)$ and V_g^* are in Kummer duality. Also, let L_g be a finite extension of $\mathbf{Q}_p$ over which the Hecke eigenvalues of g are defined. By [**BK90**, Example 3.11], the image of the induced composite map

$$\delta_{g,F} : J_s(F) \xrightarrow{\delta_F} H^1(F, V_s(1)) \longrightarrow H^1(F, V_g(1)) \tag{1.1.7}$$

lies in the Bloch–Kato "finite" subspace $H^1_f(F, V_g(1))$, and by our assumption on g, the Bloch-Kato exponential map gives an isomorphism

$$\exp_{F,V_g(1)} : \frac{D_{\rm dR}(V_g(1))}{\mathrm{Fil}^0 D_{\rm dR}(V_g(1))} \longrightarrow H^1_f(F, V_g(1)) \tag{1.1.8}$$

whose inverse will be denoted by $\log_{F,V_g(1)}$.

Our aim in this section is to compute the images of certain degree 0 divisors on X_s under the p-adic Abel–Jacobi map $\delta^{(p)}_{g,F}$, defined as the composition

$$J_s(F) \xrightarrow{\delta_{g,F}} H^1_f(F, V_g(1)) \xrightarrow{\log_{F,V_g(1)}} \frac{D_{\rm dR}(V_g(1))}{\mathrm{Fil}^0 D_{\rm dR}(V_g(1))} \xrightarrow{\sim} (\mathrm{Fil}^0 D_{\rm dR}(V_g^*))^\vee, \tag{1.1.9}$$

where the last identification arises from the de Rham pairing

$$\langle\ ,\ \rangle_{\rm dR} : D_{\rm dR}(V_g(1)) \times D_{\rm dR}(V_g^*) \longrightarrow D_{\rm dR}(\mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L_g \cong F \otimes_{\mathbf{Q}_p} L_g \tag{1.1.10}$$

with respect to which $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$ and $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g^*)$ are exact annihilators of each other. A basic ingredient for this computation will be the following alternate description of the logarithm map $\log_{F,V_g(1)}$.

Recall the interpretation of $H^1(F, V_g(1))$ as the space $\mathrm{Ext}^1_{\underline{\mathrm{Rep}}(G_F)}(L_g, V_g(1))$ of extensions of $V_g(1)$ by L_g in the category of p-adic G_F-representations. Since F contains $\mathbf{Q}_p(\zeta_s)$, V_g is a crystalline G_F-representation in the sense of Fontaine, and under that interpretation the Bloch–Kato "finite" subspace corresponds to those extensions which are crystalline (see [**Nek93**, Prop. 1.26], for example):

$$H^1_f(F, V_g(1)) \cong \mathrm{Ext}^1_{\underline{\mathrm{Rep}}_{\mathrm{cris}}(G_F)}(L_g, V_g(1)). \tag{1.1.11}$$

Now consider a crystalline extension

$$0 \longrightarrow V_g(1) \longrightarrow W \longrightarrow L_g \longrightarrow 0. \tag{1.1.12}$$

Since $D_{\mathrm{cris}}(V_g(1))^{\varphi=1} = 0$ by our assumptions, the resulting extension of φ-modules

$$0 \longrightarrow D_{\mathrm{cris}}(V_g(1)) \longrightarrow D_{\mathrm{cris}}(W) \longrightarrow F_0 \otimes_{\mathbf{Q}_p} L_g \longrightarrow 0 \tag{1.1.13}$$

admits a unique section $s_W^{\mathrm{frob}} : F_0 \otimes_{\mathbf{Q}_p} L_g \longrightarrow D_{\mathrm{cris}}(W)$ with $s_W^{\mathrm{frob}}(1)$ landing in the φ-invariant subspace $D_{\mathrm{cris}}(W)^{\varphi=1}$. Extending scalars from F_0 to F in (1.1.13) and taking Fil^0-parts, we take an arbitrary section $s_W^{\mathrm{fil}} : F \otimes_{\mathbf{Q}_p} L_g \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(W)$ of the resulting exact sequence of F-vector spaces

$$0 \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1)) \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(W) \longrightarrow F \otimes_{\mathbf{Q}_p} L_g \longrightarrow 0 \tag{1.1.14}$$

and form the difference

$$t_W := s_W^{\mathrm{fil}}(1) - s_W^{\mathrm{frob}}(1),$$

which can be seen in $D_{\mathrm{dR}}(V_g(1))$, and whose image modulo $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$ is well-defined.

LEMMA 1.1.4. *Under the identification* (1.1.11)*, the above assignment*

$$0 \to V_g(1) \to W \to L_g \to 0 \quad \rightsquigarrow \quad t_W \bmod \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))$$

defines an isomorphism which agrees with the Bloch–Kato logarithm map

$$\log_{F,V_g(1)} : H^1_f(F, V_g(1)) \xrightarrow{\sim} \frac{D_{\mathrm{dR}}(V_g(1))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1))}.$$

PROOF. See [**Nek93**, Lemma 2.7], for example. □

Let $\Delta \in J_s(F)$ be the class of a degree 0 divisor on X_s with support contained in the finite set of points $S \subset X_s(F)$. The extension class $W = W_\Delta$ (1.1.12) corresponding to

$\delta_{g,F}(\Delta)$ can then be constructed from the étale cohomology of the open curve $Y_s := X_s \smallsetminus S$, as explained in [**BDP13**, §3.1]. We describe the associated $s^{\mathrm{fil}}_{W_\Delta}$ and $s^{\mathrm{frob}}_{W_\Delta}$.

By [**Tsu99**] (or also [**Fal02**]), denoting g-isotypical components by the superscript g, there is a canonical isomorphism of $F_0 \otimes_{\mathbf{Q}_p} L_g$-modules

$$D_{\mathrm{cris}}(V_g) \cong H^1_{\log-\mathrm{cris}}(\mathscr{X}_s)^g \tag{1.1.15}$$

compatible with φ-actions and inducing an $F \otimes_{\mathbf{Q}_p} L_g$-module isomorphism

$$D_{\mathrm{dR}}(V_g) \cong H^1_{\mathrm{dR}}(X_s/F)^g \tag{1.1.16}$$

after extension of scalars.

Writing $\Delta = \sum_{Q\in S} n_Q.Q$ for some $n_Q \in \mathbf{Z}$, we assume from now on that S contains the cusps, and that the reductions of the points $Q \in S$ are smooth and pair-wise distinct. We also assume that the reduction of S in the special fiber is stable under the absolute Frobenius. Like $H^1_{\mathrm{dR}}(X_s/F)$, the F-vector space $H^1_{\mathrm{dR}}(Y_s/F)$ is equipped with a canonical F_0-structure

$$H^1_{\log-\mathrm{cris}}(\mathscr{Y}_s) \hookrightarrow H^1_{\mathrm{dR}}(Y_s/F), \tag{1.1.17}$$

a Frobenius operator still denoted by φ, and a Hecke action compatible with that in (1.1.4). Thus for $W = W_\Delta$ the exact sequence (1.1.13) is obtained as the pullback

$$\begin{tikzcd} D_{\mathrm{cris}}(V_g(1)) \arrow[r, hook] \arrow[d, equal] & D_{\mathrm{cris}}(W_\Delta) \arrow[r, two heads, "\rho"] \arrow[d] & F_0 \otimes_{\mathbf{Q}_p} L_g \arrow[d, "\Delta"] \\ H^1_{\log-\mathrm{cris}}(\mathscr{X}_s)^g(1) \arrow[r, hook] & H^1_{\log-\mathrm{cris}}(\mathscr{Y}_s)^g(1) \arrow[r, two heads, "\oplus \mathrm{res}_Q"] & (F_0 \otimes_{\mathbf{Q}_p} L_g)_0^{\oplus S} \end{tikzcd} \tag{1.1.18}$$

of the bottom extension of φ-modules with respect to the $F_0 \otimes_{\mathbf{Q}_p} L_g$-linear map sending $1 \mapsto (n_Q)_{Q\in S}$, where the subscript 0 indicates taking the degree 0 subspace.

On the other hand, after extending scalars from F_0 to F and taking Fil^0-parts, (1.1.14) is given by the pullback[2]

$$\begin{tikzcd} \mathrm{Fil}^0 D_{\mathrm{dR}}(V_g(1)) \arrow[r, hook] \arrow[d, equal] & \mathrm{Fil}^0 D_{\mathrm{dR}}(W_\Delta) \arrow[r, two heads, "\rho"] \arrow[d] & F \otimes_{\mathbf{Q}_p} L_g \arrow[d, "\Delta"] \\ \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X_s/F)^g \arrow[r, hook] & \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y_s/F)^g \arrow[r, two heads, "\oplus \mathrm{res}_Q"] & (F \otimes_{\mathbf{Q}_p} L_g)_0^{\oplus S} \end{tikzcd} \tag{1.1.19}$$

of the bottom exact sequence of free $F \otimes_{\mathbf{Q}_p} L_g$-modules with respect to the $F \otimes_{\mathbf{Q}_p} L_g$-linear map sending $1 \mapsto (n_Q)_{Q\in S}$.

[2]Notice the effect of the Tate twist on the filtrations.

Let $\varepsilon_g = \varepsilon_{g,p} \cdot \varepsilon_g^{(p)}$ be the nebentypus of g, decomposed as the product of its "wild" component $\varepsilon_{g,p}$ on $(\mathbf{Z}/p^s\mathbf{Z})^\times$ and its "tame" component $\varepsilon_g^{(p)}$ on $(\mathbf{Z}/N\mathbf{Z})^\times$. Let $g^* \in S_2(X_s)$ be the form *dual* to g, defined as the newform associated with the twist $g \otimes \varepsilon_{g,p}^{-1}$, and let $\omega_{g^*} \in H^0(X_s, \Omega^1_{X_s/F})$ be its associated differential, so that

$$\mathrm{Fil}^0 D_{\mathrm{dR}}(V_g^*) = (F \otimes_{\mathbf{Q}_p} L_g).\omega_{g^*}.$$

The image of the functional $\delta_{g,F}^{(p)}(\Delta)$ is thus determined by the value

$$\delta_{g,F}^{(p)}(\Delta)(\omega_{g^*}) = \langle t_{W_\Delta}, \omega_{g^*} \rangle_{\mathrm{dR}} \tag{1.1.20}$$

of the pairing (1.1.10), which corresponds to the Poincaré pairing on $H^1_{\mathrm{dR}}(X_s/F)$ under the identification (1.1.16). Using rigid analysis, we will now give an expression for the latter pairing that will make (1.1.20) amenable to computations.

Let $\mathcal{X}_s$ be the canonical balanced model of X_s over $\mathbf{Z}_p[\zeta_s]$ constructed by Katz and Mazur. The special fiber $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \mathbf{F}_p$ is a reduced disjoint union of Igusa curves over $\mathbf{F}_p$ intersecting at the supersingular points, with exactly two of them isomorphic to the Igusa curve $\mathrm{Ig}(\Gamma_s)$ representing the moduli problem $([\Gamma_1(N)], [\Gamma_1(p^s)])$ over $\mathbf{F}_p$ (see [**KM85**, §13]); we let I_∞ be the one that contains the reduction of $\mathcal{W}_1(p^s) \times_{\mathbf{Q}_p} \mathbf{Q}_p(\zeta_s)$, and let I_0 be the other. (We note that these components are the two "good" components in the terminology of [**MW86**].)

By the universal property of the regular minimal model, there exists a morphism

$$\mathscr{X}_s \longrightarrow \mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \mathcal{O}_F \tag{1.1.21}$$

which reduces to a sequence of blow-ups on the special fiber. Letting κ be the residue field of F, define $\mathcal{W}_\infty \subset X_s$ (resp. $\mathcal{W}_0 \subset X_s$) to be the inverse image under the reduction map via $\mathscr{X}_s$ of the unique irreducible component of $\mathscr{X}_s \times_{\mathcal{O}_F} \kappa$ mapping bijectively onto $I_\infty \times_{\mathbf{F}_p} \kappa$ (resp. $I_0 \times_{\mathbf{F}_p} \kappa$) in $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \kappa$ via the reduction of (1.1.21). Similarly, define $\mathcal{U} \subset X_s$ by considering the irreducible components of $\mathcal{X}_s \times_{\mathbf{Z}_p[\zeta_s]} \kappa$ different from $I_\infty \times_{\mathbf{F}_p} \kappa$ and $I_0 \times_{\mathbf{F}_p} \kappa$. Letting SS denote (the degree of) the supersingular divisor of $\mathrm{Ig}(\Gamma_s)$, one can show that $\mathcal{U}$ intersects $\mathcal{W}_\infty$ (resp. $\mathcal{W}_0$) in a union of SS supersingular annuli.

Since they reduce to smooth points, the residue class D_Q of each $Q \in S$ is conformal to the open unit disc $D \subset \mathbf{C}_p$. Fix an isomorphism $h_Q : D_Q \xrightarrow{\sim} D$ that sends Q to 0, and for a real number $r_Q < 1$ in $p^{\mathbf{Q}}$, denote by $\mathcal{V}_Q \subset D_Q$ the annulus consisting of the points $x \in D_Q$ with $r_Q < |h_Q(x)|_p < 1$.

Attached to any (oriented) annulus $\mathcal{V}$, there is a *p-adic annular residue map*

$$\mathrm{Res}_{\mathcal{V}} : \Omega^1_{\mathcal{V}} \longrightarrow \mathbf{C}_p$$

defined by expanding $\omega \in \Omega^1_{\mathcal{V}}$ as $\omega = \sum_{n\in\mathbf{Z}} a_n T^n \frac{dT}{T}$ for a fixed uniformizing parameter T on $\mathcal{V}$ (compatible with the orientation), and setting $\mathrm{Res}_{\mathcal{V}}(\omega) = a_0$. This descends to a linear functional on $\Omega^1_{\mathcal{V}}/d\mathcal{O}_{\mathcal{V}}$. (See [**Col89**, Lemma 2.1].)

For any *basic wide-open* $\mathcal{W}$ (as in [**Buz03**, p. 34]), define

$$H^1_{\mathrm{rig}}(\mathcal{W}) := \mathbb{H}^1(\mathcal{W}, \Omega^\bullet(\log Z)) \cong \Omega^1_{\mathcal{W}}/d\mathcal{O}_{\mathcal{W}}, \tag{1.1.22}$$

where $\Omega^\bullet(\log Z)$ denotes the complex of rigid analytic sheaves on $\mathcal{W}$ deduced from (1.1.1) by analytification and pullback, and consider the basic wide-opens

$$\widetilde{\mathcal{W}}_\infty := \mathcal{W}_\infty \smallsetminus \bigcup_{Q\in S}(D_Q \smallsetminus \mathcal{V}_Q) \qquad \widetilde{\mathcal{W}}_0 := \mathcal{W}_0 \smallsetminus \bigcup_{Q\in S}(D_Q \smallsetminus \mathcal{V}_Q).$$

The space $H^1_{\mathrm{dR}}(X_s/F)$ is equipped with a natural action of the diamond operators, and following [**Col97a**, §2] we define $H^1_{\mathrm{dR}}(X_s/F)^{\mathrm{prim}}$ to be the subspace of $H^1_{\mathrm{dR}}(X_s/F)$ spanned by (the pullbacks of) the classes in $H^1_{\mathrm{dR}}(X_r/F)$, for $0 \le r \le s$, with primitive nebentypus at p. Also, we define $H^1_{\mathrm{dR}}(Y_s/F)^{\mathrm{prim}}$ to be the image of $H^1_{\mathrm{dR}}(X_s/F)^{\mathrm{prim}}$ under the natural restriction map $H^1_{\mathrm{dR}}(X_s/F) \longrightarrow H^1_{\mathrm{dR}}(Y_s/F)$. Finally, let $H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)^*$ be the subspace of $H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)$ consisting of classes ω with $\mathrm{res}_{\mathcal{V}_x}(\omega) = 0$ for all supersingular annuli $\mathcal{V}_x$ and $\mathrm{res}_{\mathcal{V}_Q}(\omega) = 0$ for all $Q \in S$, and define $H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)^*$ in the analogous manner.

Lemma 1.1.5 (Coleman). *The natural restriction maps induce an isomorphism*

$$H^1_{\mathrm{dR}}(Y_s/F)^{\mathrm{prim}} \xrightarrow{\ \sim\ } H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)^* \oplus H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)^*,$$

and if η and ω are any two classes in $H^1_{\mathrm{dR}}(X_s/F)^{\mathrm{prim}}$, their Poincaré pairing is given by

$$\langle \eta, \omega\rangle_{\mathrm{dR}} = \sum_{x\in S\cup SS} \mathrm{Res}_{\mathcal{V}_x}(F_{\omega_\infty|_{\mathcal{V}_x}} \cdot \eta_\infty|_{\mathcal{V}_x}) + \sum_{x\in S\cup SS} \mathrm{Res}_{\mathcal{V}_x}(F_{\omega_0|_{\mathcal{V}_x}} \cdot \eta_0|_{\mathcal{V}_x}), \tag{1.1.23}$$

where for each annulus $\mathcal{V}_x$, $F_{\omega_{\mathcal{V}_x}}$ denotes any solution to $dF_{\omega_{\mathcal{V}_x}} = \omega_{\mathcal{V}_x}$ on $\mathcal{V}_x$.

Proof. By an excision argument, the first assertion follows from [**Col97a**, Thm. 2.1], and the second is shown by adapting the arguments in [**Col96**, §5] for each of the two components, as done in [**Col94a**, Prop. 1.3] for $s = 1$. (See also [**Col97a**, §3].) □

1.1.3. Coleman p-adic integration. Coleman's theory of p-adic integration provides a coherent choice of local primitives that will allow us to we compute (1.1.20) using the formula (1.1.23). The key idea is to exploit the action of Frobenius.

Recall the lift of Frobenius $\phi : \mathcal{W}_2(p^s) \longrightarrow \mathcal{W}_1(p^s)$ described in Section 1.1.1, where $\mathcal{W}_i(p^s)$ are the strict neighborhoods of the connected component $X_s(0)$ of the ordinary locus of X_s containing the cusp ∞ described there. Recall also the wide open space $\mathcal{W}_\infty$ described in the preceding section, which also contains $X_s(0)$ by construction.

PROPOSITION 1.1.6 (Coleman). *Let $g = \sum_{n>0} b_n q^n \in S_2(X_s)$ be a normalized newform with primitive nebentypus of p-power conductor, so that b_p is such that $U_p g = b_p g$. There exists a unique locally analytic function F_{ω_g} on $\mathcal{W}_\infty$ with the following three properties:*

- *$dF_{\omega_g} = \omega_g$ on $\mathcal{W}_\infty$,*
- *$F_{\omega_g} - \frac{b_p}{p}\phi^* F_{\omega_g} \in M_0^{\mathrm{rig}}(X_s)$, and*
- *F_{ω_g} vanishes at ∞.*

PROOF. This follows from the general result of Coleman [**Col94b**, Thm. 10.1]. Indeed, a computation on q-expansions shows that the action of the Frobenius lift ϕ on differentials agrees with that of pV, with V the map acting as $q \mapsto q^p$ on q-expansions, in the sense that $\phi^*\omega_g = p\omega_{Vg}$ on $\mathcal{W}'_\infty := \phi^{-1}(\mathcal{W}_\infty \cap \mathcal{W}_1(p^s))$. Since the differential $\omega_{g^{[p]}} = \omega_g - b_p\omega_{Vg}$ attached to

$$g^{[p]} = \sum_{(n,p)=1} b_n q^n$$

becomes exact upon restriction to $\mathcal{W}'_\infty$, this shows that the polynomial $L(T) = 1 - \frac{b_p}{p}T$ is such that $L(\phi^*)\omega_g = 0$. Finally, since g has primitive nebentypus, b_p has complex absolute value $p^{1/2}$, and hence [**Col94b**, Thm. 10.1] can be applied with $L(T)$ as above. $\square$

Attached to a primitive p^s-th root of unity ζ, there is an automorphism w_ζ of X_s which interchanges the components $\mathcal{W}_\infty$ and $\mathcal{W}_0$. (See [**BE10**, Lemma 4.4.3].)

COROLLARY 1.1.7. *Set $\phi' := w_\zeta \circ \phi \circ w_\zeta$. With hypotheses as in Proposition 1.1.6, there exists a unique locally analytic function F'_{ω_g} on $\mathcal{W}_0$ with the following three properties:*

- *$dF'_{\omega_g} = \omega_g$ on $\mathcal{W}_0$,*
- *$F'_{\omega_g} - \frac{b_p}{p}(\phi')^* F'_{\omega_g}$ is rigid analytic on a wide-open neighborhood $\mathcal{W}'_0$ of $w_\zeta X_s(0)$ in $\mathcal{W}_0$, and*
- *F'_{ω_g} vanishes at 0.*

PROOF. Proposition 1.1.6 applied to the differential $\omega'_g := w_\zeta^*\omega_g$ gives the existence of a locally analytic function $F_{\omega'_g}$ with $F'_{\omega_g} := w_\zeta^* F_{\omega'_g}$ having the desired properties. The uniqueness of F'_{ω_g} follows immediately from that of $F_{\omega'_g}$. $\square$

We refer to the locally analytic function F_{ω_g} (resp. F'_{ω_g}) appearing in Proposition 1.1.6 as the *Coleman primitive* of g on $\mathcal{W}_\infty$ (resp. $\mathcal{W}_0$).

Let $g = \sum_{n>0} b_n q^n$ be as in Proposition 1.1.6. The q-expansion $\sum_{(n,p)=1} \frac{b_n}{n} q^n$ corresponds to a p-adic modular form g' vanishing at ∞ and satisfying $dg' = g^{[p]}$, where d is the operator described at the end of Section 2.1, which here corresponds to the differential operator $\mathcal{O}_{\mathcal{W}} \longrightarrow \Omega^1_{\mathcal{W}}$ for any subspace $\mathcal{W} \subset X_s$. Set $d^{-1}g^{[p]} := g'$.

Corollary 1.1.8. *If F_{ω_g} is the Coleman primitive of g on $\mathcal{W}_\infty$, then*

$$F_{\omega_g} - \frac{b_p}{p}\phi^* F_{\omega_g} = d^{-1}g^{[p]}.$$

Proof. Since $d^{-1}g^{[p]}$ is an overconvergent rigid analytic primitive of $\omega_{g^{[p]}}$, and the operator $L(\phi^*) = 1 - \frac{b_p}{p}\phi^*$ acting on the space of locally analytic functions on $\mathcal{W}_\infty$ is invertible, we see that $L(\phi^*)^{-1}(d^{-1}g^{[p]})$ satisfies the defining properties of F_{ω_g}. Since $d^{-1}g^{[p]}$ vanishes at ∞, the result follows. $\square$

We can now give an explicit formula for the p-adic Abel–Jacobi images of certain degree 0 divisors on X_s. Note that this formula it is key in all what follows.

Proposition 1.1.9. *Assume that $s > 1$. Let $g \in S_2(X_s)$ be a normalized newform with primitive nebentypus of p-power conductor, let P be an F-rational point of X_s factoring through $X_s(0) \subset X_s$, and let $\Delta \in J_s(F)$ be the divisor class of $(P) - (\infty)$. Then*

$$\delta^{(p)}_{g,F}(\Delta)(\omega_{g^*}) = F_{\omega_{g^*}}(P), \tag{1.1.24}$$

where $F_{\omega_{g^*}}$ is the Coleman primitive of ω_{g^*} on $\mathcal{W}_\infty$.

Proof. By (1.1.20), we must evaluate $\langle t_{W_\Delta}, \omega_{g^*}\rangle_{\mathrm{dR}}$, where (with a slight abuse of notation) $t_{W_\Delta} = s^{\mathrm{fil}}_{W_\Delta} - s^{\mathrm{frob}}_{W_\Delta}$ with

- $s^{\mathrm{fil}}_{W_\Delta} \in \mathrm{Fil}^1 D_{\mathrm{dR}}(W_\Delta)$ is such that $\rho(s^{\mathrm{fil}}_{W_\Delta}) = 1$ in (1.1.19), and
- $s^{\mathrm{frob}}_{W_\Delta} \in D_{\mathrm{cris}}(W_\Delta)^{\varphi=1}$ is such that $\rho(s^{\mathrm{frob}}_{W_\Delta}) = 1$ in (1.1.18).

By Lemma 1.1.5, we see that these can be represented, respectively, by

- $\eta^{\mathrm{fil}}_\Delta$ a section of $\Omega^1_{X_s/F}$ over Y_s with simple poles at P and ∞ and with
  - $\mathrm{Res}_P(\eta^{\mathrm{fil}}_\Delta) = 1$,
  - $\mathrm{Res}_\infty(\eta^{\mathrm{fil}}_\Delta) = -1$,
  - $\mathrm{Res}_Q(\eta^{\mathrm{fil}}_\Delta) = 0$ for all $Q \in S - \{P, \infty\}$;
- $\eta^{\mathrm{frob}}_\Delta = (\eta^{\mathrm{frob}}_\infty, \eta^{\mathrm{frob}}_0) \in \Omega^1_{\widetilde{\mathcal{W}_\infty}} \times \Omega^1_{\widetilde{\mathcal{W}_0}}$ with
  - $(\phi^*\eta^{\mathrm{frob}}_\infty, (\phi')^*\eta^{\mathrm{frob}}_\Delta) = (p \cdot \eta^{\mathrm{frob}}_\infty + dG_\infty, p \cdot \eta^{\mathrm{frob}}_0 + dG_0)$ with G_∞ and G_0 rigid analytic on $\phi^{-1}\widetilde{\mathcal{W}_\infty}$ and $(\phi')^{-1}\widetilde{\mathcal{W}_0}$, respectively,
  - $\mathrm{Res}_{\mathcal{V}_x}(\eta^{\mathrm{frob}}_\Delta) = 0$ for all supersingular annuli $\mathcal{V}_x$, and
  - $\mathrm{Res}_{\mathcal{V}_Q}(\eta^{\mathrm{frob}}_\Delta) = \mathrm{Res}_Q(\eta^{\mathrm{fil}}_\Delta)$ for all $Q \in S$.

The arguments in [**BDP13**, Prop. 3.21] can now be straightforwardly adapted to deduce the result. Indeed, using the defining properties of the Coleman primitives $F_{\omega_{g^*}}$ and $F'_{\omega_{g^*}}$ of ω_{g^*} on $\mathcal{W}_\infty$ and $\mathcal{W}_0$, respectively, one first shows that

$$\sum_{x\in S\cup SS} \mathrm{Res}_{\mathcal{V}_x}(F_{\omega_{g^*}} \cdot \eta_\infty^{\mathrm{frob}}) = 0 \quad \text{and} \quad \sum_{x\in S\cup SS} \mathrm{Res}_{\mathcal{V}_x}(F'_{\omega_{g^*}} \cdot \eta_0^{\mathrm{frob}}) = 0 \tag{1.1.25}$$

as in [*loc.cit.*, Lemma 3.20]. On the other hand, using the same primitives, one shows as in [*loc.cit.*, Lemma 3.19] that

$$\sum_{x\in S\cup SS} \mathrm{Res}_{\mathcal{V}}(F_{\omega_{g^*}} \cdot \eta_\Delta^{\mathrm{fil}}) = F_{\omega_{g^*}}(P) \quad \text{and} \quad \sum_{x\in S\cup SS} \mathrm{Res}_{\mathcal{V}}(F'_{\omega_{g^*}} \cdot \eta_\Delta^{\mathrm{fil}}) = 0. \tag{1.1.26}$$

Substituting (1.1.26) and (1.1.25) into the formula (1.1.23) for the Poincaré pairing (and using that $s > 1$, so that there is no overlap between the supersingular annuli in $\widetilde{\mathcal{W}_\infty}$ and the supersingular annuli in $\widetilde{\mathcal{W}_0}$), the result follows. □

1.2. Generalised Heegner cycles

Let $X_1(N)$ be the compactified modular curve of level $\Gamma_1(N)$ defined over $\mathbf{Q}$, and let $\mathcal{E}$ be the universal generalized elliptic curve over $X_1(N)$. (Recall that $N > 4$.) For $r > 1$, denote by W_r the $(2r-1)$-dimensional Kuga–Sato variety[3], defined as the canonical desingularization of the $(2r-2)$-nd fiber product of $\mathcal{E}$ with itself over $X_1(N)$. By construction, the variety W_r is equipped with a proper morphism

$$\pi_r : W_r \longrightarrow X_1(N)$$

whose fibers over a noncuspidal closed point of $X_1(N)$ corresponding to an elliptic curve E with $\Gamma_1(N)$-level structure is identified with $2r-2$ copies of E. (For a more detailed description, see [**BDP13**, §2.1].)

Let K be an imaginary quadratic field of odd discriminant $-D < 0$. It will be assumed throughout that K satisfies the following hypothesis:

Assumptions 1.2.1. *All the prime factors of N split in K.*

Denote by $\mathcal{O}_K$ the ring of integers of K, and note that by this assumption we may choose an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with

$$\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}$$

that we now fix once and for all.

[3]Perhaps most commonly denoted by W_{2r-2}; cf. [**Zha97**] and [**Nek95**], for example.

Let A be a fixed elliptic curve with CM by $\mathcal{O}_K$. The pair $(A, A[\mathfrak{N}])$ defines a point P_A on $X_0(N)$ rational over H, the Hilbert class field of K. Choose one of the square-roots $\sqrt{-D} \in \mathcal{O}_K$, let $\Gamma_{\sqrt{-D}} \subset A \times A$ be the graph of $\sqrt{-D}$, and define

$$\Upsilon^{\mathrm{heeg}}_{A,r} := \Gamma_{\sqrt{-D}} \times \overset{(r-1)}{\cdots} \times \Gamma_{\sqrt{-D}}$$

viewed inside W_r by the natural inclusion $(A \times A)^{r-1} \longrightarrow W_r$ as the fiber of π_r over a point on $X_1(N)$ lifting P_A. Let ϵ_W be the projector from [**BDP13**, (2.1.2)], and set

$$\Delta^{\mathrm{heeg}}_{A,r} := \epsilon_W \Upsilon^{\mathrm{heeg}}_{A,r}, \tag{1.2.1}$$

which is an $(r-1)$-dimensional null-homologous cycle on W_r defining an H-rational class in the Chow group $\mathrm{CH}^r(W_r)_0$ (taken with $\mathbf{Q}$-coefficients, as always here) which is independent of the chosen lift of P_A.

These cycles (1.2.1) are the so-called *Heegner cycles* (of conductor one, weight $2r$), and they share with classical Heegner points many of their arithmetic properties (see [**Nek92, Nek95, Zha97**]).

We next recall a variation of the previous construction introduced in the recent work [**BDP13**] of Bertolini, Darmon, and Prasanna. Let A be the CM elliptic curve fixed above, and consider the variety[4]

$$X_r := W_r \times A^{2r-2}.$$

For each class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, represented by an ideal $\mathfrak{a} \subset \mathcal{O}_K$ prime to N, let $A_{\mathfrak{a}} := A/A[\mathfrak{a}]$ and denote by $\varphi_{\mathfrak{a}}$ the degree $\mathrm{N}\mathfrak{a}$-isogeny

$$\varphi_{\mathfrak{a}} : A \longrightarrow A_{\mathfrak{a}}.$$

The pair $\mathfrak{a} * (A, A[\mathfrak{N}]) := (A_{\mathfrak{a}}, A_{\mathfrak{a}}[\mathfrak{N}])$ defines a point $P_{A_{\mathfrak{a}}}$ in $X_0(N)$ rational over H. Let $\Gamma^t_{\varphi_{\mathfrak{a}}} \subset A_{\mathfrak{a}} \times A$ be the transpose of the graph of $\varphi_{\mathfrak{a}}$, and set

$$\Upsilon^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} := \Gamma^t_{\varphi_{\mathfrak{a}}} \times \overset{(2r-2)}{\cdots} \times \Gamma^t_{\varphi_{\mathfrak{a}}} \ \subset \ (A_{\mathfrak{a}} \times A)^{2r-2} = A_{\mathfrak{a}}^{2r-2} \times A^{2r-2} \xrightarrow{(\imath_{\mathfrak{a}}, \mathrm{id}_A)} X_r,$$

where $\imath_{\mathfrak{a}}$ is the natural inclusion $A_{\mathfrak{a}}^{2r-2} \longrightarrow W_r$ as the fiber of π_r over a point on $X_1(N)$ lifting $P_{A_{\mathfrak{a}}}$. Letting ϵ_A be the projector from [**BDP13**, (1.4.4)], the cycles

$$\Delta^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} := \epsilon_A \epsilon_W \Upsilon^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r} \tag{1.2.2}$$

define classes in $\mathrm{CH}^{2r-1}(X_r)_0(H)$ and are referred to as *generalised Heegner cycles*.

We will assume for the rest of this paper that K also satisfies the following:

[4]Notice that our indices differ from those in [**BDP13**].

Assumptions 1.2.2. *The prime p splits in K.*

Let $g \in S_{2r}(X_0(N))$ be a normalized newform, and let V_g be the p-adic Galois representation associated to g by Deligne. By the Künneth formula, there is a map

$$H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p(2r-1)) \longrightarrow H^{2r-1}_{\text{ét}}(\overline{W}_r, \mathbf{Q}_p(1)) \otimes \operatorname{Sym}^{2r-2} H^1_{\text{ét}}(\overline{A}, \mathbf{Q}_p(1)),$$

which composed with the natural Galois-equivariant projection

$$H^{2r-1}_{\text{ét}}(\overline{W}_r, \mathbf{Q}_p(1)) \otimes \operatorname{Sym}^{2r-2} H^1_{\text{ét}}(\overline{A}, \mathbf{Q}_p(1)) \xrightarrow{\pi_g \otimes \pi_{\mathrm{N}^{r-1}}} V_g(r)$$

induces a map

$$\pi_{g,\mathrm{N}^{r-1}} : H^1(F, H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p(2r-1))) \longrightarrow H^1(F, V_g(r))$$

over any number field F. In the following we fix a number field F containing H.

