THE VALUES OF THE DEDEKIND–RADEMACHER COCYCLE
AT REAL MULTIPLICATION POINTS

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ABSTRACT. The values of the so-called *Dedekind–Rademacher cocycle* at certain real quadratic arguments are shown to be global p-units in the narrow Hilbert class field of the associated real quadratic field, as predicted by the conjectures of [DD06] and [DV21]. The strategy for proving this result combines the approach of [DPV21] with one crucial extra ingredient: the study of infinitesimal deformations of irregular Hilbert Eistenstein series of weight one in the anti-parallel direction, building on the techniques of [BDP].

CONTENTS

INTRODUCTION

Let $\mathcal{H}_p$ denote Drinfeld's p-adic upper half plane, and let $\mathcal{M}^\times$ denote the multiplicative group of non-zero rigid meromorphic functions on $\mathcal{H}_p$, equipped with the translation action of the discrete group $\mathrm{SL}_2(\mathbb{Z}[1/p])$ by Möbius transformations. A *rigid meromorphic cocycle* on a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z}[1/p])$ is a class in $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$. If $\tau \in \mathcal{H}_p$ is a *real multiplication*, or RM, point, i.e., generates a real quadratic extension of $\mathbb{Q}$, the *value* of J at τ is defined to be

$$(1) \qquad J[\tau] := J(\gamma_\tau)(\tau) \in \mathbb{C}_p \cup \{\infty\},$$

where $\gamma_\tau \in \Gamma$ is the *automorph* of τ, a suitably normalised generator of the stabiliser of τ in Γ. The relevance of the RM values of rigid meromorphic cocycles to explicit class field theory for real quadratic fields has been explored in [Da01], [DD06] [DV21], and [DV], where it is conjectured, broadly speaking, that they behave in many key respects just like the values of classical modular functions at CM points, and in particular that they belong to, and often generate, narrow ring class fields of real quadratic fields.

Theorem B below gives some theoretical evidence for this general conjecture in the simplest case where $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$ and J is *analytic*, i.e., takes values in the subgroup $\mathcal{A}^\times \subset \mathcal{M}^\times$ of rigid analytic functions.

Strictly speaking, there are no interesting rigid analytic cocycles: the group $\mathrm{H}^1(\Gamma, \mathcal{A}^\times)$ is generated, up to torsion, by the class J_{triv} given by

$$J_{\mathrm{triv}} \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) (z) = cz + d,$$

whose RM values are units in the associated quadratic order – hence, algebraic, but not in an interesting way for explicit class field theory.

There is a less trivial class in $\mathrm{H}^1(\Gamma, \mathcal{A}^\times / p^{\mathbb{Z}})$ arising from the classical *Dedekind–Rademacher homomorphism* $\varphi_{\mathrm{DR}} : \Gamma_0(p) \longrightarrow \mathbb{Z}$ describing the periods of the weight two Eisenstein series

$$(2) \quad E_2^{(p)}(q) = \mathrm{dlog}\left(\frac{\Delta(q^p)}{\Delta(q)} \right) = \left(p - 1 + 24 \sum_{n=1}^\infty \sigma^{(p)}(n) q^n \right) \frac{dq}{q}, \qquad \text{where } \sigma^{(p)}(n) := \sum_{p \nmid d | n} d,$$

and given by

$$(3) \qquad\qquad\qquad \varphi_{\mathrm{DR}}(\gamma) := \frac{1}{2\pi i} \int_{z_0}^{\gamma z_0} 2 E_2^{(p)}(z) dz.$$

More precisely, the description of Γ as an amalgamated product of two conjugate copies of $\mathrm{SL}_2(\mathbb{Z})$ intersecting in $\Gamma_0(p)$ leads to an injection

$$\mathrm{H}^1(\Gamma_0(p), \mathbb{Z}) \hookrightarrow \mathrm{H}^2(\Gamma, \mathbb{Z}).$$

Let $\alpha_{\mathrm{DR}} \in Z^2(\Gamma, \mathbb{Z})$ be a two-cocycle whose cohomology class is the image of φ_{DR} under this map. Refining a construction of [DD06], Theorem A below asserts that the cocycle $p^{\alpha_{\mathrm{DR}}}$ with values in $p^{\mathbb{Z}}$ is trivialised in the larger group $\mathcal{A}^\times \supset p^{\mathbb{Z}}$:

Theorem A. *There is a one-cochain $J_{\mathrm{DR}} \in C^1(\Gamma, \mathcal{A}^\times)$ satisfying*

$$\gamma_1 J_{\mathrm{DR}}(\gamma_2) \div J_{\mathrm{DR}}(\gamma_1 \gamma_2) \times J_{\mathrm{DR}}(\gamma_1) = p^{\alpha_{\mathrm{DR}}(\gamma_1, \gamma_2)}, \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma.$$

The essential triviality of $\mathrm{H}^1(\Gamma, \mathcal{A}^\times)$ shows that J_{DR} is uniquely determined up to coboundaries and powers of the cocycle J_{triv} above. The proof of Theorem A is given in § 1, and constructs an explicit cochain J_{DR} which is well defined up to coboundaries. The natural image of J_{DR} in $\mathrm{H}^1(\Gamma, \mathcal{A}^\times / p^{\mathbb{Z}})$ is the *Dedekind–Rademacher cocycle* of the title. The rigid analytic cocycles of higher level studied in [DD06] are all multiplicative combinations of $\mathrm{GL}_2(\mathbb{Q})$-translates of this basic cocycle. The proof of Theorem A complements the approach of [DD06], producing a more canonical object in level 1 which can be envisaged as an avatar of the Eisenstein series E_2 in the setting of rigid meromorphic cocycles. The RM values of J_{DR} are well defined modulo $p^{\mathbb{Z}}$, and it therefore makes sense to enquire about their algebraicity, and their factorisation away from p.

An RM point $\tau \in \mathcal{H}_p$ is said to be of *discriminant D* if it satisfies an equation of the form $Q(\tau, 1) = 0$, where $Q(x, y) = Ax^2 + Bxy + Cy^2$ is a primitive integral binary quadratic form of discriminant D. The set $\mathcal{H}_p^D$ of τ of a fixed discriminant D is non-empty precisely when p is inert or ramified in the quadratic field $F = \mathbb{Q}(\sqrt{D})$, and is preserved by the action of $\mathrm{SL}_2(\mathbb{Z})$. The orbit set $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}_p^D$ is in natural bijection with the class group $\mathrm{Cl}(D)$, by sending the orbit of τ to the narrow equivalence class $\mathcal{C}_\tau$ of the fractional ideal generated by τ and 1 when $\tau - \tau'$ is positive. The reciprocity map

$$\mathrm{rec} : \mathrm{Cl}(D) \longrightarrow \mathrm{Gal}(H/F)$$

of global class field theory identifies $\mathrm{Cl}(D)$ with the Galois group of the narrow ring class field of H over F attached to D. If $\mathcal{H}_p^D$ is non-empty and $p \nmid D$, then the prime p is inert in $F/\mathbb{Q}$ and splits completely in

H/F. The choice of an embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$ hence determines a prime $\mathfrak{p}$ of H above p, which is fixed once and for all. Fix also a complex embedding $\bar{\mathbb{Q}} \subset \mathbb{C}$ and write $x \mapsto \bar{x}$ for the action of complex conjugation on H (which is independent of the choice of embedding). Let

$$\mathcal{O}_H[1/p]_-^\times$$

be the group of p-units of H which are in the minus-eigenspace for the action of complex conjugation. By the Dirichlet S-unit theorem, it is a $\mathbb{Z}$-module of rank $[H : F]/2$ if F does not possess a unit of negative norm, and is finite otherwise. In particular, there is a unique element $u_\tau \in (\mathcal{O}_H[1/p]_-^\times) \otimes \mathbb{Q}$ satisfying

(4) $$\operatorname{ord}_{\mathfrak{p}^\sigma}(u_\tau) = -L(F, \mathcal{C}_\tau^\sigma, 0), \quad \text{for all } \sigma \in \operatorname{Gal}(H/F),$$

where $L(F, \mathcal{C}_\tau^\sigma, s)$ is the partial zeta function of the narrow ideal class $\mathcal{C}_\tau^\sigma$ (cf. [Gr82, Prop. 3.8]). The p-unit u_τ is called the *Gross–Stark unit* attached to H/F (and the prime $\mathfrak{p}$). The Brumer–Stark conjecture implies that u_τ^{12} belongs to $\mathcal{O}_H[1/p]^\times$ rather than to the tensor product of this group with $\mathbb{Q}$. The proof by Samit Dasgupta and Mahesh Kakde of (the prime to 2 part of) the Brumer–Stark conjecture [DKa] in this setting shows that u_τ^{12} belongs to $(\mathcal{O}_H[1/p]^\times) \otimes \mathbb{Z}[1/2]$.

