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Towards a p-adic theory of harmonic weak Maass forms

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Abstract

Harmonic weak Maass forms are instances of real analytic modular forms which have recently found applications in several areas of mathematics. They provide a framework for Ramanujan's theory of mock modular forms ([Ono08]), arise naturally in investigating the surjectivity of Borcherds' singular theta lift ([BF04]), and their Fourier coefficients seem to encode interesting arithmetic information ([BO]). Until now, harmonic weak Maass forms have been studied solely as complex analytic objects. The aim of this thesis is to recast their definition in more conceptual, algebro-geometric terms, and to lay the foundations of a p-adic theory of harmonic weak Maass forms analogous to the theory of p-adic modular forms formulated by Katz in the classical context. This thesis only discusses harmonic weak Maass forms of weight 0. The treatment of more general integral weights requires no essentially new idea but involves further notational complexities which may obscure the main features of our approach. This more general theory is presented in the article [CD], to which this thesis may serve as a motivated introduction.

Abrégé

Les formes de Maass faiblement harmoniques ont récemment trouvé des applications dans plusieurs domaines des mathmatiques. Elles fournissent un cadre pour la thorie des "Mock Modular forms" de Ramanujan ([Ono08]), surviennent naturellement dans l'étude de la surjectivité de la correspondance de Borcherds ([BF04]), et leurs coefficients de Fourier semblent donner des informations arithmétiques sur les dériveées des tordues quadratiques de certaines fonctions L associées aux formes modulaires ([BO]). Jusqu'à présent, les formes de Maass faiblement harmoniques ont uniquement été étudiées en tant qu' objets analytiques sur les nombres complexes. L'objectif de cette thése est de les décrire dans un cadre algébrique plus conceptuel, et de jeter les bases d'une théorie p-adique des formes de Maass faiblement harmoniques, par analogie avec le point de vue géomeétrique de Katz sur la théorie des formes modulaires p-adiques. Cette thèse traite uniquement du cas des formes de Maass faiblement harmoniques de poids 0. Le traitement plus gééral des formes de poids entier négatif, qui ne nécessite aucune idée essentiellement nouvelle, sera décrit dans l'article ([CD]), auquel cette thèse peut servir d'introduction.

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CONTENTS

INTRODUCTION

The theory of harmonic weak Maass forms has its roots in Ramanujan's work on the partition function $p(n)$. This arithmetic function is defined on $\mathbb{Z}_{\geq 1}$ by:

$$p(0) = 1$$
$$p(n) = \#\{\text{non increasing sequences of positive integers whose members add up to } n\}.$$

Leonhard Euler was the first to consider the generating function

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots,$$

for which he derived the formal identity

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

If we let $q = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$ a variable in the complex upper half-plane, then the Euler product expansion of $P(q)$ ties together this purely combinatorial object with the classical theory of modular forms. More precisely, we have

$$\frac{1}{\eta(\tau)} = \sum_{n=0}^{\infty} p(n)q^{n-\frac{1}{24}}$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=0}^{\infty} (1-q^n)$$

is Dedekind's eta function, a weight 1/2 modular form of level 1.

Almost two centuries after Euler, Srinivasa Ramanujan became interested in the combinatorics of the partition function. He derived the identity

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^m)^2}, \tag{0.1}$$

whose proof can be found, for example, in Ken Ono's survey [Ono08]. Just as the Euler product expansion connects the partition function to the theory of modular forms, Ramanujan expected the right-hand side of (0.1) to belong to a meaningful family of complex analytic functions. In his last notebook, he began studying q-series of the form

$$\Omega(t, q) = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-tq)^2(1-tq^2)^2\cdots(1-tq^m)^2},$$

in search of an underlying theory analogous to that of modular forms. He named these infinite series, which do not in general satisfy the transformation laws of modular forms, *mock modular forms*.

After Ramanujan's premature death the notion of mock modular forms continued to baffle mathematicians for several decades. The resemblance with the theory of modular forms suggested a relation between the two, but there was no obvious way to relate the transformation properties of one family to another. It was Sanders Zwegers, in his 2002 Ph.D. thesis written under the supervision of Don Zagier, who first saw a way to relate mock modular forms to the classical theory of modular forms. Roughly speaking, Zwegers takes a mock modular form and adds to it a suitable anti-holomorphic period integral of a cusp form. The result is a non-holomorphic, but still smooth modular form which satisfies certain hyperbolic Laplacian differential equations. These harmonic modular forms turn out to be examples of harmonic weak Maass forms, which were first introduced by Bruinier and Funke [BF04] in 2004.

Zwegers's work was seminal in establishing the first connection between mock modular forms and classical modular forms. Subsequently, the theory of harmonic weak Maass forms has found several applications, described in the survey article [Ono08]. In particular, Bruinier and Ono discovered in [BO] that the Fourier coefficients of the holomorphic part of certain harmonic weak Maass forms of weight 1/2 associated to a weight 2 cusp form f encode information about the vanishing of the derivatives of L-series attached to f. These kind of results motivate a further study of the arithmetic properties of the Fourier coefficients of harmonic weak Maass forms, which is the aim of this thesis.

In Chapter 1 we state the background notions from algebraic geometry that are necessary to understand our geometric theory of harmonic weak Maass forms. The reader who is not familiar with those notions is encouraged to look at the numerous references provided, especially those about the Hodge filtration and the Hodge decomposition Theorem. Chapter 2 is devoted to the complex analytic theory of harmonic weak Maass forms of weight 0. In Section 2.2 we present our geometric point of view, and then proceed in re-interpreting some known results in the literature under this new point of view. In particular, Theorem 2.6 and Theorem 2.25 answer in the affirmative two conjectures made by the authors of [BOR08]. In Chapter 3 we systematically develop a theory of p-harmonic weak Maass forms of weight 0. These forms are a subspace of the space of overconvergent p-adic modular forms of weight 0. As an application, we re-prove a theorem of [GKO09] in the weight 2 case. In the final chapter, titled 'Further directions' we indicate how our geometric theory of harmonic weak Maass forms can be extended to more general weights and what kind of applications would such a theory entertain. The contents of this final chapter will also be the subject of further joint work with Henri Darmon.

1 BACKGROUND NOTIONS

In this chapter we recall basic facts from algebraic geometry. In particular, we recall the Hodge filtration on the de Rham cohomology of algebraic curves and the Hodge decomposition Theorem for compact Riemann surfaces. These will be the main ingredients of our description of harmonic weak Maass forms.

1.1 ALGEBRAIC DE RHAM COHOMOLOGY

Let X be a nonsingular algebraic variety defined over a field k of characteristic 0 (by this we mean a smooth integral scheme of finite type over $\mathrm{Spec}(k)$). If $\mathcal{F}$ is a sheaf on X and $U \subset X$ is a Zariski open subset we denote by $\mathcal{F}(U)$ the sections of $\mathcal{F}$ over U and by $H^i(X, -)$ the usual sheaf cohomology functors. Sometimes we will denote by $\Gamma(X, -)$ the global sections functor.

Denote by $\mathcal{O}_{X/k}$ the structure sheaf of X and by $\Omega^i_{X/k}$ the sheaf of regular differential i-forms on X. The de Rham complex

$$0 \to \mathcal{O}_{X/k} \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \to \dots$$

gives rise to the *algebraic de Rham cohomology* groups (k-vector spaces, in fact)

$$H^i_{\mathrm{dR}}(X/k) := (\mathbb{R}^i\Gamma(X, -))(0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \to \dots)$$

obtained by applying the i-th right hyperderived functor to the global sections of the de Rham complex.

In practice, we can compute these groups as follows. Fix an affine Zariski open cover $\mathcal{U} = \{U_\alpha\}$ for X. For each sheaf $\Omega^i_{X/k}$ denote by $(C^\bullet(\Omega^i_{X/k}), \delta)$ the Cech resolution of $\Omega^i_{X/k}$ with respect to $\mathcal{U}$. The exterior derivative d induces differentials

$$d : C^j(\Omega^i_{X/k}) \longrightarrow C^j(\Omega^{i+1}_{X/k})$$

for all i and j and we obtain a double complex of abelian groups $(C^\bullet(\Omega^\bullet_{X/k}), d, \delta)$:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
C^2(\mathcal{O}_{X/k}) & \longrightarrow & C^2(\Omega^1_{X/k}) & \longrightarrow & C^2(\Omega^2_{X/k}) & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
C^1(\mathcal{O}_{X/k}) & \longrightarrow & C^1(\Omega^1_{X/k}) & \longrightarrow & C^1(\Omega^2_{X/k}) & \longrightarrow & \dots \\
\delta\uparrow & & \uparrow & & \uparrow & & \\
C^0(\mathcal{O}_{X/k}) & \xrightarrow{d} & C^0(\Omega^1_{X/k}) & \longrightarrow & C^0(\Omega^2_{X/k}) & \longrightarrow & \dots
\end{array}$$

Given $C^\bullet(\Omega^\bullet_{X/k})$, define the *total complex* $\mathrm{Tot}(C^\bullet(\Omega^\bullet_{X/k}))$ to be the complex of abelian groups

$$\mathrm{Tot}^n(C^\bullet(\Omega^\bullet_{X/k})) := \bigoplus_{i+j=n} C^j(\Omega^i_{X/k})$$

with differentials $D = d + (-1)^p\delta$. We have the following basic theorem:

Theorem 1.1. *Let X/k be an algebraic variety over a field k and let $\Omega^\bullet_{X/k}$ be its de Rham complex. Then*

$$H^i_{\mathrm{dR}}(X/k) = \frac{\ker(D : \mathrm{Tot}^i(X) \to \mathrm{Tot}^{i+1}(X))}{\mathrm{im}(D : \mathrm{Tot}^{i-1}(X) \to \mathrm{Tot}^i(X))}.$$

Proof. See, for example, Chapter 8 of [Voi02]. □

Using the description given by Theorem 1.1, it is easy to compute explicitly the first two de Rham cohomology groups of X/k. When $i = 0$, we have:

$$H^0_{\mathrm{dR}}(X/k) = \ker\left(H^0(X, \mathcal{O}_{X/k}) \xrightarrow{d} H^0(X, \Omega^1_{X/k})\right).$$

When $i = 1$,

$$H^1_{\mathrm{dR}}(X/k) = \frac{Z^1(X/k)}{B^1(X/k)}$$

where

$$Z^1(X/k) = \Big\{(\{\omega_\alpha\}, \{f_{\alpha\beta}\}) \in \prod_\alpha \Omega^1(U_\alpha) \times \prod_{\alpha<\beta} \mathcal{O}_X(U_\alpha \cap U_\beta) : d\omega_\alpha = 0, \omega_\alpha - \omega_\beta = df_{\alpha\beta}$$

$$\text{and } f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} = 0\}$$

and

$$B^1(X/k) = \{(\{dg_\alpha\}, \{g_\alpha - g_\beta\}) : g_\alpha \in \mathcal{O}_X(U_\alpha \cap U_\beta)\}.$$

Example 1.2. When X/k is a complete curve we can always find an open affine cover of X by two open subsets $U = X - \{P\}$ and $V = X - \{Q\}$ and so for $i \geq 2$ all the i-th algebraic de Rham cohomology groups vanish. Using Theorem 1.1 we have:

$$H^1_{\mathrm{dR}}(X/k) = \frac{\{(\omega_U, \omega_V, f_{UV}) \in \Omega^1(U) \times \Omega^1(V) \times \mathcal{O}(U \cap V) : \omega_U - \omega_V|_{U\cap V} = df_{UV}\}}{\{(dx_U, dx_V, x_U|_{U\cap V} - x_V|_{U\cap V}) : x_U \in \mathcal{O}(U), x_V \in \mathcal{O}(V)\}}.$$

1.2 The Hodge filtration

As with any double complex, the total complex $T := \mathrm{Tot}(C^\bullet(\Omega^\bullet_{X/k}))$ appearing in the statement of Theorem 1.1 comes equipped with a natural structure of a filtered complex. The filtration F is defined by:

$$F^r(T^n) := \bigoplus_{i+j=n, i\geq r} C^j(\Omega^i_{X/k}).$$

The *Hodge to de Rham* spectral sequence:

$$E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k)$$

is the spectral sequence associated to the filtered complex (T, F) (see [Voi02] Section 8.3 for details).

Theorem 1.3 (Deligne-Illusie)**.** *Let X/k be a complete algebraic variety over a field k of characteristic 0. Then the Hodge to de Rham spectral sequence of X degenerates at the E_1 term.*

Proof. See the original article [DI87]. □

As a result of Theorem 1.3 we get an exact sequence

$$0 \to H^0(X, \Omega^1_{X/k}) \to H^1_{\mathrm{dR}}(X/k) \to H^1(X, \mathcal{O}_{X/k}) \to 0 \tag{1.1}$$

of k-vector spaces called the *Hodge filtration* of X/k, and an isomorphism

$$H^2_{\mathrm{dR}}(X/k) \simeq H^1(X, \Omega^1_{X/k}).$$

Example 1.4. In terms of the description of $H^1_{\mathrm{dR}}(X/k)$ given in Example 1.2 the two maps in the exact sequence (1.1) are given by

$$\omega \in H^0(X, \Omega^1_{X/k}) \longmapsto [\omega|_U, \omega|_V, 0] \in H^1_{\mathrm{dR}}(X/k)$$
$$[\omega_U, \omega_V, f] \in H^1_{\mathrm{dR}}(X/k) \longmapsto [f] \in H^1(X, \mathcal{O}_{X/k}).$$

1.3 Residues and duality

Let X be a complete nonsingular curve over a field k of characteristic 0 and assume for simplicity that k is algebraically closed. Let $k(X)$ be the function field of X and let $\Omega_{k(X)}$ be the 1-dimensional $k(X)$-vector space of meromorphic differentials. For each closed point P let $\mathcal{O}_{X,P}$ denote the stalk of $\mathcal{O}_X$ at P. If $t \in \mathcal{O}_{X,P}$ is a uniformizer, then any element $\eta \in \Omega_{k(X)}$ can be written as:

$$\eta = (\sum_{n<0} a_n t^n + h) \cdot dt$$

with $h \in \mathcal{O}_{X,P}$. One can prove that the coefficient a_{-1} is independent of the choice of uniformizer t.

Definition 1.5. The *residue of* η *at* P is the unique element of k given by $\mathrm{res}_P(\eta) := a_{-1}$.

We have the following important local-global theorem, the so called 'Residue Theorem'.

Theorem 1.6 (Residue Theorem)**.** *For any* $\eta \in \Omega_{k(X)}$ *we have* $\sum_{P \in X} \mathrm{res}_P(\eta) = 0$.

Proof. See [Har77] III.7.14.2 □

A regular differential ω will have $\mathrm{res}_P(\omega) = 0$ for all P. However, a meromorphic differential might have zero residues everywhere and not be regular.

Definition 1.7. If $\mathrm{res}_P(\eta) = 0$ for all $P \in X$ then η is called a *differential of the second kind*. We denote the vector space of all such differentials by $\Omega^{\mathrm{II}}_{k(X)}$.

Consider a cover U, V of X as in Example 1.2 and take any class $[\omega_U, \omega_V, f_{UV}] \in H^1_{\mathrm{dR}}(X/k)$. The regular differential $\omega_U \in \Omega^1_{X/k}(U)$ can be uniquely extended to a meromorphic differential in

$\Omega_{k(X)}$ and we can compute its residue at P. By definition, near P

$$\omega_U = \omega_V + df_{UV}$$

from which it follows that the residue of ω_U at P is zero, since both regular and exact differentials have zero residues everywhere. Since ω_U is regular on $U = X - \{P\}$ and it has zero residue at P, projection on the first component gives a linear map:

$$\begin{aligned} H^1_{\mathrm{dR}}(X/k) &\longrightarrow \frac{\Omega^{\mathrm{II}}_{X/k}}{dk(X)} \\ [(\omega_U, \omega_V, f_{UV})] &\longmapsto [\omega_U] \end{aligned}$$

into classes of differentials of the second kind. Using the Riemann-Roch Theorem we can show that this map is in fact an isomorphism.