Now consider the étale Abel–Jacobi map

$$\Phi^{\text{ét}}_F : \mathrm{CH}^{2r-1}(X_r)_0(F) \longrightarrow H^1(F, H^{4r-3}_{\text{ét}}(\overline{X}_r, \mathbf{Q}_p)(2r-1))$$

constructed in [**Nek00**]. Let F_p be the completion of $\imath_p(F)$, and denote by loc_p the induced localization map from G_F to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_p)$. Then we may define the *p-adic Abel–Jacobi map* AJ_{F_p} by the commutativity of the diagram

$$(1.2.3)\qquad \begin{tikzcd}
\mathrm{CH}^{2r-1}(X_r)_0(F) \arrow[r, "\pi_{g,\mathrm{N}^{r-1}} \circ \Phi^{\text{ét}}_F"] \arrow[ddrr, dotted] \arrow[dddrr, bend right, "\mathrm{AJ}_{F_p}"'] & H^1(F, V_g(r)) \arrow[r, "\mathrm{loc}_p"] & H^1(F_p, V_g(r)) \\
 & & \cup \\
 & & H^1_f(F_p, V_g(r)) \arrow[d, "\log_{F_p, V_g(r)}"] \\
 & & \mathrm{Fil}^1 D_{\mathrm{dR}}(V_g(r-1))^\vee,
\end{tikzcd}$$

where the existence of the dotted arrow follows from [**Nek00**, Thm.(3.1)(i)], and the vertical map is given by the logarithm map of Bloch–Kato of $V_g(r)$ as G_{F_p}-representation, similarly as it appeared in (1.1.9) for $r = 1$. Using the comparison isomorphism of Faltings [**Fal89**], the map AJ_{F_p} may be evaluated at the class $\omega_g \otimes e_\zeta^{\otimes r-1}$, with e_ζ an F_p-basis of $D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \cong F_p$.

The main result of [**BDP13**] yields the following formula for the p-adic Abel–Jacobi images of the generalised Heegner cycles (1.2.2) which we will need.

THEOREM 1.2.3 (Bertolini–Darmon–Prasanna). *Let $g = \sum_n b_n q^n \in S_{2r}(X_0(N))$ be a normalized newform of weight $2r \geq 2$ and level N prime to p. Then*

$$(1 - b_p p^{-r} + p^{-1}) \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} \mathrm{N}\mathfrak{a}^{1-r} \cdot \mathrm{AJ}_{F_p}(\Delta^{\mathrm{bdp}}_{\varphi_{\mathfrak{a}},r})(\omega_g \otimes e_\zeta^{\otimes r-1})$$
$$= (-1)^{r-1}(r-1)! \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-r} g^{[p]}(\mathfrak{a} * (A, A[\mathfrak{N}])),$$

where $g^{[p]} = \sum_{(n,p)=1} b_n q^n$ is the p-depletion of g.

PROOF. See the proof of [**BDP13**, Thm. 5.13]. $\square$

We end this section by relating the images of Heegner cycles and of generalised Heegner cycles under the p-adic height pairing. (Cf. [**BDP13**, §2.4].) Consider $\Pi_r := W_r \times A^{r-1}$ seen as a subvariety of $W_r \times X_r = W_r \times W_r \times (A^2)^{r-1}$ via the map

$$(\mathrm{id}_{W_r}, \mathrm{id}_{W_r}, (\mathrm{id}_A, \sqrt{-D})^{r-1}).$$

Denoting by π_W and π_X the projections onto the first and second factors of $W_r \times X_r$, the rational equivalence class of the cycle Π_r gives rise to a map on Chow groups

$$\Pi_r : \mathrm{CH}^{2r-1}(X_r) \longrightarrow \mathrm{CH}^{r+1}(W_r)$$

induced by $\Pi_r(\Delta) = \pi_{W,*}(\Pi_r \cdot \pi_X^* \Delta)$.

LEMMA 1.2.4. *We have*

$$\langle \Delta^{\mathrm{heeg}}_{A,r}, \Delta^{\mathrm{heeg}}_{A,r} \rangle_{W_r} = (4D)^{r-1} \cdot \langle \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}, \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r} \rangle_{X_r},$$

where $\langle , \rangle_{W_r}$ and $\langle , \rangle_{X_r}$ are the p-adic height pairings of [**Nek93**] *on $\mathrm{CH}^{r+1}(W_r)_0$ and $\mathrm{CH}^{2r-1}(X_r)_0$, respectively.*

PROOF. The image $\Phi^{\text{ét}}_F(\Delta^{\mathrm{heeg}}_{A,r})$ remains unchanged if we replace the cycle $\Gamma_{\sqrt{-D}}$ by the modification

$$Z_A := \Gamma_{\sqrt{-D}} - (A \times \{0\}) - D(\{0\} \times A)$$

(see [**Nek95**, §II(3.6)]). Since clearly $Z_A \cdot Z_A = -2D$, we thus see from the construction of Π_r that

$$\Phi^{\text{ét}}_F(\Delta^{\mathrm{heeg}}_{A,r}) = (-2D)^{r-1} \cdot \Phi^{\text{ét}}_F(\Pi_r(\Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r})). \tag{1.2.4}$$

On the other hand, if $\langle , \rangle_A$ denotes the Poincaré pairing on $H^1_{\mathrm{dR}}(A/F)$, we have

$$\langle (\sqrt{-D})^* \omega, (\sqrt{-D})^* \omega' \rangle_A = D \cdot \langle \omega, \omega' \rangle_A,$$

for all $\omega, \omega' \in H^1_{\mathrm{dR}}(A/F)$. By the definition of the p-adic height pairings $\langle , \rangle_{W_r}$ and $\langle , \rangle_{X_r}$ (factoring through $\Phi_F^{\text{ét}}$), we thus see that

$$\langle \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}, \Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}\rangle_{X_r} = D^{r-1} \cdot \langle \Pi_r(\Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r}), \Pi_r(\Delta^{\mathrm{bdp}}_{\mathrm{id}_A,r})\rangle_{W_r}. \tag{1.2.5}$$

Combining (1.2.4) and (1.2.5), the result follows. □

1.3. The big logarithm map

Let

$$\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

be a Hida family passing through (the ordinary p-stabilization of) a p-ordinary newform $f_o \in S_k(X_0(N))$ as described in the Introduction. We begin this section by recalling the definition of a certain twist of $\mathbf{f}$ such that all of its specializations at arithmetic primes of *even weight* correspond to p-adic modular forms with trivial weight-nebentypus.

Decompose the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}}$ as the product

$$\varepsilon_{\mathrm{cyc}} = \omega \cdot \epsilon : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^\times = \boldsymbol{\mu}_{p-1} \times \Gamma.$$

Since k is even, the character ω^{k-2} admits a square root $\omega^{\frac{k-2}{2}} : G_{\mathbf{Q}} \longrightarrow \boldsymbol{\mu}_{p-1}$, and in fact two different square roots, corresponding to the two different lifts of $k-2 \in \mathbf{Z}/(p-1)\mathbf{Z}$ to $\mathbf{Z}/2(p-1)\mathbf{Z}$. Fix for now a choice of $\omega^{\frac{k-2}{2}}$, and define the *critical character* to be

$$\Theta := \omega^{\frac{k-2}{2}} \cdot [\epsilon^{1/2}] : G_{\mathbf{Q}} \longrightarrow \Lambda_{\mathcal{O}}^\times, \tag{1.3.1}$$

where $\epsilon^{1/2} : G_{\mathbf{Q}} \longrightarrow \Gamma$ denotes the unique square root of ϵ taking values in Γ.

Remark 1.3.1. As noted in [**How07b**, Rem. 2.1.4], the above choice of Θ is for most purposes largely indistinguishable from the other choice, namely $\omega^{\frac{p-1}{2}}\Theta$, where

$$\omega^{\frac{p-1}{2}} : \mathrm{Gal}(\mathbf{Q}(\sqrt{p^*})/\mathbf{Q}) \xrightarrow{\sim} \{\pm 1\} \qquad (p^* = (-1)^{\frac{p-1}{2}} p).$$

Nonetheless, for a given f_o as above, our main result (Theorem 1.4.12) will specifically apply *to only one* of the two possible choices for the critical character.

The *critical twist* of $\mathbb{T}$ is then defined to be the module

$$\mathbb{T}^\dagger := \mathbb{T} \otimes_{\mathbb{I}} \mathbb{I}^\dagger \tag{1.3.2}$$

equipped with the diagonal $G_{\mathbf{Q}}$-action, where $\mathbb{I}^\dagger = \mathbb{I}(\Theta^{-1})$ is the free $\mathbb{I}$-module of rank one equipped with the $G_{\mathbf{Q}}$-action via the character $G_{\mathbf{Q}} \xrightarrow{\Theta^{-1}} \Lambda_{\mathcal{O}}^\times \longrightarrow \mathbb{I}^\times$.

LEMMA 1.3.2. *Let $\rho_{\mathbb{T}^\dagger} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbb{T}^\dagger)$ be the Galois representation carried by $\mathbb{T}^\dagger$. Then for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$ we have*

$$\nu(\rho_{\mathbb{T}^\dagger}) \cong \rho_{\mathbf{f}'_\nu} \otimes \varepsilon_{\mathrm{cyc}}^{r_\nu},$$

where $\mathbf{f}'_\nu$ is a character twist of $\mathbf{f}_\nu$ of weight k_ν and with trivial nebentypus. In other words, defining $\mathbb{V}^\dagger_\nu := \mathbb{T}^\dagger \otimes_{\mathbb{I}} F_\nu$ and letting $V_{\mathbf{f}'_\nu}$ be the representation space of $\rho_{\mathbf{f}'_\nu}$, we have

$$\mathbb{V}^\dagger_\nu \cong V_{\mathbf{f}'_\nu}(r_\nu), \tag{1.3.3}$$

and in particular $\mathbb{V}^\dagger_\nu$ is isomorphic to its Kummer dual.

PROOF. This follows from a straightforward computation explained in [**NP00**, (3.5.2)] for example (where $\mathbb{T}^\dagger$ is denoted by T). □

Let $\theta : \mathbf{Z}_p^\times \longrightarrow \Lambda_{\mathcal{O}}^\times$ be such that $\Theta = \theta \circ \varepsilon_{\mathrm{cyc}}$. It follows from the preceding lemma that the formal q-expansion

$$\mathbf{f}^\dagger = \mathbf{f} \otimes \theta^{-1} := \sum_{n>0} \theta^{-1}(n)\mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

(where we put $\theta^{-1}(n) = 0$ whenever p divides n) is such that, for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight, $\mathbb{V}^\dagger_\nu$ is the Galois representation attached to the specialization $\mathbf{f}_\nu \otimes \theta_\nu^{-1}$ of $\mathbf{f}^\dagger$, which by Lemma 1.1.3 is a p-adic modular form of weight 0 and trivial nebentypus.

We next recall some of the local properties of the big Galois representation $\mathbb{T}$. Let $I_w \subset D_w \subset G_{\mathbf{Q}}$ be the inertia and decompositon groups at the place w of $\overline{\mathbf{Q}}$ above p induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. In the following we will identify D_w with the absolute Galois group $G_{\mathbf{Q}_p}$. Then by a result of Mazur and Wiles (see [**Wil88**, Thm. 2.2.2]) there exists a filtration of $\mathbb{I}[D_w]$-modules

$$0 \longrightarrow \mathscr{F}_w^+\mathbb{T} \longrightarrow \mathbb{T} \longrightarrow \mathscr{F}_w^-\mathbb{T} \longrightarrow 0 \tag{1.3.4}$$

with $\mathscr{F}_w^\pm\mathbb{T}$ free of rank one over $\mathbb{I}$ and with the Galois action on $\mathscr{F}_w^-\mathbb{T}$ unramified, given by the character $\alpha : D_w/I_w \longrightarrow \mathbb{I}^\times$ sending an arithmetic Frobenius σ_p to $\mathbf{a}_p$. Twisting (2.2.1) by Θ^{-1} we define $\mathscr{F}_w^\pm\mathbb{T}^\dagger$ in the natural manner.

Let $\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathbb{T}, \mathbb{I})$ be the contragredient[5] of $\mathbb{T}$, and consider the $\mathbb{I}$-module

$$\mathbb{D} := (\mathscr{F}_w^+\mathbb{T}^* \widehat{\otimes}_{\mathbf{Z}_p} \widehat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}, \tag{1.3.5}$$

where $\mathscr{F}_w^+\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathscr{F}_w^-\mathbb{T}, \mathbb{I}) \subset \mathbb{T}^*$, and $\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$ is the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$.

[5]So that $\mathbb{T}^* \otimes_{\mathbb{I}} F_\nu \cong V_{\mathbf{f}_\nu}$ for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

Fix a compatible system $\zeta = (\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity $\zeta_s \in \overline{\mathbf{Q}}_p$, and let e_ζ be the basis of $D_{\mathrm{dR}}(\mathbf{Q}_p(1))$ corresponding to $1 \in \mathbf{Q}_p$ under the resulting identification $D_{\mathrm{dR}}(\mathbf{Q}_p(1)) = \mathbf{Q}_p$.

Lemma 1.3.3. *The module $\mathbb{D}$ is free of rank one over $\mathbb{I}$, and for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$ there is a canonical isomorphism*

$$\mathbb{D}_\nu \otimes D_{\mathrm{dR}}(\mathbf{Q}_p(r_\nu)) \xrightarrow{\sim} \frac{D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}. \tag{1.3.6}$$

Proof. Since the action on $\mathscr{F}_w^+\mathbb{T}^*$ is unramified, the first claim follows from [**Och03**, Lemma 3.3] in light of the definition (2.2.3) of $\mathbb{D}$. The second can be deduced from [**Och03**, Lemma 3.2] as in the proof of [**Och03**, Lemma 3.6]. □

With the same notations as in Lemma 1.3.3, we denote by $\langle\ ,\ \rangle_{\mathrm{dR}}$ the pairing

$$\langle\ ,\ \rangle_{\mathrm{dR}} : \mathbb{D}_\nu \otimes D_{\mathrm{dR}}(\mathbf{Q}_p(r_\nu)) \times \mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^*}(r_\nu - 1)) \longrightarrow F_\nu \tag{1.3.7}$$

deduced from the usual de Rham pairing

$$\frac{D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}(r_\nu))} \times \mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu}^*(1-r_\nu)) \longrightarrow F_\nu$$

via the identification (1.3.6) and the isomorphism $V_{\mathbf{f}_\nu}^* \cong V_{\mathbf{f}_\nu^*}(k_\nu - 1)$.

Theorem 1.3.4 (Ochiai). *Assume that the residual representation $\bar{\rho}_{f_o}$ is irreducible, fix an $\mathbb{I}$-basis η of $\mathbb{D}$, and set $\lambda := \mathbf{a}_p - 1$. There exists an $\mathbb{I}$-linear map*

$$\mathrm{Log}^\eta_{\mathbb{T}^\dagger} : H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \longrightarrow \mathbb{I}[\lambda^{-1}]$$

such that for every $\mathfrak{Y} \in H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger)$ and every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$ with $k_\nu \equiv k \pmod{2(p-1)}$, we have

$$\nu(\mathrm{Log}^\eta_{\mathbb{T}^\dagger}(\mathfrak{Y})) = \frac{(-1)^{r_\nu - 1}}{(r_\nu - 1)!} \times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^{-1} \left(1 - \frac{\nu(\mathbf{a}_p)}{p^{r_\nu}}\right) \langle \log_{V^\dagger_{\mathbf{f}_\nu}}(\mathfrak{Y}_\nu), \eta'_\nu\rangle_{\mathrm{dR}} & \textit{if } \vartheta_\nu = \mathbb{1}; \\ G(\vartheta_\nu^{-1})^{-1} \left(\frac{\nu(\mathbf{a}_p)}{p^{r_\nu - 1}}\right)^{s_\nu} \langle \log_{V^\dagger_{\mathbf{f}_\nu}}(\mathfrak{Y}_\nu), \eta'_\nu\rangle_{\mathrm{dR}} & \textit{if } \vartheta_\nu \neq \mathbb{1}, \end{cases} \tag{1.3.8}$$

where

- $\log_{V^\dagger_{\mathbf{f}_\nu}}$ *is the Bloch–Kato logarithm map for the representation $V^\dagger_{\mathbf{f}_\nu}$ over $\mathbf{Q}_p$,*
- $\eta'_\nu \in \mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^*}(r_\nu - 1))$ *is such that $\langle \eta_\nu \otimes e_\zeta^{\otimes r_\nu}, \eta'_\nu\rangle_{\mathrm{dR}} = 1$ under (1.3.7),*
- $\vartheta_\nu : \mathbf{Z}_p^\times \longrightarrow F_\nu^\times$ *is the finite order character $z \longmapsto \theta_\nu(z) z^{1-r_\nu}$,*
- $s_\nu > 0$ *is such that the conductor of ϑ_ν is p^{s_ν}, and*

- $G(\vartheta_\nu^{-1})$ *is the Gauss sum* $\sum_{x \bmod p^{s_\nu}} \vartheta_\nu^{-1}(x)\zeta_{s_\nu}^x$.

Proof. Let $\Lambda(C_\infty) := \mathbf{Z}_p[[C_\infty]]$, where C_∞ is the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and let Λ_{cyc} be the module $\Lambda(C_\infty)$ equipped with the natural action of $G_{\mathbf{Q}_p}$ on group-like elements. Also, let γ_o be a topological generator of C_∞ and define

$$\mathcal{I} := (\lambda, \gamma_o - 1),$$

seen as an ideal of height 2 inside $\mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty) \cong \mathbb{I}[[C_\infty]]$.

Consider the $\mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty)$-modules

$$\mathcal{D} := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty), \qquad \mathscr{F}_w^+\mathcal{T}^* := \mathscr{F}_w^+\mathbb{T}^*\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}} \otimes \omega^{\frac{k-2}{2}},$$

the latter being equipped with the diagonal action of $G_{\mathbf{Q}_p}$. By [**Och03**, Prop. 5.3] there exists an *injective* $\mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(C_\infty)$-linear map

$$\mathrm{Exp}_{\mathscr{F}_w^+\mathcal{T}^*} : \mathcal{I}\mathcal{D} \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^*),$$

with cokernel killed by λ, which interpolates the Bloch–Kato exponential map over the arithmetic primes of $\mathbb{I}$ and of $\Lambda(C_\infty)$.

As in (1.3.1), let $\epsilon^{1/2} : C_\infty \longrightarrow \Gamma \subset \mathbf{Z}_p^\times$ be the unique square-root of the wild component of the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}}$, and let

$$\mathrm{Tw}^\dagger : \mathbb{I}[[C_\infty]] \longrightarrow \mathbb{I}[[C_\infty]]$$

be the $\mathbb{I}$-algebra isomorphism given by $\mathrm{Tw}^\dagger([\gamma]) = \epsilon_{\mathrm{w}}^{1/2}(\gamma)[\gamma]$ for all $\gamma \in C_\infty$. Then letting $\mathscr{F}_w^+\mathcal{T}^\dagger$ be the $\mathbb{I}[[C_\infty]]$-module $\mathscr{F}_w^+\mathcal{T}^*$ with the C_∞-action twisted by $\epsilon^{1/2}$, there is a natural projection $\mathrm{Cor} : \mathscr{F}_w^+\mathcal{T}^\dagger \longrightarrow \mathscr{F}_w^+\mathbb{T}^\dagger$ induced by the augmentation map $\mathbb{I}[[C_\infty]] \longrightarrow \mathbb{I}$.

Setting $\mathbb{D}^\dagger := \mathcal{I}\mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathbb{I}[[C_\infty]]/(\epsilon^{1/2}(\gamma_o)[\gamma_o] - 1)$, the composition

$$\mathcal{I}\mathcal{D} \xrightarrow{(\mathrm{Tw}^\dagger)^{-1}} \mathcal{I}\mathcal{D} \xrightarrow{\mathrm{Exp}_{\mathscr{F}_w^+\mathcal{T}^*}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^*) \xrightarrow{\otimes\epsilon^{1/2}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathcal{T}^\dagger) \xrightarrow{\mathrm{Cor}} H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger),$$

factors through an injective $\mathbb{I}$-linear map

$$\mathrm{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger} : \mathbb{D}^\dagger \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \tag{1.3.9}$$

making, for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ as in the statement, the diagram

$$\begin{tikzcd} \mathbb{D}^\dagger \arrow[r, "\mathrm{Exp}_{\mathscr{F}_w^+\mathbb{T}^\dagger}"] \arrow[d, "\mathrm{Sp}_\nu"] & H^1(\mathbf{Q}_p, \mathscr{F}_w^+\mathbb{T}^\dagger) \arrow[d, "\mathrm{Sp}_\nu"] \\ D_{\mathrm{dR}}(\mathscr{F}_w^+V_{\mathbf{f}_\nu}^\dagger) \arrow[r] & H^1(\mathbf{Q}_p, \mathscr{F}_w^+V_{\mathbf{f}_\nu}^\dagger) \end{tikzcd}$$

commutative, where the bottom horizontal arrow is given by

$$(-1)^{r_\nu-1}(r_\nu-1)!\times\begin{cases}\left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)\left(1-\frac{\nu(\mathbf{a}_p)}{p^{r_\nu}}\right)^{-1}\exp_{V^\dagger_{\mathbf{f}_\nu}} & \text{if } \vartheta_\nu=\mathbb{1};\\ \left(\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right)^{s_\nu}\exp_{V^\dagger_{\mathbf{f}_\nu}} & \text{if } \vartheta_\nu\neq\mathbb{1},\end{cases}$$

with $\exp_{V^\dagger_{\mathbf{f}_\nu}}$ the Bloch–Kato exponential map for $V^\dagger_{\mathbf{f}_\nu}$, which factors as

$$\frac{D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu})}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\mathbf{f}_\nu})}\xleftarrow{\sim} D_{\mathrm{dR}}(\mathscr{F}^+_w V^\dagger_{\mathbf{f}_\nu})\xrightarrow{\exp_{V^\dagger_{\mathbf{f}_\nu}}} H^1(\mathbf{Q}_p,\mathscr{F}^+_w V^\dagger_{\mathbf{f}_\nu})\longrightarrow H^1(\mathbf{Q}_p,V^\dagger_{\mathbf{f}_\nu}).$$

Now if $\mathfrak{Y}$ is an arbitrary class in $H^1(\mathbf{Q}_p,\mathscr{F}^+_w\mathbb{T}^\dagger)$, then $\lambda\cdot\mathfrak{Y}$ lands in the image of the map $\mathrm{Exp}_{\mathscr{F}^+_w\mathbb{T}^\dagger}$ and so

$$\mathrm{Log}_{\mathbb{T}^\dagger}(\mathfrak{Y}):=\lambda^{-1}\cdot\mathrm{Exp}^{-1}_{\mathscr{F}^+_w\mathbb{T}^\dagger}(\lambda\cdot\mathfrak{Y})$$

is a well-defined element in $\mathbb{I}[\lambda^{-1}]\otimes_{\mathbb{I}}\mathbb{D}^\dagger$. Thus defining $\mathrm{Log}^\eta_{\mathbb{T}^\dagger}(\mathfrak{Y})\in\mathbb{I}[\lambda^{-1}]$ by the relation

$$\mathrm{Log}_{\mathbb{T}^\dagger}(\mathfrak{Y})=\mathrm{Log}^\eta_{\mathbb{T}^\dagger}(\mathfrak{Y})\cdot\eta\otimes 1,$$

the result follows. □

1.4. The big Heegner point

In this section we prove the main results of this paper, relating the étale Abel–Jacobi images of Heegner cycles to the specializations at higher even weights of the big Heegner point $\mathfrak{Z}$ (whose definition is recalled below). Their proof is based on two key ingredients: the properties of the big logarithm map deduced from the work of Ochiai as explained in the preceding section, and the local study of (almost all) the weight 2 specializations of $\mathfrak{Z}$ taken up in the following.

1.4.1. Weight two specializations. As in Section 1.2, let K be a fixed imaginary quadratic field in which all prime factors of N split, equipped with an ideal $\mathfrak{N}\subset\mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N}\cong\mathbf{Z}/N\mathbf{Z}$. We also assume that p splits in K, and let $\mathfrak{p}$ be the prime of K above p induced by our fixed embedding $\imath_p$, and by $\bar{\mathfrak{p}}$ the other. Finally, A is a fixed elliptic curve with CM by $\mathcal{O}_K$ defined over the Hilbert class field H of K, and recall that in Section 1.3 we fixed a compatible system $\zeta=(\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity $\zeta_s\in\overline{\mathbf{Q}}_p$.

Let $R_0=\widehat{\mathbf{Z}}^{\mathrm{nr}}_p$ be the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$, which we view as an overfield of H via $\imath_p$. Since p splits in K, A admits a trivialization

$$\imath_A:\hat{A}\longrightarrow\hat{\mathbf{G}}_m$$

over R_0 with $\imath_A^{-1}(\boldsymbol{\mu}_{p^s}) = A[\mathfrak{p}^s]$ for every $s > 0$. Letting α_A be the cyclic N-isogeny on A with kernel $A[\mathfrak{N}]$, the triple $(A, \alpha_A, \imath_A)$ thus defines a trivialized elliptic curve with $\Gamma_0(N)$-level structure defined over R_0.

Set $A_0 := A/A[\mathfrak{p}^s]$ and let $(A_0, \alpha_{A_0}, \imath_{A_0})$ be the trivialized elliptic curve deduced from $(A, \alpha_A, \imath_A)$ via the projection $A \longrightarrow A_0$. Let $C \subset A_0[p^s]$ be any étale subgroup of order p^s, and set $A_s := A_0/C$. Finally, let $(A_s, \alpha_{A_s}, \imath_{A_s})$ be the trivialized elliptic curve with $\Gamma_0(N)$-level structure deduced from $(A_0, \alpha_{A_0}, \imath_{A_0})$ via the projection $A_0 \longrightarrow A_s$, and consider

$$h_s = (A_s, \alpha_{A_s}, \imath_{A_s}(\zeta_s)), \tag{1.4.1}$$

which defines an algebraic point on the modular curve X_s.

Write $p^* = (-1)^{\frac{p-1}{2}}p$, and let ϑ be the unique continuous character

$$\vartheta : G_{\mathbf{Q}(\sqrt{p^*})} \longrightarrow \mathbf{Z}_p^\times/\{\pm 1\} \tag{1.4.2}$$

such that $\vartheta^2 = \varepsilon_{\mathrm{cyc}}$. Notice the inclusion $G_{H_{p^s}} \subset G_{\mathbf{Q}(\sqrt{p^*})}$ for any $s > 0$, where H_{p^s} denotes the ring class field of K of conductor p^s.

LEMMA 1.4.1. *The curve A_s has CM by the order $\mathcal{O}_{p^s}$ of K of conductor p^s, and the point h_s is rational over $L_{p^s} := H_{p^s}(\boldsymbol{\mu}_{p^s})$. In fact we have*

$$h_s^\sigma = \langle \vartheta(\sigma) \rangle \cdot h_s \tag{1.4.3}$$

for all $\sigma \in \mathrm{Gal}(L_{p^s}/H_{p^s})$.

PROOF. The first assertion is clear, and immediately from the construction we also see that α_{A_s} is the cyclic N-isogeny on A_s with kernel $A_s[\mathfrak{N} \cap \mathcal{O}_{p^s}]$. It follows that the point (1.4.1) gives rise to precisely the point $h_s \in X_s(\mathbf{C})$ in [**How07b**, Eq. (4)]. The result thus follows from [*loc.cit.*, Cor. 2.2.2]. $\square$

If ν is an arithmetic prime of $\mathbb{I}$, we let ψ_ν denote its *wild character*, defined as the composition of $\nu : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p$ with the structure map $\Gamma = 1 + p\mathbf{Z}_p \longrightarrow \mathbb{I}^\times$, which we view as a Dirichlet character of p-power conductor in the obvious manner. The nebentypus of $\mathbf{f}_\nu$ is then given by

$$\varepsilon_{\mathbf{f}_\nu} = \psi_\nu \omega^{k - k_\nu},$$

where $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \longrightarrow \boldsymbol{\mu}_{p-1} \subset \mathbf{Z}_p^\times$ is the Teichmüller character.

Recall the critical characters Θ and θ from Section 1.3, and for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2, consider the $F_\nu^\times$-valued Hecke character of K given by

$$\chi_\nu(x) = \Theta_\nu(\mathrm{art}_{\mathbf{Q}}(\mathrm{N}_{K/\mathbf{Q}}(x))) \tag{1.4.4}$$

for all $x \in \mathbb{A}_K^\times$. Notice that since χ_ν has finite order, it may alternately be seen as character on G_K via the Artin reciprocity map $\mathrm{art}_K : \mathbb{A}_K^\times \longrightarrow G_K^{\mathrm{ab}}$.

For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, after fixing an embedding $F_\nu \longrightarrow \overline{\mathbf{Q}}_p$, the form $\mathbf{f}_\nu \in S_{k_\nu}(X_{s_\nu})$ defines a p-adic modular form $\mathbf{f}_\nu \in \mathbf{M}(N)$. Finally, recall the dual form $\mathbf{f}_\nu^*$ defined as in the paragraph before (1.1.20).

Lemma 1.4.2. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have weight 2 and non-trivial wild character, and let $s > 1$ be the p-power of the conductor of ψ_ν. Then*

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = \frac{u}{G(\theta_\nu^{-1})} \sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}), \tag{1.4.5}$$

where $u = |\mathcal{O}_K^\times|/2$, $G(\theta_\nu^{-1}) = \sum_{x \bmod p^s} \theta_\nu^{-1}(x)\zeta_s^x$ is a usual Gauss sum, and for every $\sigma \in \mathrm{Gal}(H_{p^s}/H)$, $\tilde{\sigma}$ is any lift of σ to $\mathrm{Gal}(L_{p^s}/H)$.

Remark 1.4.3. Since ν has weight $k_\nu = 2$, we have $\theta_\nu = \vartheta_\nu$, where ϑ_ν is the finite order character in the statement of Theorem 1.3.4.

Proof. We begin by noting that the expression in the right hand side of (1.4.5) does not depend on the choice of lifts $\tilde{\sigma}$. Indeed, as explained in [**How07a**, p. 808] the character $\chi_{0,\nu} := \chi_\nu|_{\mathbb{A}_{\mathbf{Q}}^\times}$, seen as a Dirichlet character in the usual manner, is such that $\chi_{0,\nu}^{-1} = \theta_\nu^2$. But since the weight of ν is 2, we have

$$\theta_\nu^2 = \varepsilon_{\mathbf{f}_\nu} = \varepsilon_{\mathbf{f}_\nu^*}^{-1}$$

(see [**How07a**, p. 806]), and our claim thus follows immediately from (1.4.3).

To compute the above value of the twist $d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu$ we follow Definition 1.1.2. The integer $s > 1$ in the statement is such that θ_ν factors through $(\mathbf{Z}/p^s\mathbf{Z})^\times$, therefore

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) &= \sum_{a \bmod p^s} \theta_\nu(a) \left(\int_{a + p^s\mathbf{Z}_p} d\mu_{\mathrm{Gou}}(x) \right) (d^{-1}\mathbf{f}_\nu^{*[p]})(A, \alpha_A, \imath_A) \\ &= \frac{1}{p^s} \sum_{a \bmod p^s} \theta_\nu(a) \sum_C \zeta_C^{-a} \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_0/C, \alpha_C, \imath_C), \end{aligned} \tag{1.4.6}$$

where as before $A_0 := A/\imath_A^{-1}(\boldsymbol{\mu}_{p^s}) = A/A[\mathfrak{p}^s]$ and the sum is over the étale subgroups $C \subset A_0[p^s]$ of order p^s. Letting γ_s be a generator of $\mathbf{Z}/p^s\mathbf{Z}$, these subgroups correspond bijectively with the cyclic subgroups $C_u = \langle \zeta_s^u . \gamma_s \rangle \subset \boldsymbol{\mu}_{p^s} \times \mathbf{Z}/p^s\mathbf{Z}$, with u running over the integers modulo p^s, and we set $\zeta_{C_u} = \zeta_s^u$.

Since θ_ν does not factor through $(\mathbf{Z}/p^{s-1}\mathbf{Z})^\times$, we have $\sum_{a \bmod p^s} \theta_\nu(a)\zeta_s^{-ua} = 0$ whenever $u \notin (\mathbf{Z}/p^s\mathbf{Z})^\times$. Continuing from (1.4.6), we thus obtain

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) &= \frac{1}{p^s} \sum_{a \bmod p^s} \theta_\nu(a) \sum_{u \bmod p^s} \zeta_s^{-ua} \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u}, \alpha_{C_u}, \imath_{C_u}) \\ &= \frac{1}{p^s} \sum_{u \in (\mathbf{Z}/p^s\mathbf{Z})^\times} d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u}, \alpha_{C_u}, \imath_{C_u}) \sum_{a \bmod p^s} \theta_\nu(a)\zeta_s^{-ua} \\ &= \frac{1}{G(\theta_\nu^{-1})} \sum_{u \in (\mathbf{Z}/p^s\mathbf{Z})^\times} \theta_\nu^{-1}(u) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u}, \alpha_{C_u}, \imath_{C_u}), \end{aligned}$$

with the last equality obtained by a change of variables. The result thus follows from the relation

$$\sum_{u \in (\mathbf{Z}/p^s\mathbf{Z})^\times} \theta_\nu^{-1}(u) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(A_{C_u}, \alpha_{C_u}, \imath_{C_u}) = u \sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}),$$

where $u = |\mathcal{O}_K^\times|/2$, and for each $\sigma \in \mathrm{Gal}(H_{p^s}/H)$, $\tilde{\sigma} \in \mathrm{Gal}(L_{p^s}/H)$ lifts σ. $\square$

Keeping the above notations, let $\Delta_s \in J_s(L_{p^s})$ be the divisor class of $(h_s) - (\infty)$, and consider the element in $J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu$ given by

$$\widetilde{Q}_{\chi_\nu} := \sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \Delta_s^{\tilde{\sigma}} \otimes \chi_\nu^{-1}(\tilde{\sigma}), \tag{1.4.7}$$

where for every $\sigma \in \mathrm{Gal}(H_{p^s}/H)$, $\tilde{\sigma}$ is any lift to $\mathrm{Gal}(L_{p^s}/H)$.