The principal conjecture of [DD06], and its refinement covering the Dedekind–Rademacher cocycle itself, asserts that $J_{\mathrm{DR}}[\tau]$ is equal, up to a small torsion ambiguity and powers of p, to an integer power of the Gross–Stark unit u_τ. The weaker equality

(5) $$\operatorname{Norm}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}(J_{\mathrm{DR}}[\tau]) = \operatorname{Norm}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}(u_\tau^{12}) \pmod{(\mathbb{Q}_p^\times)_{\mathrm{tors}}, p^{\mathbb{Z}}}$$

involving the norms to $\mathbb{Q}_p^\times$ of these invariants was shown in [DD06] to follow from Gross's p-adic analogue of the Stark conjecture on p-adic abelian L-series of totally real fields at $s = 0$ – at least, after replacing $J_{\mathrm{DR}}[\tau]$ by the closely allied quantities denoted $u(\alpha, \tau)$ in [DD06], which depend on the choice of a suitable modular unit $\alpha \in \mathcal{O}_{Y_1(N)}^\times$ with auxiliary level structure. The Gross–Stark conjecture was then proved in [DDP11]. An important recent work of Samit Dasgupta and Mahesh Kakde [DKb] has significantly refined the approach of [DDP11] to prove Gross's *tame refinement* of the Gross–Stark conjecture, for arbitrary totally real fields. Specialising this result to the case of a real quadratic field leads to the refinement

(6) $$J_{\mathrm{DR}}[\tau] = u_\tau^{12} \pmod{(\mathbb{Q}_{p^2}^\times)_{\mathrm{tors}}, \ p^{\mathbb{Z}}}$$

of (5) in which the norm is removed. The removal of this ambiguity is crucial for a truly satisfying approach to explicit class field theory for real quadratic fields.

The main contribution of this paper is an independent and more direct proof of (6) for fundamental discriminants:

Theorem B. *Let $D > 0$ be a fundamental discriminant that is prime to p. If τ is an RM point in $\mathcal{H}_p$ of discriminant D, then $J_{\mathrm{DR}}[\tau]$ is equal to the Gross–Stark unit u_τ^{12}, up to torsion in $\mathbb{Q}_{p^2}$ and powers of p, and in particular belongs to $(\mathcal{O}_H[1/p]^\times) \otimes \mathbb{Z}[1/2]$.*

To situate the approach of this paper in the context of previous works, note that Dasgupta and Kakde tackle Theorem B by studying Mazur–Tate style "tame refinements" of the techniques of [DDP11], leading to a proof of Gross's tame refinement of his p-adic Stark conjecture (known as the "tower of fields conjecture" [Gr88]). They then show that this tame refinement implies Theorem B. Like [DKb], the present work rests on the careful study of deformations of Galois representations that was also exploited in [DDP11], but otherwise differs in its approach to Theorem B by avoiding the recourse to tame deformations. Its key idea is to package the RM values of J_{DR} as the coefficients of certain modular generating series. The resulting identities (cf. Theorem C below) are of interest in their own right and enrich the tapestry of analogies between RM values of rigid meromorphic cocycles and CM values of modular functions.

The Dedekind–Rademacher cocycle, taken modulo $\mathbb{C}_p^\times$ rather than $p^{\mathbb{Z}}$, is a prototypical instance of a *rigid analytic theta-cocycle*: a function $J : \Gamma \longrightarrow \mathcal{A}^\times$ which satisfies the one-cocycle relation, but only up to multiplicative scalars. The proof of Theorem B rests on the study of another theta-cocycle, the so-called *winding cocycle*

$$\tag{7} J_w \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times / \mathbb{C}_p^\times),$$

whose key properties are recalled in § 2. The notion of RM value can be extended to theta cocycles by noting that, if the RM point τ has discriminant prime to p, then its automorph γ_τ belongs to $\mathrm{SL}_2(\mathbb{Z})$. The groups $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$ and $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$ are finite of order dividing 12, which implies that the restriction of J^{12} to $\mathrm{SL}_2(\mathbb{Z})$ admits an essentially unique lift $\tilde{J} \in \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times)$, and the value $J[\tau]$ can then be defined as in (1), with J replaced by $\tilde{J}^{1/12}$ on the right hand side. Although there is some torsion ambiguity in the resulting RM values, the p-adic logarithms of these RM values are well-defined.