Proposition 1.8. *There is a canonical isomorphism:*

$$H^1_{\mathrm{dR}}(X/k) \simeq \frac{\Omega^{\mathrm{II}}_{X/k}}{dk(X)}$$

Proof. Given any differential of the second kind ϕ on X, we can invoke the Riemann-Roch Theorem to find exact differentials dg, dh regular on $U \cap V$ such that $\phi + dg$ is regular on U and $\phi + dh$ is regular on V. The inverse map is then given by $[\phi] \mapsto (\phi + dg, \phi + dh, g - h)$. $\square$

Given any differential of the second kind ϕ, and a point P on X, consider the localization $\phi_P \in (\Omega^{\mathrm{II}}_{k(X)})_P$. Since the residue of ϕ at P is zero, we can find a local antiderivative $\widetilde{\phi}_P$ in the complete local ring $k(X)_P$.

Definition 1.9. Let $\phi^1, \phi^2 \in \Omega^{\mathrm{II}}_{X/k}$. For each point P, let $\widetilde{\phi^1}_P \in k(X)_P$ be such that $d\widetilde{\phi^1}_P = \phi^1_P$. We define the *Poincaré pairing by*

$$\langle \phi^1, \phi^2 \rangle := \sum_{P \in X} \mathrm{res}_P(\widetilde{\phi^1}_P \cdot \phi^2_P)$$

Remark 1.10. Since ϕ^1 and ϕ^2 are regular at all but finitely many points of X the sum on the right is finite. Also, a different choice of local antiderivative for ϕ^1 yields the same pairing, since any two antiderivatives differ by a constant and $\mathrm{res}_P(\phi^2_P) = 0$. Therefore the pairing is well-defined.

By the residue Theorem, adding a global exact differential to either ϕ^1 or ϕ^2 leaves the pairing unchanged and therefore the Poincaré pairing gives a well-defined pairing

$$\langle .,. \rangle : H^1_{dR}(X/k) \times H^1_{dR}(X/k) \longrightarrow k$$

on the first algebraic de Rham cohomology group. This pairing in fact endows $H^1_{\mathrm{dR}}(X/k)$ with a symplectic structure, as the two following theorems show.

Theorem 1.11. *The Poincaré pairing is alternating.*

Proof. Let ϕ^1, ϕ^2 be differentials of the second kind representing classes in cohomology. Let $\widetilde{\phi^i}_P$ be the local antiderivative of ϕ^i at P for $i = 1, 2$. Then

$$\begin{aligned}
\left\langle \phi^1, \phi^2 \right\rangle + \left\langle \phi^2, \phi^1 \right\rangle &= \sum_{P \in X} \mathrm{res}_P(\widetilde{\phi^1}_P \cdot \phi^2_P) + \mathrm{res}_P(\widetilde{\phi^2}_P \cdot \phi^1_P) \\
&= \sum_{P \in X} \mathrm{res}_P(\widetilde{\phi^1}_P \cdot \phi^2_P + \widetilde{\phi^2}_P \cdot \phi^1_P) \\
&= \sum_{P \in X} \mathrm{res}_P(d(\widetilde{\phi^1}_P \cdot \widetilde{\phi^2_P})) = 0
\end{aligned}$$

as any exact differential must have zero residue. This shows that the pairing is alternating. $\square$

The non-degeneracy of the pairing is part of the Serre duality Theorem, which we now state without proof.

Theorem 1.12 (Serre Duality)**.** *Let k be a field and X/k a complete curve. Then*

(a) $H^1(X, \Omega^1_{X/k}) \simeq k$.

(b) The Poincaré pairing induces a perfect pairing of k-vector spaces:

$$\langle .,. \rangle : H^0(X, \Omega^1_{X/k}) \times H^1(X, \mathcal{O}_{X/k}) \to H^1(X, \Omega^1_{X/k})$$

Proof. This is in [Har77] Section III.7. $\square$

Combined with part (a), part (b) says that there is an isomorphism

$$H^0(X, \Omega^1_{X/k}) \xrightarrow{\simeq} H^1(X, \mathcal{O}_{X/k})^\vee$$

where the symbol $\vee$ indicates the linear dual of a vector space. If we let $g := \dim H^1(X, \mathcal{O}_{X/k})$ be the genus of X/k then by Serre duality we immediately deduce that

$$\dim H^0(X, \Omega^1_{X/k}) = g, \quad \dim H^1_{\mathrm{dR}}(X/k) = 2g$$

as k-vector spaces.

1.4 Splitting the Hodge filtration

The exact sequence (1.1) gives a canonical isomorphism of $H^0(X, \Omega^1_{X/k})$ with a subspace $V \subset H^1_{\mathrm{dR}}(X/k)$. This space is isotropic (i.e. the form $\langle ., . \rangle$ is identically 0 on $V \times V$) as one can see from the explicit description of the Poincaré pairing, and of dimension g, which is half the dimension of $H^1_{\mathrm{dR}}(X/k)$. By the general theory of symplectic vector spaces, given such a V it is always possible to find a k-linear isomorphism:

$$H^1_{\mathrm{dR}}(X/k) \simeq V \oplus V^\vee. \tag{1.2}$$

By Serre duality, this amounts to finding a linear map Φ:

$$\Phi : H^1(X, \mathcal{O}_{X/k}) \to H^1_{\mathrm{dR}}(X/k) \tag{1.3}$$

splitting the exact sequence (1.1). As the following example shows, there are many ways of finding Φ.

Example 1.13. Suppose $g = 1$. Then any choice of a nonzero $\omega \in H^0(X, \Omega^1_{X/k})$ gives a basis for this space. Corresponding to ω we can find a dual element $\alpha \in H^1(X, \mathcal{O}_{X/k})$ characterized uniquely by $\langle \omega, \alpha \rangle = 1$. The splitting map Φ is completely determined by a choice of $\Phi(\alpha)$.

Now if we view $\omega \in H^1_{\mathrm{dR}}(X/k)$ then we can find a basis ω, η for $H^1_{\mathrm{dR}}(X/k)$. By rescaling we can assume $\langle \omega, \eta \rangle = 1$. Under this choice of basis, Φ is uniquely determined by the value:

$$\Phi(\alpha) = a\omega + b\eta$$

for $a, b \in k$. However, we also require that Φ is a splitting homomorphism for (1.1) which in this case means

$$\langle \omega, \Phi(\alpha) \rangle = 1 \Longleftrightarrow b = 1.$$

On the other hand, any value of a will give a splitting map. In other words, adding to Φ an element $a \in \mathrm{Hom}_k(H^1(X, \mathcal{O}_{X/k}), H^0(X, \Omega^1_{X/k})) \simeq k$ gives another splitting map Φ'.

Remark 1.14. Generalizing example 1.13 to any g, if Φ is a linear map splitting the exact sequence (1.1), then so is $\Phi + a$, for any $a \in \mathrm{Hom}_k(H^1(X, \mathcal{O}_{X/k}), H^0(X, \Omega^1_{X/k})) \simeq M_{g\times g}(k)$. Conversely, any two splittings of (1.1) differ by an element of $\mathrm{Hom}_k(H^1(X, \mathcal{O}_{X/k}), H^0(X, \Omega^1_{X/k}))$.

For general X and k there is no natural choice of a splitting map Φ. However, in special circumstances we can exploit extra structures on the de Rham cohomology to find a *canonical* splitting of the Hodge filtration.

1.5 The Hodge decomposition

Suppose now that $k = \mathbb{C}$ is the field of complex numbers. If X is a projective algebraic curve over $\mathbb{C}$ then the points $X(\mathbb{C})$ are a compact complex analytic manifold of dimension 1, i.e. a compact Riemann surface. By Serre's GAGA theorems the global sections $H^0(X, \Omega^1_{X/\mathbb{C}})$ are precisely the holomorphic 1-forms on $X(\mathbb{C})$. We also have a notion of de Rham cohomology in terms of smooth (i.e. infinitely differentiable, or C^∞) differential forms. Denote by $A^1(X)$ the space of such forms and define:

$$H^1_{\mathrm{dR}}(X(\mathbb{C})) := \frac{\{\zeta \in A^1(X) : d\zeta = 0\}}{\{\zeta \in A^1(X) : \zeta = df \text{ for some } f \in C^\infty(X)\}}$$

By the following theorem, this notion coincides with the algebraic de Rham cohomology of the curve $X/\mathbb{C}$.

Theorem 1.15. *There is a canonical isomorphism:*

$$H^1_{\mathrm{dR}}(X/\mathbb{C}) \simeq H^1_{\mathrm{dR}}(X(\mathbb{C}))$$

where the left-hand side is the algebraic de Rham cohomology of the curve $X/\mathbb{C}$ and on the right-hand side is the de Rham cohomology of the Riemann surface $X(\mathbb{C})$.

Proof. See [Voi02] Remark 8.31. □

The isomorphism can be used to obtain an analytic formula for the Poincaré pairing.

Proposition 1.16. *Let η_1, η_2 be two classes in $H^1_{\mathrm{dR}}(X/\mathbb{C})$. Let ζ_1, ζ_2 be smooth closed 1-forms on $X(\mathbb{C})$ such that $\eta_i = [\zeta_i]$ for $i = 1, 2$. Then:*

$$\langle \eta_1, \eta_2 \rangle = \frac{1}{2\pi i} \int_{X(\mathbb{C})} \zeta_1 \wedge \zeta_2. \tag{1.4}$$

Proof. Write $\zeta_i = \phi_i + dF_i$ for some differential of second kind representing η_i and some meromorphic function F_i and apply Cauchy's residue formula. $\square$

Let z denote the coordinate map of a complex analytic chart of $X(\mathbb{C})$ at a point x. Viewing $X(\mathbb{C})$ as a (real) differentiable manifold of dimension 2, the cotangent space $T^*_{X,x}$ at x decomposes as:

$$T^*_{X,x} = \mathbb{R} \cdot dz \oplus \mathbb{R} \cdot d\overline{z} \tag{1.5}$$

into eigenspaces for the action of multiplication by i. The pullback of the conjugation map $z \mapsto \overline{z}$ on $T^*_{X,x}$ interchanges these two spaces.

Define:

$$\begin{aligned} A^{1,0}(X) &= \{\zeta \in A^1(X) : \zeta(x) = a \cdot dz \text{ for all } x \in X(\mathbb{C})\} \\ A^{0,1}(X) &= \{\zeta \in A^1(X) : \zeta(x) = a \cdot d\overline{z} \text{ for all } x \in X(\mathbb{C})\}. \end{aligned}$$

Note that if $\zeta(x) = a \cdot dz$ in a neighborhood of x, then the same must be true for every $x \in X(\mathbb{C})$. This is because the charts of $X(\mathbb{C})$ are *holomorphic*, i.e. they preserve the decomposition (1.5) of the cotangent space at a point. Consequently, the action of i and of conjugation can be glued to give global endomorphisms of $A^1(X)$ and the decomposition (1.5) induces a decomposition:

$$A^1(X) = A^{1,0}(X) \oplus A^{0,1}(X)$$

where we can also write $A^{0,1}(X) = \overline{A^{1,0}(X)}$.

Define:

$$\begin{aligned} H^{1,0}(X/\mathbb{C}) &= \frac{\{\zeta \in A^{1,0}(X) : d\zeta = 0\}}{\{\zeta \in A^{1,0}(X) : \zeta = df \text{ for some } f \in C^\infty(X)\}} \\ H^{0,1}(X/\mathbb{C}) &= \frac{\{\zeta \in A^{0,1}(X) : d\zeta = 0\}}{\{\zeta \in A^{0,1}(X) : \zeta = df \text{ for some } f \in C^\infty(X)\}} \end{aligned}$$

A deep theorem of Hodge shows that these two spaces are complementary inside $H^1_{\mathrm{dR}}(X/\mathbb{C})$.

Theorem 1.17 (Hodge Decomposition). *Let X be a compact Riemann surface. There is a canonical decomposition*

$$H^1_{\mathrm{dR}}(X/\mathbb{C}) = H^{1,0}(X/\mathbb{C}) \oplus H^{0,1}(X/\mathbb{C})$$

and $H^{0,1}(X) = \overline{H^{1,0}(X)}$.

Proof. See [Voi02] Chapter 2. Note that conjugation acts on H^1_{dR} since its action on $A^1(X)$, defined via pullback, commutes with d. $\square$

Remark 1.18. The proof of the Hodge decomposition uses the Hodge Theorem, which identifies $H^1_{\mathrm{dR}}(X/\mathbb{C})$ with the space of harmonic 1-forms on X. The need for these harmonic representatives for the classes in $H^1_{\mathrm{dR}}(X/\mathbb{C})$ makes it impossible to translate the proof into the algebraic setting, since the condition of being harmonic is purely analytic.

We can relate the spaces $H^{1,0}(X)$ and $H^{0,1}(X)$ to sheaf cohomology using the following theorem:

Theorem 1.19 (Dolbeaux)**.** *Let X be a compact Riemann surface. Then, for $0 \leq i, j \leq 1$, there is a canonical isomorphism:*

$$H^{i,j}(X) \simeq H^j(X, \Omega^i_{X/\mathbb{C}})$$

where the right-hand side is sheaf cohomology.

Proof. See [Voi02] Corollary 4.37. $\square$

In particular, we have

$$\begin{aligned} H^{1,0}(X) &= H^0(X, \Omega^1_{X/\mathbb{C}}) \\ H^{0,1}(X) &= H^1(X, \mathcal{O}_{X/\mathbb{C}}) = \overline{H^0(X, \Omega^1_{X/\mathbb{C}})} \end{aligned}$$

and so the Hodge Decomposition gives a canonical splitting linear map

$$\Phi_{\mathrm{Hodge}} : H^1(X, \mathcal{O}_{X/\mathbb{C}}) \xrightarrow{\simeq} \overline{H^0(X, \Omega^1_{X/\mathbb{C}})} \subset H^1_{\mathrm{dR}}(X/\mathbb{C})$$

of the Hodge filtration over $\mathbb{C}$.

Example 1.20. Let $g = 1$ and let ω, α as in Example 1.13. If we let

$$\eta = \frac{\overline{\omega}}{\langle \omega, \overline{\omega} \rangle}$$

then $\eta \in \overline{H^0(X, \Omega^1_{X/\mathbb{C}})}$ and therefore $\{\omega, \eta\}$ form a basis for $H^1_{\mathrm{dR}}(X/\mathbb{C})$ such that $\langle \omega, \eta \rangle = 1$. Setting

$$\Phi_{\mathrm{Hodge}}(\alpha) = \eta$$

gives a canonical splitting of the exact sequence (1.1) when $k = \mathbb{C}$, called the *Hodge splitting*.

1.6 CM ABELIAN VARIETIES

In Section 1.5 we used the complex structure of a compact Riemann surface X to find a canonical splitting of the Hodge filtration of $H^1_{\mathrm{dR}}(X)$. In general, this construction is not available for projective varieties over an arbitrary field. For a special class of abelian varieties, however, namely those possessing 'extra' endomorphisms, it is possible to define a decomposition analogous to the Hodge decomposition.

Consider first the case when $X = E/L$ is an elliptic curve defined over a number field $L/\mathbb{Q}$. Assume further that E has complex multiplication by an order $\mathcal{O}$ of a quadratic imaginary field $K/\mathbb{Q}$, and that $K \subset L$. If $a \in \mathcal{O}$, we denote by $[a]$ the corresponding element of $\mathrm{End}_L(E)$.

The ring $\mathrm{End}_L(E)$ acts via $[a]^*$ on the L-vector spaces $H^0(E, \Omega^1_{E/L})$, $H^1_{\mathrm{dR}}(E/L)$ and $H^1(E, \mathcal{O}_{E/L})$ in a way which is compatible with the Hodge filtration. The action on $H^0(E, \Omega^1_{E/L})$ gives an embedding $K \hookrightarrow L$ such that:

$$[a]^*(\omega) = a \cdot \omega \tag{1.6}$$

where ω is a generator for $H^0(E, \Omega^1_{E/L})$. Using this fact we can also compute the action of $[a]^*$ on $H^1(E, \mathcal{O}_{E/L})$.

Proposition 1.21. *Let α be a generator of $H^1(E, \mathcal{O}_{E/L})$. For each $a \in \mathcal{O}$ such that $a \notin \mathbb{Z}$ we have:*

$$[a]^*(\alpha) = \sigma(a) \cdot \alpha.$$

where σ is the generator of $\mathrm{Gal}(K/\mathbb{Q})$.