Let F_s be the completion of $\imath_p(L_{p^s})$, and consider the p-adic Abel–Jacobi map $\delta^{(p)}_{\mathbf{f}_\nu, F_s}$ defined in (1.1.9) which we extend by F_ν-linearity to a map

$$\delta^{(p)}_{\mathbf{f}_\nu, F_s} : J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu \longrightarrow (\mathrm{Fil}^1 D_{\mathrm{dR}}(V_{\mathbf{f}_\nu^*}))^\vee.$$

Proposition 1.4.4. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ and $s > 1$ be as in Lemma 1.4.2. Then*

$$\sum_{\sigma \in \mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde{\sigma}) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde{\sigma}}) = \delta^{(p)}_{\mathbf{f}_\nu, F_s}(\widetilde{Q}_{\chi_\nu})(\omega_{\mathbf{f}_\nu^*}). \tag{1.4.8}$$

Proof. The integer $s > 1$ in the statement is so that the nebentypus $\varepsilon_{\mathbf{f}_\nu}$ of $\mathbf{f}_\nu$ is primitive modulo p^s. Moreover, since p splits in K, we see from the construction that the point h_s lies in the connected component $X_s(0)$ of the ordinary locus of X_s containing the cusp ∞. Thus Proposition 1.1.9 applies, giving

$$\delta^{(p)}_{\mathbf{f}_\nu, F_s}(\Delta_s)(\omega_{\mathbf{f}_\nu^*}) = F_{\omega_{\mathbf{f}_\nu^*}}(h_s),$$

where $F_{\omega_{\mathbf{f}_\nu^*}}$ is the Coleman primitive of $\omega_{\mathbf{f}_\nu^*}$ from Proposition 1.1.6, and by linearity

$$\sum_{\sigma\in\mathrm{Gal}(H_{p^s}/H)} \chi_\nu^{-1}(\tilde\sigma)\cdot F_{\omega_{\mathbf{f}_\nu}}(h_s^{\tilde\sigma}) = \delta^{(p)}_{\mathbf{f}_\nu,F_s}(\widetilde{Q}_{\chi_\nu})(\omega_{\mathbf{f}_\nu^*}). \tag{1.4.9}$$

Since ϕ lifts the Deligne–Tate map to X_s, we see that ϕh_s is defined over the subfield $H_{p^{s-1}}(\zeta_s)\subset L_{p^s}$. If b_p denotes the U_p-eigenvalue of $\mathbf{f}_\nu^*$, by Corollary 1.1.8 we obtain

$$\begin{aligned}\sum_\sigma \chi_\nu^{-1}(\tilde\sigma)\cdot d^{-1}\mathbf{f}_\nu^{*[p]}(h_s^{\tilde\sigma}) &= \sum_\sigma \chi_\nu^{-1}(\tilde\sigma)\cdot F_{\omega_{\mathbf{f}_\nu^*}}(h_s^{\tilde\sigma}) - \frac{b_p}{p}\sum_\sigma \chi_\nu^{-1}(\tilde\sigma)\cdot F_{\omega_{\mathbf{f}_\nu^*}}(\phi h_s^{\tilde\sigma})\\ &= \sum_\sigma \chi_\nu^{-1}(\tilde\sigma)\cdot F_{\omega_{\mathbf{f}_\nu^*}}(h_s^{\tilde\sigma}),\end{aligned}$$

where all the sums are over $\sigma\in\mathrm{Gal}(H_{p^s}/H)$, and the second equality follows immediately from the fact θ_ν is primitive modulo p^s. The result thus follows from (1.4.9). $\square$

Still with the same notations, recall Hida's ordinary projector (1.0.2) and set $y_s := e^{\mathrm{ord}}h_s$, which naturally lies in $e^{\mathrm{ord}}J_s(L_{p^s})$ (see [**How07b**, p.100]). Equation (1.4.3) then amounts to the fact that

$$y_s^\sigma = \Theta(\sigma)\cdot y_s \tag{1.4.10}$$

for all $\sigma\in\mathrm{Gal}(L_{p^s}/H_{p^s})$, where Θ is the critical character (1.3.1). Denoting by $J_s^{\mathrm{ord}}(L_{p^s})^\dagger$ the module $e^{\mathrm{ord}}J_s(L_{p^s})$ with the Galois action twisted by Θ^{-1}, and by $y_s^\dagger$ the point y_s seen in this new module, (1.4.10) translates into the statement that

$$y_s^\dagger \in H^0(H_{p^s}, J_s^{\mathrm{ord}}(L_{p^s})^\dagger).$$

Lemma 1.4.5 (Howard). *The classes*

$$x_s := \mathrm{Cor}_{H_{p^s}/H}(y_s^\dagger)\in H^0(H, J_s^{\mathrm{ord}}(L_{p^s})^\dagger) \tag{1.4.11}$$

are such that

$$\alpha_* x_{s+1} = U_p\cdot x_s, \qquad \textit{for all } s>0$$

under the Albanese maps induced from the degeneracy maps $\alpha : X_{s+1}\longrightarrow X_s$.

Proof. This is shown in the course of the proof of [**How07b**, Lemma 2.2.4]. $\square$

Abbreviate by $\mathrm{Ta}_p^{\mathrm{ord}}(J_s)$ the module $e^{\mathrm{ord}}(\mathrm{Ta}_p(J_s)\otimes_{\mathbf{Z}_p}\mathcal{O})$ from the Introduction, and denote by $\mathrm{Ta}_p^{\mathrm{ord}}(J_s)^\dagger$ this same module with the Galois action twisted by Θ^{-1}. By the Galois and Hecke-equivariance of the twisted Kummer map

$$\mathrm{Kum}_s : H^0(H, J_s^{\mathrm{ord}}(L_{p^s})^\dagger)\longrightarrow H^1(H, \mathrm{Ta}_p^{\mathrm{ord}}(J_s)^\dagger)$$

constructed in [**How07b**, p. 101], Lemma 1.4.5 implies that the cohomology classes $\mathfrak{X}_s := \mathrm{Kum}_s(x_s)$ are such that $\alpha_* \mathfrak{X}_{s+1} = U_p \cdot \mathfrak{X}_s$, for all $s > 0$.

DEFINITION 1.4.6 (Howard). The *big Heegner point* of conductor one is the cohomology class $\mathfrak{X}$ given by the image of

$$\varprojlim_s U_p^{-s} \cdot \mathfrak{X}_s$$

under the natural map induced by the $\mathfrak{h}^{\mathrm{ord}}[G_{\mathbf{Q}}]$-linear projection $\varprojlim_s \mathrm{Ta}_p^{\mathrm{ord}}(J_s)^\dagger \longrightarrow \mathbb{T}^\dagger$.

Our object of study is in fact the big cohomology class

$$\mathfrak{Z} := \mathrm{Cor}_{H/K}(\mathfrak{X}), \tag{1.4.12}$$

which is predicted to be non-trivial by [**How07b**, Conj. 3.4.1].

CONJECTURE 1.4.7 (Howard). *The class $\mathfrak{Z}$ is not $\mathbb{I}$-torsion.*

Recall from [**How07b**, §2.4] that the *strict Greenberg Selmer group* $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$ is defined to be the subspace of $H^1(K, \mathbb{T}^\dagger)$ consisting of classes c which are unramified outside the places above p and such that $\mathrm{loc}_v(c)$ lies in the kernel of the natural map

$$H^1(K_v, \mathbb{T}^\dagger) \longrightarrow H^1(K_v, \mathscr{F}_w^- \mathbb{T}^\dagger)$$

for all $v|p$.

For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2, let $L(s, \mathbf{f}_\nu, \chi_\nu)$ be the Rankin–Selberg convolution L-function of [**How09**, §1]. In the spirit of the classical Gross–Zagier theorem, one has the following criterion for the non-triviality of (the specializations of) $\mathfrak{Z}$.

THEOREM 1.4.8 (Howard). *The class $\mathfrak{Z}$ belongs to $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$, and if there is some $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that $L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0$, then Conjecture 1.4.7 holds.*

PROOF. The first assertion is shown in [**How07b**, Prop. 2.4.5]. The second is shown in [**How07a**, Prop. 3] and follows from the equivalence

$$\mathfrak{Z}_{\nu'} \neq 0 \quad \Longleftrightarrow \quad L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0 \tag{1.4.13}$$

combined with the freeness of $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger) \otimes_{\mathbb{I}} \mathcal{O}_\nu$ ([**Nek06**, Prop. 12.7.13.4(iii)]) for any $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$. □

The following result shows that Proposition 1.4.4 may be reformulated as giving an explicit formula, for all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2, for the image of the classes $\mathfrak{Z}_\nu$ under the inverse of the Bloch–Kato exponential map.

For any class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, taking a representative $\mathfrak{a} \subset \mathcal{O}_K$ prime to Np, define

$$\mathfrak{a} * (A, \alpha_A, \imath_A) := (A_\mathfrak{a}, \alpha_{A_\mathfrak{a}}, \imath_{A_\mathfrak{a}}),$$

where $A_\mathfrak{a} = A/A[\mathfrak{N}]$, $\alpha_{A_\mathfrak{a}} = A_\mathfrak{a}[\mathfrak{N}]$, and $\imath_{A_\mathfrak{a}}$ is the trivialization $\hat{A}_\mathfrak{a} \xrightarrow{\hat{\varphi}_\mathfrak{a}^{-1}} \hat{A} \xrightarrow{\imath_A} \hat{\mathbf{G}}_m$ induced by the projection $\varphi_\mathfrak{a} : A \longrightarrow A_\mathfrak{a}$.

THEOREM 1.4.9. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ have weight 2 and non-trivial wild character ψ_ν, and let $s > 1$ be the p-power of the conductor of ψ_ν. Then*

$$\sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}(\mathfrak{a} * (A, \alpha_A, \imath_A)) = u\frac{\nu(\mathbf{a}_p)^s}{G(\theta_\nu^{-1})}\log_{s,V_{\mathbf{f}_\nu}(1)}(\mathrm{loc}_\mathfrak{p}(\mathfrak{Z}_\nu))(\omega_{\mathbf{f}_\nu^*}), \tag{1.4.14}$$

where $u = |\mathcal{O}_K^\times|/2$, and $G(\theta_\nu^{-1})$ is the Gauss sum $\sum_{x \bmod p^s} \theta_\nu^{-1}(x)\zeta_s^x$.

PROOF. Since clearly

$$d^{-1}\mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1} = d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu,$$

letting F_s be the completion of $\imath_p(L_{p^s})$ it suffices to establish the equality

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = u\frac{\nu(\mathbf{a}_p)^s}{G(\theta_\nu^{-1})}\log_{F_s,V_{\mathbf{f}_\nu}(1)}(\mathrm{loc}_\mathfrak{p}(\mathfrak{X}_\nu))(\omega_{\mathbf{f}_\nu^*}). \tag{1.4.15}$$

Combining the formulas from Lemma 1.4.2 and Proposition 1.4.4, we have

$$d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \theta_\nu(A, \alpha_A, \imath_A) = \frac{u}{G(\theta_\nu^{-1})}\delta^{(p)}_{\mathbf{f}_\nu, F_s}(\widetilde{Q}_{\chi_\nu}). \tag{1.4.16}$$

Now the integer $s > 1$ is such that the natural map $\mathbb{T} \longrightarrow \mathbb{V}_\nu$ can be factored as

$$\mathbb{T} \longrightarrow \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \longrightarrow \mathbb{V}_\nu, \tag{1.4.17}$$

and we have $\mathbb{V}_\nu^\dagger \cong \mathbb{V}_\nu$ as $G_{L_{p^s}}$-modules. Tracing through the construction of $\mathfrak{X}$, we see that the image of $U_p^s \cdot \mathfrak{X}_\nu$ in $H^1(L_{p^s}, \mathbb{V}_\nu^\dagger)$ agrees with the image of $\widetilde{Q}_{\chi_\nu}$ under the composite map (where the unlabelled arrow is induced by (1.4.17))

$$\begin{aligned} J_s(L_{p^s}) \otimes_\mathbf{Z} F_\nu &\xrightarrow{\mathrm{Kum}_s} H^1(L_{p^s}, \mathrm{Ta}_p(J_s) \otimes_\mathbf{Z} F_\nu) \xrightarrow{e^{\mathrm{ord}}} H^1(L_{p^s}, \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \otimes_\mathbf{Z} F_\nu) \longrightarrow \\ &\longrightarrow H^1(L_{p^s}, \mathbb{V}_\nu) \cong H^1(L_{p^s}, \mathbb{V}_\nu^\dagger). \end{aligned} \tag{1.4.18}$$

Since U_p acts on $\mathbb{V}_\nu^\dagger$ as multiplication by $\nu(\mathbf{a}_p)$, we thus arrive at the equality

$$\mathrm{Kum}_s(e^{\mathrm{ord}}\widetilde{Q}_{\chi_\nu}) = \nu(\mathbf{a}_p)^s \cdot \mathrm{res}_{L_{p^s}/H}(\mathfrak{X}_\nu) \in H^1(L_{p^s}, \mathbb{V}_\nu). \tag{1.4.19}$$

By [**Rub00**, Prop. 1.6.8], this shows that the restriction to $\mathrm{loc}_\mathfrak{p}(\mathfrak{X}_\nu)$ to G_{F_s} is contained in the Bloch–Kato finite subspace $H^1_f(F_s, \mathbb{V}_\nu) \cong H^1_f(F_s, \mathbb{V}_\nu^\dagger)$. Since the map $\delta^{(p)}_{\mathbf{f}_\nu, F_s}$ is defined by the commutativity of the diagram

$$\begin{tikzcd}
J_s(L_{p^s}) \otimes_{\mathbf{Z}} F_\nu \arrow[r, "(2.3.17)"] \arrow[rrd, dotted] \arrow[rrdd, bend right, "\delta^{(p)}_{\mathbf{f}_\nu, F_s}"'] & H^1(L_{p^s}, \mathbb{V}_\nu) \arrow[r, "\mathrm{loc}_{\mathfrak{p}}"] & H^1(F_s, \mathbb{V}_\nu) \\
& & H^1_f(F_s, \mathbb{V}_\nu) \arrow[u, phantom, "\cup"] \arrow[d, "\log_{F_s, \mathbb{V}_\nu}"] \\
& & (\mathrm{Fil}^0 D_{\mathrm{dR}}(\mathbb{V}_\nu))^\vee,
\end{tikzcd}$$

we thus see that (1.4.15) follows from (1.4.16) and (1.4.19). □

Corollary 1.4.10. *Assume that there is some $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that $L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0$. Then, for all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, the localization map*

$$\mathrm{loc}_{\mathfrak{p}} : \mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}_\nu^\dagger) \longrightarrow H^1(\mathbf{Q}_p, \mathbb{V}_\nu^\dagger) \tag{1.4.20}$$

is injective.

Proof. By Howard's Theorem 1.4.8, our nonvanishing assumption implies that $\mathfrak{Z}$ is not $\mathbb{I}$-torsion, and hence by the exact sequence

$$0 \longrightarrow \frac{\widetilde{H}^1_f(K, \mathbb{T}^\dagger)_\nu}{\nu \cdot \widetilde{H}^1_f(K, \mathbb{T}^\dagger)_\nu} \longrightarrow \widetilde{H}^1_f(K, \mathbb{V}_\nu^\dagger) \longrightarrow \widetilde{H}^2_f(K, \mathbb{T}^\dagger)_\nu[\nu] \longrightarrow 0$$

(see [**How07b**, Cor. 3.4.3]), combined with the finite generation over $\mathbb{I}$ of $\widetilde{H}^1_f(K, \mathbb{T}^\dagger)$ and [**How07b**, Lemma 2.1.6], it implies that the image of $\mathfrak{Z}$ in $\widetilde{H}_f(K, \mathbb{V}_\nu^\dagger)$ is nonzero for almost all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$; in particular, for all but finitely many ν of weight 2 and non-trivial nebentypus, $\mathfrak{Z}_\nu \neq 0$ in $\widetilde{H}_f(K, \mathbb{V}_\nu^\dagger) \cong \mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}_\nu^\dagger)$ (see [**How07b**, Eq. (22)] for the comparison).

Now, by [**How07b**, Cor. 3.4.3] it follows that $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$ has rank one, and hence

$$\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}_\nu^\dagger) = \mathfrak{Z}_\nu . F_\nu \tag{1.4.21}$$

for almost all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$. Thus to prove the result it suffices to show that the implication

$$\mathfrak{Z}_\nu \neq 0 \quad \Longrightarrow \quad \mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu) \neq 0$$

holds for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus. (Indeed, by (1.4.21) this will show that the localization map (1.4.20) is injective at infinitely many ν, and by [**How07b**, Lemma 2.1.7] it will follow that the kernel of the localization map

$$\mathrm{loc}_{\mathfrak{p}} : \widetilde{H}^1_f(K, \mathbb{T}^\dagger) \longrightarrow H^1(\mathbf{Q}_p, \mathbb{T}^\dagger)$$

must be $\mathbb{I}$-torsion, hence supported only at a finite number of arithmetic primes.)

Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ and $s > 1$ be as in Theorem 1.4.9, and assume that $\mathfrak{Z}_\nu \neq 0$. Since the restriction map

$$H^1(K, \mathbb{V}_\nu^\dagger) \xrightarrow{\mathrm{res}_{L_{p^s}}} H^1(L_{p^s}, \mathbb{V}_\nu^\dagger) \cong H^1(L_{p^s}, \mathbb{V}_\nu)$$

is injective, the class $\mathrm{res}_{L_{p^s}}(\mathfrak{Z}_\nu)$ is non-zero, and it arises as the image of a necessarily non-torsion point in $J_s(L_{p^s})$ under the composite map (cf. (2.3.17))

$$J_s(L_{p^s}) \longrightarrow H^1(L_{p^s}, \mathrm{Ta}_p^{\mathrm{ord}}(J_s)) \longrightarrow H^1(L_{p^s}, \mathbb{V}_\nu), \tag{1.4.22}$$

where the first arrow is the Kummer map composed with the ordinary projector e^{ord}. Let $\mathcal{L}_{p^s}$ be the completion of $\imath_p(L_{p^s}) \subset \overline{\mathbf{Q}}_p$. Then both the natural map

$$J_s(L_{p^s}) \otimes \mathbf{Q} \longrightarrow J_s(\mathcal{L}_{p^s}) \otimes \mathbf{Q}_p$$

and the local Kummer map

$$J_s(\mathcal{L}_{p^s}) \otimes \mathbf{Q}_p \longrightarrow H^1(\mathcal{L}_{p^s}, \mathrm{Ta}_p(J_s) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$$

are injective, and hence by the commutativity of the resulting diagram

$$\begin{tikzcd}
J_s(L_{p^s}) \otimes \mathbf{Q} \arrow[r, "(2.3.21)\otimes\mathbf{Q}"] \arrow[d] & H^1(L_{p^s}, \mathbb{V}_\nu) \arrow[d, "\mathrm{loc}_\mathfrak{p}"] \\
J_s(\mathcal{L}_{p^s}) \otimes \mathbf{Q}_p \arrow[r] & H^1(\mathcal{L}_{p^s}, \mathbb{V}_\nu),
\end{tikzcd}$$

it follows that $\mathrm{loc}_\mathfrak{p}(\mathrm{res}_{L_{p^s}}(\mathfrak{Z}_\nu)) = \mathrm{res}_{\mathcal{L}_{p^s}}(\mathrm{loc}_\mathfrak{p}(\mathfrak{Z}_\nu)) \neq 0$, whence $\mathrm{loc}_\mathfrak{p}(\mathfrak{Z}_\nu) \neq 0$ as desired. $\square$

1.4.2. Higher weight specializations. Now we can prove our main result. Recall from the introduction that f_o is a p-ordinary newform of level N prime to p, even weight $k \geq 2$ and trivial nebentypus, and that $\mathbf{f} = \sum_{n>0} \mathbf{a}_n q^n \in \mathbb{I}[[q]]$ is the Hida family passing through the ordinary p-stabilization of f_o. Let ν_o be the arithmetic prime of $\mathbb{I}$ such that $\mathbf{f}_{\nu_o}$ is the ordinary p-stabilization of f_o, and let $\mathbb{T}^\dagger = \mathbb{T} \otimes \Theta^{-1}$ be the critical twist of $\mathbb{T}$ such that ϑ_{ν_o} is the trivial character (as opposed to $\omega^{\frac{p-1}{2}}$.)

If $\mathbf{f}_\nu$ is the ordinary p-stabilization of a p-ordinary newform $\mathbf{f}_\nu^\sharp$ of even weight $2r_\nu > 2$ and trivial nebentypus, the Heegner cycle $\Delta_{A,r_\nu}^{\mathrm{heeg}}$ has been defined in Section 1.2 (attached to a suitable choice of an imaginary quadratic field K), and by [**Nek00**, Thm. (3.1)(i)] the class

$$\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Delta_{r_\nu}^{\mathrm{heeg}}) := \mathrm{Cor}_{H/K}(\Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, H}(\Delta_{A,r_\nu}^{\mathrm{heeg}})) \tag{1.4.23}$$

lies in the Bloch–Kato Selmer group $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$.

On the other hand, by [**How07b**, Prop. 2.4.5], the big Heegner point $\mathfrak{X}$ lies in the strict Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(H, \mathbb{T}^\dagger)$ (defined in [*loc.cit.*, Def. 2.4.2]), and since $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{V}_\nu^\dagger) \cong H^1_f(K, \mathbb{V}_\nu^\dagger)$ as explained in [**How07b**, p. 114]) and $\mathbb{V}_\nu^\dagger \cong V_{\mathbf{f}_\nu^\sharp}(r_\nu)$ by Lemma 1.3.2, the class

$$\mathfrak{Z}_\nu = \mathrm{Cor}_{H/K}(\mathfrak{X}_\nu)$$

naturally lies in $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$ as well. Our main result relates these two classes.

ASSUMPTIONS 1.4.11. (1) *The residual representation $\bar{\rho}_{f_o}$ is absolutely irreducible,*
(2) *$\bar{\rho}_{f_o}|_{G_{\mathbf{Q}_p}}$ has non-scalar semi-simplication,*
(3) *The prime p splits in K,*
(4) *Every prime divisor of N splits in K.*

THEOREM 1.4.12. *Together with Assumptions 1.4.11, suppose that there exists some $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus such that*

$$L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0. \tag{1.4.24}$$

Then for all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight $2r_\nu > 2$ with $2r_\nu \equiv k \pmod{2(p-1)}$, we have

$$\langle \mathfrak{Z}_\nu, \mathfrak{Z}_\nu \rangle_K = \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^4 \cdot \frac{\langle \Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Delta^{\mathrm{heeg}}_{r_\nu}), \Phi^{\text{ét}}_{\mathbf{f}_\nu^\sharp, K}(\Delta^{\mathrm{heeg}}_{r_\nu}) \rangle_K}{u^2 (4D)^{r_\nu - 1}}, \tag{1.4.25}$$

where $\langle , \rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K, V_{\mathbf{f}_\nu^\sharp}(r_\nu))$, $u = |\mathcal{O}_K^\times|/2$, and $-D < 0$ is the discriminant of K.

PROOF. By [**How07b**, Prop. 2.4.5] the class $\mathfrak{Z}$ lies in the strict Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$ (note that under our assumptions we may take $\lambda = 1$ in Howard's result), and hence its restriction $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})$ at $\mathfrak{p}$ lies in the kernel of the natural map

$$H^1(\mathbf{Q}_p, \mathbb{T}^\dagger) \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}_w^- \mathbb{T}^\dagger)$$

induced by (2.2.1) (twisted by Θ^{-1}). Since $H^0(\mathbf{Q}_p, \mathscr{F}_w^- \mathbb{T}^\dagger) = 0$ by [**How07b**, Lemma 2.4.4], the class $\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})$ can therefore be seen as sitting inside $H^1(\mathbf{Q}_p, \mathscr{F}_w^+ \mathbb{T}^\dagger)$. Thus upon taking an $\mathbb{I}$-basis ℓ of $\mathbb{D}$, we can form

$$\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger) := u \cdot \mathrm{Log}^{\ell}_{\mathbb{T}^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z})) \in \mathbb{I}[\lambda^{-1}] \qquad (\lambda := \mathbf{a}_p - 1).$$

On the other hand, consider the continuous function on $\mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$ given by

$$\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger) : \nu \longmapsto \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} d^{-1} \mathbf{f}_\nu^{[p]} \otimes \theta_\nu^{-1}(\mathfrak{a} * (A, \alpha_A, \imath_A)).$$

(Its continuity can be checked by staring at the q-expansion of $d^{-1}\mathbf{f}_\nu^{[p]}\otimes\theta_\nu^{-1}$ and appealing to the results in [**Gou88**, §I.3.5], for example.)

By the specialization property (1.3.8) of the map $\mathrm{Log}^\ell_{\mathbb{T}^\dagger}$, we immediately see that Theorem 1.4.9 can be reformulated as follows: For every $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial wild character, there exists a unit $\Omega_\nu^\eta\in\mathcal{O}_\nu^\times$ such that

$$\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)\big)=\Omega_\nu^\eta\cdot\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger)\big). \tag{1.4.26}$$

In fact,

$$\Omega_\nu^\eta=\langle\eta_\nu\otimes e_\zeta^{\otimes r_\nu},\omega_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}} \tag{1.4.27}$$

under the pairing (1.3.7), so that $\omega_{\mathbf{f}_\nu^*}=\Omega_\nu^\eta\cdot\eta_\nu'$ with η_ν' as defined in Theorem 1.3.4. (That Ω_ν^η, which a priori just lies in F_ν, is indeed a unit is shown in [**Och06**, Prop. 6.4].) Since both $\mathfrak{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger)$ and $\mathfrak{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)$ are continuous functions of ν, (1.4.26) shows that the map $\nu\longmapsto\Omega_\nu^\eta$ is continuous, and hence the relation (1.4.27) is valid for all $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$.

Now let $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be as in the statement. Then $\theta_\nu(z)=z^{r_\nu-1}\vartheta_\nu(z)=z^{r_\nu-1}$ as characters on $\mathbf{Z}_p^\times$, from where if follows that

$$\begin{aligned}\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)\big)&=\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)}d^{-1}\mathbf{f}_\nu^{[p]}\otimes\theta_\nu^{-1}(\mathfrak{a}*(A,\alpha_A,\imath_A))\\&=\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)}d^{-r_\nu}\mathbf{f}_\nu^{[p]}(\mathfrak{a}*(A,A[\mathfrak{N}])).\end{aligned}\tag{1.4.28}$$

By Theorem 1.2.3, setting

$$\Delta_{r_\nu}^{\mathrm{bdp}}:=\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)}\mathrm{N}\mathfrak{a}^{1-r}\cdot\Delta_{\varphi_{\mathfrak{a}},r_\nu}^{\mathrm{bdp}}\in\mathrm{CH}^{2r_\nu-1}(X_{r_\nu})_0(K),\tag{1.4.29}$$

the equation (1.4.28) can be rewritten as

$$\begin{aligned}\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{analy}}(\mathbf{f}^\dagger)\big)&=\mathcal{E}_\nu(r_\nu)\mathcal{E}_\nu^*(r_\nu)\frac{(-1)^{r_\nu-1}}{(r_\nu-1)!}\cdot\mathrm{AJ}_{\mathbf{Q}_p}(\Delta_{r_\nu}^{\mathrm{bdp}})(\omega_{\mathbf{f}_\nu^\sharp}\otimes e_\zeta^{\otimes r_\nu-1})\\&=\mathcal{E}_\nu(r_\nu)\mathcal{E}_\nu^*(r_\nu)\frac{(-1)^{r_\nu-1}}{(r_\nu-1)!}\cdot\log_{\mathbb{V}_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\Phi^{\mathrm{\acute{e}t}}_{\mathbf{f}_\nu^\sharp,K}(\Delta_{r_\nu}^{\mathrm{bdp}})))(\omega_{\mathbf{f}_\nu^\sharp}\otimes e_\zeta^{\otimes r_\nu-1}),\end{aligned}\tag{1.4.30}$$

where

$$\mathcal{E}_\nu(r_\nu):=\left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)}\right),\qquad\mathcal{E}_\nu^*(r_\nu):=\left(1-\frac{\nu(\mathbf{a}_p)}{p^{r_\nu}}\right),$$

and $\Phi^{\mathrm{\acute{e}t}}_{\mathbf{f}_\nu^\sharp,K}:=\pi_{\mathbf{f}_\nu^\sharp,\mathrm{N}^{r_\nu-1}}\circ\Phi_K^{\mathrm{\acute{e}t}}$ with notations as in the diagram (1.2.3) defining $\mathrm{AJ}_{\mathbf{Q}_p}$.

On the other hand, by the specialization property of the map $\mathrm{Log}^{\ell}_{\mathbb{T}^\dagger}$ we have

$$\nu\big(\mathcal{L}_{\mathfrak{p}}^{\mathrm{arith}}(\mathbf{f}^\dagger)\big)=u\frac{(-1)^{r_\nu-1}}{(r_\nu-1)!}\mathcal{E}_\nu(r_\nu)^{-1}\mathcal{E}^*_\nu(r_\nu)\cdot\log_{\mathbb{V}^\dagger_\nu}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu))(\eta'_\nu). \tag{1.4.31}$$

Comparing (1.4.31) and (1.4.30), we thus conclude form (1.4.26) that

$$\log_{\mathbb{V}^\dagger_\nu}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu))(\omega_{\mathbf{f}^\sharp_\nu}\otimes e_\zeta^{\otimes r_\nu-1})=\frac{1}{u}\mathcal{E}_\nu(r_\nu)^2\cdot\log_{\mathbb{V}^\dagger_\nu}(\mathrm{loc}_{\mathfrak{p}}(\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\mathrm{bdp}}_{r_\nu})))(\omega_{\mathbf{f}^\sharp_\nu}\otimes e_\zeta^{\otimes r_\nu-1}).$$

Since $\mathrm{Fil}^1D_{\mathrm{dR}}(V_{\mathbf{f}^\sharp_\nu}(r_\nu-1))$ is spanned by $\omega_{\mathbf{f}^\sharp_\nu}\otimes e_\zeta^{\otimes r_\nu-1}$, it follows that

$$\log_{\mathbb{V}^\dagger_\nu}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu))=\frac{1}{u}\mathcal{E}_\nu(r_\nu)^2\cdot\log_{\mathbb{V}^\dagger_\nu}(\mathrm{loc}_{\mathfrak{p}}(\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\mathrm{bdp}}_{r_\nu}))),$$

and since $\log_{\mathbb{V}^\dagger_\nu}$ is an isomorphism, that

$$\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_\nu)=\frac{1}{u}\mathcal{E}_\nu(r_\nu)^2\cdot\mathrm{loc}_{\mathfrak{p}}(\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\mathrm{bdp}}_{r_\nu})). \tag{1.4.32}$$

Our nonvanishing assumption (1.4.24) implies by Corollary 1.4.10 that the localization map $\mathrm{loc}_{\mathfrak{p}}$ on $\mathrm{Sel}_{\mathrm{Gr}}(K,\mathbb{V}^\dagger_\nu)\cong\mathrm{Sel}_{\mathrm{Gr}}(K,V_{\mathbf{f}^\sharp_\nu}(r_\nu))$ is injective for almost all ν, and hence

$$\mathfrak{Z}_\nu=\frac{1}{u}\mathcal{E}_\nu(r_\nu)^2\cdot\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\mathrm{bdp}}_{r_\nu}) \tag{1.4.33}$$

for all but finitely many $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight $k_\nu=2r_\nu$ as in the statement. In particular, we each such ν we have

$$\begin{aligned}\langle\mathfrak{Z}_\nu,\mathfrak{Z}_\nu\rangle_K&=\frac{1}{u^2}\mathcal{E}_\nu(r_\nu)^4\cdot\langle\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\mathrm{bdp}}_{r_\nu}),\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\mathrm{bdp}}_{r_\nu})\rangle_K\\&=\frac{1}{u^2}\mathcal{E}_\nu(r_\nu)^4\cdot\frac{\langle\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\mathrm{heeg}}_{r_\nu}),\Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\mathrm{heeg}}_{r_\nu})\rangle_K}{(4D)^{r_\nu-1}},\end{aligned}$$

where the last equality follows from Lemma 1.2.4 in light of (1.4.23) and (1.4.29). The result follows. ∎

Remark 1.4.13. As shown in the course of the proof of Theorem 1.4.12, we deduce in fact the equality (1.4.33) of global cohomology classes for almost all ν as in the statement.

1.5. $\mathbb{I}$-adic Gross–Zagier formula

Let $\mathscr{G}_\infty$ be the Galois group of the unique $\mathbf{Z}_p^2$-extension of K, and denote by

$$\mathcal{L}_p(\mathbf{f}\otimes K)\in\mathbb{I}[[\mathscr{G}_\infty]]$$

the "three-variable" p-adic L-function attached to $\mathbf{f}$ over K constructed in [**SU13**, §12.3]. Letting D_∞ (resp. C_∞) denote the Galois group of the anticyclotomic (resp. cyclotomic)

$\mathbf{Z}_p$-extension of K, we identify $\mathbb{I}[[\mathscr{G}_\infty]]$ with $\mathbb{I}_\infty[[C_\infty]]$ where $\mathbb{I}_\infty := \mathbb{I}[[D_\infty]]$, and choosing a generator γ_o of C_∞, we may thus expand

$$\mathcal{L}_p(\mathbf{f}\otimes K) = \mathcal{L}_{\mathbf{f},K} + \mathcal{L}'_{\mathbf{f},K}(\gamma_o - 1) + \mathcal{L}''_{\mathbf{f},K}(\gamma_o-1)^2 + \cdots \tag{1.5.1}$$

with coefficients $\mathcal{L}^{(i)}_{\mathbf{f},K} \in \mathbb{I}_\infty$.

Recall that the big Heegner point $\mathfrak{Z}$ lies in the strict Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(K, \mathbb{T}^\dagger)$, and that (as explined in [**How07b**, p. 113] for example) this group identified with Nekovář's extended Selmer group $\widetilde{H}^1_f(K, \mathbb{T}^\dagger)$.

By [**Nek06**, §11], there exists an $\mathbb{I}$-bilinear "height pairing"

$$\langle\ ,\ \rangle_{K,\mathbb{T}^\dagger} : \widetilde{H}^1_f(K,\mathbb{T}^\dagger) \times \widetilde{H}^1_f(K,\mathbb{T}^\dagger) \longrightarrow \mathbb{I}$$

such that

$$\nu\left(\langle \mathfrak{Y}, \mathfrak{Y}'\rangle_{K,\mathbb{T}^\dagger}\right) = \langle \mathfrak{Y}_\nu, \mathfrak{Y}'_\nu\rangle_K \tag{1.5.2}$$

for all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ and $\mathfrak{Y}, \mathfrak{Y}' \in \widetilde{H}^1_f(K, \mathbb{T}^\dagger)$.