The explicit nature of J_w can be parlayed into a proof of the following result:

Theorem C. *Let τ be as in Theorem B. There is a classical modular form G_τ of weight two on $\Gamma_0(p)$ with p-adic Fourier coefficients, whose q-expansion is given by*

$$G_\tau(q) = \log(u_\tau) + \sum_{n=1}^\infty \log((T_n J_w)[\tau]) q^n,$$

where $\log : \mathcal{O}_{\mathbb{C}_p}^\times \longrightarrow \mathbb{C}_p$ *is the p-adic logarithm. The modular form G_τ is non-trivial if and only if $\mathbb{Q}(\sqrt{D})$ does not admit a unit of norm -1.*

The modular generating series of Theorem C is constructed from the diagonal restriction of a nearly ordinary deformation of a weight one Hilbert Eisenstein series for $\mathrm{SL}_2(\mathcal{O}_F)$ in the anti-parallel direction. The logarithm of the global p-unit u_τ enters into the proof as the eigenvalue of the Frobenius at p on a quotient of the associated p-adic Galois representation, via a calculation which exploits the reciprocity law of global class field theory, thereby leveraging class field theory for H into explicit class field theory for F. An essential ingredient in the proof of Theorem C is the study of p-adic deformations of irregular Hilbert Eisenstein series of weight one, which is explained in § 3 and forms the technical core of this article. This approach is inspired by the study of the local geometry of the modular eigenvariety in the neighbourhood of irregular Eisenstein points of weight one carried out in [BDP], and its extension to the Hilbert setting in [BDS].

Derivatives of p-adic families of (classical, or Hilbert) modular forms can be viewed as p-adic counterparts of incoherent Eisenstein series in the sense of Kudla, and provide a protoypical instance of what might be envisaged as *p-adic mock modular forms*. Deformations of weight one Hilbert modular Eisenstein series in the parallel weight direction and their diagonal restrictions are studied in [DPV21], where they are related to the *norms to $\mathbb{Q}_p$* of $J_{\mathrm{DR}}[\tau]$. Because of the loss of information inherent in taking the norm, Theorem C represents a significant strengthening of the main theorem of [DPV21], just as Theorem B strengthens the equality (5) resulting from the proof of Gross–Stark conjecture in the setting of odd ring class characters of real quadratic fields.

In § 1 the Dedekind–Rademacher cocycle is constructed, thereby proving Theorem A. The definition and main properties of the winding cocycle appear in § 2, where Theorem B is reduced to Theorem C. The modular generating series G_τ of Theorem C is constructed in § 3–4. The pivotal § 3 studies infinitesimal p-adic deformations of weight one Hilbert Eisenstein series and their Fourier expansions. Finally, through a calculation carried out in § 4, the form G_τ is obtained from the ordinary projection of the diagonal restriction of this infinitesimal deformation.

1. The Dedekind–Rademacher cocycle

This section constructs a one-cochain satisfying Theorem A, which is well-defined up to coboundaries and whose image in $\mathrm{H}^1(\Gamma, \mathcal{A}^\times / p^{\mathbb{Z}})$ is the Dedekind–Rademacher cocycle J_{DR} of the introduction.

1.1. **Siegel units.** Let $\mathcal{O}_{\mathcal{H}}^\times$ denote the multiplicative group of nowhere vanishing holomorphic functions on the Poincaré upper half-plane, endowed with the right action of $\mathrm{SL}_2(\mathbb{R})$ given by

$$h|\gamma(z) = h(\gamma z),$$

where γz denotes the usual action of γ by Möbius transformations.

The construction of J_{DR} rests on the Siegel units $_c g_{\alpha,\beta} \in \mathcal{O}_{\mathcal{H}}^\times$ indexed by pairs $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2 - \{(0,0)\}$ of order $N > 1$, depending on an auxiliary integer c which is relatively prime to $6N$. They satisfy the transformation properties

$$(8) \qquad _c g_{v\gamma} = {}_c g_v | \gamma \qquad \text{for all } v = (\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^2, \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

(Cf. [Ka04, Lemma 1.7(1)].) In particular, $_c g_{\alpha,\beta}$ is a unit on the open modular curve attached to the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that fixes (α, β), and hence belongs to $\mathcal{O}^\times(Y_0(N))$. The Siegel units also satisfy the distribution relations:

$$(9) \qquad \prod_{m\alpha'=\alpha} {}_c g_{\alpha',\beta}(z) = {}_c g_{\alpha,\beta}(z/m), \qquad \prod_{m\beta'=\beta} {}_c g_{\alpha,\beta'}(z) = {}_c g_{\alpha,\beta}(mz),$$

which together imply that

$$(10) \qquad \prod_{m(\alpha',\beta')=(\alpha,\beta)} {}_c g_{\alpha',\beta'}(z) = {}_c g_{\alpha,\beta}(z).$$

(Cf. [Ka04, Lemma 1.7(2)] or [LLZ14, Prop. 2.2.1 and 2.2.2].)