Proof. The key fact used here is that $[a]^*$ acts on $H^2_{\mathrm{dR}}(E/L) \simeq L$ as multiplication by the degree of $[a]$. With this in mind, recall from Theorem 1.12 that there is a perfect pairing:

$$H^0(E, \Omega^1_{E/L}) \times H^1(E, \mathcal{O}_{E/L}) \to H^2_{\mathrm{dR}}(E/L)$$

which we also denote by $\langle ., . \rangle$ since it coincides with the Poincaré pairing when lifted to $H^1_{\mathrm{dR}}(E/L)$. Let $\alpha \in H^1(E, \mathcal{O}_{E/L})$ be nonzero. Since $H^1(E, \mathcal{O}_{E/L})$ is 1-dimensional there is a $\lambda \in L$ such that $[a]^*(\alpha) = \lambda \cdot \alpha$. We need to show that $\lambda = \sigma(a)$. Let $\omega \in H^0(E, \Omega^1_{E/L})$ be nonzero. Then

$$\deg([a]) \cdot \langle \omega, \alpha \rangle = [a]^* \langle \omega, \alpha \rangle = \langle [a]^*\omega, [a]^*\alpha \rangle = a\lambda \cdot \langle \omega, \alpha \rangle .$$

But now

$$\deg([a]) = N(a)_{K/\mathbb{Q}} = a \cdot \sigma(a)$$

and therefore $\lambda = \sigma(a)$. In other words, $[a]^*$ acts by $\sigma(a)$ on $H^1(E, \mathcal{O}_{E/L})$. $\square$

The action of $\mathrm{End}_L(E)$ on the 2-dimensional L-vector space $H^1_{\mathrm{dR}}(E/L)$ forms a ring of commuting linear operators $[a]^*$. Therefore, for any $a \in \mathcal{O}$, we can study the eigenspace decomposition of $H^1_{\mathrm{dR}}(E/L)$ with respect to $[a]^*$. If $a \in \mathbb{Z}$ we obtain nothing new, since the characteristic polynomial of $[a]^*$ (i.e. the minimal polynomial of a) has a as a root with multiplicity two. But when $a \notin \mathbb{Z}$ then the characteristic polynomial of $[a]^*$ has a and $\sigma(a)$ as distinct roots. Hence $H^1_{\mathrm{dR}}(E/L)$ decomposes canonically as:

$$H^1_{\mathrm{dR}}(E/L) \simeq H^{1,0}(E/L) \oplus H^{0,1}(E/L) \tag{1.7}$$

where $H^{1,0}(E/L)$ is the a-eigenspace of $[a]^*$ and $H^{0,1}(E/L)$ is the $\sigma(a)$-eigenspace. We know from Equation (1.6) that $H^0(E, \Omega^1_{E/L}) \subseteq H^{1,0}(E/L)$ and since they are both 1-dimensional spaces $H^0(E, \Omega^1_{E/L}) = H^{1,0}(E/L)$. Similarly by Proposition 1.21 we know that $H^1(E, \mathcal{O}_{E/L}) = H^{0,1}(E/L)$. We therefore obtain a canonical splitting:

$$\Phi_{\mathrm{CM}} : H^1(E, \mathcal{O}_{E/L}) \xrightarrow{\simeq} H^{0,1}(E/L) \subset H^1_{\mathrm{dR}}(E/L)$$

of the Hodge filtration of $H^1_{\mathrm{dR}}(E/L)$. This splitting map is defined over L and does not require any embedding of L into $\mathbb{C}$.

One can, however, embed $L \subset \mathbb{C}$ and consider the space $H^1_{\mathrm{dR}}(E/L) \otimes \mathbb{C} = H^1_{\mathrm{dR}}(E/\mathbb{C})$, which is now equipped with two canonical decompositions: one coming from the Hodge splitting Φ_{Hodge} of Theorem 1.17 and one given by $\Phi_{\mathrm{CM}} \otimes \mathbb{C}$. It is natural to ask then whether the two coincide, a question that is answered by Katz in [Kat76].

Proposition 1.22. *Let $E/\mathbb{C}$ be the curve obtained from E/L by embedding $L \subset \mathbb{C}$. Then:*

$$H^1_{\mathrm{dR}}(E/L) \cap H^{0,1}(E/\mathbb{C}) = H^{0,1}(E/L)$$

where $H^{0,1}(E/\mathbb{C})$ is the space of anti-holomorphic representatives appearing in the Hodge Decomposition (Theorem 1.17) and $H^{0,1}(E/L)$ is the $\sigma(a)$ eigenspace appearing in the decomposition (1.7).

Proof. See [Kat76] Lemma 4.0.7. □

Remark 1.23. Proposition 1.22 shows that if $E/\mathbb{C}$ has complex multiplication, then its Hodge decomposition is induced by an algebraic decomposition of $H^1_{\mathrm{dR}}(E/L)$, where L is any field large enough to contain both the field of definition of E and the ring of complex multiplication. In [Kat76] Question 4.0.8., Nick Katz asks whether the converse is true, i.e. whether any elliptic curve $E/\mathbb{C}$ whose Hodge decomposition is induced by an algebraic decomposition has complex multiplication. The question is still open.

Next, we want to generalize to the case when $X = A/L$ is a simple abelian variety of dimension d defined over a number field L, chosen to be large enough to contain all endomorphisms of A. In this case we know that $[\mathrm{End}_L(A)\otimes\mathbb{Q} : \mathbb{Q}]$ divides $2d$ ([Shi98], Section 5, Proposition 2). When the degree equals $2d$ then $\mathrm{End}_L(A)\otimes\mathbb{Q}$ is an imaginary quadratic extension K of a totally real number field ([Shi98], Chapter 5, Propositions 5 and 6). We then say that A has *complex multiplication by K*, and one can show that $H^1_{\mathrm{dR}}(A/L)$ has a canonical decomposition similar to (1.7).

Henceforth, assume that A is simple and it has complex multiplication by K and that L is large enough so that $K \subset L$. Denote by $\rho_1, \dots, \rho_{2d}$ all the automorphisms of $\mathrm{Gal}(K/\mathbb{Q})$. Since K is a quadratic imaginary extension of a totally real number field K_0 of degree d, the ρ_i are conjugate in pairs, say $\rho_i = \sigma \circ \rho_{i+d}$ for $1 \le i \le d$, where σ is the generator of $\mathrm{Gal}(K/K_0)$.

As before, the action of $\mathrm{End}_L(A)$ on the d-dimensional L-vector space $H^0(A, \Omega^1_{A/L})$ gives an embedding $K \hookrightarrow L$ and this action can be simultaneously diagonalized.

Proposition 1.24. *Let A/L have complex multiplication by K. Then there exists a basis $\omega_1, \dots, \omega_d$ for $H^0(A, \Omega^1_{A/L})$ such that, for any $a \in \mathrm{End}_L(A) \subset K$, we have:*

$$[a]^*(\omega_i) = \rho_i(a) \cdot \omega_i$$

for all i such that $1 \le i \le d$.

Proof. See [Shi98] Section 3.2. □

The action of $\mathrm{End}_L(A)$ as commuting operators on $H^1_{\mathrm{dR}}(A/L)$ gives a decomposition:

$$H^1_{\mathrm{dR}}(A/L) = H^{1,0}(A/L) \oplus H^{0,1}(A/L) \tag{1.8}$$

where $H^{1,0}(A/L)$ is the direct sum of the $\rho_i(a)$-eigenspaces of $[a]^*$ and $H^{0,1}(A/L)$ is the direct sum of the $\sigma(\rho_i(a))$-eigenspaces, for $1 \leq i \leq d$ and any $a \in \mathrm{End}_L(A)$. By Proposition 1.24 and the analog of Proposition 1.21 we then obtain a canonical splitting

$$\Phi_{\mathrm{CM}} : H^1(A, \mathcal{O}_{A/L}) \xrightarrow{\simeq} H^{0,1}(A/L) \subset H^1_{\mathrm{dR}}(A/L)$$

of the Hodge filtration of A/L and over $\mathbb{C}$, this decomposition will agree with the Hodge decomposition.

2 THE COMPLEX ANALYTIC THEORY

In this chapter we present the geometric construction of harmonic weak Maass forms. We will construct harmonic weak Maass forms of weight zero, which present the least technical difficulty. We then proceed to analyze some of their arithmetic properties and recover some of the results in [BOR08] and [GKO09].

Throughout this chapter, we let $Y := Y_1(N)/\mathbb{Q}$ be the affine modular curve corresponding to the modular group $\Gamma_1(N)$ (this is defined in Section 2.1 below). This curve has the property that for any subfield $K \subset \mathbb{C}$, the K-points of Y parametrize $\overline{K}$-isomorphism classes of pairs of the form:

$$(E, P), \quad E = \text{Elliptic curve over } K, \quad P \in E(K) \text{ such that } P \text{ has exact order } N.$$

We also denote by $X := X_1(N)/\mathbb{Q}$ the complete curve which contains $Y_1(N)$ as a (Zariski) open set. The complex points of X have the structure of a compact Riemann surface. Moreover, by Serre's complex analytic GAGA, there are canonical isomorphisms:

$$H^j(X(\mathbb{C}), \Omega^p_{\text{hol}}) = H^j(X, \Omega^p_{X/\mathbb{C}})$$

for all j and p, where Ω^p_{hol} is the sheaf of holomorphic p-forms on $X(\mathbb{C})$. Therefore we will tacitly identify the two spaces throughout. Similarly, by the comparison isomorphism of Theorem 1.15, we will identify the algebraic de Rham cohomology group $H^1_{\text{dR}}(X/\mathbb{C})$ with its complex analytic counterpart.

2.1 MODULAR FORMS

Write $\tau = u + iv$, with $u, v \in \mathbb{R}$ and $v > 0$, for an element of the complex upper half-plane $\mathbb{H}$. For any such τ we also let $q = e^{2\pi i \tau}$. We have the operators

$$\frac{\partial}{\partial \tau} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$$
$$\frac{\partial}{\partial \overline{\tau}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Under the identification $\mathbb{C} \cong \mathbb{R}^2$ these two operators span the real tangent space of $\mathbb{C}$. The cotangent space of $\mathbb{R}^2$ is spanned by du, dv or by the complex differentials

$$d\tau = du + idv \quad , \quad d\overline{\tau} = du - idv$$

which are the dual basis of $\frac{\partial}{\partial\tau}$ and $\frac{\partial}{\partial\bar{\tau}}$. A smooth (i.e. infinitely differentiable, C^∞) function $F : \mathbb{H} \to \mathbb{C}$ is said to be *holomorphic* if $\frac{\partial F}{\partial\bar{\tau}} = 0$, which is equivalent to F satisfying the Cauchy-Riemann equations. A smooth function satisfying the conjugate equation $\frac{\partial F}{\partial\tau} = 0$ is said to be *anti-holomorphic*.

Fix an integer $N \geq 1$ and consider the level N congruence subgroups:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \text{ and } a, d \equiv 1 \mod N \right\}.$$

Let $k \in \mathbb{Z}$ be an integer.

Definition 2.1. A *weakly holomorphic modular form of weight k on* $\Gamma \in \{\Gamma_0(N), \Gamma_1(N)\}$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:

(i) $\frac{\partial f}{\partial\bar{\tau}} = 0$ for all $\tau \in \mathbb{H}$ (i.e. f is holomorphic on $\mathbb{H}$).

(ii) $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(iii) For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ there exists a positive integer h, a polynomial $P_{f,\gamma} \in \mathbb{C}[q^{-1/h}]$ and an $\epsilon \in \mathbb{R}_{>0}$ such that:

$$\left|(c\tau + d)^{-k} f(\gamma\tau) - P_{f,\gamma}\right| \in O(e^{-\epsilon v})$$

as $v \to \infty$.

The set of all such f is a complex vector space denoted by $M_k^!(\Gamma)$. The polynomials $P_{f,\gamma}$ are called the *principal parts* of f.

Putting extra conditions on the $P_{f,\gamma}$ cuts out familiar subspaces of $M_k^\dagger(\Gamma)$.

Definition 2.2. A $f \in M_k^!(\Gamma)$ is said to be a *cusp form of weight k on* Γ if $P_{f,\gamma} = 0$ for all $\gamma \in \mathbf{SL}_2(\mathbb{Z})$.

Denoting the space of all cusp forms by $S_k(\Gamma)$, we have a natural inclusion:

$$S_k(\Gamma) \subset M_k^!(\Gamma)$$

valid for all k.

Consider now the space $M_k^!(\Gamma_1(N))$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ the assignment:

$$f \longmapsto (c\tau + d)^{-k} f(\gamma\tau) \tag{2.1}$$

induces an endomorphism $\langle d \rangle$ of $M_k^!(\Gamma_1(N))$ which only depends on the class of d modulo N. These endomorphisms are the *diamond operators*. Since $a \cdot d \equiv 1 \mod N$, we get an action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $M_k^!(\Gamma_1(N))$ by commuting linear operators. Consequently, there is an eigenspace decomposition:

$$M_k^!(\Gamma_1(N)) \simeq \bigoplus_\chi M_k^!(\Gamma_1(N))_\chi$$

where the direct sum runs through all the complex characters $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$.

Definition 2.3. A *weakly holomorphic modular form of weight* k *on* $\Gamma_0(N)$ *with Nebentypus* χ is an element of $M_k^!(\Gamma_1(N))_\chi$. Namely f is a form in $M_k^!(\Gamma_1(N))$ satisfying:

$$f(\gamma\tau) = \chi(d)(c\tau + d)^k f(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

We denote the space of all such forms by $M_k^!(\Gamma_0(N), \chi)$.

The χ-decomposition of $M_k^!(\Gamma_1(N))$ induces a decomposition of the space $S_k(\Gamma_1(N))$ and consequently we get analogous spaces $S_k(\Gamma_0(N), \chi)$ of *cups forms of weight* k *on* $\Gamma_0(N)$ *with Nebentypus* χ and inclusions:

$$S_k(\Gamma_0(N), \chi) \subset M_k^\dagger(\Gamma_0(N), \chi)$$

for all k and χ. Note moreover that when $\chi = 1$ is the trivial character, we have

$$M_k^\dagger(\Gamma_0(N), 1) = M_k^\dagger(\Gamma_0(N)).$$

For $k = 2$, these vector spaces of modular forms arise naturally as spaces of differential 1-forms on modular curves. The complex points $Y_1(N)(\mathbb{C})$ of the affine modular curve can be uniformized:

$$Y_1(N)(\mathbb{C}) \simeq \mathbb{H}/\Gamma_1(N)$$

as the quotient of the upper-half plane by the action of $\Gamma_1(N)$ via linear fractional transformations. Its compactification:

$$X_1(N)(\mathbb{C}) \simeq \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})/\Gamma_1(N)$$

corresponds to the complex points of the complete modular curve $X_1(N)$ and is a compact Riemann surface. We will use the notation introduced at the beginning of this chapter by letting $Y(\mathbb{C}) := Y_1(N)(\mathbb{C})$ and $X(\mathbb{C}) := X_1(N)(\mathbb{C})$.

The points of $X(\mathbb{C}) - Y(\mathbb{C})$ are called the *cusps* and they are indexed by the equivalence classes of $\mathbb{P}^1(\mathbb{Q})/\Gamma_1(N)$. For each cusp s, there is a positive integer h such that $q^{1/h} = e^{2\pi i\tau/h}$ is a local parameter near s. For any differential form ω on $X(\mathbb{C})$, a local expression near s for ω in terms of $q^{1/h}$ is called the *q-expansion at s of* ω. Note that property (iii) of the definition of a weakly holomorphic modular form (Definition 2.1) puts growth conditions on each of its q-expansion, which allows one to view weakly holomorphic modular forms in terms of the sheaf of holomorphic differentials of the Riemann surface $X(\mathbb{C})$. Denote by $\Omega^1_{X/\mathbb{C}}$ the sheaf whose sections on $U \subseteq X(\mathbb{C})$ consist of the meromorphic differentials on X that are holomorphic on U.