THEOREM 1.5.1. *With notations and assumptions as in Theorem 1.4.12, if $\mathcal{L}'_{\mathbf{f},K} \in \mathbb{I}_\infty$ is the linear term in the expansion* (1.5.1), *then*

$$\mathcal{L}'_{\mathbf{f},K}(\mathbb{1}_K) = \langle \mathfrak{Z}, \mathfrak{Z}\rangle_{K,\mathbb{T}^\dagger}$$

up to a unit in $\mathbb{I}^\times$.

PROOF. For every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight as Theorem 1.4.12, the work [**Nek95**] of Nekovář produces a two-variable p-adic L-function $\mathcal{L}_p(\mathbf{f}_\nu \otimes K) \in F_\nu[[\mathscr{G}_\infty]]$. After expanding

$$\mathcal{L}_p(\mathbf{f}_\nu\otimes K) = \mathcal{L}_{\mathbf{f}_\nu,K} + \mathcal{L}'_{\mathbf{f}_\nu,K}(\gamma_o - 1) + \mathcal{L}''_{\mathbf{f}_\nu,K}(\gamma_o-1)^2 + \cdots$$

similarly as in (1.5.1), [**SU13**, Thm. 12.3.2(ii)] implies on the one hand that

$$\nu(\mathcal{L}'_{\mathbf{f},K}(\mathbb{1}_K)) = \mathcal{L}'_{\mathbf{f}_\nu,K}(\mathbb{1}_K) \tag{1.5.3}$$

up to a unit in $\mathcal{O}_L^\times$, and on the other the main result of [**Nek95**] can be rewritten as the p-adic Gross–Zagier formula

$$\mathcal{L}'_{\mathbf{f}_\nu,K}(\mathbb{1}_K) = \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)}\right)^4 \frac{\langle \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu}), \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Delta^{\text{heeg}}_{r_\nu})\rangle_K}{u^2(4D)^{r_\nu-1}}. \tag{1.5.4}$$

Combining (1.5.2) applied to $\mathfrak{Z}$ with (1.5.3) and (1.5.4), the result follows immediately from Theorem 1.4.12. □

CHAPTER 2

p-adic L-functions and the p-adic variation of Heegner points

Summary

Let f be a newform of weight $k \geq 2$ and trivial nebentypus, let K be an imaginary quadratic field, and let χ be an anticyclotomic Hecke character of K of infinity type $(\ell, -\ell)$ with $\ell \geq k/2$. Denote by V_f the restriction to G_K of the self-dual Tate twist of the p-adic Galois representation associated to f. Specialized to $V_f \otimes \chi$, the general Bloch–Kato conjectures predict the equality between the rank of certain Galois cohomology groups associated to $V_f \otimes \chi$ and the order of vanishing of the associated L-function $L(f, \chi^{-1}, s)$ at the central critical point $s = k/2$. In this chapter[1] we prove the Bloch–Kato conjecture when $L(f, \chi^{-1}, k/2) \neq 0$ under certain assumptions, including that p is a prime of good ordinary reduction for f which splits in K, and that K satisfies the Heegner hypothesis.

[1]Revised in August 2013.

Introduction

Let p be an odd prime and let $f \in S_k(\Gamma_0(N))$ be a normalized newform of weight $k \geq 2$ and level N prime to p. Let $\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_L(V_f)$ be the self-dual Tate twist of the p-adic Galois representation associated with f, defined over a finite extension $L/\mathbf{Q}_p$ with ring of integers $\mathcal{O}$, let $T_f \subset V_f$ be a $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice, and denote by $\bar{\rho}_f$ the (semi-simple) mod p representation obtained by reducing the resulting $\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{O}}(T_f)$ modulo the maximal ideal of $\mathcal{O}$.

Let K be an imaginary quadratic field of odd discriminant. If χ is any anticyclotomic Hecke character of K with values in $\mathcal{O}$, the representation $V_{f,\chi} := (V_f|_{G_K}) \otimes \chi$ is conjugate self-dual, i.e. $V_{f,\chi}^*(1) \cong V_{f,\chi}^c$, where $V_{f,\chi}^*(1)$ denotes the Kummer dual of $V_{f,\chi}$, and $V_{f,\chi}^c$ the conjugate of $V_{f,\chi}$ by the non-trivial automorphism of K. Motivated by Dirichlet's class number formula and the celebrated Birch and Swinnerton-Dyer conjecture, which they are expected to generalize, the Bloch–Kato conjectures (see [**BK90**], [**FPR94**]) predict the equality

$$\mathrm{ord}_{s=k/2} L(f, \chi^{-1}, s) \stackrel{?}{=} \dim_L \mathrm{Sel}(K, V_{f,\chi}^c).$$

Here $\mathrm{Sel}(K, V_{f,\chi})$ is the Bloch–Kato Selmer group, and $L(f, \chi^{-1}, s) = L(V_{f,\chi}, s)$ is the Rankin–Selberg L-function of f with the theta series associated to χ^{-1}. This L-function satisfies a functional equation relating its values at s to those at $s-k$, so that $s = k/2$ is the central critical point.

Assume that K satisfies the following *Heegner hypothesis*:

(heeg) every prime factor of N splits in K,

and fix an integral ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \cong \mathbf{Z}/N\mathbf{Z}$. Then if we restrict above to characters χ of conductor dividing $\mathfrak{N}$, the sign $\epsilon = \pm 1$ in the functional equation of $L(f, \chi^{-1}, s)$, and hence the parity of $\mathrm{ord}_{s=k/2} L(f, \chi^{-1}, s)$, depends only on the behaviour of χ at ∞: if χ has infinity type $(\ell, -\ell)$ with $\ell \geq 0$, then $\epsilon = -1$ for $\ell < k/2$, whereas $\epsilon = +1$ for $\ell \geq k/2$.

Theorem. *Suppose $k \equiv 2 \pmod{p-1}$, and let χ be an $\mathcal{O}$-valued anticyclotomic Hecke character of K of conductor dividing $\mathfrak{N}$ and infinity type $(\ell, -\ell)$ with $\ell \geq k/2$ and $\ell \equiv 0 \pmod{p-1}$. In addition to* (heeg)*, assume*

(ord) *f is ordinary at p,*
(spl) *p splits in K,*
(ram) *p does not divide the class number of K,*
(irr) *$\overline{\rho}_f|_{G_K}$ is irreducible.*

If $L(f,\chi^{-1},k/2)\neq 0$*, then* $\mathrm{Sel}(K,V_{f,\chi}^{c})=0$*.*

The proof of this Theorem (Theorem 2.4.6 in the paper) is based on the study of a certain anticyclotomic p-adic L-function; we devote the rest of this introduction to describe in some detail the main ingredients that enter into the proof.

Fix a choice of embeddings $\imath_p:\overline{\mathbf{Q}}\hookrightarrow\overline{\mathbf{Q}}_p$ and $\imath_\infty:\overline{\mathbf{Q}}\hookrightarrow\mathbf{C}$, and let $\mathfrak{p}\subset\mathcal{O}_K$ be the prime above p induced by $\imath_p$. By the work [**BDP13**] of Bertolini, Darmon and Prasanna, there exists a p-adic L-function $\mathscr{L}_{\mathfrak{p}}(f)$ interpolating the *square-roots* of (the algebraic part of) the central critical values $L(f,\chi^{-1},k/2)$ for varying χ as in the statement of the Theorem above.

Let K_∞/K be the anticyclotomic $\mathbf{Z}_p$-extension of K, and let $\mathbb{T}^\dagger$ be a self-dual twist of the big Galois representation associated with a Hida family with coefficients in $\mathbb{I}$. Howard's construction [**How07b**] of *big Heegner points* produces a "big" cohomology class $\mathfrak{Z}_\infty$ with values in $\mathbb{T}^\dagger$ interpolating in weight 2 the Kummer images of Heegner points over K_∞/K. Extending Cornut and Vatsal's proof of Mazur's conjecture on the nonvanishing of Heegner points, it is shown in *loc.cit.* that $\mathfrak{Z}_\infty$ is nontorsion over the Iwasawa algebra $\mathbb{I}[[\mathrm{Gal}(K_\infty/K)]]$.

Since f satisfies (ord), there is a Hida family $\mathbf{f}\in\mathbb{I}[[q]]$ passing through it. Theorem 2.3.1, which is the technical core of the paper, shows that a "big" p-adic L-function $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger)$, extending $\mathscr{L}_{\mathfrak{p}}(f)$ over a certain twist of $\mathbf{f}$, is the image of $\mathfrak{Z}_\infty$ under a generalization of the Coleman power series map. This map is constructed in Theorem 2.2.8, and relies crucially on the assumption (spl)[2]. If $\nu_f:\mathbb{I}\longrightarrow\overline{\mathbf{Q}}_p^\times$ is the $\mathcal{O}$-algebra homomorphism such that $\nu_f(\mathbf{f})$ recovers f, and if χ be as above, it then follows from the "explicit reciprocity law" of Theorem 2.3.11 that the nonvanishing of $L(f,\chi^{-1},k/2)$ controls the nonvanishing of the twist of $\nu_f(\mathfrak{Z}_\infty)$ by χ. The proof of the Theorem above then follows easily from Fouquet's extension of Kolyvagin's methods, which proves under mild assumptions one of the divisibilities predicted by the two-variable main conjecture of [**How07b**, §3.3].

Following similar arguments, we also obtain one of the divisibilities in the Iwasawa-Greenberg main conjecture for the representation $V_{f,\chi}$ (see Corollary 2.4.10).

The organization of this paper is as follows. In the next section we briefly recall the construction of the class $\mathfrak{Z}_\infty$ and of the big p-adic L-function $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger)$. In Sect. 2.2, by combining the work of Ochiai with some ideas from the recent work of Loeffler–Zerbes, we deduce a variant of Perrin-Riou's regulator map adapted to the local situation at p that

[2]As the construction of $\mathscr{L}_{\mathfrak{p}}(f)$ in [**BDP13**] does. Note however that Howard's construction of $\mathfrak{Z}_\infty$ requires no assumption on the behaviour of p in K.

arises in our setting. Sect. 2.3 is devoted to the proof Theorem 2.3.1, relating $\mathfrak{Z}_\infty$ and $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger)$ via the big p-adic regulator map of Sect. 2.2. Finally, in Sect. 2.4 we deduce the preceding arithmetic applications.

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Unless otherwise stated, it is assumed throughout the article that f and K are as above.

2.1. Big Heegner points

2.1.1. Critical twist of a Hida family. Let $\mathbf{f} = \sum_{n=1}^\infty \mathbf{a}_n q^n \in \mathbb{I}[[q]]$ be the Hida family passing through f, where $\mathbb{I}$ is a complete noetherian local ring finite flat over $\mathcal{O}[[1+p\mathbf{Z}_p]]$, and let

$$\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbb{T})$$

be the big Galois representation associated with $\mathbf{f}$ in [**Hid86a**, Thm. 2.1]. We say that a continuous $\mathcal{O}$-algebra homomorphism $\nu : \mathbb{I} \longrightarrow \overline{\mathbf{Q}}_p^\times$ is an *arithmetic prime* of $\mathbb{I}$ if there is an integer $k_\nu \geq 2$ such that the composition $\Gamma := 1 + p\mathbf{Z}_p \longrightarrow \mathbb{I}^\times \xrightarrow{\nu} \overline{\mathbf{Q}}_p^\times$ is given by $\gamma \longmapsto \psi_\nu(\gamma)\gamma^{k_\nu - 2}$ for some finite order character $\psi_\nu : \Gamma \longrightarrow \overline{\mathbf{Q}}_p^\times$; we then say that k_ν is the *weight* ν and that ψ_ν is the *wild character* of ν. Denote by $\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ the set of arithmetic primes of $\mathbb{I}$, and for each $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, let F_ν be its residue field (which is a finite extension of $\mathbf{Q}_p$), and $\mathcal{O}_\nu$ denote its ring of integers. By [**Hid86a**, Cor. 1.3], for each $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ there is an ordinary p-stabilized newform $\mathbf{f}_\nu$ of weight k_ν whose q-expansion is given by $\nu(\mathbf{f}) \in F_\nu[[q]]$. Moreover, if $s_\nu \geq 1$ is such that ψ_ν is trivial on $\Gamma^{s_\nu} \subset \Gamma$, then $\mathbf{f}_\nu$ has level Np^{s_ν} and nebentypus $\varepsilon_{\mathbf{f}_\nu} = \psi_\nu \omega^{k - k_\nu}$, where $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \longrightarrow \mathbf{Z}_p^\times$ is the Teichmüller character.

Decompose the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^\times$ as the product $\epsilon_{\mathrm{tame}} \cdot \epsilon_{\mathrm{wild}}$ of its tame and wild components, and define the *critical character*

$$\Theta := \epsilon_{\mathrm{tame}}^{(k-2)/2} \cdot [\epsilon_{\mathrm{wild}}^{1/2}] : G_{\mathbf{Q}} \longrightarrow \mathcal{O}[[\Gamma]]^\times \longrightarrow \mathbb{I}^\times, \tag{2.1.1}$$

where $\epsilon_{\mathrm{tame}}^{(k-2)/2} : G_{\mathbf{Q}} \longrightarrow (\mathbf{Z}/p\mathbf{Z})^\times$ is any of the two possible square-roots of $\epsilon_{\mathrm{tame}}^{k-2}$, and $\epsilon_{\mathrm{wild}}^{1/2}$ is the unique square-root of ϵ_{wild} with values in Γ. Let $\mathbb{I}^\dagger$ be the free $\mathbb{I}$-module of rank 1 equipped with the $G_{\mathbf{Q}}$-action via the character Θ, and define the *critical twist* of $\mathbb{T}$ to be (cf. [**How07b**, Def. 2.1.3])

$$\mathbb{T}^\dagger := \mathbb{T}(1) \otimes_{\mathbb{I}} \mathbb{I}^\dagger$$

equipped with the diagonal $G_{\mathbf{Q}}$-action, where $\mathbb{T}(1) = \mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p(1)$ is the Tate twist of $\mathbb{T}$. For each $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, let V_ν be the p-adic Galois representation associated with $\mathbf{f}_\nu$ by Deligne. Then by [**Hid86a**, Thm. 2.1], $T_\nu := \nu(\mathbb{T})$ is a $G_{\mathbf{Q}}$-stable $\mathcal{O}_\nu$-lattice in V_ν, and by construction $V_\nu^\dagger := T_\nu^\dagger \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is a self-dual twist of V_ν.

2.1.2. Review of Howard's big Heegner points. Recall the cyclic ideal $\mathfrak{N} \subset \mathcal{O}_K$ of norm N fixed in the introduction. For each integer $s \geq 0$, let J_s be the Jacobian of the modular curve X_s of level $\Gamma_0(N) \cap \Gamma_1(p^s)$, and for each $t \geq 0$ let $H_{p^{t+1}}$ be the ring class field of K for the order $\mathcal{O}_{p^{t+1}} \subset \mathcal{O}_K$ of conductor p^{t+1}. Denote by $h_{p^{t+1},s} \in X_s(\mathbf{C})$ the point corresponding to the triple $(A_{p^{t+1},s}, \mathfrak{n}_{p^{t+1},s}, \pi_{p^{t+1},s})$, where

- $A_{p^{t+1},s}(\mathbf{C}) = \mathbf{C}/\mathcal{O}_{p^{t+1+s}}$,
- $\mathfrak{n}_{p^{t+1},s} = A_{p^{t+1},s}[\mathfrak{N} \cap \mathcal{O}_{p^{t+1+s}}]$,
- $\pi_{p^{t+1},s}$ generates the kernel of the cyclic p^s-isogeny $\mathbf{C}/\mathcal{O}_{p^{t+1+s}} \longrightarrow \mathbf{C}/\mathcal{O}_{p^{t+1}}$.

For each residue class i modulo $p-1$, let e_i be the projector to the ω^i-isotypical component of $\mathcal{O}[[\mathbf{Z}_p^\times]]$. By [**How07b**, Cor. 2.2.2],

$$y_{p^{t+1},s} := e_{k-2}e^{\mathrm{ord}}h_{p^{t+1},s}$$

defines a point on $J_s^{\mathrm{ord},\dagger}$ rational over $H_{p^{t+1+s}}$, where $J_s^{\mathrm{ord},\dagger} = e^{\mathrm{ord}}J_s^\dagger$ is the ordinary part of J_s with its natural Galois action twisted by Θ^{-1}. Let $\mathfrak{G}_{p^{t+1}}$ be the Galois group of the maximal extension of $H_{p^{t+1}}$ unramified outside the primes above Np. By [*loc.cit*, Lemma 2.2.4], the image $\mathfrak{X}_{p^{t+1},s}$ of $y_{p^{t+1},s}$ under the composite map

$$J_s^{\mathrm{ord},\dagger}(H_{p^{t+1+s}}) \xrightarrow{\mathrm{Cor}^{H_{p^{t+1+s}}}_{H_{p^{t+1}}}} J_s^{\mathrm{ord},\dagger}(H_{p^{t+1}}) \xrightarrow{\mathrm{Kum}_s} H^1(\mathfrak{G}_{p^{t+1}}, \mathrm{Ta}_p^{\mathrm{ord},\dagger}(J_s)),$$

satisfies $\alpha_*\mathfrak{X}_{p^{t+1},s} = U_p \cdot \mathfrak{X}_{p^{t+1},s-1}$, where

$$\alpha_* : H^1(\mathfrak{G}_{p^{t+1}}, \mathrm{Ta}_p^{\mathrm{ord},\dagger}(J_s)) \longrightarrow H^1(\mathfrak{G}_{p^{t+1}}, \mathrm{Ta}_p^{\mathrm{ord},\dagger}(J_{s-1}))$$

is induced by the degeneracy map $X_s \xrightarrow{\alpha} X_{s-1}$ given by $(E, C, \pi) \mapsto (E, C, p\cdot\pi)$ on non-cuspidal points.

Definition 2.1.1. The *big Heegner point of conductor* p^{t+1} is the cohomology class

$$\mathfrak{X}_{p^{t+1}} \in H^1(H_{p^{t+1}}, \mathbb{T}^\dagger)$$

defined as the image of $\varprojlim_s U_p^{-s} \cdot \mathfrak{X}_{p^{t+1},s}$ under the composite map

$$H^1(\mathfrak{G}_{p^{t+1}}, \varprojlim_s \mathrm{Ta}_p^{\mathrm{ord},\dagger}(J_s)) \longrightarrow H^1(\mathfrak{G}_{p^{t+1}}, \mathbb{T}^\dagger) \xrightarrow{\mathrm{Inf}} H^1(H_{p^{t+1}}, \mathbb{T}^\dagger).$$

Let K_∞/K be the anticyclotomic $\mathbf{Z}_p$-extension of K, and let $K_t \subset K_\infty$ be the subfield of degree p^t over K, so that $K_t \subset H_{p^{t+1}}$ by (ram). Set $D_\infty := \mathrm{Gal}(K_\infty/K)$, and $\mathbb{I}_\infty := \mathbb{I}[[D_\infty]]$.

THEOREM 2.1.2 (Howard). *Let $t > 0$ be an integer.*

(1) *The big Heegner point $\mathfrak{X}_{p^{t+1}}$ belongs to the* strict Greenberg Selmer group

$$\mathrm{Sel}_{\mathrm{Gr}}(H_{p^{t+1}}, \mathbb{T}^\dagger) \subset H^1(H_{p^{t+1}}, \mathbb{T}^\dagger),$$

and satisfies

$$\mathrm{Cor}_{H_{p^{t+1}}/H_{p^t}}(\mathfrak{X}_{p^{t+1}}) = U_p \cdot \mathfrak{X}_{p^t}.$$

(2) *Setting $\mathfrak{Z}_t := \mathrm{Cor}_{H_{p^{t+1}}/K_t}(\mathfrak{X}_{p^{t+1}})$, the class*

$$\mathfrak{Z}_\infty := \varprojlim_t U_p^{-t} \cdot \mathfrak{Z}_t \in \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger) := \varprojlim_t \widetilde{H}^1_f(K_t, \mathbb{T}^\dagger)$$

is not $\mathbb{I}_\infty$-torsion, where $\widetilde{H}^1_f(K_t, \mathbb{T}^\dagger)$ is Nekovář's extended Selmer group.

PROOF. See [**How07b**, Prop. 2.4.5, Prop. 2.3.1] for (1), and [*loc.cit.*, Thm. 3.1.1] for (2), noting that

$$\widetilde{H}^1_f(K_t, \mathbb{T}^\dagger) \cong \mathrm{Sel}_{\mathrm{Gr}}(K_t, \mathbb{T}^\dagger)$$

after [*loc.cit.*, Lemma 2.4.4]. □

2.1.3. Big p-adic Rankin L-series. Fix an elliptic curve A with CM by $\mathcal{O}_K$, defined over the Hilbert class field H of K. By (spl), A has ordinary reduction at the prime of H above $\mathfrak{p}$ induced by $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, and hence we may fix a trivialization

$$\imath_A : \hat{A} \xrightarrow{\sim} \hat{\mathbb{G}}_m$$

as formal groups over $\widehat{\mathbf{Z}}_p^{\mathrm{nr}}$, the completion of the ring of integers of the maximal unramified extension $\mathbf{Q}_p^{\mathrm{nr}}$ of $\mathbf{Q}_p$. Fix a $\Gamma_1(N)$-level structure

$$\alpha_A : \boldsymbol{\mu}_N \hookrightarrow A[\mathfrak{N}],$$

and consider the triple $(A, \alpha_A, \imath_A)$. For each class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, represented by an $\mathcal{O}_K$-ideal $\mathfrak{a} \subset K$ prime to $\mathfrak{N}p$, define

$$\mathfrak{a} * (A, \alpha_A, \imath_A) = (A_\mathfrak{a}, \varphi_\mathfrak{a} \circ \alpha_A, \imath_A \circ \hat{\varphi}_\mathfrak{a}^{-1}),$$

where $\varphi_{\mathfrak{a}} : A \longrightarrow A_{\mathfrak{a}} := A/A[\mathfrak{a}]$ is the natural projection and $\hat{\varphi}_{\mathfrak{a}} : \hat{A} \xrightarrow{\sim} \hat{A}_{\mathfrak{a}}$ denotes the isomorphism induced by $\varphi_{\mathfrak{a}}$ on the associated formal groups.

If g is a p-adic modular form with q-expansion $\sum_n b_n q^n$, the q-expansions

$$g^{[p]} := \sum_{(n,p)=1} b_n q^n, \qquad d^{-1}g = d^{-1}g^{[p]} := \sum_{(n,p)=1} n^{-1} b_n q^n$$

correspond to p-adic modular forms that will abusively be denoted by $g^{[p]}$ and $d^{-1}g$ in the following. Let $\mathrm{rec}_K : K^\times \backslash \mathbb{A}_K^\times \longrightarrow G_K^{\mathrm{ab}}$ and $\mathrm{rec}_{\mathfrak{p}} : \mathbf{Q}_p^\times \cong K_{\mathfrak{p}}^\times \longrightarrow G_K^{\mathrm{ab}}$ be the global and local-at-$\mathfrak{p}$ reciprocity maps of class field theory. Then if $\phi : D_\infty \longrightarrow \overline{\mathbf{Q}}_p^\times$ is a continuous character and $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, define $\phi_{\mathfrak{a}} : \mathbf{Z}_p^\times \longrightarrow \overline{\mathbf{Q}}_p^\times$ by $\phi_{\mathfrak{a}}(x) = \phi(\mathrm{rec}_{\mathfrak{p}}(x)\mathrm{rec}_K(\mathfrak{a}))$.

Two-variable construction. The critical character Θ defined in (2.1.1) factors through $\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_{p^\infty})/\mathbf{Q})$, and we let $\theta : \mathbf{Z}_p^\times \longrightarrow \mathbb{I}^\times$ be such that $\Theta = \varepsilon_{\mathrm{cyc}} \circ \theta$. If $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has even weight $k_\nu = 2r_\nu \geq 2$, then

$$\theta_\nu(z) = z^{r_\nu - 1}\vartheta_\nu(z)$$

for all $z \in \mathbf{Z}_p^\times$, where $\vartheta_\nu : \mathbf{Z}_p^\times \longrightarrow \overline{\mathbf{Q}}_p^\times$ is a finite order character such that $\varepsilon_{\mathbf{f}_\nu} = \vartheta_\nu^2$ (see [**How07a**, p. 808]). Thus the formal q-expansion

$$\mathbf{f}^\dagger := \sum_{n=1}^{\infty} \theta^{-1}(n)\mathbf{a}_n q^n \in \mathbb{I}[[q]]$$

has the property that for every $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight, $\nu(\mathbf{f}^\dagger) \in F_\nu[[q]]$ gives the q-expansion of the twist $\mathbf{f}_\nu^\dagger := \mathbf{f}_\nu \otimes \theta_\nu^{-1}$, which is a p-adic modular form of weight 2 with trivial nebentypus.

Definition 2.1.3. The *big anticyclotomic p-adic L-function* $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger)$ associated to $\mathbf{f}^\dagger$ is the generalized measure on $D_\infty \times \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ (in the sense of [**Hid88**, §3]) given by

$$\nu \longmapsto \mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$$

for all $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, where $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$ is the integral measure $\mathrm{d}\mu_{\mathbf{f}_\nu^\dagger}$ on D_∞ given by

$$\begin{aligned}\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)(\phi) &= \int_{D_\infty} \phi(x)\mathrm{d}\mu_{\mathbf{f}_\nu^\dagger}(x)\\ &:= \sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)} \phi^{-1}(\mathfrak{a})\phi^{-1}(\mathrm{N}\mathfrak{a}) \cdot d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi_{\mathfrak{a}}(\mathfrak{a} * (A, \alpha_A, \imath_A))\end{aligned}$$

for all $\phi \in \mathrm{Hom}_{\mathrm{cts}}(D_\infty, \overline{\mathbf{Q}}_p^\times)$.

This terminology is justified by recent works of Bertolini, Darmon and Prasanna [**BDP13**], Brakočević [**Bra11**], and Hsieh [**Hsi12**], among others. The next result just gives the coarse form of their result that will be used here.

THEOREM 2.1.4. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be an arithmetic prime of weight $k_\nu = 2r_\nu \geq 2$ with $r_\nu \equiv k/2 \pmod{p-1}$ and trivial wild character, and let ϕ be an anticyclotomic Hecke character of K of conductor dividing $\mathfrak{N}$ and infinity type $(\ell, -\ell)$ with $\ell \geq r_\nu$ and $\ell \equiv 0 \pmod{p-1}$. Then*

$$\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)(\phi) \neq 0 \quad \Longleftrightarrow \quad L(\mathbf{f}_\nu, \phi^{-1}, r_\nu) \neq 0,$$

where $L(\mathbf{f}_\nu, \phi^{-1}, r_\nu)$ is the central critical value for the Rankin–Selberg convolution of $\mathbf{f}_\nu$ with the theta series of ϕ^{-1}.

PROOF. Our assumptions on ν are so that $\vartheta_\nu = \mathbb{1}$, and since ϕ has conductor prime to p, it follows that

$$d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi = d^{\ell - r_\nu}\mathbf{f}_\nu^{[p]},$$

where d is the Atkin–Serre θ-operator $q\frac{d}{dq}$ acting on p-adic modular forms. Setting $j := \ell - r_\nu \geq 0$, we thus see that

$$\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)(\phi) = \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)} \phi^{-1}(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-j} \cdot d^j \mathbf{f}_\nu^{[p]}(\mathfrak{a} * (A, \alpha_A, \imath_A)),$$

and since the Hecke character $\phi\mathbf{N}^{r_\nu}$ has infinity type $(k_\nu + j, -j)$, the result follows from the combination of [**BDP13**, (5.2.4)] and [*loc.cit.*, Thm. 5.9]. ☐

2.2. p-adic regulator maps

In [**Och03**], Ochiai constructs a map interpolating the cyclotomic regulator of Perrin-Riou over the arithmetic specializations of a Hida family. Using ideas from the recent work of Loeffler–Zerbes [**LZ11**], in this section we show how to assemble Ochiai's construction at each of the finite layers in the unramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, deducing a two-variable regulator map for the critical twist of a Hida family.

2.2.1. Twisted nearly-ordinary deformations. Let $\mathbf{f}$ be a Hida family with associated Galois representation $\rho_{\mathbf{f}} : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{I}}(\mathbb{T})$. By a result of Mazur and Wiles ([**Wil88**, Thm. 2.2.2]), if $D_w \subset G_{\mathbf{Q}}$ is the decomposition group of a place w of $\overline{\mathbf{Q}}$ above p, there exists an exact sequence of $\mathbb{I}[[D_w]]$-modules

$$0 \longrightarrow \mathscr{F}_w^+(\mathbb{T}) \longrightarrow \mathbb{T} \longrightarrow \mathscr{F}_w^-(\mathbb{T}) \longrightarrow 0 \tag{2.2.1}$$

with each $\mathscr{F}_w^{\pm}(\mathbb{T})$ free of rank 1 over $\mathbb{I}$, and with the action of D_w on $\mathscr{F}_w^{+}(\mathbb{T})$ given by an unramified character $\alpha : D_w \longrightarrow \mathbb{I}^{\times}$ sending a geometric Frobenius Frob_p to $\mathbf{a}_p \in \mathbb{I}^{\times}$, the p-th coefficient of $\mathbf{f}$. In the following, we will take w to be the place of $\overline{\mathbf{Q}}_p$ above p induced by our fixed embedding $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, and identify the associated D_w with the absolute Galois group $G_{\mathbf{Q}_p} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$.

Denote by $\Gamma_{\mathrm{cyc}} := \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, and let Λ_{cyc} be the free $\Lambda(\Gamma_{\mathrm{cyc}}) := \mathbf{Z}_p[[\Gamma_{\mathrm{cyc}}]]$-module of rank 1 equipped with the natural $G_{\mathbf{Q}}$-action on group-like elements.

Definition 2.2.1. The *nearly-ordinary deformation* of $\mathbb{T}$ is the $\mathcal{I} := \mathbb{I}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(\Gamma_{\mathrm{cyc}})$-module

$$\mathcal{T} := \mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_{\mathrm{cyc}}$$

equipped with the diagonal $G_{\mathbf{Q}}$-action. It comes equipped with the exact sequence of $\mathcal{I}[[D_w]]$-modules

$$0 \longrightarrow \mathscr{F}_w^{+}(\mathcal{T}) \longrightarrow \mathcal{T} \longrightarrow \mathscr{F}_w^{-}(\mathcal{T}) \longrightarrow 0 \tag{2.2.2}$$

obtained by tensoring (2.2.1) with Λ_{cyc}.

Definition 2.2.2. The *big Dieudonné module of* $\mathbb{T}$ is the $\mathbb{I}$-module

$$\mathbb{D} := (\mathscr{F}_w^{+}(\mathbb{T})\widehat{\otimes}_{\mathbf{Z}_p}\widehat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}},$$

and we let $\mathcal{D} := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(\Gamma_{\mathrm{cyc}})$ be the corresponding $\mathcal{I}$-module associated with the nearly-ordinary deformation $\mathcal{T}$.

Specialization maps. Recall that Γ denotes the group of 1-units in $\mathbf{Z}_p^{\times}$, and let ϵ be the isomorphism

$$\epsilon : \Gamma_{\mathrm{cyc}} \xrightarrow{\sim} \Gamma \tag{2.2.3}$$

induced by the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^{\times}$. Similarly as $\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, define $\mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_{\mathrm{cyc}}))$ to be the set of continuous characters $\sigma : \Gamma_{\mathrm{cyc}} \longrightarrow \overline{\mathbf{Q}}_p^{\times}$ such that for some integer $\ell_\sigma \geq 0$, called the *weight* of σ, the character $\sigma_0 := \sigma \cdot \epsilon^{-\ell_\sigma}$ has finite order.

Every pair $(\nu, \sigma) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_{\mathrm{cyc}}))$ defines a continuous homomorphism $\mathcal{I} \longrightarrow \mathcal{O}_{\nu,\sigma}$, where $\mathcal{O}_{\nu,\sigma}$ is the ring obtained by adjoining to $\mathcal{O}_\nu$ the values of σ. Tensoring with $\mathcal{O}_{\nu,\sigma}$ over $\mathcal{I}$ via this map, we set

$$T_{\nu,\sigma} := \mathcal{T} \otimes_{\mathcal{I}} \mathcal{O}_{\nu,\sigma}, \qquad \mathscr{F}_w^{\pm}(T_{\nu,\sigma}) := \mathscr{F}_w^{\pm}(\mathcal{T}) \otimes_{\mathcal{I}} \mathcal{O}_{\nu,\sigma},$$

and define $V_{\nu,\sigma} := T_{\nu,\sigma}[1/p]$ and $\mathscr{F}_w^{\pm}(V_{\nu,\sigma}) := \mathscr{F}_w^{\pm}(T_{\nu,\sigma})[1/p]$. Then $T_{\nu,\sigma} \cong T_\nu(\sigma)$, where $T_\nu(\sigma)$ denotes the cyclotomic twist of T_ν by the character σ.

For every finite unramified extension $F/\mathbf{Q}_p$, let

$$\mathrm{Sp}_{\nu,\sigma} : H^1(F, \mathscr{F}_w^{+}(\mathcal{T})) \longrightarrow H^1(F, \mathscr{F}_w^{+}(T_{\nu,\sigma})) \longrightarrow H^1(F, \mathscr{F}_w^{+}(V_{\nu,\sigma})) \tag{2.2.4}$$

be the induced maps on cohomology.