The unit $_c g_{\alpha,\beta}$ is equal to $g_{\alpha,\beta}^{c^2} \cdot g_{c\alpha,c\beta}^{-1}$, where the q-expansion of $g_{\alpha,\beta} \in \mathcal{O}^\times(Y(N)) \otimes \mathbb{Q}$ is given by

$$(11) \qquad g_{\alpha,\beta}(q) = -q^w \prod_{n\geq 0}(1 - q^{n+\alpha}e^{2\pi i \beta}) \prod_{n>0}(1 - q^{n-\alpha}e^{-2\pi i \beta}),$$

where $w = 1/12 - \alpha/2 + (1/2)\alpha/N$, with $0 \leq \alpha < 1$. (Cf. [Ka04, §1.9].)

Fix a rational prime p, and assume that (α, β) is of p-power order in $(\mathbb{Q}/\mathbb{Z})^2$. To lighten notations, it will be assumed below that $p \neq 5$, and the choice $c = 5$ will be fixed. (The constructions are readily adapted to the case $p = 5$ by changing the value of c.)

1.2. **The Siegel distribution.** Let $\mathbb{X}_0 := (\mathbb{Z}_p^2)'$ be the set of vectors $(a, b) \in \mathbb{Z}_p^2$ that are primitive, i.e., satisfy $\gcd(a, b) = 1$, and let

$$(12) \qquad \mathbb{X} := (\mathbb{Q}_p^2 - \{0,0\}) = \bigcup_{j=-\infty}^{\infty} p^j \, \mathbb{X}_0 \,.$$

Let A be an $\mathrm{SL}_2(\mathbb{Z})$-module, and let $\mathrm{LC}(\mathbb{X}_0, \mathbb{Z})$ be the space of locally constant $\mathbb{Z}$-valued functions on $\mathbb{X}_0$. An A-*valued distribution* on $\mathbb{X}_0$ is a homomorphism from $\mathrm{LC}(\mathbb{X}_0, \mathbb{Z})$ to A. Because $\mathbb{X}_0$ is compact, a distribution μ is determined by its values $\mu(U)$ on the characteristic functions of compact open subsets $U \subset \mathbb{X}_0$. Let $\mathbb{D}(\mathbb{X}_0, A)$ denote the module of A-valued distributions. It is endowed with the (right) $\mathrm{SL}_2(\mathbb{Z})$-action defined by

$$(13) \qquad (\mu|\gamma)(U) = \mu(U\gamma^{-1})|\gamma, \qquad \text{for } \gamma \in \mathrm{SL}_2(\mathbb{Z}), \ U \subset \mathbb{X}_0 \,.$$

A distribution on $\mathbb{X}$ is said to be p-invariant if it is invariant under multiplication by p, i.e.,

(14) $$\mu(p^j U) = \mu(U) \text{ for all } j \in \mathbb{Z} \text{ and all compact open } U \subset \mathbb{X}.$$

Denote by $\mathbb{D}(\mathbb{X}, A)$ the module of p-invariant distributions on $\mathbb{X}$. Because $\mathbb{X}_0$ is a fundamental domain for the action of p on $\mathbb{X}$ (cf. (12)), every distribution on $\mathbb{X}_0$ extends uniquely to a p-invariant distribution, yielding an isomorphism

(15) $$\mathbb{D}(\mathbb{X}_0, A) \xrightarrow{\simeq} \mathbb{D}(\mathbb{X}, A).$$

The target space is equipped with a natural action of the larger group Γ when A is a Γ-module, defined by (13) with $\mathrm{SL}_2(\mathbb{Z})$ replaced by Γ. For all $\mu \in \mathbb{D}(\mathbb{X}, A)$ and for all locally constant, compactly supported $\mathbb{Z}$-valued functions f on $\mathbb{X}$, the Γ-action is determined by

$$\int_{\mathbb{X}} f(x,y) d(\mu|\gamma)(x,y) = \int_{\mathbb{X}} f((x,y)\gamma) d\mu(x,y).$$