Proposition 2.4. *Let $q = e^{2\pi i\tau}$. The assignment*

$$f \longmapsto \omega_f := 2\pi i f(\tau) \cdot d\tau = f(q) \cdot \frac{dq}{q}$$

induces a canonical isomorphism:

$$M_2^{\dagger}(\Gamma_1(N)) = H^0(Y, \Omega^1_{X/\mathbb{C}}).$$

Proof. By direct computation, the form ω_f is invariant under the action of $\Gamma_1(N)$. Moreover, f is holomorphic on $\mathbb{H}$ so ω_f is holomorphic on $Y(\mathbb{C})$. In fact, ω_f is meromorphic at the cusps. This follows at once from property (iii) of a weakly holomorphic modular form (Definition 2.1), since the property states that the q-expansion of f at the cusps is a finite tailed Laurent series in $q^{1/h}$ for some h corresponding to the cusp. □

Remark 2.5. Recall by the introductory discussion that we are tacitly identifying the space of global sections of holomorphic 1-forms on $X(\mathbb{C})$ with the space of global regular 1-forms on X defined over $\mathbb{C}$.

Corollary 2.6. *The assignment $f \mapsto \omega_f$ of Proposition 2.4 induces a canonical isomorphism:*

$$S_2(\Gamma_1(N)) = H^0(X, \Omega^1_{X/\mathbb{C}}).$$

Proof. By definition, all the q-expansions of the cusp form f begin with a linear term at least. Therefore the form $\omega_f = f(q)dq/q$ has a Laurent series expansion with no negative powers everywhere on $X(\mathbb{C})$ and it defines a holomorphic 1-form on $X(\mathbb{C})$. □

The modular curve $X = X_1(N)$ has the structure of an algebraic curve defined over $\mathbb{Q}$ in such a way that the cusp at ∞ is also defined over $\mathbb{Q}$. Using the isomorphism of Corollary 2.6 we can try to carry over this rational structure to the space of cusp forms. For any field $K \subset \mathbb{C}$, we can consider the space of regular differential forms on X/K:

$$H^0(X, \Omega^1_{X/K}) := H^0(X, \Omega^1_{X/\mathbb{Q}}) \otimes K.$$

All these spaces sit inside $H^0(X, \Omega^1_{X/\mathbb{C}})$ and in particular inside the space of weight 2 cusp forms $S_2(\Gamma_1(N))$. The following theorem identifies their images in $S_2(\Gamma_1(N))$.

Theorem 2.7 (q-expansion principle)**.** *Let $f \in S_2(\Gamma_1(N))$ and let ω_f be the corresponding element of $H^0(X, \Omega^1_{X/\mathbb{C}})$ under the isomorphism of Corollary 2.6. Then ω_f belongs to $H^0(X, \Omega^1_{X/K})$ if and only if all the coefficients of the q-expansion of f at ∞ belong to K.*

Proof. See [Kat73] Corollary 1.6.2. □

We will denote by $S_2(\Gamma_1(N), K)$ the subspace of $S_2(\Gamma_1(N))$ consisting of those cusp forms whose coefficients of the q-expansion at ∞ belong to K. The q-expansion principle can then be rephrased as saying that the assignment of Corollary 2.6 gives a canonical isomorphism:

$$H^0(X, \Omega^1_{X/K}) = S_2(\Gamma_1(N), K).$$

Going back to the complex points $X(\mathbb{C})$ we make the operators $\langle d\rangle$ act on $H^0(X, \Omega^1_{X/\mathbb{C}})$ via the isomorphism of Corollary 2.6. As before we get a decomposition

$$H^0(X, \Omega^1_{X/\mathbb{C}}) \simeq \bigoplus_{\chi} H^0(X, \Omega^1_{X/\mathbb{C}})_\chi$$

into χ-eigenspaces. Consequently we have canonical isomorphisms:

$$H^0(X, \Omega^1_{X/\mathbb{C}})_\chi = S_2(\Gamma_0(N), \chi).$$

Moreover, the operators $\langle d\rangle$ act by duality on the space $H^1(X, \mathcal{O}_{X/\mathbb{C}})$ and on $H^1_{\mathrm{dR}}(X/\mathbb{C})$. In particular, we have a decomposition

$$H^1_{\mathrm{dR}}(X/\mathbb{C}) \simeq \bigoplus_{\chi} H^1_{\mathrm{dR}}(X/\mathbb{C})_\chi$$

into χ-eigenspaces for the action of the diamond operators.

2.2 HARMONIC WEAK MAASS FORMS

We now present the basic geometric construction of harmonic weak Maass forms of weight zero. This construction will have to be refined later for arithmetic applications, but for clarity we describe the basic idea first without worrying about the rational structures on the spaces of modular forms involved.

Let $f \in S_2(\Gamma_0(N), \chi)$ and let ω_f be the associated holomorphic 1-form $\omega_f \in H^0(X, \Omega^1_{X/\mathbb{C}})_\chi$. Using the Hodge filtration (1.1) of $X(\mathbb{C})$ we can view ω_f as an element of $H^1_{\mathrm{dR}}(X/\mathbb{C})$. By the Hodge decomposition (Theorem 1.17), this space decomposes canonically as:

$$H^1_{\mathrm{dR}}(X/\mathbb{C}) \simeq H^0(X, \Omega^1_{X/\mathbb{C}}) \oplus \overline{H^0(X, \Omega^1_{X/\mathbb{C}})}.$$

Therefore, the anti-holomorphic differential $\overline{\omega_f}$ associated to ω_f represents a class in $H^1_{\mathrm{dR}}(X/\mathbb{C})$ such that:

$$\mathrm{span}(\overline{\omega_f}) \cap H^0(X, \Omega^1_{X/\mathbb{C}}) = 0.$$

On the other hand, from the algebraic description of Proposition 1.8 we have

$$H^1_{\mathrm{dR}}(X/\mathbb{C}) \simeq \frac{\Omega^{\mathrm{II}}_{X/\mathbb{C}}}{d\mathbb{C}(X)}$$

so that the class corresponding to $\overline{\omega}_f \in H^1_{\mathrm{dR}}(X/\mathbb{C})$ can be represented by a differential of the second kind ϕ. As $\overline{\omega}_f$ lies in the $\overline{\chi}$-eigenspace of $H^1_{\mathrm{dR}}(X/\mathbb{C})$, the differential ϕ can be chosen to be a $\overline{\chi}$-eigenvector for the action of the $\langle d\rangle$'s. Moreover, by Proposition 1.8, this ϕ can be chosen to be regular everywhere outside the cusps of $X(\mathbb{C})$.

The class of $\phi - \overline{\omega}_f$ vanishes in $H^1_{\mathrm{dR}}(X/\mathbb{C}) \subset H^1_{\mathrm{dR}}(Y/\mathbb{C})$. We can therefore find a smooth function $F \in C^\infty(Y(\mathbb{C}))$ such that:

$$dF = \phi - \overline{\omega_f}. \tag{2.2}$$

The Fourier expansion of F at ∞ has finitely many negative powers of q. These can be computed as the Eichler integral (formal antiderivative obtained by integration term by term) of the principal part of ϕ at ∞, and similarly at the other cusps. Moreover, F is harmonic by construction, being a linear combination of holomorphic and anti-holomorphic functions. In fact, F is the prototypical example of a *harmonic weak Maass form of weight 0 and Nebentypus* $\overline{\chi}$. To characterize these forms among all the smooth functions on $Y(\mathbb{C})$ we follow Bruinier and Funke [BF04]:

Definition 2.8. A *harmonic weak Maass form of weight* 0 *on* $\Gamma_0(N)$ *with Nebentypus* χ is a smooth function $F : \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:

(i) $\frac{\partial^2 F}{\partial\tau\partial\overline{\tau}} = 0$ for all $\tau \in \mathbb{H}$ (i.e. F is harmonic on $\mathbb{H}$).

(ii) $F(\gamma\tau) = \chi(d)F(\tau)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

(iii) For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ there exists a positive integer h, a polynomial $P_{F,\gamma} \in \mathbb{C}[q^{-1/h}]$ and an $\epsilon \in \mathbb{R}_{>0}$ such that:

$$\left|\chi(d)^{-1}F(\gamma\tau) - P_{F,\gamma}\right| \in O(e^{-\epsilon v})$$

as $v \to \infty$.

We denote the $\mathbb{C}$-vector space of such harmonic weak Maass forms of weight 0 and character χ by $\mathcal{H}_0(\Gamma_0(N), \chi)$. Note that the definition differs from that of weakly holomorphic modular forms only in that the property of being holomorphic has been relaxed to that of being harmonic. Consequently we have natural inclusions:

$$M_0^\dagger(\Gamma_0(N),\chi) \subset \mathcal{H}_0(\Gamma_0(N),\chi).$$

Equation (2.2) associates to ω_f a smooth function $F \in C^\infty(Y(\mathbb{C}))$ such that:

$$dF = \phi - \overline{\omega_f}$$

where ϕ is a differential of the second kind on $X(\mathbb{C})$ which is regular outside the cusps and ω_f is the differential 1-form on $X(\mathbb{C})$ corresponding to f under the isomorphism of Proposition 2.6. We claimed that F belongs to $\mathcal{H}_0(\Gamma_0(N),\overline{\chi})$. To prove it, choose any base point $P \in Y(\mathbb{C})$ and write

$$F(\tau) = \int_P^\tau \phi - \overline{\omega_f}.$$

From this expression it is immediate to check that F is harmonic. Moreover, it is modular of weight 0 by construction and the principal parts of F at the cusps are simply the local antiderivatives of the principal parts of ϕ, a weakly holomorphic modular form. The differential ϕ was chosen to represent a class in the $\overline{\chi}$-eigenspace of $H^1_{\mathrm{dR}}(X/\mathbb{C})$, and the form $\overline{\omega_f}$ also represents a class in the same eigenspace, since $f \in S_2(\Gamma_0(N),\overline{\chi})$. Therefore $F \in \mathcal{H}_0(\Gamma_0(N),\overline{\chi})$.

Harmonic weak Maass forms arise naturally as 'anti-derivatives' of cusp forms, as Equation (2.2) suggests. Namely, if we define the differential operator:

$$\xi_0 := 2i \cdot \overline{\frac{\partial}{\partial \overline{\tau}}}$$

(the reason for the $2i$ factor will be clear) then we have:

Theorem 2.9. *The map*

$$\xi_0 : \mathcal{H}_0(\Gamma_0(N),\chi) \longrightarrow S_2(\Gamma_0(N),\overline{\chi})$$

is surjective.

Proof. Let F be an element of $\mathcal{H}_0(\Gamma_0(N),\chi)$. To see that the map is well-defined, we need to check three properties of $\xi_0(F)$.

(i) *$\xi_0(F)$ is holomorphic on $\mathbb{H}$*. This follows by noticing:

$$\frac{\partial \xi_0(F)}{\partial \overline{\tau}} = \frac{\partial}{\partial \overline{\tau}}\left(2i \cdot \frac{\partial \overline{F}}{\partial \tau}\right) = 2i \cdot \frac{\partial^2 \overline{F}}{\partial \tau \partial \overline{\tau}} = 0$$

since if F is harmonic so is $\overline{F}$.

(ii) *$\xi_0(F)$ is modular of weight 2 on $\Gamma_0(N)$ and Nebentypus $\overline{\chi}$*. The function F is modular of weight 0 and Nebentypus χ, therefore:

$$\overline{F}(\gamma\tau) = \overline{\chi(d)} \cdot \overline{F}(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Applying $\frac{\partial}{\partial \tau}$ to both sides of the equation we get:

$$(c\tau + d)^{-2}\frac{\partial \overline{F}}{\partial \tau}(\gamma\tau) = \overline{\chi(d)}\frac{\partial \overline{F}}{\partial \tau}(\tau)$$

which shows that $\frac{\partial \overline{F}}{\partial \tau} = \overline{\frac{\partial F}{\partial \overline{\tau}}}$ is modular of weight 2 and Nebentypus $\overline{\chi}$. The same then holds for the operator ξ_0.

(iii) *The principal parts of $\xi_0(F)$ at the cusps vanish*. The principal parts of F are formal antiderivatives of the tails of the Laurent expansions of the differential of the second kind ϕ. As such, they are annihilated by the operator $\frac{\partial}{\partial \overline{\tau}}$ and therefore by ξ_0.

To show surjectivity, we can follow the construction done at the beginning of this section to find a harmonic weak Maass form $F \in \mathcal{H}_0(\Gamma_0(N))$ such that:

$$dF = \phi - \overline{\omega_f}.$$

for a differential of the second kind ϕ. By construction, F has an holomorphic part (whose Fourier expansion is the formal antiderivative of the Fourier expansion of ϕ) and a anti-holomorphic part (whose Fourier expanions is the formal antiderivative of the Fourier expansion of $\overline{\omega_f}$). The operator ξ_0 annihilates the holomorphic part and conjugates $\overline{\omega_f}$. Throwing in the constants, we see that $\xi_0(\frac{1}{4\pi}F) = f$, and surjectivity follows. $\square$

Remark 2.10. Note that $\xi_0(F) = 0$ if and only if F is holomorphic, i.e. F is a weakly holomorphic modular form of weight 0. Therefore, we have an exact sequence:

$$0 \to M_0^{\dagger}(\Gamma_0(N), \chi) \longrightarrow \mathcal{H}_0(\Gamma_0(N), \chi) \xrightarrow{\xi_0} S_2(\Gamma_0(N), \overline{\chi}) \longrightarrow 0.$$

Remark 2.11. Theorem 2.9 is the weight $k = 0$ case of [BF04] Theorem 3.7.

Note that around the cusp ∞ we can write $F = F^+ + F^-$ where

$$dF^+ = \phi$$

and

$$dF^- = -\overline{\omega}_f.$$

We recall the terminology of [GKO09].

Definition 2.12. The Laurent series F^+ is called the *holomorphic part* of F. The cusp form $\xi_0(F)$ is called the *shadow* of F. We write:

$$F^+ = \sum_{n \gg -\infty} c^+(n) q^n$$

for the q-expansion at ∞ of F^+.

Remark 2.13. The holomorphic part F^+ is simply the local antiderivative of the weakly holomorphic modular form corresponding to ϕ. As such, it does not in general extend to give a weight zero modular form on all of $X(\mathbb{C})$, since the class of $[\phi]$ is not necessarily trivial in cohomology. It is however an example of Ramanujan's *mock modular forms* as described in the introduction.

In [BO] the authors relate the arithmetic properties of the $c^+(n)$ to the vanishing of the first derivatives of L-series attached to $\xi_0(F)$, which suggests that these coefficients may carry valuable arithmetic information. The geometric point of view of this section can be refined to study some of these properties. The refinement is described in Section 2.5, Theorem 2.18, but before we take on that task we need to recall a few notions about the hermitian structure of $S_2(\Gamma_0(N), \chi)$ and the action of Hecke and diamond operators on these spaces.

2.3 The Petersson and Poincaré pairings

Classically (see for example [Ser73]) the *Petersson scalar product*, which we denote by $(\cdot,\cdot)$, is defined by:

$$(f,g) = \int_{w\in\mathcal{F}} f\cdot\overline{g}\cdot dx\wedge dy$$

for $f, g \in S_2(\Gamma_1(N))$, where $w = x+iy$ and $\mathcal{F}$ is a fundamental domain for $\mathbb{H}/\Gamma_1(N)$. It is a non-degenerate hermitian inner product on $S_2(\Gamma_1(N))$. For each $f \in S_2(\Gamma_1(N))$, let

$$\|f\| := \sqrt{(f,f)}$$

be the *Petersson norm* of f.