In parallel to (2.2.4), attached to every pair (ν, σ) there are specialization maps

$$\mathrm{Sp}_{\nu,\sigma} : \mathcal{D} \otimes_{\mathbf{Z}_p} \mathcal{O}_F \longrightarrow D_{\mathrm{dR}}^{(F)}(\mathscr{F}_w^{+}(V_{\nu,\sigma})) \tag{2.2.5}$$

dependent upon the choice of a compatible system $(\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity $\zeta_s \in \overline{\mathbf{Q}}_p$, and where $D_{\mathrm{dR}}^{(F)}(\mathscr{F}_w^{+}(V_{\nu,\sigma})) := (\mathscr{F}_w^{+}(V_{\nu,\sigma}) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{G_F}$ is the de Rham Dieudonné module associated with the p-adic G_F-representation $V_{\nu,\sigma}$. Since

$$D_{\mathrm{dR}}^{(F)}(\mathscr{F}_w^{+}(V_{\nu,\sigma})) \cong D_{\mathrm{dR}}(\mathscr{F}_w^{+}(V_{\nu,\sigma})) \otimes_{\mathbf{Z}_p} \mathcal{O}_F,$$

the definition of the maps (2.2.5) is reduced to the case where $F = \mathbf{Q}_p$, for which see [**Och03**, Def. 3.12]. (Note that the definition of these maps will not be needed in the following.)

2.2.2. Going up the unramified $\mathbf{Z}_p$-extension. Let F_∞ be the unramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and denote by $\mathcal{O}_\infty$ its ring of integers. Identify $U := \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$ with $\mathbf{Z}_p$ by sending a geometric Frobenius Frob_p to 1, and for every $n \geq 0$, let F_n be the subfield of F_∞ with $\mathrm{Gal}(F_n/\mathbf{Q}_p) \cong \mathbf{Z}/p^n\mathbf{Z}$.

Let $\mathcal{O}_n$ be the ring of integers of F_n. Setting $U_n := \mathrm{Gal}(F_\infty/F_n)$, the group ring $\mathcal{O}_n[U/U_n]$ is equipped with two natural commuting actions of U, one on the coefficients and the other on the group-like elements, and we let $\mathcal{S}_n$ be the $\mathbf{Z}_p$-submodule of $\mathcal{O}_n[U/U_n]$ where these two actions agree. Thus

$$\mathcal{S}_n := \left\{ \sum_{\sigma \in U/U_n} a_\sigma.\sigma \in \mathcal{O}_n[U/U_n] \ : \ \tau a_\sigma = a_{\tau^{-1}\sigma} \quad \text{for all } \tau \in U \right\}.$$

If $x_n \in \mathcal{O}_n$, the element $y_n(x_n) = \sum_{\sigma\in U/U_n} x_n^{\sigma^{-1}}.\sigma$ lies in $\mathcal{S}_n$, and the resulting map $y_n : \mathcal{O}_n \longrightarrow \mathcal{S}_n$ is easily seen to be an isomorphism of $\mathbf{Z}_p[U/U_n]$-modules. These maps fit into commutative diagrams

$$\begin{tikzcd} \mathcal{O}_{n+1} \arrow[r, "y_{n+1}"] \arrow[d, "\mathrm{Tr}_{F_{n+1}/F_n}"'] & \mathcal{S}_{n+1} \arrow[d] & \subset & \mathcal{O}_{n+1}[U/U_{n+1}] \\ \mathcal{O}_n \arrow[r, "y_n"] & \mathcal{S}_n & \subset & \mathcal{O}_n[U/U_n], \end{tikzcd}$$

where the right vertical arrow is induced by the projection $U/U_{n+1} \twoheadrightarrow U/U_n$.

Lemma 2.2.3 (Loeffler–Zerbes). *The* Yager module $\mathcal{S}_\infty := \varprojlim_n \mathcal{S}_n$ *is free of rank* 1 *over* $\mathbf{Z}_p[[U]]$. *More precisely, the maps* y_n *induce an isomorphism*

$$\varprojlim_n \mathcal{O}_n \xrightarrow{\sim} \mathcal{S}_\infty$$

of $\mathbf{Z}_p[[U]]$*-modules.*

Proof. See [**LZ11**, Prop. 3.3] and the discussion in [*loc.cit.*, §3.2]. ☐

Corollary 2.2.4. *The module* $\mathcal{D}_\infty := \mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{S}_\infty$ *is free of rank* 1 *over* $\mathcal{I}_\infty := \mathcal{I}[[U]]$.

Proof. By Lemma 2.2.3, we have $\mathcal{D}_\infty \cong \varprojlim_n(\mathcal{D}\otimes_{\mathbf{Z}_p}\mathcal{O}_n)$, where the limit is taken with respect to the maps $1\otimes\mathrm{Tr}_{F_{n+1}/F_n}$. Since $\mathcal{D}$ is free of rank 1 over $\mathcal{I}$ by [**Och03**, Lemma 3.3], the result follows. ☐

The $\mathbf{Z}_p^2$*-extension of* $\mathbf{Q}_p$. Let $\mathbf{Q}_{p,\infty}$ be the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$, and let L_∞ be the compositum $F_\infty\mathbf{Q}_{p,\infty}$, so that

$$G := \mathrm{Gal}(L_\infty/\mathbf{Q}_p) \cong \mathbf{Z}_p^2.$$

By local class field theory, L_∞ contains many distinguished $\mathbf{Z}_p$-extensions obtained from the torsion points of height 1 Lubin–Tate formal groups over $\mathbf{Q}_p$. More precisely, if $k_\infty/\mathbf{Q}_p$ is a ramified $\mathbf{Z}_p$-extension contained in L_∞, and if $\varpi \in \mathbf{Z}_p$ is a generator of the group of universal norms of the extension $k_\infty(\boldsymbol{\mu}_p)/\mathbf{Q}_p$, there is a height 1 Lubin–Tate formal group $\mathfrak{F}_\varphi$ over $\mathbf{Q}_p$ (associated with a "lift of Frobenius" $\varphi \in \mathbf{Z}_p[[X]]$ corresponding to ϖ) such that

$$k_n(\boldsymbol{\mu}_p) = \mathbf{Q}_p(\mathfrak{F}_\varphi[\varpi^n])$$

for all $n > 0$, where $k_n \subset k_\infty$ is the n-th layer of $k_\infty/\mathbf{Q}_p$ (so that $\mathrm{Gal}(k_n/\mathbf{Q}_p) \cong \mathbf{Z}/p^n\mathbf{Z}$), and $\mathfrak{F}_\varphi[\varpi^n]$ denotes the ϖ^n-torsion of $\mathfrak{F}_\varphi$. Upon fixing an isomorphism

$$\eta : \mathfrak{F}_\varphi \xrightarrow{\sim} \widehat{\mathbb{G}}_m$$

over $\widehat{\mathbf{Q}}_p^{\mathrm{nr}}$, we may then define a basis $(\xi_s)_{s\geq 0}$ of the ϖ-adic Tate module of $\mathfrak{F}_\varphi$ corresponding to $(\zeta_s)_{s\geq 0}$ by setting $\xi_s := \eta^{\sigma_p^{-s}}(\zeta_s - 1)$, where $\sigma_p \in \mathrm{Gal}(\widehat{\mathbf{Q}}_p^{\mathrm{nr}}/\mathbf{Q}_p)$ is an arithmetic Frobenius. This induces the horizontal isomorphisms

$$\begin{array}{ccc} \mathrm{Gal}(k_\infty(\boldsymbol{\mu}_p)/\mathbf{Q}_p) & \xrightarrow{\sim} & \mathrm{Gal}(\mathbf{Q}_p(\boldsymbol{\mu}_{p^\infty})/\mathbf{Q}_p) \\ \downarrow & & \downarrow \\ \Gamma_\varpi := \mathrm{Gal}(k_\infty/\mathbf{Q}_p) & \xrightarrow{\sim} & \mathrm{Gal}(\mathbf{Q}_{p,\infty}/\mathbf{Q}_p) = \Gamma_{\mathrm{cyc}}, \end{array}$$

and we let

$$\kappa_\varpi : \mathrm{Gal}(k_\infty(\boldsymbol{\mu}_p)/\mathbf{Q}_p) \xrightarrow{\sim} \mathbf{Z}_p^\times \qquad (\text{resp. } \kappa : \Gamma_\varpi \xrightarrow{\sim} \Gamma) \tag{2.2.6}$$

be the composition of the top (resp. bottom) isomorphism with $\varepsilon_{\mathrm{cyc}}$ (resp. (2.2.3)).

Set $\Lambda(\Gamma_\varpi) := \mathbf{Z}_p[[\Gamma_\varpi]]$. Similarly as $\mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_{\mathrm{cyc}}))$, define $\mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ to be the set of continuous characters $\phi : \Gamma_\varpi \longrightarrow \overline{\mathbf{Q}}_p^\times$ such that for some integer $\ell_\phi \geq 0$, called the *weight* of ϕ, the character $\phi_0 := \phi \cdot \kappa^{-\ell_\phi}$ has finite order.

Define the modules

$$\mathcal{D}_\varpi := \mathbb{D}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda(\Gamma_\varpi), \qquad \mathcal{T}_\varpi := \mathbb{T}\widehat{\otimes}_{\mathbf{Z}_p}\Lambda_\varpi,$$

where Λ_ϖ is the free $\Lambda(\Gamma_\varpi)$-module of rank 1 equipped with the natural action of $G_{\mathbf{Q}_p}$. Then, in analogy with (2.2.4) and (2.2.5), we may define specialization maps

$$\mathrm{Sp}_{\nu,\phi} : \mathcal{D}_\varpi \longrightarrow D_{\mathrm{dR}}(\mathscr{F}_w^+(V_{\nu,\phi}))$$

and

$$\mathrm{Sp}_{\nu,\phi} : H^1(\mathbf{Q}_p, \mathscr{F}_w^+(\mathcal{T}_\varpi)) \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}_w^+(V_{\nu,\phi}))$$

for every pair $(\nu,\phi) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$.

For an abelian extension $\mathcal{K}/\mathbf{Q}_p$ with $\mathrm{Gal}(\mathcal{K}/\mathbf{Q}_p) \cong \mathbf{Z}_p^d$, and a $\mathbf{Z}_p$-module M equipped with a continuous linear action of $G_{\mathbf{Q}_p}$, we set

$$H^1_{\mathrm{Iw}}(\mathcal{K}, M) := \varprojlim_{K'} H^1(K', M),$$

where the limit is over the finite subextensions $K'/\mathbf{Q}_p$ contained in $\mathcal{K}$ with respect to corestriction. We then continue to denote by $\mathrm{Sp}_{\nu,\phi}$ the composition of the preceding specialization maps with the natural projections $\mathcal{D}_\infty \longrightarrow \mathcal{D}_\varpi$ and $H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}_w^+(\mathbb{T})) \longrightarrow H^1(\mathbf{Q}_p, \mathscr{F}_w^+(\mathcal{T}_\varpi))$, the latter arising from the identification

$$H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}_w^+(\mathbb{T})) \xrightarrow{\sim} H^1_{\mathrm{Iw}}(F_\infty, \mathscr{F}_w^+(\mathcal{T}_\varpi))$$

given by Shapiro's lemma.

2.2.3. Construction of big p-adic regulator maps. For $F/\mathbf{Q}_p$ a finite extension, and V a p-adic $G_{\mathbf{Q}_p}$-representation, let

$$\exp_{F,V} : D_{\mathrm{dR}}^{(F)}(V) \longrightarrow H^1(F,V)$$

be the Bloch–Kato exponential map, and let $H^1_e(F,V) \subset H^1(F,V)$ denote its image. This map through $D_{\mathrm{dR}}^{(F)}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}^{(F)}(V)$, and when the induced map is an injection we let

$$\log_{F,V} := \exp_{F,V}^{-1} : H^1_e(F,V) \longrightarrow D_{\mathrm{dR}}^{(F)}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}^{(F)}(V)$$

be the logarithm map of Bloch–Kato. As usual, we use the notational abbreviations $D_{\mathrm{dR}}(V)$, $\exp_V$ and $\log_V$ when $F = \mathbf{Q}_p$.

Definition 2.2.5. We say that an arithmetic prime $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ is *exceptional* if it has weight $k_\nu = 2$, $\nu(\mathbf{a}_p) = \pm 1$, and the wild character ψ_ν of ν is trivial.

Let $\widetilde{\alpha} := \lim_{n\to\infty} \mathbf{a}_p^{p^n} \in \boldsymbol{\mu}_{p-1}(\mathbb{I}^\times)$ be the Teichmüller lift of $\mathbf{a}_p$, and define the ideal of $\mathcal{I}[[U]] \cong \mathbb{I}[[G]]$

$$\mathcal{J}_\infty := (\widetilde{\alpha} - 1, \gamma_o - 1),$$

where $\gamma_o \in \Gamma_{\mathrm{cyc}}$ is any fixed topological generator.

Theorem 2.2.6. *Fix a compatible system $(\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity. There exists an injective $\mathbb{I}[[G]]$-linear map*

$$\mathrm{Exp}^G_{\mathscr{F}_w^+(\mathbb{T})} : \mathcal{J}_\infty \mathcal{D}_\infty \longrightarrow H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}_w^+(\mathbb{T}))$$

with pseudo-null cokernel and with the following property: If $k_\infty/\mathbf{Q}_p$ is any ramified $\mathbf{Z}_p$-extension contained in L_∞, then for every pair $(\nu,\phi) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ with $1 \leq \ell_\phi < k_\nu$, the diagram

$$\begin{tikzcd}
\mathcal{J}_\infty \mathcal{D}_\infty \arrow[r, "\mathrm{Exp}^G_{\mathscr{F}_w^+(\mathbb{T})}"] \arrow[d, "\mathrm{Sp}_{\nu,\phi}"] & H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}_w^+(\mathbb{T})) \arrow[d, "\mathrm{Sp}_{\nu,\phi}"] \\
D_{\mathrm{dR}}(\mathscr{F}_w^+(V_{\nu,\phi})) \arrow[r] & H^1(\mathbf{Q}_p, \mathscr{F}_w^+(V_{\nu,\phi}))
\end{tikzcd}$$

commutes, where the bottom horizontal arrow is given by

$$(-1)^{\ell_\phi-1}(\ell_\phi-1)!\times\begin{cases}\left(1-\frac{\varpi^{\ell_\phi}}{\nu(\mathbf{a}_p)p}\right)\left(1-\frac{\nu(\mathbf{a}_p)}{\varpi^{\ell_\phi}}\right)^{-1}\exp & \textit{if } \phi_0=\mathbb{1},\\ \\ \left(\frac{\nu(\mathbf{a}_p)p}{\varpi^{\ell_\phi}}\right)^{-t_\phi}\exp & \textit{if } \phi_0\neq\mathbb{1},\end{cases}$$

with $t_\phi>0$ the p-order of the conductor of ϕ_0, and exp *the Bloch–Kato exponential map.*

REMARK 2.2.7. If $(\nu,\phi)\in\mathcal{X}_{\rm arith}(\mathbb{I})\times\mathcal{X}_{\rm arith}(\Lambda(\Gamma_\varpi))$ is a pair as in Theorem 2.2.6, there is a commutative diagram

$$\begin{tikzcd} D_{\rm dR}(\mathscr{F}_w^+(V_{\nu,\phi})) \arrow[r, "\exp_{\mathscr{F}_w^+(V_{\nu,\phi})}"] \arrow[d, "\cong"'] & H^1_e(\mathbf{Q}_p,\mathscr{F}_w^+(V_{\nu,\phi})) \arrow[d] \\ D_{\rm dR}(V_{\nu,\phi})/\mathrm{Fil}^0D_{\rm dR}(V_{\nu,\phi}) \arrow[r, "\exp_{V_{\nu,\phi}}"] & H^1_e(\mathbf{Q}_p,V_{\nu,\phi}), \end{tikzcd}$$

where the vertical maps are induced by the inclusion $\mathscr{F}_w^+(V_{\nu,\phi})\subset V_{\nu,\phi}$, and the left one is shown to be an isomorphism as in [**Och03**, Lemma 3.2]. Hence in the statement of Theorem 2.2.6 we let exp denote either of the horizontal maps in the preceding diagram. (Note that the same remarks apply with $\mathbf{Q}_p$ replaced by any finite *unramified* extension $F/\mathbf{Q}_p$.)

PROOF OF THEOREM 2.2.6. For every $n\geq 0$, consider the height 2 ideal of $\mathcal{I}$

$$\mathcal{J}_n:=(\alpha(\mathrm{Frob}_p|_{F_n}),\gamma_o-1)=(\mathbf{a}_p^{p^n}-1,\gamma_o-1).$$

By [**Och03**, Prop. 5.3] (which is stated over $\mathbf{Q}_p$, but the same arguments work over the unramified extensions $F_n/\mathbf{Q}_p$, changing Frob_p to $\mathrm{Frob}_p|_{F_n}=\mathrm{Frob}_p^{p^n}$), there exists an injective $\mathcal{I}$-linear map

$$\mathrm{Exp}^{(n)}_{\mathscr{F}_w^+(\mathcal{T})}:\mathcal{J}_n(\mathcal{D}\otimes_{\mathbf{Z}_p}\mathcal{O}_n)\longrightarrow H^1(F_n,\mathscr{F}_w^+(\mathcal{T}))$$

with pseudo-null cokernel and such that for every $(\nu,\sigma)\in\mathcal{X}_{\rm arith}(\mathbb{I})\times\mathcal{X}_{\rm arith}(\Lambda(\Gamma_{\rm cyc}))$ with $1\leq\ell_\sigma<k_\nu$ the diagram

$$\begin{tikzcd} \mathcal{J}_n(\mathcal{D}\otimes_{\mathbf{Z}_p}\mathcal{O}_n) \arrow[r, "\mathrm{Exp}^{(n)}_{\mathscr{F}_w^+(\mathcal{T})}"] \arrow[d, "\mathrm{Sp}_{\nu,\sigma}"] & H^1(F_n,\mathscr{F}_w^+(\mathcal{T})) \arrow[d, "\mathrm{Sp}_{\nu,\sigma}"] \\ D^{(F_n)}_{\rm dR}(\mathscr{F}_w^+(V_{\nu,\sigma})) \arrow[r] & H^1(F_n,\mathscr{F}_w^+(V_{\nu,\sigma})) \end{tikzcd}$$

commutes, where the bottom horizontal arrow is given by the map

$$(-1)^{\ell_\sigma-1}(\ell_\sigma-1)!\times\begin{cases}\left(1-\frac{p^{\ell_\sigma-1}}{\nu(\mathbf{a}_p)}\right)\left(1-\frac{\nu(\mathbf{a}_p)}{p^{\ell_\sigma}}\right)^{-1}\exp^{(n)} & \text{if } \sigma_0=\mathbb{1},\\[2ex] \left(\frac{\nu(\mathbf{a}_p)}{p^{\ell_\sigma-1}}\right)^{-s_\sigma}\exp^{(n)} & \text{if } \sigma_0\neq\mathbb{1},\end{cases}$$

with s_σ the p-order of the conductor of σ_0 and $\exp^{(n)}$ the Bloch–Kato exponential map over F_n (see Remark 2.2.7). The map $\mathrm{Exp}^{(n)}_{\mathscr{F}_w^+(\mathcal{T})}$ is obtained by multiplying by $\mathcal{J}_n$ the restriction

$$\left(\widehat{\mathbf{Z}}_p^{\mathrm{nr}}[[\Gamma_{\mathrm{cyc}}]]\widehat{\otimes}_{\mathbf{Z}_p}\mathscr{F}_w^+(\mathbb{T})\right)^{G_{F_n}}\longrightarrow\left(\frac{H^1(\mathbf{Q}_p^{\mathrm{nr}},\Lambda_{\mathrm{cyc}})}{H^0(\mathbf{Q}_p^{\mathrm{nr}},\mathbf{Z}_p)}\widehat{\otimes}_{\mathbf{Z}_p}\mathscr{F}_w^+(\mathbb{T})\right)^{G_{F_n}}$$

to the G_{F_n}-invariants of the $\mathcal{I}$-linear isomorphism of [*loc.cit.*, Prop. 5.11], obtained in turn by taking the formal tensor product $\widehat{\otimes}_{\mathbf{Z}_p}\mathscr{F}_w^+(\mathbb{T})$ of the large exponential map of Perrin-Riou [**PR94**, Thm. 3.2.3] for the representation $\mathbf{Q}_p(1)$ of $G_{\widehat{\mathbf{Q}}_p^{\mathrm{nr}}}$.

Now define

$$\mathrm{Exp}^G_{\mathscr{F}_w^+(\mathbb{T})}:=\varprojlim_n\mathrm{Exp}^{(n)}_{\mathscr{F}_w^+(\mathcal{T})}:\mathcal{J}_\infty\mathcal{D}_\infty\longrightarrow H^1_{\mathrm{Iw}}(F_\infty,\mathscr{F}_w^+(\mathcal{T}))\cong H^1_{\mathrm{Iw}}(L_\infty,\mathscr{F}_w^+(\mathbb{T})),$$

using the identification $\varprojlim_n\mathcal{D}\otimes_{\mathbf{Z}_p}\mathcal{O}_n\cong\mathcal{D}_\infty(:=\mathcal{D}\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{S}_\infty)$ from Lemma 2.2.3. By the discussion in [**LZ11**, §6.4.3], we see that for every $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ the specialization at ν of the map $\mathrm{Exp}^G_{\mathscr{F}_w^+(\mathbb{T})}$ so constructed interpolates over F_∞ the large exponential map of $\mathscr{F}_w^+(V_\nu)$ for any ramified $\mathbf{Z}_p$-extension $k_\infty/\mathbf{Q}_p$ contained in L_∞, and hence the result follows. □

Recall from §2.1.1 the definition of the critical twist $\mathbb{T}^\dagger$ of a Hida family, and set $\lambda:=\mathbf{a}_p-1$.

Theorem 2.2.8. *Fix an $\mathbb{I}$-basis η of $\mathbb{D}$ and a compatible system $(\zeta_s)_{s\geq0}$ of primitive p^s-th roots of unity. Let $k_\infty/\mathbf{Q}_p$ be a ramified $\mathbf{Z}_p$-extension contained in L_∞, and assume $k_\infty\neq\mathbf{Q}_{p,\infty}$. There exists a $\mathbb{I}[[\Gamma_\varpi]]$-linear map*

$$\mathrm{Log}^\eta_{\mathscr{F}_w^+(\mathbb{T}^\dagger)}:H^1_{\mathrm{Iw}}(k_\infty,\mathscr{F}_w^+(\mathbb{T}^\dagger))\longrightarrow\mathbb{I}[[\Gamma_\varpi]]\otimes_{\mathbb{I}}\mathbb{I}[\lambda^{-1}]$$

with the following property: For any non-exceptional $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu=2r_\nu\geq2$, and any $\phi\in\mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ of weight $\ell_\phi\equiv0\pmod{p-1}$ with $\ell_\phi<r_\nu$, if

$\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}^+_w(\mathbb{T}^\dagger))$ *then*

$$\nu(\mathrm{Log}^{\eta}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\mathfrak{Y}_\infty))(\phi) = \frac{(-1)^{r_\nu-\ell_\phi-1}}{(r_\nu-\ell_\phi-1)!}$$
$$\times \begin{cases} \left(1-\frac{p^{r_\nu-1}}{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}\right)^{-1}\left(1-\frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu}}\right)\langle \log_{V^\dagger_{\nu,\phi}}(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \phi_0 = \mathbb{1}, \\ \\ G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu-1}}\right)^{t_\phi}\langle \log_{V^\dagger_{\nu,\phi}}(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \phi_0 \neq \mathbb{1}, \end{cases}$$

where $t_\phi > 0$ *is* p*-order of the conductor of* ϕ_0*,*

$$G(\phi_0^{-1}) := \sum_{v \bmod p^{t_\phi+1}} \phi_0^{-1}(v)\zeta^v_{t_\phi+1},$$

$\nu(\mathfrak{Y}_\infty)^\phi := \mathrm{Sp}_{\nu,\phi}(\mathfrak{Y}_\infty)$*, and* $\eta'_\nu \in \mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_\nu(\kappa_\varpi^{-\ell_\phi}))$ *is the dual to* $\eta^\dagger_{\nu,\ell_\phi}$ *under the de Rham pairing*

$$\frac{D_{\mathrm{dR}}(V^\dagger_\nu(\kappa_\varpi^{\ell_\phi}))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_\nu(\kappa_\varpi^{\ell_\phi}))} \times \mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_\nu(\kappa_\varpi^{-\ell_\phi})) \longrightarrow F_\nu. \tag{2.2.7}$$

Proof. The proof is similar to that of [**Cas13a**, Thm. 3.4]. Let $\gamma_1 \in \mathrm{Gal}(\mathbf{Q}_{p,\infty}/\mathbf{Q}_p)$ be such that $\varepsilon_{\mathrm{cyc}}(\gamma_1) = 1+p$, and set

$$\mathbb{D}^\dagger := \mathcal{D} \otimes_{\mathcal{I}} \mathcal{I}/(\gamma_o - (1+p)[\epsilon^{1/2}_{\mathrm{wild}}(\gamma_1)]).$$

Consider the map $\mathrm{Tw}_{\Theta_1} : \mathbb{I}[[G]] \longrightarrow \mathbb{I}[[G]]$ defined by the commutativity of the diagram

$$\begin{tikzcd} \mathbb{I}[[G]] \arrow[r, "\mathrm{Tw}_{\Theta_1}"] \arrow[d] & \mathbb{I}[[G]] \arrow[d] \\ \mathcal{I} \arrow[r, "\gamma\mapsto[\epsilon^{1/2}_{\mathrm{wild}}(\gamma)]\gamma"] & \mathcal{I}. \end{tikzcd} \tag{2.2.8}$$

The bottom horizontal map is an $\mathbb{I}$-linear isomorphism, and since $k_\infty \neq \mathbf{Q}_{p,\infty}$, the map Tw_{Θ_1} is $\mathbb{I}[[\Gamma_\varpi]]$-linear. Use the same notation to denote the induced $\mathbb{I}[[\Gamma_\varpi]]$-module isomorphism $\mathrm{Tw}_{\Theta_1} : \mathcal{J}_\infty\mathcal{D}_\infty \longrightarrow \mathcal{J}_\infty\mathcal{D}_\infty$ and let

$$\mathrm{Tw}_\Theta : H^1_{\mathrm{Iw}}(L_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T}))^\Delta \longrightarrow H^1_{\mathrm{Iw}}(L_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T}^\dagger))^\Delta$$

be the result of taking the $\Delta := \mathrm{Gal}(\mathbf{Q}_p(\boldsymbol{\mu}_p)/\mathbf{Q}_p)$-invariants of the composite map

$$H^1_{\mathrm{Iw}}(L_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T})) \xrightarrow{\otimes_{\mathbb{I}}\mathbb{I}^\dagger} H^1_{\mathrm{Iw}}(L_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T})) \otimes_{\mathbb{I}} \mathbb{I}^\dagger \cong H^1(L_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T}^\dagger))$$

(see [**Rub00**, Prop. 6.2.1(i)] for a proof of the last isomorphism).

Let $\mathrm{pr}_\varpi : H^1_{\mathrm{Iw}}(L_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T}^\dagger)) \longrightarrow H^1_{\mathrm{Iw}}(k_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T}^\dagger))$ be the projection induced by $\mathbb{I}[[G]] \longrightarrow \mathbb{I}[[\Gamma_\varpi]]$, and define

$$\mathcal{J}^\dagger_\varpi \mathcal{D}^\dagger_\varpi := \mathcal{J}_\infty \mathcal{D}_\infty \otimes_{\mathbb{I}[[G]]} \mathbb{I}[[G]]/(\gamma_o - (1+p)[\epsilon^{1/2}_{\mathrm{wild}}(\gamma_1)]) \otimes_{\mathbb{I}[[G]]} \mathbb{I}[[\Gamma_\varpi]].$$

By the interpolation property of the map $\mathrm{Exp}^G_{\mathscr{F}^+_w(\mathbb{T})}$ of Theorem 2.2.6, the composition

$$\begin{aligned} \mathcal{J}_\infty \mathcal{D}_\infty \xrightarrow{\mathrm{Tw}^{-1}_{\Theta_1}} \mathcal{J}_\infty \mathcal{D}_\infty & \xrightarrow{\mathrm{Exp}^G_{\mathscr{F}^+_w(\mathbb{T})}} H^1_{\mathrm{Iw}}(L_\infty, \mathscr{F}^+_w(\mathbb{T})) \\ & \xrightarrow{\mathrm{Res}} H^1_{\mathrm{Iw}}(L_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T}))^\Delta \xrightarrow{\mathrm{Tw}_\Theta} H^1_{\mathrm{Iw}}(L_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T}^\dagger))^\Delta \\ & \xrightarrow{\mathrm{pr}_\varpi} H^1_{\mathrm{Iw}}(k_\infty(\boldsymbol{\mu}_p), \mathscr{F}^+_w(\mathbb{T}^\dagger))^\Delta \cong H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}^+_w(\mathbb{T}^\dagger)) \end{aligned}$$

is easily seen to factor through an injective $\mathbb{I}[[\Gamma_\varpi]]$-linear map

$$\mathrm{Exp}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)} : \mathcal{J}^\dagger_\varpi \mathcal{D}^\dagger_\varpi \longrightarrow H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}^+_w(\mathbb{T}^\dagger))$$

such that for every pair $(\nu, \phi) \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \times \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ as in the statement, the diagram

$$\begin{tikzcd} \mathcal{J}^\dagger_\varpi \mathcal{D}^\dagger_\varpi \arrow[r, "\mathrm{Exp}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}"] \arrow[d, "\mathrm{Sp}_{\nu,\phi}"] & H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}^+_w(\mathbb{T}^\dagger)) \arrow[d, "\mathrm{Sp}_{\nu,\phi}"] \\ D_{\mathrm{dR}}(\mathscr{F}^+_w(V^\dagger_{\nu,\phi})) \arrow[r] & H^1(\mathbf{Q}_p, \mathscr{F}^+_w(V^\dagger_{\nu,\phi})) \end{tikzcd}$$

commutes, where the bottom horizontal arrow is given by

$$(-1)^{r_\nu - \ell_\phi - 1}(r_\nu - \ell_\phi - 1)! \times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}\right)\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu}}\right)^{-1} \exp & \text{if } \phi_0 = \mathbb{1}, \\ \\ \left(\frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu - 1}}\right)^{-t_\phi} \exp & \text{if } \phi_0 \neq \mathbb{1}, \end{cases}$$

where $t_\phi > 0$ is the p-order in the conductor of ϕ_0.

Now if $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}^+_w(\mathbb{T}^\dagger))$ then $\lambda \cdot \mathfrak{Y}_\infty$ is in the image of $\mathrm{Exp}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}$ and so

$$\mathrm{Log}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\mathfrak{Y}_\infty) := \lambda^{-1} \cdot \mathrm{Exp}^{-1}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\lambda \cdot \mathfrak{Y}_\infty)$$

is a well-defined element in $\mathbb{I}[\lambda^{-1}] \otimes_{\mathbb{I}} \mathcal{J}^\dagger_\varpi \mathcal{D}^\dagger_\varpi \subset \mathbb{I}[\lambda^{-1}] \otimes_{\mathbb{I}} \mathcal{D}^\dagger_\varpi$. Finally, the chosen $\mathbb{I}$-basis η of $\mathbb{D}$ induces an $\mathbb{I}[[\Gamma_\varpi]]$-basis $\widetilde{\eta}$ of $\mathcal{D}^\dagger_\varpi$, and defining $\mathrm{Log}^\eta_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\mathfrak{Y}_\infty)$ to be the element in $\mathbb{I}[[\Gamma_\varpi]] \otimes_{\mathbb{I}} \mathbb{I}[\lambda^{-1}]$ determined by the relation

$$\mathrm{Log}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\mathfrak{Y}_\infty) = \mathrm{Log}^\eta_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\mathfrak{Y}_\infty) \cdot (\widetilde{\eta} \otimes 1),$$

the result follows. $\square$

2.2.4. Explicit reciprocity formula. By construction, the big p-adic regulator map $\mathrm{Log}^{\eta}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}$ of Theorem 2.2.6 interpolates the Bloch–Kato logarithm maps $\log_{V^\dagger_{\nu,\phi}}$ in a range where $\ell_\phi < k_\nu/2$. However, the arithmetic applications in Section 2.4 will be entirely concerned with cases where $\ell_\phi \geq k_\nu/2$, and hence knowing the interpolation property of the map $\mathrm{Log}^{\eta}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}$ in this extended range will be important for us. This is the content of the next result, based on the work of Shaowei Zhang [**Zha04**] adapting Colmez's work [**Col98**] on the p-adic regulator map of Perrin-Riou to a general height 1 Lubin–Tate formal groups over $\mathbf{Z}_p$.

Recall that if V is a p-adic representation of $G_{\mathbf{Q}_p}$ with coefficients in a finite extension $L/\mathbf{Q}_p$, the Bloch–Kato dual exponential map

$$\exp^*_{V^*(1)} : H^1(\mathbf{Q}_p, V) \longrightarrow \mathrm{Fil}^0 D_{\mathrm{dR}}(V)$$

is defined by the commutativity of the diagram

$$\begin{array}{ccccc}
H^1(\mathbf{Q}_p,V) & \times & H^1(\mathbf{Q}_p,V^*(1)) & \xrightarrow{(\ ,\ )} & L \\
\downarrow{\scriptstyle \exp^*_{V^*(1)}} & & \uparrow{\scriptstyle \exp_{V^*(1)}} & & \| \\
\mathrm{Fil}^0 D_{\mathrm{dR}}(V) & \times & \frac{D_{\mathrm{dR}}(V^*(1))}{\mathrm{Fil}^0 D_{\mathrm{dR}}(V^*(1))} & \xrightarrow{\langle\ ,\ \rangle_{\mathrm{dR}}} & L,
\end{array}$$

where $(\ ,\ )$ is the local Tate pairing.