As was implicitly observed in the work of Kubert and Lang, the collection of Siegel units of p-power level are conveniently packaged into a distribution on $\mathbb{X}_0$, by setting

$$\mu_{\mathrm{Siegel}}\left((a,b) + p^n(\mathbb{Z}_p^2)\right) :=_c g_{\frac{a}{p^n}, \frac{b}{p^n}}, \quad \text{for all } (a,b) \in (\mathbb{Z}^2)'.$$

Since every compact open subset of $\mathbb{X}_0$ is a union of sets of the form $(a,b) + p^n(\mathbb{Z}_p^2)$, the above rule determines μ_{Siegel} on all compact open subsets of $\mathbb{X}_0$. The fact that it is well-defined follows from the distribution relation (10) with $m = p$.

View μ_{Siegel} as an element of $\mathbb{D}(\mathbb{X}, \mathcal{O}_{\mathcal{H}}^{\times})$ via (15). A key feature of μ_{Siegel} is its invariance under $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$, and even under the full group $\mathrm{GL}_2^+(\mathbb{Z}[1/p])$ of invertible matrices with coefficients in $\mathbb{Z}[1/p]$ and positive determinant.

Theorem 1.1. *The distribution* μ_{Siegel} *satisfies*

(16) $$\mu_{\mathrm{Siegel}}(U\gamma) = \mu_{\mathrm{Siegel}}(U)|\gamma,$$

for all compact open subsets $U \subset \mathbb{X}$ *and all* $\gamma \in \mathrm{GL}_2^+(\mathbb{Z}[1/p])$.

Proof. Let $(\alpha, \beta) = (\frac{a}{p^n}, \frac{b}{p^n})$ be an element of order p^n in $(\mathbb{Q}/\mathbb{Z})^2$. Since the sets $U_{\alpha,\beta} = (a,b) + p^n\mathbb{Z}_p^2$ and their translates under multiplication by p form a basis for the topology on $\mathbb{X}$, it suffices to prove the theorem for the sets of this form. The equivariance (16) for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ follows directly from (8). Since $\mathrm{GL}_2^+(\mathbb{Z}[1/p])$ is generated by $\mathrm{SL}_2(\mathbb{Z})$ and the matrix $T := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, one is reduced to showing the relation

$$\mu_{\mathrm{Siegel}}(U_{\alpha,\beta}T) = \mu_{\mathrm{Siegel}}(U_{\alpha,\beta})|T.$$

To see this, note that

$$\begin{aligned} U_{\alpha,\beta}T &= (pa + p^{n+1}\mathbb{Z}_p) \times (b + p^n\mathbb{Z}_p) \\ &= \bigcup_{b' \equiv b(p^n)} (pa + p^{n+1}\mathbb{Z}_p) \times (b' + p^{n+1}\mathbb{Z}_p) = \bigcup_{p\beta'=\beta} U_{\alpha,\beta'}. \end{aligned}$$

It then follows from (9) that

$$\mu_{\mathrm{Siegel}}(U_{\alpha,\beta}T) = \prod_{p\beta'=\beta} {}_c g_{\alpha,\beta'}(z) = {}_c g_{\alpha,\beta}(pz) = \mu_{\mathrm{Siegel}}(U_{\alpha,\beta})|T,$$

as was to be shown. $\qquad\square$

The invariance of μ_{Siegel} under translation by the full p-arithmetic group Γ, which is hinted at in [LLZ14, Rem. 2.2.3], combines the $\mathrm{SL}_2(\mathbb{Z})$-invariance properties (8) and norm compatibility relations (9), (10) satisfied by the Siegel units of p-power level into a single unified statement.

The following Lemma evaluates the Siegel distribution at some distinguished open subsets of $\mathbb{X}$.

Lemma 1.2. *The distribution μ_{Siegel} satisfies*

$$(17) \qquad \begin{array}{rcll} \mu_{\mathrm{Siegel}}(\mathbb{X}_0) & = & 1 & (\mathrm{mod}\ \pm p^{\mathbb{Z}}), \\ \mu_{\mathrm{Siegel}}(p\mathbb{Z}_p \times \mathbb{Z}_p^{\times}) & = & (\Delta(q^p)/\Delta(q))^2 & (\mathrm{mod}\ \pm p^{\mathbb{Z}}). \end{array}$$