Using the isomorphism of Proposition 2.4 we can also take the 1-form $\omega_f \in H^0(X, \Omega^1_{X/\mathbb{C}})$ corresponding to f, find its 'complementary' element $\overline{\omega_f}$ in $H^1_{\mathrm{dR}}(X/\mathbb{C})$ using the Hodge decomposition, and compute the Poincaré pairing $\langle\omega_f, \overline{\omega_f}\rangle$. By the formula of Proposition 1.16 for the Poincaré pairing on a Riemann surface, we have

$$\langle\omega_f,\overline{\omega_f}\rangle = \frac{1}{2\pi i}\int_{X(\mathbb{C})}\omega_f\wedge\overline{\omega_f}.$$

The relationship between the two pairings can be stated as follows

Lemma 2.14. *Let $f \in S_2(\Gamma_1(N))$ and let ω_f be its corresponding holomorphic 1-form on $X(\mathbb{C})$. Then*

$$\langle\omega_f,\overline{\omega_f}\rangle = -4\pi\cdot\|f\|^2$$

Proof. By direct computation:

$$\begin{aligned}\langle\omega_f,\overline{\omega_f}\rangle &= \left\langle 2\pi i f\cdot d\tau, -2\pi i\overline{f}\cdot d\overline{\tau}\right\rangle\\ &= -(2\pi i)^2\left\langle f\cdot d\tau, \overline{f}\cdot d\overline{\tau}\right\rangle\\ &= -2\pi i\int_{X(\mathbb{C})} f\cdot d\tau\wedge\overline{f}\cdot d\overline{\tau}.\end{aligned}$$

This integral can be evaluated by choosing a fundamental domain $\mathcal{F}$ for $X(\mathbb{C})$. Letting $w = x+iy$ be a variable in $\mathcal{F}$, we have:

$$
\begin{aligned}
-2\pi i \int_{X(\mathbb{C})} f \cdot d\tau \wedge \overline{f} \cdot d\overline{\tau} &= -2\pi i \int_{\mathcal{F}} f \cdot \overline{f}(-2i \cdot dx \wedge dy) \\
&= 2\pi i \cdot 2i \cdot \|f\|^2.
\end{aligned}
$$

□

2.4 HECKE AND DIAMOND OPERATORS

The Hecke operators T_ℓ, for ℓ a prime with $\ell \nmid N$, are linear endomorphisms of the space $S_2(\Gamma_1(N))$. If $f \in S_2(\Gamma_1(N))$ has q-expansion at ∞ given by $\sum_{n=1}^{\infty} a_n(f)q^n$ then the action of T_ℓ is given by the formula:

$$
T_\ell(\sum_{n=1}^{\infty} a_n(f)q^n) = \sum_{n=1}^{\infty} a_{n\ell}(f)q^n + \ell \cdot \sum_{n=1}^{\infty} a_n(\langle \ell \rangle f)q^{n\ell} \tag{2.3}
$$

where $\langle \ell \rangle$ is the diamond operator of Equation (2.1). By this explicit formula we see that the diamond and Hecke operators commute and therefore the T_ℓ's preserve the spaces $S_2(\Gamma_0(N), \chi)$ of modular forms with Nebentypus.

In Section 2.3 we saw how the space $S_2(\Gamma_1(N))$ is equipped with a non-degenerate hermitian pairing $(\cdot, \cdot)$. The hermitian adjoints of the operators T_ℓ and $\langle d \rangle$ are given by:

$$
\langle d \rangle^t = \langle d^{-1} \rangle \quad \text{(inverse taken mod } N\text{)}
$$

and

$$
T_\ell^t = \langle \ell^{-1} \rangle T_\ell \tag{2.4}
$$

from which we conclude that the T_ℓ's and the $\langle d \rangle$'s commute with their adjoints, i.e. they are normal operators. From the classical theory of hermitian vector spaces, this means that each T_ℓ and $\langle d \rangle$ can be diagonalized, and since they commute this can be done simultaneously. Therefore each space $S_2(\Gamma_0(N), \chi)$ has a basis of cusp forms f such that, for all $\ell \nmid N$,

$$
T_\ell(f) = c_\ell \cdot f
$$

for some $c_\ell \in \mathbb{C}$. We call these cusp forms *eigenforms*. The set $\{c_\ell\}$ is a *system of eigenvalues* for the eigenform f.

Definition 2.15. Let $f \in S_2(\Gamma_0(N), \chi)$ be an eigenform with corresponding system of eigenvalues $\{c_\ell\}$. We say that f is a *newform* if any other eigenform in $S_2(\Gamma_0(N), \chi)$ with the same system of eigenvalues is a scalar multiple of f. We say that f is *normalized* if $a_1(f) = 1$.

If $f \in S_2(\Gamma_0(N), \chi)$ is a normalized newform with q-expansion at ∞ given by $\sum_{n=1}^{\infty} a_n(f) q^n$, then:

$$T_\ell(f) = a_\ell(f) \cdot f$$

or, in other words, the coefficients $a_\ell(f)$ are the system of eigenvalues of f.

Proposition 2.16. *Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N), \chi)$ be an eigenform and let K_f be the field generated by the $a_n(f)$ and by $\chi(d)$ for $(d, N) = 1$. Then K_f is a finite extension of $\mathbb{Q}$, and it is generated by the $a_n(f)$ alone.*

Proof. See [Rib04] Corollary 3.1. □

If f is a normalized newform as in Proposition 2.16, the q-expansion principle gives:

$$\omega_f \in H^0(X, \Omega^1_{X/K_f})$$

where ω_f is the regular differential attached to f. Moreover, by definition the T_ℓ and $\langle d \rangle$ operators preserve the field of definition of the coefficients of the q-expansions of modular forms, so they preserve the space $H^0(X, \Omega^1_{X/K_f})$. One can then speak of the *f-isotypical component*

$$H^0(X, \Omega^1_{X/K_f})_f \subset H^0(X, \Omega^1_{X/K_f})$$

attached to f. This is the space of differential forms corresponding to the 1-dimensional K_f-subspace of $S_2(\Gamma_0(N), \chi)$ of all eigenforms with coefficients in K_f and system of eigenvalues $\{a_p(f)\}$.

The Hecke and diamond operators extend naturally to $H^1_{\mathrm{dR}}(X/\mathbb{C})$ and $H^1(X, \mathcal{O}_{X/\mathbb{C}})$ and they respect the Hodge filtration. For a newform $f \in S_2(\Gamma_0(N), \chi)$ defined over K_f we therefore get a Hodge filtration associated to the f-isotypical component of $H^1_{\mathrm{dR}}(X/K_f)$:

$$0 \to H^0(X, \Omega^1_{X/K_f})_f \longrightarrow H^1_{\mathrm{dR}}(X/K_f)_f \longrightarrow H^1(X, \mathcal{O}_{X/K_f})_f \longrightarrow 0. \tag{2.5}$$

As a sequence of K_f vector spaces, we can always find a splitting of this exact sequence. Fix such a map and denote it by:

$$\Phi_{\mathrm{alg}} : H^1(X, \mathcal{O}_{X/K_f})_f \longrightarrow H^1_{\mathrm{dR}}(X/K_f)_f. \tag{2.6}$$

The subscript 'alg' reminds us that Φ_{alg}, albeit non-canonical, is always defined over K_f.

2.5 GOOD LIFTS

By Remark 2.10 we know that the kernel of the differential operator ξ_0 is an infinite dimensional vector space corresponding to the space of weakly holomorphic modular forms of weight 0. It is then reasonable to ask whether we can find a specific set of preimages of f which encode useful arithmetic information.

Recent work of Bruinier, Ono and Rhoades ([BOR08]) suggests that the following class of 'good' harmonic weak Maass forms can indeed be useful for arithmetic applications.

Definition 2.17. Let $f \in S_2(\Gamma_0(N), \overline{\chi})$ be a normalized newform. A harmonic weak Maass form $F \in \mathcal{H}_0(\Gamma_0(N), \chi)$ is *good for* f if it satisfies the following properties:

(i) The principal part of F at the cusp ∞ belongs to $K_f[q^{-1}]$.

(ii) The principal parts of F at the other cusps are constant.

(iii) $\xi_0(F) = (f, f)^{-1} \cdot f$

The existence of such 'good' lifts is guaranteed by the following theorem.

Theorem 2.18. *Let $f \in S_2(\Gamma_0(N), \overline{\chi})$ be a normalized newform. Then there is a harmonic weak Maass form $F \in \mathcal{H}_0(\Gamma_0(N), \chi)$ which is good for f.*

Proof. The construction of F is similar to the one in the proof of Theorem 2.9. For ease of notation, let $K := K_f$, the field obtained by adjoining the coefficients of the q-expansion of f at ∞.

Since f is a newform defined over K, the corresponding regular differential ω_f generates the f-isotypical component $H^0(X, \Omega^1_{X/K})_f$, which fits in the exact sequence

$$0 \to H^0(X, \Omega^1_{X/K})_f \longrightarrow H^1_{\mathrm{dR}}(X/K)_f \longrightarrow H^1(X, \mathcal{O}_{X/K})_f \longrightarrow 0.$$

described in (2.5).

Let $\alpha_f \in H^1(X, \mathcal{O}_{X/K})_f$ be the element dual to ω_f, i.e. $\langle \omega_f, \alpha_f \rangle = 1$. Using the algebraic splitting (2.6) we can lift α_f inside $H^1_{\mathrm{dR}}(X/K)_f$ as $\Phi_{\mathrm{alg}}(\alpha_f)$. Using the algebraic description of Proposition 1.8 we can find a differential of the second kind ϕ such that:

$$[\phi] = \Phi_{\mathrm{alg}}(\alpha_f) \in H^1_{\mathrm{dR}}(X/K)_f.$$

Moreover, by Proposition 1.8 we can always choose a representative for the class $[\phi]$ so that ϕ has a pole at ∞ and it is regular everywhere else on X/K.

We now extend scalars to $\mathbb{C}$ and apply the transcendental methods of Section 1.5. The Hodge decomposition gives a canonical decomposition of $H^1_{\mathrm{dR}}(X/\mathbb{C})$:

$$H^1_{\mathrm{dR}}(X/\mathbb{C}) = H^0(X, \Omega^1_{X/\mathbb{C}}) \oplus \overline{H^0(X, \Omega^1_{X/\mathbb{C}})}$$

and gives a canonical basis $\omega_f, \overline{\omega_f}$ for $H^1_{\mathrm{dR}}(X/\mathbb{C})_f$. Using the notation of Example 1.20, let

$$\eta_f := \Phi_{\mathrm{Hodge}}(\alpha_f) = \frac{\overline{\omega_f}}{\langle \omega_f, \overline{\omega_f} \rangle}$$

be the unique element in the line spanned by $\overline{\omega_f}$ such that $\langle \omega_f, \eta_f \rangle = 1$. Viewing $[\phi] = \Phi_{\mathrm{alg}}(\alpha_f)$ as representing a class inside $H^1_{\mathrm{dR}}(X/\mathbb{C})_f$, we can write the class $[\phi]$ in terms of the basis ω_f, η_f:

$$[\phi] = a \cdot [\omega_f] + b \cdot [\eta_f]$$

for some $a, b \in \mathbb{C}$. The relationship $\langle \omega_f, \phi \rangle = 1$ gives $b = 1$, but the value of a cannot be determined exactly: each value of a corresponds to a choice of a splitting Φ_{alg}.

The class of $\phi - a \cdot \omega_f - \eta_f$ is trivial in the cohomology group $H^1_{\mathrm{dR}}(X/\mathbb{C}) \subset H^1_{\mathrm{dR}}(Y/\mathbb{C})$, so there exists a $F \in C^\infty(Y(\mathbb{C}))$ such that

$$dF = \phi - a \cdot \omega_f - \eta_f.$$

We claim that F is good for f. To show that F belongs to $\mathcal{H}_0(\Gamma_0(N), \chi)$ we appeal to the proof of Theorem 2.9, which goes through unchanged in this case. For properties (i)-(ii) of Definition 2.17, note that the principal parts of F correspond to the principal parts of ϕ. This differential of the second kind ϕ is defined over K so its principal parts must have coefficients in K. Moreover, ϕ was chosen to be regular everywhere but at the cusp ∞. Finally, for property (iii) we compute directly using Lemma 2.14:

$$\xi_0(F) = 2i \overline{\frac{\partial F}{\partial \overline{\tau}}} = \frac{2i}{\langle \omega_f, \overline{\omega_f} \rangle} \frac{\partial}{\partial \tau} \int_{w=\tau_0}^{w=\tau} 2\pi i f \cdot dw = \frac{f}{(f, f)}$$

□

Remark 2.19. Theorem 2.18 is the $k = 2$ case of [BOR08] Proposition 5.1.

2.6 Vanishing of Hecke eigenvalues

In the next two sections we show how the geometric construction of Theorem 2.18 can be used to analyze some of the arithmetic properties of harmonic weak Maass forms. In particular, in this section we improve upon [BOR08] Theorem 1.4 for the weight 2 case and in the next section we improve upon the weight 2 case of [BOR08] Theorem 1.3.

Recall from Definition 2.12 that we write:

$$F^+ = \sum_{n \gg -\infty} c^+(n) q^n$$

for the q-expansion at ∞ of the holomorphic part of some $F \in \mathcal{H}_0(\Gamma_0(N), \chi)$ which is good for f, a normalized newform in $S_2(\Gamma_0(N), \overline{\chi})$.

The following theorem relates the arithmetic properties of the coefficients $c^+(n)$ with those of f.

Theorem 2.20. *Suppose that $f = \sum_{n=1}^{\infty} b(n) q^n \in S_2(\Gamma_0(N), \overline{\chi})$ is the q-expansion at ∞ of a normalized newform and suppose that $F \in \mathcal{H}_0(\Gamma_0(N), \chi)$ is good for f. If $\ell \nmid N$ is a prime for which $b(\ell) = 0$ then $c^+(n)$ belongs to K_f whenever $\mathrm{ord}_\ell(n)$ is odd.*

Proof. This follows directly from the construction of F given in Theorem 2.18. Using the same notation, write:

$$dF^+ = \phi - a \cdot \omega_f$$

where ϕ is a differential of the second kind whose q-expansion at ∞ is of the form

$$h = \sum_{n \gg -\infty} h(n) q^n dq$$

and all the $h(n)$ belong to K_f. For any prime $\ell \nmid N$,

$$c^+(\ell) = \frac{h(\ell) + a \cdot b(\ell)}{\ell}$$

and since $h(\ell) \in K_f$ the theorem follows once we apply the multiplicative properties of the coefficients of the q-expansion of f. $\square$

Remark 2.21. In [BOR08] Theorem 1.4 the authors prove an analogous statement which holds for cusp forms of arbitrary integer weight. Namely they show that the $c^+(n)$ are algebraic and they belong to an abelian extension of K_f. They further raise the question of whether the $c^+(n)$ belong in fact to K_f. Theorem 2.6 answers this question in the affirmative, at least in the weight 2 case.

The formula expressing the $c^+(n)$ in terms of the coefficients of f and ϕ immediately gives the following theorem, which is the case $k = 2$ of [GKO09] Theorem 1.1.

Theorem 2.22. *Let $K_f(\{c^+(n)\})$ be the field obtained by adjoining all the coefficients $c^+(n)$ to K_g. Then the transcendence degree of $K_f(\{c^+(n)\})$ over K_f is at most 1.*

Proof. This is a simple consequence of the formula:

$$c^+(n) = \frac{h(n) + a \cdot b(n)}{n}$$

derived in the proof of Theorem 2.6. $\square$

Remark 2.23. Using the notation of the proof of Theorem 2.18, the value of a is determined by a choice of splitting Φ_{alg}. This was explained in detail in Example 1.13. Following the same argument, we can see that:

$$a \in \mathrm{Hom}_{\mathbb{C}}(H^1(X, \mathcal{O}_{X/K_f}) \otimes \mathbb{C}, H^0(X, \Omega^1_{X/\mathbb{C}})) \simeq \mathbb{C}.$$

2.7 Modular forms with CM

Let $L = \mathbb{Q}(\sqrt{D})$ be a quadratic imaginary field of discriminant $D < 0$. Denote by $\mathcal{O}_L$ the ring of integers of L and let $\chi_L := \left(\frac{D}{\bullet}\right)$ be the quadratic character attached to it. Let $\mathfrak{c} \subset \mathcal{O}_L$ be an integral ideal of L and denote by $I(\mathfrak{c})$ the group of all fractional ideals prime to $\mathfrak{c}$. Consider a Hecke character:

$$c : I(\mathfrak{c}) \longrightarrow \mathbb{C}^\times$$

such that $c(\alpha\mathcal{O}_L) = \alpha$ for all $\alpha \in L^\times$ such that $\alpha \equiv 1 \mod \mathfrak{c}$. Let ω_c be the Dirichlet character modulo $N(\mathfrak{c})$, the norm of $\mathfrak{c}$, defined by:

$$\omega_c(n) := \frac{c(n\mathcal{O}_L)}{n}.$$

The power series:

$$f_{L,c} = \sum_{\mathfrak{a}} c(\mathfrak{a})q^N(\mathfrak{a})$$

taken over all integral ideals $\mathfrak{a} \subset \mathcal{O}_L$ prime to $\mathfrak{c}$, is a cusp form ([Zag04] Page 93)

$$f_{L,c} \in S_2(\Gamma_0(|D| \cdot N(\mathfrak{c})), \chi_L \cdot \omega_c).$$

Definition 2.24. The cusp form $f_{\mathfrak{c},L}$ is called a *cusp form with complex multiplication by L*, or *CM by L* in short.