Corollary 2.2.9. *Let the notations be as in Theorem 2.2.8. For any non-exceptional $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of even weight $k_\nu = 2r_\nu \geq 2$, and any $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ of weight $\ell_\phi \equiv 0 \pmod{p-1}$ with $\ell_\phi \geq r_\nu$, if $\mathfrak{Y}_\infty \in H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}^+_w(\mathbb{T}^\dagger))$ then*

$$\begin{aligned}
\nu(\mathrm{Log}^{\eta}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\mathfrak{Y}_\infty))(\phi) &= (\ell_\phi - r_\nu)! \\
&\times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}\right)^{-1}\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu}}\right)\langle \exp^*(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \textit{if } \phi_0 = \mathbb{1}, \\[2ex] G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu - 1}}\right)^{t_\phi}\langle \exp^*(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \textit{if } \phi_0 \neq \mathbb{1}. \end{cases}
\end{aligned}$$

Remark 2.2.10. Dualizing (the twist by ϑ_ν of) the diagram in Remark 2.2.7, we obtain

$$\begin{array}{ccc}
D_{\mathrm{dR}}(\mathscr{F}^+_w(V^\dagger_{\nu,\phi})) & \xrightarrow{\exp^*_{\mathscr{F}^-_w(V^\dagger_{\nu,\phi^{-1}})}} & H^1(\mathbf{Q}_p, \mathscr{F}^+_w(V^\dagger_{\nu,\phi})) \\
\downarrow & & \downarrow \\
D_{\mathrm{dR}}(V^\dagger_{\nu,\phi})/\mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_{\nu,\phi}) & \xrightarrow{\exp^*_{V^\dagger_{\nu,\phi^{-1}}}} & H^1(\mathbf{Q}_p, V^\dagger_{\nu,\phi}).
\end{array}$$

Hence, similarly as in Theorem 2.2.6, in the statement of Corollary 2.2.9 we let $\exp^*$ denote either of the horizontal maps in the preceding diagram.

Proof of Corollary 2.2.9. From the combination of [**Zha04**, Thm. 3.6] and [*loc.cit.*, Thm. 6.4] there exists an $\mathcal{O}_\nu[[\Gamma_\varpi]]$-linear map

$$\mathfrak{L}^{\eta_\nu}_{\mathscr{F}_w^+(V_\nu^\dagger)} : H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}_w^+(V_\nu^\dagger)) \longrightarrow \mathcal{O}_\nu[[\Gamma_\varpi]]$$

such that for any $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ of weight $\ell_\phi \equiv 0 \pmod{p-1}$:

(*i*) if $\ell_\phi \geq r_\nu$, then

$$\mathfrak{L}^{\eta_\nu}_{\mathscr{F}_w^+(V_\nu^\dagger)}(\nu(\mathfrak{Y}_\infty))(\phi) = (\ell_\phi - r_\nu)!$$
$$\times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}\right)^{-1}\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu}}\right)\langle \exp^*_{\mathscr{F}_w^-(V^\dagger_{\nu,\phi^{-1}})}(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \phi_0 = \mathbb{1}, \\ G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu - 1}}\right)^{t_\phi}\langle \exp^*_{\mathscr{F}_w^-(V^\dagger_{\nu,\phi^{-1}})}(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \phi_0 \neq \mathbb{1}, \end{cases}$$

(*ii*) if $\ell_\phi < r_\nu$, then

$$\mathfrak{L}^{\eta_\nu}_{\mathscr{F}_w^+(V_\nu^\dagger)}(\nu(\mathfrak{Y}_\infty))(\phi) = \frac{(-1)^{r_\nu - \ell_\phi - 1}}{(r_\nu - \ell_\phi - 1)!}$$
$$\times \begin{cases} \left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}\right)^{-1}\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu}}\right)\langle \log_{\mathscr{F}_w^+(V^\dagger_{\nu,\phi})}(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \phi_0 = \mathbb{1}, \\ G(\phi_0^{-1})^{-1}\left(\frac{\nu(\mathbf{a}_p)\varpi^{\ell_\phi}}{p^{r_\nu - 1}}\right)^{t_\phi}\langle \log_{\mathscr{F}_w^+(V^\dagger_{\nu,\phi})}(\nu(\mathfrak{Y}_\infty)^\phi), \eta'_\nu\rangle_{\mathrm{dR}} & \text{if } \phi_0 \neq \mathbb{1}. \end{cases}$$

Comparing these with the formulae in Theorem 2.2.8 for the map $\mathrm{Log}^{\eta}_{\mathscr{F}_w^+(\mathbb{T}^\dagger)}$, we see that the map $\mathrm{Log}^{\eta_\nu}_{\mathscr{F}_w^+(V_\nu^\dagger)}$ defined by the commutativity of the diagram

$$\text{(2.2.9)} \qquad \begin{array}{ccc} H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}_w^+(\mathbb{T}^\dagger)) & \xrightarrow{\mathrm{Log}^{\eta}_{\mathscr{F}_w^+(\mathbb{T}^\dagger)}} & \mathbb{I}[[\Gamma_\varpi]] \otimes_{\mathbb{I}} \mathbb{I}[\lambda^{-1}] \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ H^1_{\mathrm{Iw}}(k_\infty, \mathscr{F}_w^+(V_\nu^\dagger)) & \xrightarrow{\mathrm{Log}^{\eta_\nu}_{\mathscr{F}_w^+(V_\nu^\dagger)}} & \mathcal{O}_\nu[[\Gamma_\varpi]] \otimes_{\mathcal{O}_\nu} \mathcal{O}_\nu[\lambda_\nu^{-1}] \end{array}$$

is such that $\mathrm{Log}^{\eta_\nu}_{\mathscr{F}_w^+(V_\nu^\dagger)} = \mathfrak{L}^{\eta_\nu}_{\mathscr{F}_w^+(V_\nu^\dagger)}$, since both maps have the same interpolation properties at all $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ of weight $\ell_\phi \equiv 0 \pmod{p-1}$ with $\ell_\phi < r_\nu$, and these are enough to uniquely determine either of them. By the interpolation properties of the map $\mathfrak{L}^{\eta_\nu}_{\mathscr{F}_w^+(V_\nu^\dagger)}$ in the range where $\ell_\phi \geq r_\nu$, the result follows. $\square$

2.3. Main result

Recall that K_∞/K denotes the anticyclotomic $\mathbf{Z}_p$-extension of K, and let $K_{\infty,\mathfrak{p}}$ be the completion of K_∞ at the unique prime above $\mathfrak{p}$. Since K satisfies the hypotheses (ram) and (spl) from the Introduction, the field $K_{\infty,\mathfrak{p}}$ is a totally ramified $\mathbf{Z}_p$-extension of $K_\mathfrak{p}\cong\mathbf{Q}_p$. Moreover, if $\pi\in\mathcal{O}_K$ is a generator of the principal ideal $\mathfrak{p}^{h_K}$ ($h_K:=\#\mathrm{Pic}(\mathcal{O}_K)$), it follows from local class field theory that the uniformizer

$$\varpi:=\pi/\bar{\pi}$$

of $\mathcal{O}_{K_\mathfrak{p}}\cong\mathbf{Z}_p$ is a universal norm of $K_{\infty,\mathfrak{p}}/\mathbf{Q}_p$. As explained in §2.2.2, it follows that the $\mathbf{Z}_p^\times$-extension $K_{\infty,\mathfrak{p}}(\boldsymbol{\mu}_p)/\mathbf{Q}_p$ can be obtained from the torsion points of a height 1 Lubin–Tate formal group over $\mathbf{Z}_p$ associated with ϖ, and hence attached to this local situation we have a big p-adic regulator map

$$\mathrm{Log}^\eta_{\mathscr{F}_w^+(\mathbb{T}^\dagger)}:H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}_w^+(\mathbb{T}^\dagger))\longrightarrow\mathbb{I}[[\Gamma_\varpi]]\otimes_\mathbb{I}\mathbb{I}[\lambda^{-1}]$$

as constructed in Theorem 2.2.6. Also, in the following we will repeatedly use the resulting identification $D_\infty\cong\Gamma_\varpi$, so that the local component at $\mathfrak{p}$ of an anticyclotomic Hecke character ϕ of K of infinity type $(\ell,-\ell)$, with $\ell\geq 0$, will be identified with an arithmetic prime $\phi\in\mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ of weight ℓ.

Recall the big cohomology class $\mathfrak{Z}_\infty\in\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)$ constructed from Howard's big Heegner points, as in Theorem 2.1.2, and the big p-adic Rankin L-series $\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger)$ of Proposition 2.1.4. Since $\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)\cong\mathrm{Sel}_{\mathrm{Gr}}(K_t,\mathbb{T}^\dagger)$ as recalled before, the image of $\mathfrak{Z}_\infty$ under the localization map at $\mathfrak{p}$ lies in $H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}_w^+(\mathbb{T}^\dagger))$.

In this section we prove the following result.

THEOREM 2.3.1. *Fix an $\mathbb{I}$-basis η of $\mathbb{D}$ and a compatible system $(\zeta_s)_{s\geq 0}$ of primitive p^s-th roots of unity. There exists a unit $\alpha_\eta\in\mathbb{I}^\times$ such that the composite map*

$$\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)\xrightarrow{\mathrm{loc}_\mathfrak{p}}H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}_w^+(\mathbb{T}^\dagger))\xrightarrow{\mathrm{Log}^\eta_{\mathscr{F}_w^+(\mathbb{T}^\dagger)}}\mathbb{I}[[\Gamma_\varpi]]\otimes_\mathbb{I}\mathbb{I}[\lambda^{-1}]$$

sends $\alpha_\eta\cdot\mathfrak{Z}_\infty$ to $\mathscr{L}_\mathfrak{p}(\mathbf{f}^\dagger)$.

PROOF. Denote by $\mathcal{X}^{\mathrm{good}}_{\mathrm{arith},2}(\mathbb{I})$ the set of arithmetic primes of $\mathbb{I}$ of weight 2 and non-trivial wild character. (Recall that if $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has weight 2, then the corresponding p-stabilized newform $\mathbf{f}_\nu$ has nebentypus $\varepsilon_{\mathbf{f}_\nu}=\psi_\nu\omega^{k-2}=\vartheta_\nu^2$, where ψ_ν is the wild character of ν.)

We divide the proof of Theorem 2.3.1 into the following three steps:

I: For each $\nu \in \mathcal{X}^{\mathrm{good}}_{\mathrm{arith},2}(\mathbb{I})$, there exists a class $\mathfrak{Z}^{\eta}_{\infty,\nu} \in \widetilde{H}^1_{\mathrm{Iw}}(K_\infty, V^\dagger_\nu)$ such that

$$\mathrm{Log}^{\eta_\nu}_{\mathscr{F}^+_w(V^\dagger_\nu)}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}^{\eta}_{\infty,\nu})) = \mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger_\nu),$$

where $\mathrm{Log}^{\eta_\nu}_{\mathscr{F}^+_w(V^\dagger_\nu)}$ is the $\mathcal{O}_\nu$-linear map (2.2.9) obtained from $\mathrm{Log}^{\eta}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}$ by specialization at ν.

II: The classes $\mathfrak{Z}^{\eta}_{\infty,\nu}$ from Step I can be patched together, i.e. there exists a class $\mathfrak{Z}^{\eta}_{\infty} \in \widetilde{H}^1_{\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger)$ such that

$$\nu(\mathfrak{Z}^{\eta}_{\infty}) = \mathfrak{Z}^{\eta}_{\infty,\nu}$$

for all $\nu \in \mathcal{X}^{\mathrm{good}}_{\mathrm{arith},2}(\mathbb{I})$.

III: There exists a unit $\alpha_\eta \in \mathbb{I}^\times$ such that $\mathfrak{Z}^{\eta}_{\infty} = \alpha_\eta \cdot \mathfrak{Z}_\infty$.

Their proof is given in the next three subsections, respectively.

2.3.1. Step I: Weight 2 specializations. For each $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2, denote by $\mathbf{f}^*_\nu$ the primitive form associated with the twist $\mathbf{f}_\nu \otimes \varepsilon^{-1}_{\mathbf{f}_\nu}$, so that $\mathbf{f}^\dagger_\nu = \mathbf{f}^*_\nu \otimes \vartheta_\nu$, and for each class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, set $x(\mathfrak{a}) := \mathfrak{a} * (A, \alpha_A, \imath_A)$, where $(A, \alpha_A, \imath_A)$ is the trivialized elliptic curve with $\Gamma_1(N)$-level structure fixed in §2.1.3.

Proposition 2.3.2. *Let $\nu \in \mathcal{X}^{\mathrm{good}}_{\mathrm{arith},2}(\mathbb{I})$, let $\phi = \phi_0 \in \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ have weight 0, and assume that $t_\phi > s_\nu$, where t_ϕ (resp. s_ν) is the p-order of the conductor of ϕ_0 (resp. ϑ_ν). Then*

$$\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)} \phi^{-1}(\mathfrak{a})\phi^{-1}(\mathrm{N}\mathfrak{a}) \cdot d^{-1}\mathbf{f}^\dagger_\nu \otimes \phi_{\mathfrak{a}}(x(\mathfrak{a})) = \pm\frac{\nu(\mathbf{a}_p)^{t_\phi}}{G(\phi_0^{-1})}\langle \log_{V^\dagger_{\nu,\phi}}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty))^{\phi}), \omega^\dagger_{\mathbf{f}^*_\nu}\rangle_{\mathrm{dR}},$$

where $\pm = \vartheta_\nu(-1)$, $G(\phi_0^{-1}) = \sum_{v \bmod p^{t_\phi+1}} \phi_0^{-1}(v)\zeta^v_{t_\phi+1}$, is the Gauss sum of ϕ_0^{-1}, and $\omega^\dagger_{\mathbf{f}^*_\nu} \in \mathrm{Fil}^0 D_{\mathrm{dR}}(V^\dagger_\nu)$ is the class associated with the twist $\mathbf{f}^*_\nu \otimes \vartheta_\nu$.

Proof. This follows from a calculation similar to [**Cas13a**, §5.1], and all the references in this proof are to that paper. In order to avoid a too clustered notation, we set $s = s_\nu$ and $t = t_\phi$ in the following.

By Definition 2.2,

$$d^{-1}\mathbf{f}^\dagger_\nu \otimes \phi_{\mathfrak{a}}(x(\mathfrak{a})) = \frac{\phi(\mathfrak{a})}{p^{t+1}} \sum_{v \bmod p^{t+1}} \phi(v) \sum_{H(\mathfrak{a})} \zeta^{-v}_{H(\mathfrak{a})} \cdot d^{-1}\mathbf{f}^\dagger_\nu(x(\mathfrak{a}\mathfrak{p}^{t+1})/H(\mathfrak{a})), \tag{2.3.1}$$

where the second sum is over the étale cyclic subgroups $H(\mathfrak{a}) \subset A_{\mathfrak{a}\mathfrak{p}^{t+1}}[p^{t+1}]$ of order p^{t+1}, and for each $H(\mathfrak{a})$, $\zeta_{H(\mathfrak{a})}$ is the primitive p^{t+1}-st root of unity determined as follows. The trivialization $\imath_{A_{\mathfrak{a}}}$ defines an inclusion $\boldsymbol{\mu}_{p^{t+1}} \longrightarrow A_{\mathfrak{a}}[p^{t+1}]$ whose image (which gives the *canonical subgroup of* $A_{\mathfrak{a}}$ *of order* p^{t+1}, in the sense of [**Buz03**, Def. 3.4]) is identified with $A_{\mathfrak{a}}[\mathfrak{p}^{t+1}] \subset A_{\mathfrak{a}}[p^{t+1}]$. Since $A_{\mathfrak{a}\mathfrak{p}^{t+1}} = A_{\mathfrak{a}}/A_{\mathfrak{a}}[\mathfrak{p}^{t+1}]$, it follows that the kernel of the dual $\check{\pi}_{\mathfrak{a},t+1}$ to the projection $\pi_{\mathfrak{a},t+1} : A_{\mathfrak{a}} \twoheadrightarrow A_{\mathfrak{a}\mathfrak{p}^{t+1}}$ gives an inclusion

$$\jmath_{\mathfrak{a},t+1} : \mathbf{Z}/p^{t+1}\mathbf{Z} \hookrightarrow A_{\mathfrak{a}\mathfrak{p}^{t+1}}[p^{t+1}] \tag{2.3.2}$$

whose image is identified with $A_{\mathfrak{a}\mathfrak{p}^{t+1}}[\mathfrak{p}^{t+1}] \subset A_{\mathfrak{a}\mathfrak{p}^{t+1}}[p^{t+1}]$. Since $\check{\pi}_{\mathfrak{a},t+1}$ is étale, it induces the trivialization $\hat{A}_{\mathfrak{a}\mathfrak{p}^{t+1}} \xrightarrow{\sim} \hat{A}_{\mathfrak{a}} \xrightarrow{\imath_{A_{\mathfrak{a}}}} \hat{\mathbb{G}}_m$, giving rise to an inclusion

$$\imath_{\mathfrak{a},t+1} : \boldsymbol{\mu}_{p^{t+1}} \hookrightarrow A_{\mathfrak{a}\mathfrak{p}^{t+1}}[p^{t+1}] \tag{2.3.3}$$

normalized so that $\imath_{\mathfrak{a},t+1}(\zeta_{t+1}) = \pi_{\mathfrak{a},t+1}(\alpha_{\mathfrak{a}})$, with $\alpha_{\mathfrak{a}} \in A_{\mathfrak{a}\mathfrak{p}^{t+1}}[\bar{\mathfrak{p}}^{t+1}]$ such that

$$e_{p^{t+1},\mathfrak{a}}(\alpha_{\mathfrak{a}}, \jmath_{\mathfrak{a},t+1}(m)) = \zeta_{t+1}^m \qquad \text{for all } m \in \mathbf{Z}/p^{t+1}\mathbf{Z},$$

where $e_{p^{t+1},\mathfrak{a}} : A_{\mathfrak{a}\mathfrak{p}^{t+1}}[p^{t+1}] \times A_{\mathfrak{a}\mathfrak{p}^{t+1}}[p^{t+1}] \longrightarrow \boldsymbol{\mu}_{p^{t+1}}$ is the Weil pairing, and ζ_{t+1} is our fixed (in the statement of Theorem 2.3.1) primitive p^{t+1}-st root of unity. The combination of (2.3.2) and (2.3.3) gives an isomorphism

$$\boldsymbol{\mu}_{p^{t+1}} \times \mathbf{Z}/p^{t+1}\mathbf{Z} \longrightarrow A_{\mathfrak{a}\mathfrak{p}^{t+1}}[p^{t+1}]. \tag{2.3.4}$$

In particular, applied to the unit ideal $\mathfrak{a} = \mathcal{O}_K$, we may use (2.3.4) to define a bijection between the above cyclic subgroups $H(\mathcal{O}_K) \subset A_{\mathfrak{p}^{t+1}}[p^{t+1}]$ and the invertible elements $u \in (\mathbf{Z}/p^{t+1}\mathbf{Z})^\times$, so that

$$\zeta_{H_u(\mathcal{O}_K)} = \zeta_{t+1}^u.$$

For an arbitrary class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, the subgroups $H(\mathfrak{a}) \subset A_{\mathfrak{a}\mathfrak{p}^{t+1}}[p^{t+1}]$ are the image of the subgroups $H(\mathcal{O}_K)$ under the $\mathrm{N}\mathfrak{a}$-isogeny $\varphi_{\mathfrak{a}} : A_{\mathfrak{p}^{t+1}} \longrightarrow A_{\mathfrak{a}\mathfrak{p}^{t+1}}$. Since

$$\begin{aligned} e_{t+1,\mathfrak{a}}(\alpha_{\mathfrak{a}}, \jmath_{\mathfrak{a},t+1}(m)) &= e_{t+1,\mathfrak{a}}(\varphi_{\mathfrak{a}}(\alpha_{\mathcal{O}_K}), \varphi_{\mathfrak{a}}(\jmath_{\mathcal{O}_K,t+1}(m))) \\ &= e_{t+1,\mathcal{O}_K}(\alpha_{\mathcal{O}_K}, \jmath_{\mathcal{O}_K,t+1}(m))^{\mathrm{N}\mathfrak{a}}, \end{aligned}$$

we see that $\imath_{\mathfrak{a},t+1}(\zeta_{t+1}) = \imath_{\mathcal{O}_K,t+1}(\zeta_{t+1}^{\mathrm{N}\mathfrak{a}^{-1}})$, where the inverse $\mathrm{N}\mathfrak{a}^{-1}$ is computed in $(\mathbf{Z}/p^{t+1}\mathbf{Z})^\times$, and hence

$$\zeta_{H_u(\mathfrak{a})} = \zeta_{t+1}^{u\cdot\mathrm{N}\mathfrak{a}^{-1}}.$$

We have thus defined the units $\zeta_{H(\mathfrak{a})}$ appearing in (2.3.1). Moreover, if $\mathfrak{b} \subset K$ is a fractional ideal such that $(A_{\mathfrak{a}\mathfrak{p}^{t+1}}/H(\mathfrak{a}))(\mathbf{C}) = \mathbf{C}/\mathfrak{b}$, it is easy to see that $\mathfrak{b}$ is a proper $\mathcal{O}_{p^{t+1}}$-ideal such that $\mathfrak{b}\mathcal{O}_K = \mathfrak{a}\bar{\mathfrak{p}}^{t+1}$, and we let $x(\mathfrak{b}) = x(\mathfrak{a}\mathfrak{p}^{t+1})/H(\mathfrak{a})$ denote the trivialized elliptic curve with $\Gamma_1(N)$-level structure deduced from $(A_{\mathfrak{a}}, \alpha_{A_{\mathfrak{a}}}, \imath_{A_{\mathfrak{a}}})$ after first dividing by $A_{\mathfrak{a}}[\mathfrak{p}^{t+1}]$ and then the resulting quotient by $H(\mathfrak{a})$.

Putting everything together, we thus see that (2.3.1) may be rewritten as

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi_{\mathfrak{a}}(x(\mathfrak{a})) &= \frac{\phi(\mathfrak{a})}{p^{t+1}} \sum_{v \bmod p^{t+1}} \phi(v) \sum_{\substack{[\mathfrak{b}]\in \mathrm{Pic}(\mathcal{O}_{p^{t+1}}) \\ \mathfrak{b}\mathcal{O}_K = \mathfrak{a}\bar{\mathfrak{p}}^{t+1}}} \zeta_{t+1}^{-u_{\mathfrak{b}}\cdot \mathrm{N}\mathfrak{a}^{-1}\cdot v} \cdot d^{-1}\mathbf{f}_\nu^\dagger(x(\mathfrak{b})) \\ &= \frac{\phi(\mathfrak{a})\phi(\mathrm{N}\mathfrak{a})}{G(\phi^{-1})} \sum_{\substack{[\mathfrak{b}]\in \mathrm{Pic}(\mathcal{O}_{p^{t+1}}) \\ \mathfrak{b}\mathcal{O}_K = \mathfrak{a}\bar{\mathfrak{p}}^{t+1}}} \phi^{-1}(u_{\mathfrak{b}}) \cdot d^{-1}\mathbf{f}_\nu^\dagger(x(\mathfrak{b})), \end{aligned}$$

where the second equation follows from the relation $G(\phi)G(\phi^{-1}) = \phi(-1)p^{t+1}$, and where $u_{\mathfrak{b}} \in (\mathbf{Z}/p^{t+1}\mathbf{Z})^\times$ is such that $\mathfrak{b}$ corresponds to $H_{u_{\mathfrak{b}}}(\mathfrak{a})$ as described above. Summing over the classes $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$, we thus arrive at

(2.3.5)
$$\sum_{[\mathfrak{a}]\in \mathrm{Pic}(\mathcal{O}_K)} \phi^{-1}(\mathfrak{a})\phi^{-1}(\mathrm{N}\mathfrak{a}) \cdot d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi_{\mathfrak{a}}(x(\mathfrak{a})) = \frac{1}{G(\phi^{-1})} \sum_{[\mathfrak{b}]\in \mathrm{Pic}(\mathcal{O}_{p^{t+1}})} \phi^{-1}(u_{\mathfrak{b}}) \cdot d^{-1}\mathbf{f}_\nu^\dagger(x(\mathfrak{b})).$$

If $x = (E, \alpha_E, \imath_E)$ is a trivialized elliptic curve with $\Gamma_1(N)$-level structure, define

$$\mathrm{Frob}(x) := (E/\imath_E^{-1}(\boldsymbol{\mu}_p), \lambda_E \circ \alpha_E, \check{\lambda}_E \circ \imath_E),$$

where $\lambda_E : E \longrightarrow E/\imath_E^{-1}(\boldsymbol{\mu}_p)$ is the natural projection. The Frobenius operator Frob on the space $\mathcal{M}(N)$ of p-adic modular forms of tame level N is then defined by setting $\mathrm{Frob}(g)(x) := g(\mathrm{Frob}(x))$ for all $g \in \mathcal{M}(N)$.

Now fix a class $[\mathfrak{b}] \in \mathrm{Pic}(\mathcal{O}_{p^{t+1}})$, and let $x(\mathfrak{b}) = (A_{\mathfrak{b}}, \alpha_{\mathfrak{b}}, \imath_{\mathfrak{b}})$ be the corresponding trivialized elliptic curve with $\Gamma_1(N)$-level structure. Again by Definition 2.2 of the character twist, but applied to $d^{-1}\mathbf{f}_\nu^\dagger = d^{-1}\mathbf{f}_\nu^{*[p]} \otimes \vartheta_\nu$, we have

$$d^{-1}\mathbf{f}_\nu^\dagger(x(\mathfrak{b})) = \frac{1}{p^s} \sum_{u \bmod p^s} \vartheta_\nu(u) \sum_{C_{\mathfrak{b}}} \zeta_{C_{\mathfrak{b}}}^{-u} \cdot \mathrm{Frob}^s(d^{-1}\mathbf{f}_\nu^{*[p]})(x(\mathfrak{b})/C_{\mathfrak{b}}), \tag{2.3.6}$$

where the second sum is over the étale cyclic subgroups $C_{\mathfrak{b}} \subset A_{\mathfrak{b}}[p^s]$ of order p^s. Similarly as before, these subgroups are in one-to-one correspondence with the elements $w \in (\mathbf{Z}/p^s\mathbf{Z})^\times$, and we set

$$\zeta_{C_{\mathfrak{b}}} = \zeta_s^w \tag{2.3.7}$$

if $C_{\mathfrak{b}}$ corresponds to w.

Let $F_{\omega_{\mathbf{f}_\nu^*}}$ be the Coleman primitive of $\omega_{\mathbf{f}_\nu^*}$ which vanishes at ∞ and satisfies

$$F_{\omega_{\mathbf{f}_\nu^*}} - \frac{\nu(\mathbf{a}_p)}{p}\mathrm{Frob}(F_{\omega_{\mathbf{f}_\nu^*}}) = d^{-1}\mathbf{f}_\nu^{*[p]}. \tag{2.3.8}$$

(See Corollary 2.8.) Using this relation, it is immediately seen that the equalities $U_p \circ \mathrm{Frob} = \mathrm{id}$ as operators on $\mathcal{M}(N)$ and $U_p F_{\omega_{\mathbf{f}^*_\nu}} = \frac{\nu(\mathbf{a}_p)}{p} F_{\omega_{\mathbf{f}^*_\nu}}$ imply that

$$\mathrm{Frob}^s(d^{-1}\mathbf{f}_\nu^{*[p]})(x(\mathfrak{b})/C_{\mathfrak{b}}) = \frac{p^s}{\nu(\mathbf{a}_p)^s} d^{-1}\mathbf{f}_\nu^{*[p]}(x(\mathfrak{b})/C_{\mathfrak{b}}). \tag{2.3.9}$$

Writing $(A_{\mathfrak{b}}/C_{\mathfrak{b}})(\mathbf{C}) = \mathbf{C}/\mathfrak{c}$ for a fractional ideal $\mathfrak{c} \subset K$, we see that $\mathfrak{c}$ is a proper $\mathcal{O}_{p^{t+1+s}}$-ideal such that $\mathfrak{c}\mathcal{O}_{p^{t+1}} = \mathfrak{b}$.

Writing $x(\mathfrak{b})/C_{\mathfrak{b}} = x(\mathfrak{c}_w)$ if $C_{\mathfrak{b}}$ corresponds to w, and substituting (2.3.7) and (2.3.9) into (2.3.6), we thus arrive at

$$\begin{aligned} d^{-1}\mathbf{f}_\nu^{\dagger}(x(\mathfrak{b})) &= \frac{1}{\nu(\mathbf{a}_p)^s} \sum_{u \bmod p^s} \vartheta_\nu(u) \sum_{w \in (\mathbf{Z}/p^s\mathbf{Z})^\times} \zeta_s^{-uw} \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(x(\mathfrak{c}_w)) \\ &= \pm \frac{G(\vartheta_\nu)}{\nu(\mathbf{a}_p)^s} \sum_{w \in (\mathbf{Z}/p^s\mathbf{Z})^\times} \vartheta_\nu^{-1}(w) \cdot d^{-1}\mathbf{f}_\nu^{*[p]}(x(\mathfrak{c}_w)), \end{aligned} \tag{2.3.10}$$

where $\pm = \vartheta_\nu(-1)$.

For $\mathfrak{c} = \mathcal{O}_{p^{t+1+s}}$, we have

$$d^{-1}\mathbf{f}_\nu^{*[p]}(x(\mathcal{O}_{p^{t+1+s}})) = d^{-1}\mathbf{f}_\nu^{*[p]}(h_{p^{t+1},s}),$$

where $h_{p^{t+1},s} \in X_s(\mathbf{C})$ is the CM point with $\Gamma_0(N) \cap \Gamma_1(p^s)$-level structure appeared in §2.1.2, which by [**How07b**, Cor. 2.2.2] is rational over $L_{p^{t+1},s} := H_{p^{t+1+s}}(\boldsymbol{\mu}_{p^s})$. Let F_s be a finite extension of the closure of $\imath_p(L_{p^{t+1},s}) \subset \overline{\mathbf{Q}}_p$ such that the base change $X_s \times_{\mathbf{Q}_p} F_s$ admits a stable model. The calculation in Proposition 2.9 applies to the pair $\mathbf{f}_\nu^*$ and $\Delta_{p^{t+1},s} := (h_{p^{t+1},s}) - (\infty) \in J_s(F_s)$, yielding the formula

$$\log_{\omega_{\mathbf{f}^*_\nu}}(\Delta_{p^{t+1},s}) = F_{\omega_{\mathbf{f}^*_\nu}}(h_{p^{t+1},s}). \tag{2.3.11}$$

Let $S_{t+1} \subset \mathrm{Gal}(L_{p^{t+1},s}/H_{p^{t+1}})$ be a set of lifts of $\mathrm{Gal}(H_{p^{t+1+s}}/H_{p^{t+1}})$ fixing ζ_s. Then, for $\mathfrak{b} = \mathcal{O}_{p^{t+1}}$,

$$\left\{d^{-1}\mathbf{f}_\nu^{*[p]}(x(\mathfrak{c}_w)) \,:\, w \in (\mathbf{Z}/p^s\mathbf{Z})^\times\right\} = \left\{d^{-1}\mathbf{f}_\nu^{*[p]}(h^\sigma_{p^{t+1},s}) \,:\, \sigma \in S_{t+1}\right\}; \tag{2.3.12}$$

for an arbitrary $[\mathfrak{b}] \in \mathrm{Pic}(\mathcal{O}_{p^{t+1}})$, taking a proper $\mathcal{O}_{p^{t+1+s}}$-ideal $\mathfrak{c} \subset K$ with $\mathfrak{c}\mathcal{O}_{p^{t+1}} = \mathfrak{b}$ and defining ${}_{\mathfrak{b}}h_{p^{t+1},s} \in X_s(\mathbf{C})$ by the triple $({}_{\mathfrak{b}}A_{p^{t+1},s}, \mathfrak{n}'_{p^{t+1},s}, \pi'_{p^{t+1},s})$ where

- ${}_{\mathfrak{b}}A_{p^{t+1},s}(\mathbf{C}) = \mathbf{C}/\mathfrak{c}$,
- $\mathfrak{n}'_{p^{t+1},s} = \ker(\mathbf{C}/\mathfrak{c} \longrightarrow \mathbf{C}/(\mathfrak{N} \cap \mathfrak{c})^{-1})$,
- $\pi'_{p^{t+1},s}$ generates the kernel of the cyclic p^s-isogeny $\mathbf{C}/\mathfrak{c} \longrightarrow \mathbf{C}/\mathfrak{b}$,

the analogue of (2.3.12) with $h_{p^{t+1},s}$ replaced by ${}_{\mathfrak{b}}h_{p^{t+1},s}$ holds, and hence (2.3.10) may be rewritten as

$$d^{-1}\mathbf{f}_\nu^\dagger(x(\mathfrak{b})) = \pm\frac{G(\vartheta_\nu)}{\nu(\mathbf{a}_p)^s}\sum_{\sigma\in S_{t+1}}\chi_\nu^{-1}(\sigma)\cdot d^{-1}\mathbf{f}_\nu^{*[p]}({}_{\mathfrak{b}}h^\sigma_{p^{t+1},s}). \tag{2.3.13}$$

Moreover, it is easily seen that

$$\mathrm{Frob}({}_{\mathfrak{b}}h_{p^{t+1},s}) = {}_{\mathfrak{b}}h_{p^t,s}. \tag{2.3.14}$$

Now for each class $[\mathfrak{b}]\in\mathrm{Pic}(\mathcal{O}_{p^{t+1}})$ define

$${}_{\mathfrak{b}}Q^{\chi_\nu}_{p^{t+1},s} := \sum_{\sigma\in S_{t+1}} {}_{\mathfrak{b}}\Delta^\sigma_{p^{t+1},s}\otimes\chi_\nu^{-1}(\sigma)\in J_s(L_{p^{t+1},s})\otimes_{\mathbf{Z}} F_\nu, \tag{2.3.15}$$

where ${}_{\mathfrak{b}}\Delta_{p^{t+1},s} := ({}_{\mathfrak{b}}h_{p^{t+1},s})-(\infty)$. Combining (2.3.5) and (2.3.13), we obtain

$$\begin{aligned}
&\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)}\phi^{-1}(\mathfrak{a})\phi^{-1}(\mathrm{N}\mathfrak{a})\cdot d^{-1}\mathbf{f}_\nu^\dagger\otimes\phi_{\mathfrak{a}}(x(\mathfrak{a}))\\
&\qquad= \pm\nu(\mathbf{a}_p)^{-s}\frac{G(\vartheta_\nu)}{G(\phi^{-1})}\sum_{\sigma\in S_{t+1}}\chi_\nu^{-1}(\sigma)\sum_{[\mathfrak{b}]\in\mathrm{Pic}(\mathcal{O}_{p^{t+1}})}\phi^{-1}(\sigma_{\mathfrak{b}})\cdot d^{-1}\mathbf{f}_\nu^{*[p]}({}_{\mathfrak{b}}h^\sigma_{p^{t+1},s})\\
&\qquad= \pm\nu(\mathbf{a}_p)^{-s}\frac{G(\vartheta_\nu)}{G(\phi^{-1})}\sum_{\sigma\in S_{t+1}}\chi_\nu^{-1}(\sigma)\sum_{[\mathfrak{b}]\in\mathrm{Pic}(\mathcal{O}_{p^{t+1}})}\phi^{-1}(\sigma_{\mathfrak{b}})\cdot F_{\omega_{\mathbf{f}_\nu^*}}({}_{\mathfrak{b}}h^\sigma_{p^{t+1},s})\\
&\qquad= \pm\nu(\mathbf{a}_p)^{-s}\frac{G(\vartheta_\nu)}{G(\phi^{-1})}\sum_{[\mathfrak{b}]\in\mathrm{Pic}(\mathcal{O}_{p^{t+1}})}\phi^{-1}(\sigma_{\mathfrak{b}})\cdot\log_{\omega_{\mathbf{f}_\nu^*}}({}_{\mathfrak{b}}Q^{\chi_\nu}_{p^{t+1},s}),
\end{aligned}\tag{2.3.16}$$

where the second equality follows from the combination of (2.3.8) and (2.3.14), since ϕ has conductor p^{t+1}, and the last equality follows from (2.3.11).