Proof. The first assertion follows from the fact that $\mathbb{X}_0$ is stabilised by $\mathrm{SL}_2(\mathbb{Z})$, and therefore that its associated Siegel unit is a unit on the open modular curve $Y_0(1)$ of level 1, which contains no non-constant elements. More precisely, $\mu_{\mathrm{Siegel}}(\mathbb{X}_0)$ belongs to $\mathcal{O}^{\times}(Y_0(1)_{\mathbb{Z}[1/p]}) = \pm p^{\mathbb{Z}}$. (Cf. [LLZ14, Prop.2.3.2] for instance.) The second assertion follows from the calculation

$$\mu_{\mathrm{Siegel}}(p\mathbb{Z}_p \times \mathbb{Z}_p^{\times}) = \prod_{i=1}^{p-1} \mu_{\mathrm{Siegel}}((0,i) + p\mathbb{Z}_p^2) = \prod_{i=1}^{p-1} {}_c g_{0,i/p} = \pm p^{c^2-1}(\Delta(q^p)/\Delta(q))^{(c^2-1)/12},$$

where the last equality can be read off from the q-expansions of the Siegel units given in (11). The result now follows, since $c = 5$. $\qquad\square$

1.3. **The Dedekind–Rademacher distributions.** The following general Lemmas concerning p-invariant distributions will be useful later.

Lemma 1.3. *Let μ be any element of $\mathbb{D}(\mathbb{X}, A)$. If Λ is any $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$, and Λ' is its set of primitive vectors, then $\mu(\Lambda') = \mu(\mathbb{X}_0)$.*

Proof. By compactness, there is an integer $N \geq 0$ for which $p^{-N}\mathbb{Z}_p^2 \subset \Lambda \subset p^N\mathbb{Z}_p^2$, and hence each $v \in \Lambda'$ belongs to a translate $p^j\mathbb{X}_0$ for a unique $j \in [-N, N]$. Hence one may write

$$\Lambda' = p^{m_1}U_1 \sqcup \cdots \sqcup p^{m_t}U_t,$$

for a suitable decomposition

$$\mathbb{X}_0 = U_1 \sqcup \cdots \sqcup U_t$$

of $\mathbb{X}_0$ as a disjoint union of compact open subsets. The additivity properties of μ combined with its p-invariance implies that $\mu(\Lambda') = \mu(\mathbb{X}_0)$, as claimed. $\qquad\square$

Lemma 1.4. *The rule which to A associates $\mathbb{D}(\mathbb{X}, A)$ is an exact (covariant) functor from the category of Γ-modules to itself.*

Proof. The issue is right exactness. If $\varphi : A \longrightarrow B$ is a surjective module homomorphism and $\mu \in \mathbb{D}(\mathbb{X}, B)$ is a B-valued, p-invariant distribution on $\mathbb{X}$, one can construct a distribution $\tilde{\mu} \in \mathbb{D}(\mathbb{X}, A)$ that maps to it by choosing, for each successsive $n \geq 1$ and for each primitive vector $v = (\mathbb{Z}/p^n\mathbb{Z})'$, the value $\tilde{\mu}(v + p^n\mathbb{Z}_p^2) \in A$ satisfying $\varphi(\tilde{\mu}(v + p^n\mathbb{Z}_p^2)) = \mu(v + p^n\mathbb{Z}_p^2)$, taking care at each stage that the additivity relations required of distributions be satisfied. One obtains in this way an element of $\mathbb{D}(\mathbb{X}_0, B)$, giving rise to the desired lift in $\mathbb{D}(\mathbb{X}, B)$ via (15). $\qquad\square$

Thanks to Lemma 1.4, the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\mathcal{H}} \xrightarrow{e^{2\pi i z}} \mathcal{O}_{\mathcal{H}}^{\times} \longrightarrow 1$$

induces a short exact sequence

$$1 \to \mathbb{D}(\mathbb{X}, \mathbb{Z}) \longrightarrow \mathbb{D}(\mathbb{X}, \mathcal{O}_{\mathcal{H}}) \longrightarrow \mathbb{D}(\mathbb{X}, \mathcal{O}_{\mathcal{H}}^{\times}) \to 1$$

of Γ-modules. Let

$$\delta : \mathrm{H}^0(\Gamma, \mathbb{D}(\mathbb{X}, \mathcal{O}_{\mathcal{H}}^\times)) \longrightarrow \mathrm{H}^1(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}))$$

be the connecting homomorphism arising from the resulting long exact sequence in Γ-cohomology. The image

$$\mu_{\mathrm{DR}} := \delta(\mu_{\mathrm{Siegel}}) \in \mathrm{H}^1(\Gamma, \mathbb{D}(\mathbb{X}, \mathbb{Z}))$$

is a one-cocycle on Γ, i.e., it satisfies the relation

$$\mu_{\mathrm{DR}}(\gamma_1 \gamma_2) = \mu_{\mathrm{DR}}(\gamma_1) + \mu_{\mathrm{DR}}(\gamma_2)|\gamma_1^{-1}.$$