The interesting property of CM forms is that the harmonic weak Maass forms which are good for them have holomorphic parts with algebraic coefficients.

Theorem 2.25. *Let $f \in S_2(\Gamma_0(N), \overline{\chi})$ be a normalized newform with complex multiplication. If $F \in \mathcal{H}_0(\Gamma_0(N), \chi)$ is good for f, then all of the coefficients of F^+ belong to K_f.*

Before we begin the proof, we need to recall the basics of the Eichler-Shimura construction, which associates to f an abelian variety A_f defined over $\mathbb{Q}$. For details see [Dar04] Chapter 2. Let $J := J_0(N)$ be the Jacobian of the modular curve X. The algebra $\mathbf{T}$ of Hecke operators acts via correspondences on J. For a normalized newform $f \in S_2(\Gamma_0(N))$ let

$$I(f) := \{T_n \in \mathbf{T} : T_n(f) = 0\}$$

be the ideal of $\mathbf{T}$ generated by the Hecke operators which annihilate f. Define:

$$A_f := \frac{J}{I(f)J}.$$

This is an abelian variety defined over $\mathbb{Q}$ with $\mathrm{End}(A_f) \otimes \mathbb{Q} \simeq K_f$, in such a way that the action of $a_n(f) \in K_f$ corresponds to the action of T_n on A_f. Moreover there is a canonical isomorphism, compatible with the Hodge filtration, given by:

$$ES : H^1_{\mathrm{dR}}(A_f/K_f) \xrightarrow{\simeq} \bigoplus_{\rho} H^1_{\mathrm{dR}}(X/K_f)_{f^\rho} \tag{2.7}$$

where ρ runs through all the elements of $\mathrm{Gal}(K_f/\mathbb{Q})$ and f^ρ is the normalized newform obtained by applying ρ to the coefficients of f. We will denote by $H^1_{\mathrm{dR}}(A_f/K_f)_f$ the 2-dimensional subspace of $H^1_{\mathrm{dR}}(A_f/K_f)$ corresponding to the f-isotypical component of $H^1_{\mathrm{dR}}(X/K_f)$ under the isomorphism ES.

Proof of Theorem 2.25. For ease of notation, let $K := K_f$. Let $\omega_f \in H^0(X, \Omega^1_{X/K})$ be the regular differential associated to f and let α_f be the unique element of $H^1(X, \mathcal{O}_{X/K})$ such that $\langle \omega_f, \alpha_f \rangle = 1$. Now if f has CM by L, then K is a CM field and the abelian variety $A := A_f$ associated to f has CM by K. This is because the field K necessarily contains the values of the character $\chi_L \cdot \omega_c$ associated to f (see [Rib04] Corollary 3.1). By the Eichler-Shimura isomorphism of Equation (2.7) one can view $\Omega_f := ES^{-1}(\omega_f)$ inside the f-isotypical component $H^1_{\mathrm{dR}}(A/K)_f$ of $H^1_{\mathrm{dR}}(A/K)$.

By the theory of CM abelian varieties, described in Section 1.6, there is a canonical decomposition (Equation 1.7)

$$H^1_{\mathrm{dR}}(A/K)_f = H^{1,0}(A/K)_f \oplus H^{0,1}(A/K)_f.$$

Let Θ_f be the unique element of $H^{0,1}(A/K)_f$ with $\langle \Omega_f, \Theta_f \rangle = 1$. The element:

$$\eta_f := ES(\Theta_f)$$

is a class in $H^1_{\mathrm{dR}}(X/K)_f$ with $\langle \omega_f, \eta_f \rangle = 1$, and the assignment $\alpha_f \mapsto \eta_f$ gives a canonical splitting:

$$\Phi_{\mathrm{CM}} : H^1(X, \mathcal{O}_K)_f \longrightarrow H^1_{\mathrm{dR}}(X/K)_f$$

of the Hodge filtration of $H^1_{\mathrm{dR}}(X/K)_f$, and this splitting *is defined over* K. Therefore, one can choose a differential of the second kind ϕ with poles only at the cusp at ∞, and such that:

$$[\phi] = \eta_f = \Phi_{\mathrm{CM}}(\alpha_f).$$

Moreover, since Φ_{CM} is defined over K, all the coefficients of ϕ belong to K. The theorem follows by constructing F as in the proof of Theorem 2.18. $\square$

Remark 2.26. A similar statement was proven in [BOR08] Theorem 1.3 for arbitrary weight. The authors show that F^+ must have coefficients in some abelian extension of K, analogous to their Theorem 1.4, and they further conjecture that this abelian extension is in fact trivial. Theorem 2.25, just like our result in the previous section about the vanishing of Hecke eigenvalues, confirms their conjecture in the weight 2 case.

3 THE p-ADIC THEORY

In the previous chapter we saw how harmonic weak Maass forms arise naturally from the canonical complex analytic splitting

$$\Phi_{\mathrm{Hodge}} : H^1(X_1(N), \mathcal{O}_{X_1(N)/\mathbb{C}}) \longrightarrow H^1_{\mathrm{dR}}(X_1(N)/\mathbb{C})$$

of the Hodge filtration of the complete modular curve $X_1(N)(\mathbb{C})$. The existence of the map Φ_{Hodge} depends on the Hodge decomposition of the de Rham cohomology of $X_1(N)(\mathbb{C})$, viewed as a compact complex analytic manifold.

In this chapter we switch our attention to $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$. We fix a valuation v on $\mathbb{Q}_p$ such that $v(p) = 1$ and an absolute value $|.|$ on $\mathbb{C}_p$ which is compatible with v. For simplicity, we assume that $\chi = 1$ is trivial, so that we will work with the $\Gamma_0(N)$ modular curve

$$X := X_0(N).$$

The set $X(\mathbb{C}_p)$ is naturally endowed with the structure of a rigid analytic space and we can define sheaves of rigid analytic modular forms on it analogous to the sheaves of smooth modular forms found in the complex analytic case. For a newform $f \in S_2(\Gamma_0(N))$, the f-isotypical component of the de Rham cohomology $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$ admits a canonical decomposition analogous to the Hodge decomposition. The analog of Φ_{Hodge} in the rigid analytic context then gives rise to a notion of *p-harmonic weak Maass forms*.

We assume a basic knowledge of rigid geometry at the level of Bosch's *Lectures on formal and rigid geometry* ([Bos05]). Only the basic definitions of affinoid subdomains and affinoid algebras are required, and any specific result needed will be stated as a proposition, with references provided to its proof. In particular, we will make use of the following terminology:

- A *closed disk* is a rigid analytic space isomorphic to $\{x \in \mathbb{C}_p : |x| \leq 1\}$
- An *open disk* is a rigid analytic space isomorphic to $\{x \in \mathbb{C}_p : |x| < 1\}$
- An *open annulus* is a rigid analytic space isomorphic to $\{x \in \mathbb{C}_p : r_1 < |x| < r_2\}$ for some $r_1, r_2 \in |\mathbb{C}_p^\times|$.

Throughout the chapter, we assume that p is a rational prime such that $p \nmid N$. Moreover, it will be convenient to assume $p > 3$, although the theory can be extended with minor modifications to the cases $p = 2, 3$. We will point to references on how to do this as we go along.

3.1 p-ADIC MODULAR FORMS

We start by recalling the definition of p-adic modular forms as first introduced by Katz in [Kat73] and further developed by Coleman in [Col96].

When $p \nmid N$ the modular curve $X = X_0(N)/\mathbb{Q}$ has good reduction at p (i.e. it has a smooth and proper model $\mathcal{X}$ over $\mathbb{Z}_p$) and there is a natural map:

$$\mathrm{red} : X(\mathbb{C}_p) \longrightarrow \mathcal{X}_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$$

where $\mathcal{X}_{\mathbb{F}_p}$ is the specialization of $\mathcal{X}$ mod p. Denote by $P_1, \ldots, P_t$ the points of $\mathcal{X}_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$ which correspond to supersingular elliptic curves under the modular interpretation of $X_0(N)$.

Proposition 3.1. *The subsets $D_i = \mathrm{red}^{-1}(P_i) \subset X(\mathbb{C}_p)$ are open disks. Each D_i is called the* residue disk *of P_i.*

Proof. From [Col89] Lemma 3.2, this is the case if the P_i are smooth, which they are since $\mathcal{X}_{\mathbb{F}_p}$ is smooth. □

The set

$$X^{\mathrm{ord}} = X(\mathbb{C}_p) - \{D_1 \cup \ldots \cup D_t\}$$

obtained by removing from $X(\mathbb{C}_p)$ the residue disks corresponding to the P_i, is an affinoid space.

The affinoid algebra $A(X^{\mathrm{ord}})$ has a norm inherited from the supremum norm on all p-adically convergent power series and consists of limits of rational functions on $X(\mathbb{C}_p)$ with poles supported inside the D_i. Similarly, we can consider the $A(X^{\mathrm{ord}})$-module $\Omega^1(X^{\mathrm{ord}})$ of rigid analytic differentials of X^{ord}.

Now we can also express X^{ord} as:

$$X^{\mathrm{ord}} = \{x \in X(\mathbb{C}_p) : |E_{p-1}(x)| \geq 1\}$$

where E_{p-1} is the Eisenstein series of weight $p - 1$. This is because, mod p, E_{p-1} is equal to the Hasse invariant, whose zeroes are precisely the supersingular points. Let $R = |\mathbb{C}_p|$ be the value group of $\mathbb{C}_p$. Pick any $r \in R$, $0 < r \leq 1$, and consider the sets

$$X_r = \{x \in X(\mathbb{C}_p) : |E_{p-1}(x)| \geq r\}.$$

These spaces are also affinoids and we can consider the ring $A(X_r)$ of rigid analytic functions on X_r and the $A(X_r)$-module $\Omega^1(X_r)$ of rigid analytic differentials.

Definition 3.2. A *p-adic modular form of weight 0 (resp. weight 2) and growth rate r* is an element of $A(X_r)$ (resp. $\Omega^1(X_r)$).

For any r such that $0 < r < 1$ we have obvious inclusions $X^{\mathrm{ord}} = X_1 \subset X_r \subset X$ and corresponding inclusions of p-adic modular forms going in the opposite directions. In particular,

$$S_2(\Gamma_0(N), \mathbb{C}_p) \simeq \Omega^1(X_0) \subseteq \Omega^1(X_r)$$

for any such r, so that p-adic modular forms contain the classical spaces of cusp forms in a natural way. Moreover, since the cusps of X are contained in X_r for any r, p-adic modular forms have q-expansions, just by considering the power series expansion of a rigid analytic differential in terms of local parameters at the cusps.

3.2 The U and V operators

Let k be a field of characteristic $p > 0$. Any such k possesses an injective endomorphism σ, the *Frobenius endomorphism*, defined by:

$$\begin{aligned} \sigma : & k \longrightarrow k \\ & x \longmapsto x^p. \end{aligned}$$

If C is an algebraic variety defined over k, denote by C^σ the variety obtained by base change with respect to σ. For example, if C is affine then C^σ is obtained by raising to the p-th power all the coefficients of the equations defining X. In characteristic p, there is a morphism:

$$\mathrm{Fr} : C \longrightarrow C^\sigma \tag{3.1}$$

which we refer to as the *relative Frobenius morphism* of C. On local coordinates $(x_1, \ldots, x_n)$, Fr acts simply by $x_i \mapsto x_i^p$. Note that if $k = \mathbb{F}_p$ then Fr is an endomorphism of C.

For example, when $C = \mathcal{X}_{\mathbb{F}_p}$ is the special fiber of $\mathcal{X}$ at p, then the relative Frobenius is an endomorphism of $\mathcal{X}_{\mathbb{F}_p}$. The goal of this section is to describe a 'canonical lift' of $\mathrm{Fr} : \mathcal{X}_{\mathbb{F}_p} \to \mathcal{X}_{\mathbb{F}_p}$ to the affinoid X^{ord}.

Consider first an elliptic curve E defined over a complete discretely valued subfield $K \subset \mathbb{C}_p$ with residue field k. Suppose E has a smooth proper model $\mathcal{E}$ over the ring of integers $\mathcal{O}_K$ and that the special fiber $\mathcal{E}_k$ is ordinary. In this situation, we are able to find a *canonical subgroup scheme* of $\mathcal{E}$ which only depends on the isomorphism class of $\mathcal{E}$.

Theorem 3.3. *Let E/K be an elliptic curve with good ordinary reduction. Then there is a unique connected subgroup scheme $H \subset \mathcal{E}$, finite and flat of rank p. We call H the canonical subgroup of E.*

Proof. We sketch the proof, which can be found in [Kat73] Section 3.1. If the special fiber $\mathcal{E}_k$ is ordinary, then the dual of the relative Frobenius morphism (3.1) is separable of degree p, so its kernel can be lifted to $\mathcal{O}_K$ (Hensel's Lemma). One then takes H to be the Cartier dual of this lift. □

Remark 3.4. By construction, when reduced to $\mathcal{E}_k$ the canonical subgroup H coincides with the scheme-theoretic kernel of the relative Frobenius morphism.

Now the points of X^{ord} classify pairs (E, C) of ordinary elliptic curves with a cyclic subgroup of order N. The map:

$$\varphi : (E, C) \longmapsto (E/H, C + H/H)$$

where H is the canonical subgroup of E, defines an endomorphism $\varphi : A(X^{\mathrm{ord}}) \to A(X^{\mathrm{ord}})$ of X^{ord}, since $p \nmid N$.

Definition 3.5. Define the V *operator* to be the pullback $V := \varphi^*$ on the space $\Omega^1(X^{\mathrm{ord}})$ of rigid analytic differentials.

Remark 3.6. By Remark 3.4, the endomorphism φ is a *canonical lift* to the ordinary locus X^{ord} of the Frobenius endomorphism acting on $\mathcal{X}_{\mathbb{F}_p}$. The existence of such canonical lift is peculiar to modular curves, and no such canonical lift is available for curves in general.

Proposition 3.7. *At the level of q-expansions, V acts by:*

$$V\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} a_n q^{np}.$$

Proof. In the modular interpretation of $X_0(N)$, the cusps correspond to pairs of generalized elliptic curves $(\mathrm{Tate}(q) := \mathbb{G}_m/q^{\mathbb{Z}}, \alpha_N)$ of Tate curves with level N structure. For these, the canonical subgroup is just μ_p and therefore V acts by $q \mapsto q^p$ at the cusps. $\square$

The theory of canonical subgroups can be extended (see [Kat73] Chapter 3) beyond ordinary elliptic curves to elliptic curves E for which $|E_{p-1}(E)|$ is large enough. One can then extend φ to the larger affinoids X_r in the following way:

Proposition 3.8.

(a) *For any $r \in R$ with $r < \frac{p}{p+1}$, the morphism φ extends to a morphism*

$$\varphi : A(X_{r^p}) \longrightarrow A(X_r)$$

of affinoid algebras, and to a corresponding linear map $V : \Omega^1(X_{r^p}) \to \Omega^1(X_r)$.