Letting $\mathbb{T}^* := \mathrm{Hom}_{\mathbb{I}}(\mathbb{T},\mathbb{I})$ be the contragredient of $\mathbb{T}$, the map $\mathbb{T}^*\longrightarrow V_\nu^*$ can be factored as

$$\mathbb{T}^*\longrightarrow \mathrm{Ta}_p^{\mathrm{ord}}(J_s)\longrightarrow V_\nu^*,$$

and tracing through the definitions we see that the image of ${}_{\mathfrak{b}}Q^{\chi_\nu}_{p^{t+1},s}$ under the induced map

$$\begin{aligned}
J_s(L_{p^{t+1},s})\otimes_{\mathbf{Z}} F_\nu &\xrightarrow{e^{\mathrm{ord}}} J_s^{\mathrm{ord}}(L_{p^{t+1},s})\otimes_{\mathbf{Z}} F_\nu\\
&\xrightarrow{\mathrm{Kum}_s} H^1(L_{p^{t+1},s},\mathrm{Ta}_p^{\mathrm{ord}}(J_s)\otimes_{\mathbf{Z}} F_\nu)\\
&\longrightarrow H^1(L_{p^{t+1},s},V_\nu^*)\cong H^1(L_{p^{t+1},s},V_\nu^\dagger),
\end{aligned}\tag{2.3.17}$$

is the same as the image of $U_p^s \cdot \nu(\mathfrak{X}_{p^{t+1}}^{\sigma_{\mathfrak{b}}})$ in $H^1(L_{p^{t+1},s}, V_\nu^\dagger)$ under restriction, and hence

$$\begin{aligned}\log_{\omega_{\mathbf{f}_\nu^*}}({}_{\mathfrak{b}}Q_{p^{t+1},s}^{\chi_\nu}) &= \langle \log_{V_\nu^\dagger}(\mathrm{Kum}_s(e^{\mathrm{ord}}({}_{\mathfrak{b}}Q_{p^{t+1},s}^{\chi_\nu}))), \omega_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}} \\ &= \nu(\mathbf{a}_p)^s \cdot \langle \log_{V_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{X}_{p^{t+1}}^{\sigma_{\mathfrak{b}}}))), \omega_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}}.\end{aligned} \tag{2.3.18}$$

Substituting (2.3.18) into (2.3.16) we conclude that

$$\begin{aligned}\sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)} &\phi^{-1}(\mathfrak{a})\phi^{-1}(\mathrm{N}\mathfrak{a}) \cdot d^{-1}\mathbf{f}_\nu^\dagger \otimes \phi_{\mathfrak{a}}(x(\mathfrak{a})) \\ &= \pm\frac{G(\vartheta_\nu)}{G(\phi^{-1})} \sum_{[\mathfrak{b}]\in\mathrm{Pic}(\mathcal{O}_{p^{t+1}})} \phi^{-1}(\sigma_{\mathfrak{b}})\langle \log_{V_\nu^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{X}_{p^{t+1}}^{\sigma_{\mathfrak{b}}}))), \omega_{\mathbf{f}_\nu^*}\rangle_{\mathrm{dR}} \\ &= \pm\frac{\nu(\mathbf{a}_p)^t}{G(\phi^{-1})}\langle \log_{V_{\nu,\phi}^\dagger}(U_p^{-t} \cdot \mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{X}_{p^{t+1}}))^\phi, \omega_{\mathbf{f}_\nu^*}^\dagger\rangle_{\mathrm{dR}} \\ &= \pm\frac{\nu(\mathbf{a}_p)^t}{G(\phi^{-1})}\langle \log_{V_{\nu,\phi}^\dagger}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_t))^\phi, \omega_{\mathbf{f}_\nu^*}^\dagger\rangle_{\mathrm{dR}},\end{aligned}$$

where the last equality follows from the construction of $\mathfrak{Z}_t$ (see Theorem 2.1.2). Proposition 2.3.2 follows. ☐

DEFINITION 2.3.3. Let η be an $\mathbb{I}$-basis of $\mathbb{D}$, and let $\eta^\dagger$ denote its image in $\mathbb{D}^\dagger$. For each $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, define the p-adic period $\Omega_\nu^\eta \in F_\nu^\times$ to be the value

$$\Omega_\nu^\eta := \langle \eta_\nu^\dagger, \omega_{\mathbf{f}_\nu^*}^\dagger\rangle_{\mathrm{dR}} \tag{2.3.19}$$

under the de Rham pairing (2.2.7) (with $\ell_\phi = 0$).

REMARK 2.3.4. The period Ω_ν^η depends on the choice of a compatible system of primitive p-power roots of unity, since the definition of the specialization maps (2.2.5) depend on such choice.

REMARK 2.3.5. For $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and trivial nebentypus, it can be shown that Ω_ν^η is a p-adic unit. (See [**Och06**, Prop. 6.4].)

COROLLARY 2.3.6. *Fix an $\mathbb{I}$-basis η of $\mathbb{D}$, and let $\nu \in \mathcal{X}_{\mathrm{arith},2}^{\mathrm{good}}(\mathbb{I})$. The class*

$$\mathfrak{Z}_{\infty,\nu}^\eta := \pm\Omega_\nu^\eta \cdot \nu(\mathfrak{Z}_\infty),$$

where $\pm = \vartheta_\nu(-1)$, is such that

$$\mathrm{Log}_{\mathscr{F}_w^+(V_\nu^\dagger)}^{\eta_\nu}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}_{\infty,\nu}^\eta)) = \mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger).$$

PROOF. From the specialization properties of the map $\mathrm{Log}_{\mathscr{F}_w^+(\mathbb{T}^\dagger)}^\eta$ of Theorem 2.2.8 and the formula (2.1.3) for the values of $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_\nu^\dagger)$, we see that Proposition 2.3.2 amounts to the fact that for all but finitely many $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(\Gamma_\varpi))$ of weight 0, the values at ϕ of either of the two sides of the purported equality are the same. Since an element in $\mathcal{O}_\nu[[\Gamma_\varpi]]$ is determined by these values, the equality follows. $\square$

2.3.2. Step II: A patching argument. The following result will be a key ingredient in the argument.

LEMMA 2.3.7. *Fix a non-exceptional $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$. Then for all but finitely many $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ of weight 0, the localization map*

$$\mathrm{Sel}_{\mathrm{Gr}}(K, V^\dagger_{\nu,\phi}) \longrightarrow \bigoplus_{v|p} H^1(K_v, V^\dagger_{\nu,\phi})$$

is injective.

PROOF. This is similar to [**Cas13a**, Cor. 5.10], but we give here a complete argument. We will show that the map $\mathrm{loc}_\mathfrak{p} : \mathrm{Sel}_{\mathrm{Gr}}(K, V^\dagger_{\nu,\phi}) \longrightarrow H^1(K_\mathfrak{p}, V^\dagger_{\nu,\phi})$ is injective, the argument for $\mathrm{loc}_{\overline{\mathfrak{p}}}$ being the same.

By [**How07b**, Cor. 3.1.2], the class $\nu(\mathfrak{Z}_\infty) \in \widetilde{H}^1_{f,\mathrm{Iw}}(K, T^\dagger_\nu)$ is not $\mathcal{O}_\nu[[D_\infty]]$-torsion, and hence the image of $\nu(\mathfrak{Z}_\infty)^\phi$ in $\mathrm{Sel}_{\mathrm{Gr}}(K, V^\dagger_{\nu,\phi})$ is nonzero for all but finitely many $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$. For ϕ of weight 0, the proof of [**Hsi13**, Thm. 6.1] shows[3] that the latter nonvanishing implies that

$$\mathrm{Sel}_{\mathrm{Gr}}(K, V^\dagger_{\nu,\phi}) = F_{\nu,\phi} \cdot \nu(\mathfrak{Z}_\infty)^\phi,$$

where $F_{\nu,\phi}$ is the field extension of F_ν generated by the values of ϕ. Thus upon fixing $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ of weight 0, it suffices to show that

$$\nu(\mathfrak{Z}_\infty)^\phi \neq 0 \quad \Longrightarrow \quad \mathrm{loc}_\mathfrak{p}(\nu(\mathfrak{Z}_\infty)^\phi) \neq 0. \tag{2.3.20}$$

Let s (resp. t) be the p-order of the conductor of ϑ_ν (resp. ϕ), so that $V^\dagger_{\nu,\phi} \cong V^*_\nu$ as $G_{L_{p^{t+1},s}}$-representations, where $L_{p^{t+1},s} = H_{p^{t+1+s}}(\boldsymbol{\mu}_{p^s})$. The restriction map

$$H^1(K, V^\dagger_{\nu,\phi}) \xrightarrow{\mathrm{res}_{L_{p^{t+1},s}}} H^1(L_{p^{t+1},s}, V^\dagger_{\nu,\phi}) \cong H^1(L_{p^{t+1},s}, V^*_\nu)$$

is easily seen to be injective. By construction, the class $\mathrm{res}_{L_{p^{t+1},s}}(\nu(\mathfrak{Z}_\infty)^\phi)$ agrees with the image of (a non-zero F_ν-multiple of) a point $Q_{p^{t+1},s} \in J_s(L_{p^{t+1},s})$ under the composite map

$$\kappa_s : J_s(L_{p^{t+1},s}) \xrightarrow{e^{\mathrm{ord}} \circ \mathrm{Kum}_s} H^1(L_{p^{t+1},s}, \mathrm{Ta}^{\mathrm{ord}}_p(J_s)) \longrightarrow H^1(L_{p^{t+1},s}, V^*_\nu)$$

[3]Note that *loc.cit.* works in fact with the Bloch–Kato Selmer group $H^1_f(K, V^\dagger_{\nu,\phi})$, but since ν is non-exceptional, $\mathrm{Sel}_{\mathrm{Gr}}(K, V^\dagger_{\nu,\phi}) = H^1_f(K, V^\dagger_{\nu,\phi})$.

as described in (2.3.17), and hence

$$\nu(\mathfrak{Z}_\infty)^\phi \neq 0 \quad \Longrightarrow \quad Q_{p^{t+1},s} \otimes 1 \neq 0 \in J_s(L_{p^{t+1},s}) \otimes \mathbf{Q}. \tag{2.3.21}$$

Letting $F_{p^{t+1},s}$ denote the completion of $\imath_p(L_{p^{t+1},s}) \subset \overline{\mathbf{Q}}_p$, the composite map

$$\lambda_s : J_s(L_{p^{t+1},s}) \otimes \mathbf{Q} \longrightarrow J_s(F_{p^{t+1},s}) \otimes \mathbf{Q}_p \xrightarrow{e^{\mathrm{ord}} \circ \delta_s} H^1(F_{p^{t+1},s}, \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$$

is an injection, where δ_s is the local Kummer map. Together with the commutativity of the diagram

$$\begin{tikzcd} J_s(L_{p^{t+1},s}) \otimes \mathbf{Q} \arrow[r, "\kappa_s"] \arrow[d, "\lambda_s"] & H^1(L_{p^{t+1},s}, V_\nu^*) \arrow[d, "\mathrm{loc}_{\mathfrak{p}}"] \\ H^1(F_{p^{t+1},s}, \mathrm{Ta}_p^{\mathrm{ord}}(J_s) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p) \arrow[r] & H^1(F_{p^{t+1},s}, V_\nu^*), \end{tikzcd}$$

we deduce that

$$\begin{aligned} Q_{p^{t+1},s} \otimes 1 \neq 0 \quad &\Longrightarrow \quad \lambda_s(Q_{p^{t+1},s} \otimes 1) \neq 0 \\ &\Longrightarrow \quad \mathrm{loc}_{\mathfrak{p}}(\mathrm{res}_{L_{p^{t+1},s}}(\nu(\mathfrak{Z}_\infty)^\phi)) \neq 0, \end{aligned}$$

and combined with (2.3.21), we arrive at

$$\nu(\mathfrak{Z}_\infty)^\phi \neq 0 \quad \Longrightarrow \quad \mathrm{res}_{L_{p^{t+1},s}}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)^\phi)) \neq 0.$$

By the injectivity of the map $\mathrm{res}_{L_{p^{t+1},s}}$, this shows that the implication (2.3.20) holds, as was to be shown. $\square$

For our later reference, we record here the following immediate consequence.

Corollary 2.3.8. *For any non-exceptional $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$, the kernel of the localization map*

$$\mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T_\nu^\dagger) \longrightarrow \bigoplus_{v|p} H^1(K_{\infty,v}, \mathscr{F}_w^+(T_\nu^\dagger))$$

is $\mathcal{O}_\nu[[D_\infty]]$-torsion.

Proof. Let $\mathcal{Z}$ be the kernel of this localization map. Since $\nu(\mathfrak{Z}_\infty)$ is not $\mathcal{O}_\nu[[D_\infty]]$-torsion, $\nu(\mathfrak{Z}_\infty)^\phi \neq 0$ for all but finitely many $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$, and hence in particular $\mathrm{Sel}_{\mathrm{Gr}}(K, T_{\nu,\phi}^\dagger)$ is torsion-free. By Lemma 2.3.7, it follows that the image of $\mathcal{Z}$ under the map $\mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T_\nu^\dagger) \longrightarrow \mathrm{Sel}_{\mathrm{Gr}}(K, T_{\nu,\phi}^\dagger)$ is trivial for infinitely many ϕ, and hence $\mathcal{Z}$ is necessarily $\mathcal{O}_\nu[[D_\infty]]$-torsion, as was to be shown. $\square$

PROPOSITION 2.3.9. *Let $S \subset \mathcal{X}^{\mathrm{good}}_{\mathrm{arith},2}(\mathbb{I})$ be a finite subset, and set*

$$\mathbb{T}^\dagger_{\mathfrak{n}} := \mathbb{T}^\dagger \otimes_{\mathbb{I}} \mathbb{I}/\mathfrak{n}, \qquad \mathfrak{n} := \cap_{\nu \in S} \ker(\nu).$$

There exists a class $\mathfrak{Z}^\eta_{\infty,\mathfrak{n}} \in \widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger_{\mathfrak{n}})$ such that

$$\mathrm{Log}^{\eta_\nu}_{\mathscr{F}^+_w(V^\dagger_\nu)}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}))) = \mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger_\nu), \qquad \text{for all } \nu \in S, \tag{2.3.22}$$

where $\nu(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}})$ is the image of $\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}$ under the natural map induced by $\mathbb{T}^\dagger_S \longrightarrow V^\dagger_\nu$.

PROOF. We argue by induction on the cardinality of S, following the proof of [**Och06**, Thm. 6.11] very closely. The base case $|S| = 1$ is the content of Corollary 2.3.6. Assume that the proposition holds for a fixed S as in the statement; for any fixed $\nu' \in \mathcal{X}^{\mathrm{good}}_{\mathrm{arith},2}(\mathbb{I}) \smallsetminus S$, we will show that the proposition holds for $S \cup \{\nu'\}$ as well.

Setting $\mathfrak{n}' = \ker(\nu')$, there is an exact sequence

$$0 \longrightarrow H^1_{\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger_{\mathfrak{n}\cap\mathfrak{n}'}) \xrightarrow{\alpha} H^1_{\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger_{\mathfrak{n}}) \oplus H^1_{\mathrm{Iw}}(K_\infty, T^\dagger_{\nu'}) \xrightarrow{\beta} H^1_{\mathrm{Iw}}(K_\infty, \mathbb{T}^\dagger_{\mathfrak{n}\oplus\mathfrak{n}'}),$$

where

$$\begin{aligned} \alpha :\ & \mathfrak{Y}_{\mathfrak{n}\cap\mathfrak{n}'} \longmapsto (\mathfrak{Y}_{\mathfrak{n}\cap\mathfrak{n}'} \bmod \mathfrak{n}, \mathfrak{Y}_{\mathfrak{n}\cap\mathfrak{n}'} \bmod \mathfrak{n}'), \\ \beta :\ & (\mathfrak{Y}_{\mathfrak{n}}, \mathfrak{Y}_{\nu'}) \longmapsto (\mathfrak{Y}_{\mathfrak{n}} \bmod \mathfrak{n} \oplus \mathfrak{n}') - (\mathfrak{Y}_{\nu'} \bmod \mathfrak{n} \oplus \mathfrak{n}'). \end{aligned}$$

Let $\phi \in \mathcal{X}_{\mathrm{arith}}(\Lambda(D_\infty))$ have weight 0. Then by the definition (2.1.3) of $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}_{\nu''})$, we have on the one hand that by Corollary 2.3.6 there exists a class $\mathfrak{Z}^\eta_{\infty,\mathfrak{n}'}$ such that

$$\mathrm{Log}^{\eta_{\nu'}}_{\mathscr{F}^+_w(V^\dagger_{\nu'})}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}'}))(\phi) = \sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)} \phi^{-1}(\mathfrak{a})\phi^{-1}(\mathrm{N}\mathfrak{a}) \cdot d^{-1}\mathbf{f}^\dagger_{\nu'} \otimes \phi_{\mathfrak{a}}(x(\mathfrak{a})), \tag{2.3.23}$$

and on the other hand by assumption there exists a class $\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}$ such that

$$\mathrm{Log}^{\eta_\nu}_{\mathscr{F}^+_w(V^\dagger_\nu)}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}))(\phi) = \sum_{[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_K)} \phi^{-1}(\mathfrak{a})\phi^{-1}(\mathrm{N}\mathfrak{a}) \cdot d^{-1}\mathbf{f}^\dagger_\nu \otimes \phi_{\mathfrak{a}}(x(\mathfrak{a})), \tag{2.3.24}$$

for all $\nu \in S$. Since the q-expansions of the twists $d^{-1}\mathbf{f}^\dagger_\nu \otimes \phi_{\mathfrak{a}}$ and $d^{-1}\mathbf{f}^\dagger_{\nu'} \otimes \phi_{\mathfrak{a}}$ are congruent to each other modulo $\ker(\nu) + \ker(\nu')$, the same is true for their values at the ordinary CM points appearing in (2.3.23) and (2.3.24), and hence the class $(\phi \circ \beta)(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}, \mathfrak{Z}^\eta_{\infty,\mathfrak{n}'})$ lies in the kernel of the composite map

$$\mathrm{Sel}_{\mathrm{Gr}}(K, T^\dagger_{\nu,\phi}) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} H^1(K_{\mathfrak{p}}, \mathscr{F}^+_w(T^\dagger_{\nu,\phi})) \longrightarrow H^1(K_{\mathfrak{p}}, \mathscr{F}^+_w(V^\dagger_{\nu,\phi})) \xrightarrow{\langle \log(-), \eta'_\nu \rangle} F_{\nu,\phi}.$$

Since this map clearly factors through $\mathrm{Sel}_{\mathrm{Gr}}(K, T^\dagger_{\nu,\phi}) \longrightarrow \mathrm{Sel}_{\mathrm{Gr}}(K, V^\dagger_{\nu,\phi})$, by Lemma 2.3.7 it follows that $\beta(\mathfrak{Z}^\eta_{\infty,\mathfrak{n}}, \mathfrak{Z}^\eta_{\infty,\mathfrak{n}'}) = 0$, and hence by the exactness of the above sequence, there exists a class $\mathfrak{Z}^{\eta}_{\infty,\mathfrak{n}\cap\mathfrak{n}'}\in H^1_{\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger_{\mathfrak{n}\cap\mathfrak{n}'})$ such that

$$(\mathfrak{Z}^{\eta}_{\infty,\mathfrak{n}},\mathfrak{Z}^{\eta}_{\infty,\mathfrak{n}'})=\alpha(\mathfrak{Z}^{\eta}_{\infty,\mathfrak{n}\cap\mathfrak{n}'}).$$

By construction, this class satisfies

$$\mathrm{Log}^{\eta_\nu}_{\mathscr{F}^+_w(V^\dagger_\nu)}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}^{\eta}_{\infty,\mathfrak{n}\cap\mathfrak{n}'}))=\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger_\nu)$$

for all $\nu\in S\cup\{\nu'\}$, and the result thus follows by induction. □

2.3.3. Step III: End of proof of Theorem 2.3.1. Denote by $\mathfrak{A}$ the collection of all finite subsets of $\mathcal{X}^{\mathrm{good}}_{\mathrm{arith},2}(\mathbb{I})$ ordered by inclusion, and for each $S\in\mathfrak{A}$, let $\mathfrak{Z}^{\eta}_{\infty,\mathfrak{n}(S)}$ be the class constructed in Proposition 2.3.9. Note that if $S\subset S'$ in $\mathfrak{A}$, there is a natural map

$$H^1_{\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger_{\mathfrak{n}(S')})\longrightarrow H^1_{\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger_{\mathfrak{n}(S)}),$$

and that by construction the class $\mathfrak{Z}_{\infty,\mathfrak{n}(S')}$ is sent to $\mathfrak{Z}_{\infty,\mathfrak{n}(S)}$ under this map.

Setting

$$\mathfrak{Z}^{\eta}_{\infty}:=\varprojlim_{S}\mathfrak{Z}^{\eta}_{\infty,\mathfrak{n}(S)},\tag{2.3.25}$$

for S running over an infinite strictly ascending chain in $\mathfrak{A}$, we thus obtain a class $\mathfrak{Z}^{\eta}_{\infty}\in H^1_{\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)$ such that, for infinitely many $\nu\in\mathcal{X}^{\mathrm{good}}_{\mathrm{arith},2}(\mathbb{I})$,

$$\nu(\mathfrak{Z}^{\eta}_{\infty})\in\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,V^\dagger_\nu),\tag{2.3.26}$$

and

$$\mathrm{Log}^{\eta_\nu}_{\mathscr{F}^+_w(V^\dagger_\nu)}(\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}^{\eta}_{\infty})))=\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger_\nu).\tag{2.3.27}$$

In the proof of [**How07b**, Prop. 2.4.5] it is shown that the inclusion (2.3.26) for infinitely many $\nu\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ implies[4] that $\mathfrak{Z}^{\eta}_{\infty}\in\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)$ and from (2.3.27) it follows that

$$\mathrm{Log}^{\eta}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\mathrm{loc}_{\mathfrak{p}}(\mathfrak{Z}^{\eta}_{\infty}))=\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger),$$

as was to be shown. □

Corollary 2.3.10. *Fix an $\mathbb{I}$-basis η of $\mathbb{D}$. There exists a unit $\alpha_\eta\in\mathbb{I}^\times$ such that*

$$\mathrm{Log}^{\eta}_{\mathscr{F}^+_w(\mathbb{T}^\dagger)}(\mathrm{loc}_{\mathfrak{p}}(\alpha_\eta\cdot\mathfrak{Z}_{\infty}))=\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger).$$

Proof. Since $\mathfrak{Z}_\infty$ is not $\mathbb{I}_\infty$-torsion by Theorem 2.1.2(2), $\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)$ is torsion-free of rank 1 over $\mathbb{I}_\infty$ by [**Fou13**, Thm. 6.3], and hence the class $\mathfrak{Z}^{\eta}_{\infty}$ of Theorem 2.3.1 is

[4] With the notations of *loc.cit.*, we have $\lambda=1$ by our assumption (heeg) from the Introduction.

such that $\mathfrak{Z}_\infty^\eta = \alpha_\eta \cdot \mathfrak{Z}_\infty$ for some α_η lying in $\mathrm{Frac}(\mathbb{I}_\infty)$ a priori. Since the construction (2.3.25) of the class $\mathfrak{Z}_\infty^\eta$ shows that $\nu(\alpha_\eta)$ is a unit in $\mathcal{O}_\nu^\times$ for some $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ (see Remark 2.3.5 and Corollary 2.3.6), we see that in fact α_η lies in $\mathbb{I}^\times$, and the result follows from Theorem 2.3.1. □

2.3.4. Explicit reciprocity law. We can now deduce from Theorem 2.3.1 a result that will be one of the key ingredients to the arithmetic applications of Section 2.4.

THEOREM 2.3.11. *Let $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be a non-exceptional arithmetic prime of weight $k_\nu = 2r_\nu \geq 2$ with $r_\nu \equiv k/2 \pmod{p-1}$ and trivial wild character, and let ϕ be an anticyclotomic Hecke character of K of conductor dividing $\mathfrak{N}$ and infinity type $(\ell, -\ell)$ with $\ell \geq r_\nu$ and $\ell \equiv 0 \pmod{p-1}$. Then*

$$\mathrm{loc}_{\mathfrak{p}}(\nu(\mathfrak{Z}_\infty)^\phi) \neq 0 \quad \Longleftrightarrow \quad L(\mathbf{f}_\nu, \phi^{-1}, r_\nu) \neq 0.$$

PROOF. This follows from the combination of Corollary 2.2.9 and Corollary 2.3.10, noting that since ν is non-exceptional, neither of the factors $\left(1 - \frac{p^{r_\nu - 1}}{\nu(\mathbf{a}_p)\varpi^\ell}\right)$ or $\left(1 - \frac{\nu(\mathbf{a}_p)\varpi^\ell}{p^{r_\nu}}\right)$ appearing in the former result vanishes, and that since $\ell \geq r_\nu$, the map $\exp^*$ (see Remark 2.2.10) is bijective. □

2.4. Arithmetic applications

In this section we bound the sizes of certain Selmer groups associated to the Rankin–Selberg convolution of a cusp form with a theta series of higher weight.

2.4.1. Bounding Selmer groups. Let $f \in S_k(\Gamma_0(N))$ be a normalized p-ordinary newform of weight $k \geq 2$ and trivial nebentypus, and let $\rho_f : G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_L(V_f)$ be the self-dual twist of its associated Galois representation. Let χ be an anticyclotomic Hecke character of K of conductor dividing $\mathfrak{N}$ and of infinity type $(\ell, -\ell)$ with $\ell \geq k/2$, denote by θ_χ its associated theta series, and set

$$V_{f,\chi} := (V_f \otimes V_{\theta_\chi})|_{G_K} \cong V_f|_{G_K} \otimes \chi.$$

Since χ is anticyclotomic, the representation $V_{f,\chi}$ is *conjugate self-dual*, i.e.

$$V_{f,\chi}^*(1) \cong V_{f,\chi}^c,$$

where $V_{f,\chi}^c$ denotes the conjugate of $V_{f,\chi}$ by the non-trivial automorphism of K.

Let $T_f \subset V_f$ be a $G_{\mathbf{Q}}$-stable lattice, set $T_{f,\chi} := T_f|_{G_K} \otimes \chi$, and define $A_{f,\chi}$ by the exactness of the sequence

$$0 \longrightarrow T_{f,\chi} \longrightarrow V_{f,\chi} \longrightarrow A_{f,\chi} \longrightarrow 0. \tag{2.4.1}$$

DEFINITION 2.4.1. For every finite extension $\mathcal{K}/K$ define

$$\operatorname{Sel}_{\mathfrak{p}}(\mathcal{K}, V_{f,\chi}^c) = \ker\left(H^1(\mathcal{K}, V_{f,\chi}^c) \longrightarrow \bigoplus_v \frac{H^1(\mathcal{K}_v, V_{f,\chi}^c)}{H^1_f(\mathcal{K}_v, V_{f,\chi}^c)}\right), \tag{2.4.2}$$

where v runs over all places of $\mathcal{K}$. Here for $v \nmid p$ we put

$$H^1_f(\mathcal{K}_v, V_{f,\chi}^c) = \ker\left(H^1(\mathcal{K}_v, V_{f,\chi}^c) \longrightarrow H^1(\mathcal{K}_v^{\mathrm{nr}}, V_{f,\chi}^c)\right),$$

whereas for $v \nmid p$,

$$H^1_f(\mathcal{K}_v, V_{f,\chi}^c) = \begin{cases} H^1(\mathcal{K}_v, V_{f,\chi}^c) & \text{if } v|\bar{\mathfrak{p}}, \\ \\ 0 & \text{if } v|\mathfrak{p}. \end{cases}$$

Replacing $H^1_f(\mathcal{K}_v, V_{f,\chi}^c)$ by their images in $H^1(\mathcal{K}_v, A_{f,\chi}^c)$ (resp. preimages in $H^1(\mathcal{K}_v, T_{f,\chi}^c)$) under the map induced by (2.4.1), define $\operatorname{Sel}_{\mathfrak{p}}(\mathcal{K}, A_{f,\chi}^c) \subset H^1(\mathcal{K}, A_{f,\chi}^c)$ (resp. $\operatorname{Sel}_{\mathfrak{p}}(\mathcal{K}, T_{f,\chi}^c) \subset H^1(\mathcal{K}, T_{f,\chi})^c)$ by the corresponding analogue of (2.4.2).

In particular, the classes in $\operatorname{Sel}_{\mathfrak{p}}(K, V_{f,\chi}^c)$ are unramified outside p, satisfy no specific local condition at $\bar{\mathfrak{p}}$, and they have trivial restriction at $\mathfrak{p}$.

For $v|p$, the above local subspaces $H^1_f(\mathcal{K}_v, V_{f,\chi}^c)$ agree with

$$H^1_f(\mathcal{K}_v, V_{f,\chi}^c) := \ker\left(H^1(\mathcal{K}_v, V_{f,\chi}^c) \longrightarrow H^1(\mathcal{K}_v, V_{f,\chi}^c \otimes B_{\mathrm{cris}})\right),$$

and hence $\operatorname{Sel}_{\mathfrak{p}}(\mathcal{K}, V_{f,\chi}^c)$ is the same as the Bloch–Kato Selmer group for the $G_{\mathcal{K}}$-representation $V_{f,\chi}^c$.

CONJECTURE 2.4.2 (Bloch–Kato). $\operatorname{ord}_{s=k/2} L(f, \chi^{-1}, s) = \dim_L \operatorname{Sel}_{\mathfrak{p}}(K, V_{f,\chi}^c)$.

Using our results in Section 2.3, we are going to show how certain "rank 0" cases of Conjecture 2.4.2 follow from the following result.

THEOREM 2.4.3. *If* $\nu_f(\mathfrak{Z}_\infty)^\chi \neq 0$, *then* $\operatorname{Sel}_{\mathrm{Gr}}(K, T_{f,\chi})$ *is free of rank* 1 *over* $\mathcal{O}$.

PROOF. Since ν_f is non-exceptional, this follows from [**Fou13**, Cor. 5.21]. □

Consider the following modifications of the preceding Selmer groups, obtained by changing the local condition at the prime above p. If $v|p$, let

$$H^1_a(K_v, V_{f,\chi}) := \begin{cases} H^1(K_v, V_{f,\chi}) & \text{if } a = \emptyset, \\ H^1(K_v, \mathscr{F}_w^+(V_{f,\chi})) & \text{if } a = \mathrm{Gr}, \\ 0 & \text{if } a = 0, \end{cases}$$

and define

$$\mathrm{Sel}_{a,b}(K,V_{f,\chi}) := \ker\left(H^1(G_K^{(Np)},V_{f,\chi}) \longrightarrow \frac{H^1(K_{\mathfrak{p}},V_{f,\chi})}{H^1_a(K_{\mathfrak{p}},V_{f,\chi})} \bigoplus \frac{H^1(K_{\bar{\mathfrak{p}}},V_{f,\chi})}{H^1_b(K_{\bar{\mathfrak{p}}},V_{f,\chi})}\right),$$

where $G_K^{(Np)}$ is the Galois group of the maximal extension of K unramified outside the primes above Np. Define $\mathrm{Sel}_{a,b}(K,T_{f,\chi})$ and $\mathrm{Sel}_{a,b}(K,A_{f,\chi})$ using (2.4.1) similarly as before, and if $a=b$ set $\mathrm{Sel}_a(K,M)=\mathrm{Sel}_{a,b}(K,M)$. The same construction may be applied starting with $V^c_{f,\chi}$, so that $\mathrm{Sel}_{\mathfrak{p}}(K,V^c_{f,\chi})=\mathrm{Sel}_{0,\emptyset}(K,V^c_{f,\chi})$.

PROPOSITION 2.4.4. *If* $\mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}_\infty)^\chi)\neq 0$ *then* $\mathrm{Sel}_{\mathfrak{p}}(K,V^c_{f,\chi})=0$.

We will first need the following lemma.

LEMMA 2.4.5. *There is a noncanonical isomorphism of $\mathcal{O}$-modules*

$$\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,T^c_{f,\chi}) \cong \mathcal{O}\oplus \mathrm{Sel}_{0,\mathrm{Gr}}(K,T_{f,\chi}).$$

PROOF. Denote by Φ the quotient field of $\mathcal{O}$, and let $\pi\in\mathcal{O}$ be a uniformizer. Similarly as in [**AH06**, Prop. 1.2.3], for every $i>0$ we have a noncanonical isomorphism

$$\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,A^c_{f,\chi})[\pi^i] \cong (\Phi/\mathcal{O})^r[\pi^i]\oplus \mathrm{Sel}_{0,\mathrm{Gr}}(K,A_{f,\chi})[\pi^i], \tag{2.4.3}$$

where the integer r is given by

$$\mathrm{corank}_{\mathcal{O}} H^1(K_{\bar{\mathfrak{p}}},\mathscr{F}^+_w(A_{f,\chi})) + \mathrm{corank}_{\mathcal{O}} H^1(K_{\bar{\mathfrak{p}}},A_{f,\chi}) - \mathrm{corank}_{\mathcal{O}} H^0(K_v,A_{f,\chi})$$

in light of the Poitou–Tate duality as formulated in [**Wil95**] (see also [**DDT94**, §2.3]), and where v denotes the unique archimedean place of K. Hence $r=1$. Since the groups $\mathrm{Sel}_{a,b}(K,T_{f,\psi})$ are the π-adic Tate module of $\mathrm{Sel}_{a,b}(K,A_{f,\psi})$ taking the projective limit in (2.4.3) as $i\to\infty$ the result follows. □

PROOF OF PROPOSITION 2.4.4. It suffices to see that $\mathrm{Sel}_{\mathfrak{p}}(K,T^c_{f,\chi})$ is finite. The nonvanishing assumption clearly implies that $\nu_f(\mathfrak{Z}_\infty)^\chi\neq 0$, and hence $\mathrm{Sel}_{\mathrm{Gr}}(K,T_{f,\chi})\cong\mathcal{O}$ by Theorem 2.4.3. Also by the assumption, $\nu_f(\mathfrak{Z}_\infty)^\chi\notin \mathrm{Sel}_{0,\mathrm{Gr}}(K,T_{f,\chi})$ and since $\nu_f(\mathfrak{Z}_\infty)^\chi\in \mathrm{Sel}_{\mathrm{Gr}}(K,T_{f,\chi})$, we see that $\mathrm{Sel}_{0,\mathrm{Gr}}(K,T_{f,\chi})$ is necessarily finite. By Lemma 2.4.5, it follows that $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,T^c_{f,\chi})$ has $\mathcal{O}$-rank 1.

The class $\nu_f(\mathfrak{Z}^c_\infty)^{\chi^{-1}}$ lies in $\mathrm{Sel}_{\mathrm{Gr}}(K,T^c_f\otimes\chi^{-1})=\mathrm{Sel}_{\mathrm{Gr}}(K,T^c_{f,\chi})$, and by the "functional equation" [**How07b**, Prop. 2.3.5] we see that

$$\mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}_\infty)^\chi)\neq 0 \iff \mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}^c_\infty)^{\chi^{-1}})\neq 0.$$

Thus by our nonvanishing assumption $\nu_f(\mathfrak{Z}_\infty^c)^{\chi^{-1}}$ lies in the complement of $\mathrm{Sel}_{0,\emptyset}(K,T_{f,\chi}^c)$ in $\mathrm{Sel}_{\mathrm{Gr}}(K,T_{f,\chi}^c)\subset \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,T_{f,\chi}^c)$, and since we have shown that the latter group has $\mathcal{O}$-rank 1, it follows that $\mathrm{Sel}_{0,\emptyset}(K,T_{f,\chi}^c)$ is finite. $\square$

THEOREM 2.4.6. *Let $f\in S_k(\Gamma_0(N))$ be a normalized p-ordinary newform of weight $k\geq 2$ with $k\equiv 2 \pmod{p-1}$ and trivial nebentypus, and let χ be an anticyclotomic Hecke character of K of infinity type $(\ell,-\ell)$ with $\ell\geq k/2$ and $\ell\equiv 0 \pmod{p-1}$ and of conductor dividing $\mathfrak{N}$. If $L(f,\chi^{-1},k/2)\neq 0$ then $\mathrm{Sel}_{\mathfrak{p}}(K,V_{f,\chi}^c)=0$.*

PROOF. This is the combination of Corollary 2.3.11 and Proposition 2.4.4. $\square$

2.4.2. Iwasawa theory. Let f and ρ_f be as in §2.4.1. Let $\mathbf{f}\in\mathbb{I}[[q]]$ be the Hida family of f, and let $\nu_f\in\mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ be the arithmetic prime such that $\nu_f(\mathbf{f})$ gives the ordinary p-stabilization of f. Then there are induced $\mathcal{O}$-linear specialization maps $\mathbb{I}_\infty\longrightarrow\Lambda:=\mathcal{O}[[D_\infty]]$ and $\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,\mathbb{T}^\dagger)\longrightarrow\widetilde{H}^1_{f,\mathrm{Iw}}(K_\infty,V_f)$, both also denoted by ν_f in the following.

By Theorem 2.3.1, the "big" anticyclotomic p-adic L-function of Definition 2.1.3 is given by an element $\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger)\in\mathbb{I}_\infty\otimes_{\mathbb{I}}\mathbb{I}[\lambda^{-1}]$. In this section we give an interpretation of the p-adic L-function

$$\mathscr{L}_{\mathfrak{p}}(f):=\nu_f(\mathscr{L}_{\mathfrak{p}}(\mathbf{f}^\dagger))$$

in terms of Iwasawa theory.

For $a,b\in\{0,\mathrm{Gr},\emptyset\}$, consider the finitely generated Λ-modules

$$\mathcal{S}_{a,b}(f):=\varprojlim_t \mathrm{Sel}_{a,b}(K_t,T_f),\quad X_{a,b}(f):=\mathrm{Hom}_{\mathbf{Z}_p}(\varinjlim_t \mathrm{Sel}_{b^*,a^*}(K_t,A_f^c),\mathbf{Q}_p/\mathbf{Z}_p),$$

where the groups appearing in the right-hand sides are defined as in § 2.4.1, setting $0^*=\emptyset$, $\mathrm{Gr}^*=\mathrm{Gr}$ and $\emptyset^*=0$. If $a=b$, set $X_a(f):=X_{a,b}(f)$ and $\mathcal{S}_a(f):=\mathcal{S}_{a,b}(f)$.

The following conjecture is suggested by the generalization of Iwasawa theory developed by Greenberg in [**Gre94**]. If X is a finitely generated Λ-module, we let $\mathrm{char}_\Lambda(X)\subset\Lambda$ denote its characteristic ideal, with the convention that $\mathrm{char}_\Lambda(X)=0$ if X is not Λ-torsion. Set $X_{\mathfrak{p}}(f):=X_{0,\emptyset}(f)$.

CONJECTURE 2.4.7 (Iwasawa–Greenberg). *$X_{\mathfrak{p}}(f)$ is Λ-torsion, and*

$$\mathrm{char}_\Lambda(X_{\mathfrak{p}}(f))=(\mathscr{L}_{\mathfrak{p}}(f)^2)$$

in $\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda$.

On the other hand, Howard's extension of the Heegner point main conjecture of Perrin-Riou [**PR87a**] predicts (cf. [**How07b**, Conj. 3.3.1]):

CONJECTURE 2.4.8 (Perrin-Riou–Howard). *$\mathcal{S}_{\mathrm{Gr}}(f)$ has Λ-rank 1, and*

$$\mathrm{char}_\Lambda(X_{\mathrm{Gr}}(f)_{\mathrm{tors}}) = \mathrm{char}_\Lambda\left(\frac{\mathcal{S}_{\mathrm{Gr}}(f)}{\Lambda\cdot\nu_f(\mathfrak{z}_\infty)}\right)^2$$

in $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda$.

Following the work of Mazur–Rubin [**MR04**] and Howard [**How04a**], Fouquet has extended Kolyvagin's methods to the context of Nekovář's Selmer complexes, deducing one of the divisibilities predicted by Conjecture 2.4.8. After Theorem 2.3.1, we can relate Conjecture 2.4.7 to Conjecture 2.4.8, thus deducing from Fouquet's result one of the divisibilities predicted by the former conjecture.

THEOREM 2.4.9. *$X_{\mathfrak{p}}(f)$ is Λ-torsion, and*

$$\mathrm{char}_\Lambda(X_{\mathfrak{p}}(f)) \supseteq (\mathscr{L}_{\mathfrak{p}}(f)^2)$$

in $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda$.

PROOF. By Theorem 2.1.2(2), $\nu_f(\mathfrak{z}_\infty)$ is not Λ-torsion, and hence by [**Fou13**, Cor. 6.19], $\mathcal{S}_{\mathrm{Gr}}(f)$ has Λ-rank 1, and

$$\mathrm{char}_\Lambda(X_{\mathrm{Gr}}(f)_{\mathrm{tors}}) \supseteq \mathrm{char}_\Lambda\left(\frac{\mathcal{S}_{\mathrm{Gr}}(f)}{\Lambda\cdot\nu_f(\mathfrak{z}_\infty)}\right)^2 \quad \text{in } \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda.$$

By elementary properties of the characteristic ideal, it follows that

$$\mathrm{char}_\Lambda(X_{0,\mathrm{Gr}}(f)_{\mathrm{tors}}) \supseteq \mathrm{char}_\Lambda\left(\frac{\mathcal{S}_{\mathrm{Gr},\emptyset}(f)}{\Lambda\cdot\nu_f(\mathfrak{z}_\infty)}\right)^2 \quad \text{in } \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda. \tag{2.4.4}$$

Arguing as in [**AH06**, Thm. 1.2.2] (see also Lemma 2.4.5) we find

$$\begin{aligned}\mathrm{rank}_\Lambda(\mathcal{S}_{\emptyset,\mathrm{Gr}}(f)) &= \mathrm{rank}_\Lambda(X_{\emptyset,\mathrm{Gr}}(f)) \\ &= 1 + \mathrm{rank}_\Lambda(X_{0,\mathrm{Gr}}(f)) = 1 + \mathrm{rank}_\Lambda(\mathcal{S}_{0,\mathrm{Gr}}(f)).\end{aligned} \tag{2.4.5}$$

By Theorem 2.1.2(2) and Corollary 2.3.8, the class $\mathrm{loc}_{\bar{\mathfrak{p}}}(\nu_f(\mathfrak{z}_\infty))$ is not Λ-torsion. Since $\mathcal{S}_{\mathrm{Gr}}(f) \subset \mathcal{S}_{\emptyset,\mathrm{Gr}}(f)$ is torsion-free of Λ-rank 1, it follows from (2.4.5) that $\mathrm{rank}_\Lambda(\mathcal{S}_{0,\mathrm{Gr}}(f)) = 0$ and that $X_{0,\mathrm{Gr}}(f) = X_{0,\mathrm{Gr}}(f)_{\mathrm{tors}}$.

Poitou–Tate duality gives rise to the exact sequence

$$0 \longrightarrow \frac{\mathcal{S}_{\mathrm{Gr},\emptyset}(f)}{\mathcal{S}_{0,\emptyset}(f)} \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}}, \mathscr{F}^+_w(T_f)) \longrightarrow X_{\mathfrak{p}}(f) \longrightarrow X_{0,\mathrm{Gr}}(f) \longrightarrow 0. \tag{2.4.6}$$

Again by Theorem 2.1.2(2) and Corollary 2.3.8, $\mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}_\infty))$ is not Λ-torsion, and hence from (2.4.6), it follows that $\mathcal{S}_{0,\emptyset}(f) = 0$. Thus we arrive at the exact sequence

$$0 \longrightarrow \frac{\mathcal{S}_{\mathrm{Gr},\emptyset}(f)}{\Lambda\cdot\nu_f(\mathfrak{Z}_\infty)} \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \frac{H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+_w(T_f))}{\Lambda\cdot\mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}_\infty))} \longrightarrow X_{\mathfrak{p}}(f) \longrightarrow X_{0,\mathrm{Gr}}(f) \longrightarrow 0,$$

which shows that $X_{\mathfrak{p}}(f)$ is Λ-torsion, and together with (2.4.4) it implies that

$$\mathrm{char}_\Lambda(X_{\mathfrak{p}}(f)) \;\supseteq\; \mathrm{char}_\Lambda\left(\frac{H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+_w(T_f))}{\Lambda\cdot\mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}_\infty))}\right)^2 \quad \text{in } \mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda. \tag{2.4.7}$$

Finally, by Theorem 2.3.1 the p-adic regulator map $\mathrm{Log}^{\eta_{\nu_f}}_{\mathscr{F}^+_w(V_f)}$ sends $\mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}_\infty))$ (times a unit in Λ) to $\mathscr{L}_{\mathfrak{p}}(f)$, thus inducing a Λ-linear isomorphism

$$\frac{H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+_w(T_f))}{\Lambda\cdot\mathrm{loc}_{\mathfrak{p}}(\nu_f(\mathfrak{Z}_\infty))} \xrightarrow{\sim} \frac{\Lambda}{\Lambda\cdot\mathscr{L}_{\mathfrak{p}}(f)}, \tag{2.4.8}$$

and combining (2.4.7) with (2.4.8), Theorem 2.4.9 follows. $\square$

Let χ be as in §5.1, and let $\mathrm{Tw}_{\chi^{-1}} : \Lambda \xrightarrow{\sim} \Lambda$ be the $\mathcal{O}$-linear isomorphism given by $\gamma \mapsto \chi^{-1}(\gamma)\gamma$ for $\gamma \in D_\infty$. Define

$$\mathscr{L}_{\mathfrak{p}}(f,\chi) := \mathrm{Tw}_{\chi^{-1}}(\mathscr{L}_{\mathfrak{p}}(f))$$

and

$$X_{\mathfrak{p}}(f,\chi) := \mathrm{Hom}_{\mathbf{Z}_p}(\varinjlim_t \mathrm{Sel}_{\mathfrak{p}}(K_t, A^c_{f,\chi}), \mathbf{Q}_p/\mathbf{Z}_p).$$

Corollary 2.4.10. *$X_{\mathfrak{p}}(f,\chi)$ is Λ-torsion, and*

$$\mathrm{char}_\Lambda(X_{\mathfrak{p}}(f,\chi)) \;\supseteq\; (\mathscr{L}_{\mathfrak{p}}(f,\chi)^2) \quad \textit{in } \mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda.$$

Proof. The module $X_{\mathfrak{p}}(f,\chi)$ is the twist $X_{\mathfrak{p}}(f)\otimes\chi$ as defined in [**Rub00**, §6.1], and hence $\mathrm{char}_\Lambda(X_{\mathfrak{p}}(f,\chi)) = \mathrm{Tw}_{\chi^{-1}}(\mathrm{char}_\Lambda(X_{\mathfrak{p}}(f)))$. Thus by the commutativity of the diagram

$$\begin{tikzcd}
H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+_w(T_f)) \arrow[rr, "\mathrm{Log}^{\eta_{\nu_f}}_{\mathscr{F}^+_w(T_f)}"] & & \Lambda \arrow[d, "\mathrm{Tw}_{\chi^{-1}}"] \\
H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+_w(T_f))\otimes\chi \cong H^1_{\mathrm{Iw}}(K_{\infty,\mathfrak{p}},\mathscr{F}^+_w(T_{f,\chi})) \arrow[u, "\otimes\chi^{-1}"] \arrow[rr, "\mathrm{Log}^{\eta_{\nu_f}}_{\mathscr{F}^+_w(T_{f,\chi})}"] & & \Lambda,
\end{tikzcd}$$

the result follows from Theorem 2.4.9. $\square$

CHAPTER 3

Conclusion

Summary

In this last chapter we propose a few lines of investigation suggested by the problems and ideas explored in this thesis, and rise a number of questions and conjectures. For some of these, we are admittedly being rather speculative.

Future directions

Many results are known on the arithmetic of Heegner points, both in the classical setting of Gross–Zagier and in its subsequent generalizations, where the "Heegner hypothesis" (in which we have placed ourselves throughout this thesis) is relaxed by working on Shimura curves attached to an appropriate non-split quaternion algebra over $\mathbf{Q}$. After Howard's construction of big Heegner points, and its extension by Longo–Vigni to the quaternionic setting, a natural line of enquiry is the study of the extent to which these results may be extended to their "big" counterparts over Hida families. Some key steps in this direction were already undertaken by Howard in [**How07b**] and [**How07a**], and this thesis might be seen as a further development of this study in which p-adic L-functions are introduced in the form of two different $\mathbb{I}$-adic Gross–Zagier formulae for big Heegner points, namely Theorem 1.5.1 and Theorem 2.3.1.

In the following paragraphs we indicate some natural extensions of these results and their potential arithmetic applications, as we would like to pursue in our future work.

3.1. Specializations at exceptional primes

The study of the specializations of the big Heegner point at exceptional primes of the Hida family has been completely avoided throughout this thesis, but we expect that such study will have applications to *an anticyclotomic analogue* of the p-adic Birch and Swinnerton-Dyer conjecture of Mazur–Tate–Teitelbaum [**MTT86**] in the rank 1 case for primes p of split multiplicative reduction. As we outline below, our approach is reminiscent of the strategy taken in [**GS93**] in their proof of the rank zero case of the original (cyclotomic) conjecture, with a twisted form of the $\mathbb{I}$-adic Gross–Zagier formula of Theorem 1.5.1 playing the role of the "improved" p-adic L-function of Greenberg–Stevens.

Suppose for simplicity that the imaginary quadratic field K has class number 1. Let $\pi \in \mathcal{O}_K$ be a generator of the prime ideal $\mathfrak{p}$ of K above p, and denote by ψ and ϕ the Hecke characters of K defined by

$$\psi(\mathfrak{a}) = \alpha, \qquad \phi(\mathfrak{a}) = \alpha/\bar{\alpha}, \qquad \text{if } \mathfrak{a} = \alpha\mathcal{O}_K,$$

respectively. Denote by $\Phi : G_K \longrightarrow \mathbb{I}[[D_\infty]]^\times$ the "universal" anticyclotomic character sending each $g \in G_K$ to the group-like element in $\mathbb{I}[[D_\infty]]^\times$ associated with $\bar{g}^{1/2} \in D_\infty$, where $\bar{g}$ is the natural image of g in D_∞, and note that if $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has even weight $2r_\nu \geq 2$, then $\Phi_\nu = \phi^{r_\nu - 1}$.

For $r \geq 1$, recall from Section 1.2 the generalised Heegner cycle $\Delta_r^{\mathrm{bdp}} \in \mathrm{CH}^{2r-1}(X_r)_0(K)$ on the Kuga–Sato variety $X_r = W_r \times A^{2r-2}$. For any $0 \leq j < r$, the cycle $W_r \times A^{r-1}$, seen as a subvariety of $W_r \times X_r = W_r \times W_r \times (A^2)^{r-1}$ via the map

$$(\mathrm{id}_{W_r}, \mathrm{id}_{W_r}, (\mathrm{id}_A, \mathrm{id}_A)^j, (\mathrm{id}_A, \sqrt{-D})^{r-1-j}),$$

induces a correspondence

$$\Pi_r^j : \mathrm{CH}^{2r-1}(X_r)_0(K) \longrightarrow \mathrm{CH}^r(W_r)_0(K)$$

sending $[\Delta] \longmapsto [\pi_{W*}\pi_X^*\Delta]$.

On the other hand, if $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ has even weight $k_\nu = 2r_\nu \geq 2$, the non-vanishing of $\nu(\mathfrak{Z}) = \nu(\mathfrak{Z}_\infty)^{\mathbb{1}}$ predicted by [**How07b**, Conj. 3.4.1] should hold if and only if

$$\nu(\mathfrak{Z}_\infty)^{\phi^j} \neq 0, \qquad \text{for some } -r_\nu < j < r_\nu.$$

In that case, the results and methods exploited in this thesis would lead to the following "twisted" version of Theorem 1.4.12.

Proposed Theorem 3.1.1. *Together with Assumptions 1.4.11, assume that*

$$L'(1, \mathbf{f}_{\nu'}, \chi_{\nu'}) \neq 0$$

for some $\nu' \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight 2 and non-trivial nebentypus. Then for all but finitely many $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ of weight $2r_\nu > 2$ with $2r_\nu \equiv k \pmod{2(p-1)}$ and trivial nebentypus,

$$\langle \nu(\mathfrak{Z}_\infty^\Phi)^{\mathbb{1}}, \nu(\mathfrak{Z}_\infty^\Phi)^{\mathbb{1}}\rangle_K = \left(1 - \frac{\bar{\pi}^{2r_\nu-2}}{\nu(\mathbf{a}_p)}\right)^4 \cdot \langle \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Pi_{r_\nu}^{r_\nu-1}\Delta_{r_\nu}^{\mathrm{bdp}}), \Phi^{\text{ét}}_{\mathbf{f}^\sharp_\nu,K}(\Pi_{r_\nu}^{r_\nu-1}\Delta_{r_\nu}^{\mathrm{bdp}})\rangle_K, \tag{3.1.1}$$

where $\langle\ ,\ \rangle_K$ is the cyclotomic p-adic height pairing on $H^1_f(K, V_{\mathbf{f}^\sharp_\nu}(\psi^{2r_\nu-2}))$.

Notice that, as opposed to the p-adic multiplier appearing in (1.4.25), the factor

$$\mathcal{E}_\nu(\mathbf{f}^\natural \otimes K) := \left(1 - \frac{\bar{\pi}^{2r_\nu-2}}{\nu(\mathbf{a}_p)}\right)$$

appearing in (3.1.1) depends p-adic analytically on $\nu \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I}) \subset \mathrm{Spf}(\mathbb{I})(\overline{\mathbf{Q}}_p)$.

As expressed in [**BDP13**, §2.4], one expects that the étale Abel–Jacobi images of the generalised Heegner cycles Δ_r^{bdp} bear a relation with the p-adic L-function $\mathcal{L}_p(f_{2r} \otimes K)$ of [**Nek95**] similar to that of the classical Heegner cycles in Nekovář's p-adic Gross–Zagier formula. In fact, under the simplifying assumptions of this section we propose the following.

Conjecture 3.1.2. *Let $f \in S_{2r}(\Gamma_0(N))$ be a p-ordinary eigenform of weight $2r \geq 2$, and let $\alpha_p(f)$ be the root of $X^2 - a_p(f)X + p^{2r-1}$ which is a p-adic unit. Then*

$$\frac{d}{ds}\mathcal{L}_p(f \otimes K)(\phi^{r-1}\langle\rho_{\mathrm{cyc}}\rangle^s)|_{s=0} = \left(1 - \frac{\bar{\pi}^{2r-2}}{\alpha_p(f)}\right)^4 \langle \Phi^{\text{ét}}_{f,K}(\Pi_r^{r-1}\Delta_r^{\mathrm{bdp}}), \Phi^{\text{ét}}_{f,K}(\Pi_r^{r-1}\Delta_r^{\mathrm{bdp}})\rangle_K,$$

where $\langle\ ,\ \rangle_K$ *is the cyclotomic* p*-adic height pairing on* $H^1_f(K, V_{\mathbf{f}^\sharp_\nu}(\psi^{2r_\nu-2}))$.

Now assume $\nu_o \in \mathcal{X}_{\mathrm{arith}}(\mathbb{I})$ is exceptional with $\nu_o(\mathbf{a}_p) = 1$, so that in particular $k_{\nu_o} = 2$, and set $f_o := \nu_o(\mathbf{f})$. Then

$$\mathcal{E}_{\nu_o}(\mathbf{f}^\natural \otimes K) = 0,$$

i.e. the left-hand side of (3.1.1) has an *exceptional zero*, and one then hopes to recover the right-hand side of (3.1.1) from the *second* cyclotomic derivative of $\mathcal{L}_p(f_o \otimes K)$ at $\mathbb{1}_K$.

We believe that a proof of Conjecture 3.1.2 would follow without major difficulties from an adaptation of the methods of [**Nek95**] to generalised Heegner cycles. As a consequence, we could then show the following result.

Proposed Corollary 3.1.3. *With notations and assumptions as in Theorem 1.4.12, there is a factorization*

$$\mathcal{L}'_p(\mathbf{f}^\natural \otimes K) = \mathcal{E}(\mathbf{f}^\natural \otimes K)^4 \cdot \widetilde{\mathcal{L}}'_p(\mathbf{f}^\natural \otimes K) \pmod{\mathbb{I}^\times}.$$

Moreover, the function $\widetilde{\mathcal{L}}'_p(\mathbf{f}^\natural \otimes K)$ *is such that*

$$\nu_o(\widetilde{\mathcal{L}}'_p(\mathbf{f}^\natural \otimes K)) = \langle \Phi^{\text{ét}}_{f_o,K}(\Delta_1^{\mathrm{heeg}}), \Phi^{\text{ét}}_{f_o,K}(\Delta_1^{\mathrm{heeg}}) \rangle_K.$$

In other words, $\widetilde{\mathcal{L}}'_p(\mathbf{f}^\sharp \otimes K)$ is an "improved" derivative p-adic L-function, which one would hope to exploit, in a similar fashion as in [**GS93**], to obtain progress towards an anticyclotomic analogue of the following conjecture, deduced from the combination of the classical Birch and Swinnerton-Dyer conjecture and its p-adic variant by [**MTT86**] in the exceptional rank 1 case.

Conjecture 3.1.4. *Let* $E/\mathbf{Q}$ *be an elliptic curve with split multiplicative reduction at* p*, and assume* $\mathrm{ord}_{s=1} L(E,s) = 1$*. There exists a nontorsion point* $P_E \in E(\mathbf{Q}) \otimes \mathbf{Q}$ *such that*

$$\frac{d^2}{ds^2} L_p^{\mathrm{MTT}}(f_E, s)|_{s=1} = \mathscr{L}(f_E) \frac{\langle P_E, P_E \rangle_p}{\langle P_E, P_E \rangle_\infty} L'(E,1),$$

where $L_p^{\mathrm{MTT}}(f_E, s)$ *is the cyclotomic* p*-adic* L*-function constructed in* [**MTT86**]*,* $\langle\ ,\ \rangle_p$ *and* $\langle\ ,\ \rangle_\infty$ *are the cyclotomic and Neron–Tate height pairings on* $E(\mathbf{Q}) \otimes \mathbf{Q}$ *respectively, and* $\mathscr{L}(f_E)$ *is the* L*-invariant of* $E/\mathbf{Q}_p$.

3.2. Big Heegner points and Kato elements

Let K be an imaginary quadratic field as in the Introduction to Chapter 2, and denote by K_∞^{cyc} the unique $\mathbf{Z}_p^2$-extension of K, which can be obtained as the compositum of the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K and the cyclotomic $\mathbf{Z}_p$-extension $K^{\rm cyc}/K$. Set

$$G_\infty := \mathrm{Gal}(K_\infty^{\rm cyc}/K).$$

We also let f, V_f, $\mathbf{f}$, and $\mathbb{T}$ be as in the Introduction to Chapter 2, but we restrict now to the case where the weight of f is $k = 2$. We begin by recalling the following conjecture, largely motivated by the fascinating work [**PR95**] of Perrin-Riou.

CONJECTURE 3.2.1 (Loeffler–Zerbes). *There is a special class $\mathfrak{c}_{f,\infty} \in \bigwedge^2 H^1_{\rm Iw}(K_\infty^{\rm cyc}, V_f)$ such that*

$$(\mathcal{L}^{G_\infty}_{\mathfrak{p},V_f} \wedge \mathcal{L}^{G_\infty}_{\bar{\mathfrak{p}},V_f})(\mathfrak{c}_{f,\infty}) = \mathcal{L}_p(f \otimes K) \pmod{\mathcal{O}_L^\times},$$

where for each $v|p$ in K, $\mathcal{L}^{G_\infty}_{v,V_f}$ is the two-variable p-adic regulator map of [**LZ11**, Thm. 4.7]*, and $\mathcal{L}_p(f \otimes K) \in \mathcal{O}_L[[G_\infty]]$ is the two-variable p-adic L-function constructed in* [**PR87b**].

The ongoing work [**LLZ13**] of Lei–Loeffler–Zerbes on the construction of a cyclotomic Euler system for the Rankin–Selberg convolution of two modular forms of weight 2 is expected to yield substantial progress towards an eventual proof of Conjecture 3.2.1.

Inspired by a conjecture of Perrin-Riou [**PR93**] relating the Beilinson–Kato elements to rational points on an elliptic curve, one expects a relation between the conjectural class $\mathfrak{c}_{f,\infty}$ and the Kummer images of Heegner points.

CONJECTURE 3.2.2. *The class $\mathfrak{c}_{f,\infty}$ predicted by Conjecture 3.2.1 satisfies*

$$\mathrm{Cor}_{K_\infty^{\rm cyc}/K_\infty}(\mathfrak{c}_{f,\infty}) = \nu_f(\mathfrak{Z}_\infty^{\otimes 2})$$

up to an explicit element in $L^\times$.

It is natural to upgrade the preceding two conjectures over the entire Hida family:

CONJECTURE 3.2.3. *There exists a big special class $\mathfrak{C}_\infty \in \bigwedge^2 H^1_{\rm Iw}(K_\infty^{\rm cyc}, \mathbb{T})$ such that*

$$(\mathrm{Log}^{G_\infty}_{\mathscr{F}_{\mathfrak{p}}\mathbb{T}} \wedge \mathrm{Log}^{G_\infty}_{\mathscr{F}_{\bar{\mathfrak{p}}}\mathbb{T}})(\mathfrak{C}_\infty) = \mathcal{L}_p(\mathbf{f} \otimes K) \pmod{\mathbb{I}^\times},$$

where for each $v|p$ in K, $\mathrm{Log}^{G_\infty}_{\mathscr{F}_v\mathbb{T}}$ is a three-variable regulator map of the proof of Theorem 2.2.8, and $\mathcal{L}_p(\mathbf{f} \otimes K) \in \mathbb{I}[[G_\infty]]$ is the three-variable p-adic L-function of [**SU13**, §12.3]*. Moreover,*

$$\mathrm{Cor}_{K_\infty^{\rm cyc}/K_\infty}(\mathrm{Tw}_{\Theta_1}(\mathfrak{C}_\infty)) = \mathfrak{Z}_\infty^{\otimes 2}$$

up to an explicit element in $\mathbb{I}^\times$, where Tw_{Θ_1} is defined as in (2.2.8).

3.3. Quaternionic settings and others

Howard's construction of big Heegner points has been generalized by Longo–Vigni [**LV11**] to arbitrary quaternion algebras over $\mathbf{Q}$. A remarkable feature of their work is the ability to give constructions treating the definite and the indefinite cases on an equal footing; as expressed by the authors themselves in *loc.cit.*, this holds the promise of being a first step towards an eventual development in a Hida-theoretic context of the program carried out by Bertolini–Darmon in a series of papers[1] where the interplay between the definite and indefinite settings plays a crucial role in the arguments (cf. [**How06**]).

In an independent line of investigation, Fouquet [**Fou13**] has constructed an analogue Howard's big Heegner points for *indefinite* quaternion algebras over a totally real field F. To briefly describe his construction, recall that the (2-dimensional) Galois representation ρ_f associated to Hilbert modular eigenforms f over F is not found in the étale cohomology of a Hilbert modular variety, but rather on the étale cohomology of an appropriate Shimura curve, at least when either of the following conditions is satisfied:

- $[F:\mathbf{Q}]$ is odd, or
- there exists a finite place v of F such that $\pi(f)_v$ special or supercuspidal,

where $\pi(f)$ is the automorphic representation of $\mathbf{GL}_2(\mathbb{A}_F)$ associated with f.

Indeed, either of these conditions guarantees that $\pi(f)$ arises a the Jacquet–Langlands lift of an automorphic form on an indefinite quaternion algebra over F. One can then construct ρ_f from the étale cohomology of the associated Shimura curves, and Fouquet's construction (as well as that of Longo–Vigni in the indefinite case) is obtained by taking certain twisted Kummer images of CM points over a tower of these Shimura curves with growing $\Gamma_1(p^s)$-level structure, in complete analogy with Howard's.

Let E be a CM field extension of F, and fix a CM type Σ for E/F, i.e. a set Σ of embeddings $F \hookrightarrow \overline{\mathbf{Q}}$ with the property that

$$\Sigma \cup \overline{\Sigma} = \mathrm{Hom}(F, \overline{\mathbf{Q}}) \qquad \text{and} \qquad \Sigma \cap \overline{\Sigma} = \emptyset.$$

Under the assumption that Σ is *ordinary* at $\imath_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, meaning that $\imath_p \circ \sigma \neq \imath_p \circ \overline{\tau}$ for all $\sigma \in \Sigma$, $\overline{\tau} \in \overline{\Sigma}$, the work of Hsieh [**Hsi12**] constructs in this level of generality an analogue of the anticylotomic p-adic L-function $\mathscr{L}_{\mathfrak{p}}(f)$ of Theorem 2.1.4. On the other hand, Howard has extended in [**How04b**] his anticyclotomic Kolyvagin system arguments to prove an analogue of Perrin-Riou's main conjecture for Heegner points on Hilbert modular varieties, assuming that there is a unique prime of F above p. Thus, at least under

[1]Starting with [**BD96**], and having perhaps [**BD05**] as one of its most beautiful landmarks.

this assumption on p, it seems that many of the constructions from Section 2.2 and the arguments from Sections 2.3 and 2.4 may be extended to prove analogues of Theorems 2.4.6 and Theorem 2.4.9 for ordinary Hilbert modular forms of parallel weights, with Fouquet's big cohomology classes (or a slight modification thereof) playing the role of Howard's big Heegner points in this thesis.

In light of the well-known absence of a direct analogue of modular units[2] for Hilbert modular varieties when $F \neq \mathbf{Q}$, these hopefully future developments will represent the first (at least to our knowledge) unconditional realizations of Perrin-Riou's approach to p-adic L-functions (see [**PR95**] and [**Rub00**, §7]) beyond the cases where the base field is $\mathbf{Q}$ or an imaginary quadratic field.

Of course, motivated by Stark's conjectures, there are further conjectural realizations of this approach to p-adic L-functions over a general totally real base field F. In particular, and not quite irrelevantly to the theme of this thesis, Darmon's p-adic construction [**Dar01**] of the so-called *Stark–Heegner points* attached to real quadratic fields where p stays prime (and generalized by Matt Greenberg in [**Gre09**] to totally real fields with $[F : \mathbf{Q}] > 2$), and their higher dimensional analogue by Rotger–Seveso [**RS12**], the so-called *Darmon cycles*, are expected to have a similar connection to p-adic L-functions as we have exhibited in this thesis for classical Heegner points and Heegner cycles.

We thus feel naturally led to consider the following problem:

Give a p-adic construction of "big" Stark–Heegner points attached to Hida families, and relate their arithmetic specializations to "classical" Stark–Heegner points and Darmon cycles.

It seems to us that such a desirable construction will be preceded by an extension of the constructions of [**Dar01**] and [**RS12**], which make crucial use of the special features of the multiplicative reduction setting, to primes p of good reduction, and we would like to believe that a study of this problem[3] might lead to valuable insights into the elusive properties that Stark–Heegner points and Darmon cycles are conjectured to share with the objects of study in this thesis.

[2] And as a result, of Beilinson–Kato elements.

[3] And of related ones, such as the connection of these constructions to classical and p-adic L-functions.

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