(b) *If $r < \frac{1}{p+1}$, then φ is finite and flat of degree p.*

Proof. See [Kat73] Section 3.10. This is one of the points where the assumption $p \geq 5$ is used. $\square$

Remark 3.9. Note that V is not an endomorphism of $\Omega^1(X_r)$, but it can only be defined on the smaller space $\Omega^1(X_{r^p}) \subset \Omega^1(X_r)$. This can be seen directly from q-expansions. In fact, if $f = \sum_n a_n q^n dq \in \Omega^1(X_r)$, then we must have $|a_n| r^n \to 0$ as $n \to \infty$. This convergence condition does not suffice to ensure convergence of $V(f)$, and must be replaced with the condition $|a_n| r^{pn} \to 0$ as $n \to \infty$.

From part (b) of Proposition 3.8, we see that whenever $r < \frac{1}{p+1}$ we have a trace morphism

$$\mathrm{tr}_\varphi : A(X_r) \longrightarrow A(X_{r^p})$$

corresponding to φ.

Definition 3.10. ([Kat73] 3.11.6) The U operator is defined to be the composite:

$$U : A(X_{r^p}) \longrightarrow A(X_r) \xrightarrow{\frac{1}{p}\operatorname{tr}_\varphi} A(X_{r^p})$$

where the first map is induced by the inclusion $X_r \subset X_{r^p}$. The operator U also acts via pullback on the spaces $\Omega^1(X_r)$ and we also denote this action by U.

Proposition 3.11. *The action of U on q-expansions is given by:*

$$U\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} a_{np} q^n.$$

Proof. See [Kat73] 3.11.6. □

In Section 2.4 we defined the Hecke operators T_ℓ for $\ell \nmid N$. In particular, the operator T_p is an endomorphism of the space $\Omega^1(X)$ of weight 2 cusp forms. Its action on q-expansions is given by

$$T_p\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} a_{np} q^n + p\sum_{n=0}^{\infty} a_n q^{np}.$$

Therefore on the space $\Omega^1(X)$ of classical weight 2 cusp forms, we have the fundamental relation:

$$T_p = U + p \cdot V. \tag{3.2}$$

In particular, if $\omega \in \Omega^1(X)$ is a normalized newform for T_p, we have:

$$a_p \cdot \omega = U(\omega) + p \cdot V(\omega).$$

Moreover, by direct computations with q-expansions, one sees that:

$$UV(f) = f. \tag{3.3}$$

and

$$VU(f) = f - \sum_{(n,p)=1} a_n q^n. \tag{3.4}$$

Let θ be the operator acting on q-expansions by $\sum_n a_n q^n \mapsto \sum_n n \cdot a_n q^n$. Then we can express the above relationship as:

$$VU(f) = f - \lim_{n\to\infty} \theta^{(p-1)\cdot p^n}(f). \tag{3.5}$$

3.3 WIDE OPEN SPACES

We now recall the definition of wide open spaces and state a few facts about them. The main reference is Coleman's *Reciprocity laws on curves* ([Col89]).

Definition 3.12. Let $X/\mathbb{C}_p$ be a smooth curve. A *diskoid subdomain* is a subdomain of $X(\mathbb{C}_p)$ which is non-empty and it is a finite union of disjoint closed disks. A *wide open space* is a rigid analytic space which is isomorphic to the complement of a diskoid subdomain.

In other words, just as affinoids are rigid analytic spaces isomorphic to the complement of finitely many disjoint open disks, wide opens are complements of finitely many closed disks. Following [BDP09] it will be convenient for us to consider wide open spaces as open neighborhoods of affinoids.

Definition 3.13. Let $\mathcal{A}$ be an affinoid on a smooth curve $X(\mathbb{C}_p)$. Thus, $\mathcal{A}$ can be realized as the complement of finitely many open disks in $X(\mathbb{C}_p)$. A *wide open neighborhood* $\mathcal{W}$ of $\mathcal{A}$ is any wide open space containing $\mathcal{A}$, obtained by attaching to $\mathcal{A}$ an open annulus to each disk lying in the complement of $\mathcal{A}$.

For a wide open space $\mathcal{W}$, define

$$H^1_{\mathrm{dR}}(\mathcal{W}) := \frac{\Omega^1(\mathcal{W})}{dA(\mathcal{W})}$$

where $A(\mathcal{W})$ is the algebra of rigid analytic functions on $\mathcal{W}$ and $\Omega^1(\mathcal{W})$ is the $A(\mathcal{W})$-module of rigid analytic differentials. Then we have the basic comparison theorem:

Theorem 3.14. *Let X be a smooth curve over $\mathbb{C}_p$ and let $\mathcal{W}$ be a wide open space on it. Let $S \subset X(\mathbb{C}_p)$ be a finite subset obtained by picking exactly one point in each disk lying in the complement of $\mathcal{W}$, and consider the algebraic variety $X - S$ defined over $\mathbb{C}_p$. Then there is a canonical isomorphism:*

$$H^1_{\mathrm{dR}}(\mathcal{W}) \simeq H^1_{\mathrm{dR}}(X - S/\mathbb{C}_p)$$

where the right-hand side is the algebraic de Rham cohomology of the algebraic variety $X - S$.

Proof. See [Col89] Theorem 4.2 $\square$

In particular, we see that the de Rham cohomology of wide open spaces is finite-dimensional, in contrast to the de Rham cohomology of affinoids.

Corollary 3.15. *Let $\mathcal{A}$ be an affinoid and let $\mathcal{W} \subset \mathcal{W}'$ be wide open neighborhoods of $\mathcal{A}$. Then the natural map*

$$H^1_{\mathrm{dR}}(\mathcal{W}') \longrightarrow H^1_{\mathrm{dR}}(\mathcal{W})$$

induced by inclusion is an isomorphism.

Proof. By Theorem 3.14, both spaces are naturally isomorphic to $H^1_{\mathrm{dR}}(X - S/\mathbb{C}_p)$ as long as S is chosen to be in the complement of $\mathcal{W}'$. $\square$

Going back to the case when $X = X_0(N)$, pick an $s \in |\mathbb{C}_p|$ such that $p^{-1/(p+1)} < s < 1$ and consider the wide open space:

$$\mathcal{W}_s = \{x \in X(\mathbb{C}_p) : |E_{p-1}(x)| > s\}$$

obtained by removing from $X(\mathbb{C}_p)$ all the closed residue disks of radius s around the supersingular points $P_1, \ldots, P_t$. The space $\mathcal{W}_s$ is a wide open neighborhood of the ordinary locus X^{ord} and is contained in X. In particular, by Theorem 3.14:

$$S_2(\Gamma_0(N), \mathbb{C}_p) = \Omega^1(X) \subset H^1_{\mathrm{dR}}(X/\mathbb{C}_p) \longrightarrow H^1_{\mathrm{dR}}(X/\mathbb{C}_p - \{P_1, \ldots, P_t\}) \simeq H^1_{\mathrm{dR}}(\mathcal{W}_s)$$

we can embed the space of weight 2 cusp forms with coefficients in $\mathbb{C}_p$ inside the de Rham cohomology of $\mathcal{W}_s$. Moreover, we can endow $H^1_{\mathrm{dR}}(\mathcal{W}_s)$ with an action of U and V.

Proposition 3.16. *The U, V operators induce linear operators on the finite-dimensional vector space $H^1_{\mathrm{dR}}(\mathcal{W}_s)$, and $UV = VU = \mathrm{Id}$ on $H^1_{\mathrm{dR}}(\mathcal{W}_s)$.*

Proof. By choice of s the U operator defines an endomorphism of $A(\mathcal{W}_s)$ satisfying $p \cdot Ud = dU$. As such, it respects the classes of exact differentials and it has a well-defined action in cohomology. However, the V operator is only defined on the subspace $A(\mathcal{W}_{s^p}) \subset A(\mathcal{W}_s)$ so it is not a priori obvious why it gives a well-defined endomorphism of $H^1_{\mathrm{dR}}(\mathcal{W}_s)$. Now, by the relation $p \cdot Vd = dV$ we deduce that V does extend to a linear map $H^1_{\mathrm{dR}}(\mathcal{W}_{s^p}) \to H^1_{\mathrm{dR}}(\mathcal{W}_s)$ in cohomology. But the space $\mathcal{W}_{s^p}$ is also a wide open neighborhood of the affinod X^{ord} so by Corollary 3.15 there is a natural isomorphism:

$$H^1_{\mathrm{dR}}(\mathcal{W}_s) \simeq H^1_{\mathrm{dR}}(\mathcal{W}_{s^p}).$$

which allows us to view V as an endomorphism of $H^1_{\mathrm{dR}}(\mathcal{W}_{s^p})$ (or $H^1_{\mathrm{dR}}(\mathcal{W}_s)$).

For the relation between U and V note that $UV = \mathrm{Id}$ by Equation (3.3). Since $df = \theta f \cdot dq/q$ Equation (3.4) also shows that $VU = \mathrm{Id}$ in cohomology. □

3.4 Residues

In this section we recall the notion of a p-adic annular residue, following [Col89] Section II. Recall that an *open annulus* is a rigid analytic space isomorphic to $\{x \in \mathbb{C}_p : r < |x| < s\}$ for some $r < s \in R^*$, where $R = |\mathbb{C}_p|$ is the value group of $\mathbb{C}_p$. Any such isomorphism is called a *uniformizing parameter* for the open annulus. If V is an open annulus, denote by $A(V)$ the space of rigid analytic functions on it and by $\Omega^1(V)$ the module of rigid analytic differentials.

Proposition 3.17. *Let V be an open annulus and let z be a uniformizing parameter for it. Then there is a unique $\mathbb{C}_p$-linear map $R : \Omega^1(V) \to \mathbb{C}_p$ such that:*

$$R(dg) = 0 \quad R(dz/z) = 1$$

for any $g \in A(V)$. Moreover, for any other uniformizing parameter z', $R(dz'/z') = \pm 1$.

Proof. As in the algebraic case, one defines $R(\omega)$ to be the a_{-1} coefficient in the Laurent expansion of ω. For details, see [Col89] Lemma 2.1. □

From the proposition we see that for any given V there are two possible residue maps. We call a choice of such function an *orientation* of V and denote the residue map by

$$\mathrm{res}_V : \Omega^1(V) \longrightarrow \mathbb{C}_p.$$

Let now $\mathcal{W}$ be a wide open space. Define the set of *ends* of $\mathcal{W}$ to be:

$$\mathcal{E}(\mathcal{W}) = \operatorname{proj\,lim} CC(\mathcal{W} - \mathcal{A})$$

where $CC(\mathcal{W}-\mathcal{A})$ are the connected components (for the Grothendieck topology) of $\mathcal{W}-\mathcal{A}$, where $\mathcal{A}$ runs over all affinoid subdomains of $\mathcal{W}$. The set of ends is finite, and given any end e and an affinoid subdomain $\mathcal{X}$ we call the image of e in $CC(\mathcal{W}-\mathcal{X})$ an *open neighborhood* of e. If $\omega \in \Omega^1(\mathcal{W})$ is a differential on $\mathcal{W}$, and e is an end of $\mathcal{W}$, define

$$\operatorname{res}_e \omega = \operatorname{res}_V \omega$$

where $V \in CC(X-Y)$ is the open annulus containing e for some sufficiently large affinoid Y.

Recall now from the previous section that for X a smooth curve over $\mathbb{C}_p$ we have an embedding:

$$H^1_{\mathrm{dR}}(X/\mathbb{C}_p) \hookrightarrow H^1_{\mathrm{dR}}(\mathcal{W}).$$

For any class $[\omega] \in H^1_{\mathrm{dR}}(\mathcal{W})$, one can speak of the residue of $[\omega]$, since each member in the class differs by an exact differential, which has zero residue. By computing residues in cohomology, we can then give a criterion for when a class in $H^1_{\mathrm{dR}}(\mathcal{W})$ comes from $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$.

Theorem 3.18. *Let $X/\mathbb{C}_p$ be a smooth curve and $\mathcal{W}$ be a wide open space on X. Then the image of $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$ in $H^1_{\mathrm{dR}}(\mathcal{W})$ consists of those classes $[\omega]$ with*

$$\operatorname{res}_e(\omega) = 0$$

for all ends e of $\mathcal{W}$.

Proof. See [Col89] Proposition 4.4. □

Corollary 3.19. *Let $\omega \in \Omega^1(\mathcal{W})$ be such that its class in $H^1_{\mathrm{dR}}(\mathcal{W})$ lies in $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$. Let e be an end of $\mathcal{W}$ and let D_e be the disk in the complement of $\mathcal{W}$ corresponding to e. Then there exists an open disk U_e containing the disk D_e and no other connected component of the complement of $\mathcal{W}$ and a rigid analytic function λ_e on the open annulus $V_e = U_e - D_e$ such that $d\lambda_e = \omega$ on V_e.*

Remark 3.20. The rigid analytic function λ_e serves the purposes of a 'local antiderivative' at e for ω.

Using this notion of local antiderivative, rigid analytic residues can be employed to compute the Poincaré pairing on $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$.

Proposition 3.21. *Let η_1, η_2 denote two classes in $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$. Let ζ_1, ζ_2 be two differential forms in $\Omega^1(\mathcal{W})$, for a wide open space $\mathcal{W} \subset X$, such that $[\zeta_i] = \eta_i$ for $i = 1, 2$ Then:*

$$\langle \eta_1, \eta_2 \rangle = \sum \mathrm{res}_e \, \lambda_e \zeta_2$$

where λ_e is a local antiderivative of ζ_1 around e and the sum runs through all ends of $\mathcal{W}$.

Proof. This is Proposition 4.5 of [Col89]. □

Remark 3.22. This formula for the Poincaré pairing in terms of rigid analytic residues is the rigid analytic analog of the formula given in Proposition 1.16.

3.5 p-HARMONIC DIFFERENTIALS

We now turn to our original question of finding a p-adic analog of the Hodge Decomposition for the f-isotypical component of the de Rham cohomology $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$. In the complex analytic theory, the key is to define canonical 'harmonic' representatives for the classes in $H^1_{\mathrm{dR}}(X/\mathbb{C})$. It is natural then to begin by understanding what the right notion of 'harmonic representative' is in the p-adic setting.

Definition 3.23. A form $\eta \in \Omega^1(\mathcal{W}_s)$ is *p-harmonic* if

$$\eta \in \Omega^1(X) \oplus V(\Omega^1(X)).$$

In other words, $\eta \in \Omega^1(\mathcal{W}_s)$ is *p-harmonic* if it can be written as a linear combination of ω_1 and $V(\omega_2)$ for ω_1, ω_2 regular differentials on X.

We restrict our attention to a newform $f \in S_2(\Gamma_0(N))$ with rational coefficients which is ordinary at p, i.e. its p-th Fourier coefficient is a p-adic unit. The differentials ω_f and $V(\omega_f)$ are examples of p-harmonic differentials. Moreover, by looking at their q-expansions, we deduce that they are linearly independent. The 2-dimensional subspace W_f of $\Omega^1(\mathcal{W}_s)$ generated by ω_f and $V(\omega_f)$ has the property that the action of U on it can be diagonalized nicely.

Proposition 3.24. *Let $f = \sum_{n>0} a_n q^n$ be a newform in $S_2(\Gamma_0(N))$ with coefficients in $\mathbb{Q}$ and let $T^2 - a_p T + p = (T - \alpha)(T - \beta)$ be its corresponding Frobenius polynomial at p. Assume that*

$0 = v_p(\alpha) < v_p(\beta) = 1$ *(i.e. f is ordinary at p). Then the rigid analytic differentials:*

$$\eta_f^\alpha := \omega_f - \alpha \cdot V(\omega_f)$$
$$\eta_f^\beta := \omega_f - \beta \cdot V(\omega_f)$$

form a basis for W_f of eigenvectors of U.

Proof. We first write down the matrix of U acting on W_f with respect to the basis $\{\omega_f, V(\omega_f)\}$. This can be done using the relations (3.2) and (3.3) and noting that $T_p(\omega_f) = a_p \cdot \omega_f$. We get:

$$U = \begin{pmatrix} a_p & 1 \\ -p & 0 \end{pmatrix}$$

which shows that U has eigenvalues α and β on W_f. These are distinct by our assumption on their valuations and therefore the matrix can be diagonalized. A short computation shows that $\eta_f^\alpha, \eta_f^\beta$ are the eigenvectors corresponding to α and β respectively. □

In the complex analytic theory, harmonic differentials give rise to classes in cohomology. Similarly, we want to understand how the p-harmonic differentials η_f^α and η_f^β behave under projection onto cohomology.

Proposition 3.25. *Consider the projection map:*

$$[\cdot] : \Omega^1(\mathcal{W}_s) \longrightarrow H^1_{\mathrm{dR}}(\mathcal{W}_s).$$

Then $[\eta_f^\alpha] \neq 0$.

Proof. Let $\eta_f^\alpha = \sum_n c_n q^n \frac{dq}{q}$ and suppose that there exists a rigid analytic function $g = \sum b_n q^n \in A(\mathcal{W}_s)$ such that $dg = \eta_f^\alpha$. In terms of q-expansions this means that $\theta(\sum b_n q^n) = \sum c_n q^n$. Since η_f^α is an eigenvector for U with eigenvalue α, we have:

$$c_{pn} = \alpha \cdot c_n = \alpha \cdot n \cdot b_n.$$

But α is a p-adic unit, and therefore the coefficients of η^α would not converge to zero, which is a contradiction. □

It would be reasonable to hope that $[\eta_f^\beta] \neq 0$ as well, but this is not always true.

Conjecture 3.26 (Coleman). *With the notation of Proposition 3.25, $[\eta_f^\beta] = 0$ if and only if f has CM.*

In [Col96] Robert Coleman proves that if f has CM then $[\eta_f^\beta] = 0$, but the converse it not known.

Armed with these two facts, we are ready to give a rudimentary analog of the Hodge decomposition for $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$.

Theorem 3.27. *Let $f \in S_2(\Gamma_0(N))$ be a normalized newform with coefficients in $\mathbb{Q}$ and ordinary at p. Assume moreover that $[\eta_f^\beta] \neq 0$. Then any class in the f-isotypical component $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$ has a unique representative which is p-harmonic.*

Proof. Let ω_f be the regular differential corresponding to f. We claim that any class in $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$ has a representative which is a linear combination of ω_f and $V(\omega_f)$. First we need to check that ω_f and $V(\omega_f)$ define classes in $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$, and not just in the larger space $H^1_{\mathrm{dR}}(\mathcal{W}_s)$. For ω_f, this just comes from the Hodge filtration. For $V(\omega_f)$, we appeal to Theorem 3.18 and note that $V(\omega_f)$ has zero residues everywhere, therefore it defines a class in $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$. Next, we need to show that $[\omega_f]$ and $[V(\omega_f)]$ are linearly independent in $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$. By assumption, $[\eta_f^\beta] \neq 0$ and therefore it is an eigenvector of U acting on $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$ with eigenvalue β. Similarly the class $[\eta_f^\alpha]$ is an eigenvector of U with eigenvalue α. As $\alpha \neq \beta$, these two classes must be linearly independent, and so must $[\omega_f]$ and $[V(\omega_f)]$, since η_f^α and η_f^β are linear combinations of ω_f and $V(\omega_f)$. Now the space $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$ is 2-dimensional, so $[\omega_f]$ and $[V(\omega_f)]$ span the whole space. □

Remark 3.28. The canonical basis $\{\omega_f, V(\omega_f)\}$ of $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$ can be viewed as the p-adic analog of the canonical basis $\{\omega_f, \overline{\omega_f}\}$ of harmonic representatives for $H^1_{\mathrm{dR}}(X/\mathbb{C})_f$ which is obtained from the Hodge Decomposition.

Corollary 3.29 (Frobenius Decomposition). *Let $f \in S_2(\Gamma_0(N))$ satisfy the hypotheses of Theorem 3.27. Then there is a canonical decomposition:*

$$H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f \simeq H^{1,0}(X/\mathbb{C}_p)_f \oplus H^{0,1}(X/\mathbb{C}_p)_f$$

where $H^{1,0}(X/\mathbb{C}_p)_f = H^0(X, \Omega^1_{X/\mathbb{C}_p})_f$ is the subspace generated by $[\omega_f]$ and $H^{0,1}(X/\mathbb{C}_p)_f$ is the subspaceb generated by $[V(\omega_f)]$.

Remark 3.30. If Conjecture 3.26 were true then the decomposition would hold whenever f does not have CM. On the other hand, we know from the previous chapter that if f has CM there is an *algebraic* decomposition of $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$. It would be interesting to give a general construction that encompasses both decompositions at the same time.

The Frobenius decomposition enables us to find a canonical splitting:

$$\Phi_{\mathrm{Frob}} : H^1(X, \mathcal{O}_{\mathbb{C}_p})_f \longrightarrow H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$$

of the Hodge filtration of $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$. This splitting will enable us to produce *p-harmonic weak Maass forms*.

3.6 p-HARMONIC WEAK MAASS FORMS

In the complex analytic case, we used the splitting Φ_{Hodge} to find harmonic weak Maass forms of weight 0 over $\mathbb{C}$. In the p-adic analytic case, we can repeat the same construction using the splitting Φ_{Frob} given by Corollary 3.29.

Namely, let $f \in S_2(\Gamma_0(N))$ satisfy the hypotheses of Theorem 3.27. As usual, consider the corresponding regular differential:

$$\omega_f \in H^0(X, \Omega^1_{X/\mathbb{Q}})_f$$

and let $\alpha_f \in H^1(X, \mathcal{O}_{X/\mathbb{Q}})_f$ be the unique element such that $\langle \omega_f, \alpha_f \rangle = 1$.

The p-adic f-isotypical Hodge filtration:

$$0 \longrightarrow H^0(X, \Omega^1_{X/\mathbb{C}_p})_f \longrightarrow H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f \longrightarrow H^1(X, \mathcal{O}_{X/\mathbb{C}_p})_f \longrightarrow 0.$$

has a canonical splitting:

$$\Phi_{\mathrm{Frob}} : H^1(X, \mathcal{O}_{X/\mathbb{C}_p})_f \longrightarrow H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$$

which is a consequence of Corollary 3.29. This is defined by:

$$\eta_f := \Phi_{\mathrm{Frob}}(\alpha_f) = \frac{V(\omega_f)}{\langle \omega_f, V(\omega_f) \rangle}.$$

where the pairing is evaluated using Proposition 3.21. Using Proposition 1.8, we find a differential of the second kind $\phi \in \Omega^{\mathrm{II}}_{X/\mathbb{C}_p}(X)$ such that:

$$[\eta_f] = [\phi]$$

and such that ϕ has poles only at the cusps of X. The class of $\phi - \eta_f$ is then zero in $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)$. Using the rigid analytic embedding:

$$H^1_{\mathrm{dR}}(X/\mathbb{C}_p) \hookrightarrow H^1_{\mathrm{dR}}(\mathcal{W}_s)$$

we also have that $\phi - \eta_f$ is zero in $H^1_{\mathrm{dR}}(\mathcal{W}_s)$. It follows that there exists an element $F \in A(\mathcal{W}_s)$ such that

$$dF = \phi - \eta_f.$$

The rigid analytic function F is the prototypical example of a *p-harmonic weak Maass form of weight 0*.

3.7 Recovering the shadow

We conclude our discussion by applying the techniques developed in this chapter to prove a special case of Theorem 1.2(1) of [GKO09], which ties together the complex analytic theory of harmonic weak Maass forms and the p-adic analytic theory.

Let $f \in S_2(\Gamma_0(N))$ satisfy the hypotheses of Theorem 3.27 and suppose its q-expansion is given by

$$f = \sum_{n=1}^{\infty} a_n q^n.$$

In particular, recall that the Frobenius polynomial of f at p has roots α and β with p-adic valuations 0 and 1 respectively.

Going back to the complex analytic theory, Theorem 2.18 ensures that we can find a F in $\mathcal{H}^0(\Gamma_0(N))$ which is good for f. In the terminology of [GKO09], f is called the *shadow* of F, and can be recovered from F by

$$\xi_0(F) = \frac{f}{\|f\|^2}$$

where $\|f\|$ is the Petersson norm of f,

The harmonic weak Maass form F has a holomorphic part F^+ with the property that

$$dF^+ = \phi - a \cdot \omega_f$$

for some differential of the second kind $\phi \in \Omega^{\mathrm{II}}_{X/\mathbb{Q}}(X)$ regular on Y and some $a \in \mathbb{C}$. In [GKO09], the authors ask whether one can recover the coefficients of the shadow f from the coefficients of

F^+. This is the content of Theorem 1.2 of [GKO09], which we reprove here in a special case using the p-adic techniques developed in this section.

Define:

$$\hat{f} := f(z) - \alpha \cdot f(pz)$$

and let

$$\mathcal{F}_a = F^+(z) - a \cdot E_f(z)$$

where $E_f(z)$ is the q-expansion given by:

$$E_f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} q^n.$$

Let $D := \frac{1}{2\pi i} \cdot \frac{d}{dz}$ and write:

$$D(\mathcal{F}_a) = \sum_{n \gg -\infty} c_a(n) q^n.$$

Theorem 3.31 (Guerzhoy,Kent,Ono)**.** *Suppose f satisfies the hypotheses of Theorem 3.27. Then*

$$\lim_{w \to +\infty} \frac{U^w(D(\mathcal{F}_a))}{c_a(p^w)} = \hat{f}$$

Proof. First of all, note that $d\mathcal{F}_a = \phi$, a differential of the second kind with coefficients in $\mathbb{Q}$, and therefore the element $D(\mathcal{F}_a)$ is a weakly holomorphic modular form of weight 2 with rational coefficients and ϕ defines a class in $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$. Now by our assumptions on f the space $H^1_{\mathrm{dR}}(X/\mathbb{C}_p)_f$ has a basis $\{\eta_f^\alpha, \eta_f^\beta\}$ of eigenvectors for U. Therefore we can write

$$\phi = t_1 \cdot \eta_f^\alpha + t_2 \cdot \eta_f^\beta + dh \tag{3.6}$$

for some meromorphic function $h \in \mathbb{C}_p(X)$ and constants $t_1, t_2 \in \mathbb{C}_p$. Applying U to both sides of the equation gives:

$$U(\phi) = t_1 \cdot \alpha \cdot \eta_f^\alpha + t_2 \cdot \beta \cdot \eta_f^\beta + U(dh)$$

and therefore by induction

$$U^w(\phi) = t_1 \cdot \alpha^w \cdot \eta_f^\alpha + t_2 \cdot \beta^w \cdot \eta_f^\beta + U^w(dh).$$

Dividing by α^w

$$\alpha^{-w}U^w(\phi) = t_1 \cdot \eta_f^\alpha + t_2 \cdot \left(\frac{\beta}{\alpha}\right)^w \eta_f^\beta + \alpha^{-w}U^w(dh).$$

and taking the limit as $w \to \infty$ gives:

$$\lim_{w\to\infty} \alpha^{-w}U^w(\phi) = t_1 \cdot \eta_f^\alpha.$$

This is because $v_p(\beta/\alpha) > 0$ by the hypotheses and because the differential $U^w(dh)$ has bounded denominator but its coefficients have arbitrarily high valuation as $w \to \infty$. In fact, each application of U to dh multiplies each coefficient by p.

To determine the value of the constant t_1, write down the p-th coefficient of Equation (3.6):

$$\begin{aligned} c_a(p) &= t_1 \cdot (a_p - \alpha^{-1}p) + t_2 \cdot (a_p - \beta^{-1}p) + dh \\ &= t_1 \cdot \alpha + t_2 \cdot \beta. \end{aligned}$$

By applying the multiplicative properties of the Fourier coefficients of newforms we get:

$$c_a(p^w) = t_1 \cdot \alpha^w + t_2 \cdot \beta^w$$

and taking the limit we obtain:

$$\lim_{w\to\infty} \alpha^{-w} c_a(p^w) = t_1$$

which gives the result. □

4 FURTHER DIRECTIONS

In this final chapter we sketch how to extend the geometric theory of harmonic weak Maass forms to arbitrary integer weight. The main tools come from Katz's theory of algebraic modular forms ([Kat73]) and from Section 1 of [BDP09], where the authors produce differential operators on modular forms from splittings of the Hodge filtration of modular curves.

Let $Y := Y_1(N)$ and $X := X_1(N)$ be the $\Gamma_1(N)$ modular curves of Chapter 2 and consider them as algebraic curves defined over a field $K \subset \mathbb{C}$. In the weight 2 case, the assignment:

$$f \longmapsto f(q) \cdot \frac{dq}{q}$$

gave a natural isomorphism $M^\dagger(\Gamma_1(N), K) \simeq H^0(Y, \Omega^1_X)$. We want to find an analog for arbitrary integer weight. Following Katz, let $\pi : \mathcal{E} \to Y$ be the universal elliptic curve with $\Gamma_1(N)$-level structure and let

$$\underline{\omega} := \pi^*(\Omega^1_{\mathcal{E}/Y})$$

be the sheaf of relative differentials on $\mathcal{E}/Y$. We then have an isomorphism:

$$M^!_k(\Gamma_1(N), K) = H^0(Y, \underline{\omega}^{\otimes k})$$

which plays the analog of the isomorphism in Proposition 2.4.

Consider now the relative de Rham cohomology sheaf:

$$\mathcal{L}_1 := \mathbb{R}^1\pi_*(0 \to \mathcal{O}_{\mathcal{E}} \to \Omega^1_{\mathcal{E}/Y} \to 0).$$

For each point x of Y, the fiber of $\mathcal{L}_1$ is the algebraic de Rham cohomology of the elliptic curve corresponding to x under the modular interpretation of Y. The Poincaré pairing on each fiber induces a pairing:

$$\langle \cdot, \cdot \rangle : \mathcal{L}^1 \times \mathcal{L}^1 \to \mathcal{O}_Y$$

and the Hodge filtration on the fibers induces a filtration of sheaves:

$$0 \to \underline{\omega} \to \mathcal{L}^1 \to \underline{\omega}^{-1} \to 0.$$

The sections of $\underline{\omega}^{\otimes k}$, which are modular forms of weight k, can then be naturally embedded in the space of sections of $\mathcal{L}_1^{\otimes k}$.

On $\mathcal{L}_1$ we also have a canonical integrable connection, the *Gauss-Manin connection*:

$$\nabla : \mathcal{L}_1 \longrightarrow \mathcal{L}_1 \otimes \Omega^1_Y$$

which can be used to obtain an isomorphism of sheaves over Y (the Kodaira-Spencer isomorphism)

$$\begin{aligned}&\sigma : \underline{\omega}^2 \xrightarrow{\sim} \Omega^1_Y\\ &\sigma(\omega^2) = \langle \omega, \nabla\omega \rangle \,.\end{aligned}$$

When $K = \mathbb{C}$, we can view $\mathcal{L}_1$ as a real analytic sheaf $\mathcal{L}_1^{\mathrm{ra}}$ on the differentiable manifold $X(\mathbb{C})$ and the Hodge decomposition gives a real analytic splitting:

$$\Phi_{\mathrm{Hodge}} : \mathcal{L}_1^{\mathrm{ra}} \to \underline{\omega}.$$

Combining Φ_{Hodge} with the Gauss-Manin connection (see [BDP09]) gives rise to differential operators ξ_k which are generalizations of the ξ_0 operator on harmonic weak Maass forms of weight 0 and which correspond to the operators of Bruinier and Funke ([BF04]).

When $K = \mathbb{C}_p$, one can view $\mathcal{L}_1$ as a rigid analytic sheaf on $X(\mathbb{C}_p)$ and the rigid analytic splitting:

$$\Phi_{\mathrm{Frob}} : \mathcal{L}_1^{\mathrm{rig}} \to \underline{\omega}.$$

coming from the Frobenius decomposition also gives rise to 'differential' operators. Finding antiderivatives to these operators yields p-harmonic weak Maass forms of arbitrary integer weight.

Finally, one would like to develop a geometric description of harmonic weak Maass forms of half-integral weight. Perhaps the most immediate application of such a theory would be a geometric understanding of the results of Bruinier and Ono ([BO]) relating the coefficients of the holomorphic parts of harmonic weak Maass forms of half-integral weight to the vanishing of derivatives of L-series of cusp forms of weight 2. At present, however, there is no analog in the literature of Proposition 2.4 for modular forms of half-integral weight, and therefore the cornerstone of our geometric theory of harmonic weak Maass forms is still missing in the half-integral weight case.

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