It is obtained by lifting μ_{Siegel} to an $\mathcal{O}_{\mathcal{H}}$-valued distribution

$$\tilde{\mu}_{\mathrm{Siegel}} := \frac{1}{2\pi i} \log(\mu_{\mathrm{Siegel}}) \in \mathbb{D}(\mathbb{X}, \mathcal{O}_{\mathcal{H}}),$$

and setting

(18) $$\mu_{\mathrm{DR}}(\gamma) := \tilde{\mu}_{\mathrm{Siegel}}|\gamma^{-1} - \tilde{\mu}_{\mathrm{Siegel}}.$$

Recall the Dedekind–Rademacher homomorphism $\varphi_{\mathrm{DR}} : \Gamma_0(p) \longrightarrow \mathbb{Z}$ evoked in the introduction, which encodes the periods of the Eisenstein series $E_2^{(p)} = d\log(\Delta(pz)/\Delta(z))$ of weight two.

Lemma 1.5. *The one-cocycle μ_{DR} satisfies*

(19) $$\begin{array}{llll} \mu_{\mathrm{DR}}(\gamma)(\mathbb{X}_0) & = & 0 & \text{for all } \gamma \in \Gamma, \\ \mu_{\mathrm{DR}}(\gamma)(p\mathbb{Z}_p \times \mathbb{Z}_p^\times) & = & \varphi_{\mathrm{DR}}(\gamma) & \text{for all } \gamma \in \Gamma_0(p). \end{array}$$

Proof. Observe that, for all $\gamma \in \Gamma$,

$$\mu_{\mathrm{DR}}(\gamma)(\mathbb{X}_0) = \tilde{\mu}_{\mathrm{Siegel}}|\gamma^{-1}(\mathbb{X}_0) - \tilde{\mu}_{\mathrm{Siegel}}(\mathbb{X}_0) = \tilde{\mu}_{\mathrm{Siegel}}(\mathbb{X}_0\,\gamma)|\gamma^{-1} - \tilde{\mu}_{\mathrm{Siegel}}(\mathbb{X}_0).$$

Lemma (1.3) implies that $\tilde{\mu}_{\mathrm{Siegel}}(\mathbb{X}_0\,\gamma) = \tilde{\mu}_{\mathrm{Siegel}}(\mathbb{X}_0)$, and Lemma 1.2 shows that this common value is a constant function on $\mathcal{H}$. The first assertion follows. As for the second, equation (18) implies that

$$\mu_{\mathrm{DR}}(\gamma)(p\mathbb{Z}_p \times \mathbb{Z}_p^\times) = (\tilde{\mu}_{\mathrm{Siegel}}|\gamma^{-1} - \tilde{\mu}_{\mathrm{Siegel}})(p\mathbb{Z}_p \times \mathbb{Z}_p^\times).$$

By Lemma 1.2,

$$\tilde{\mu}_{\mathrm{Siegel}}(p\mathbb{Z}_p \times \mathbb{Z}_p^\times) = \frac{2}{2\pi i} \log\left(\Delta(pz)/\Delta(z)\right) \pmod{\mathbb{C}}.$$

Since $\gamma \in \Gamma_0(p)$ preserves the region $p\mathbb{Z}_p \times \mathbb{Z}_p^\times$, it follows that

$$(\tilde{\mu}_{\mathrm{Siegel}}|\gamma^{-1} - \tilde{\mu}_{\mathrm{Siegel}})(p\mathbb{Z}_p \times \mathbb{Z}_p^\times) = \frac{2}{2\pi i} \int_{z_0}^{\gamma^{-1}z_0} d\log(\Delta(pz)/\Delta(z)) = \varphi_{\mathrm{DR}}(\gamma),$$

as was to be shown. $\square$

1.4. **The multiplicative Poisson transform.** Because a distribution $\mu \in \mathbb{D}(\mathbb{X}, \mathbb{Z})$ is $\mathbb{Z}$-valued, and hence p-adically bounded, it also gives rise to a *measure*: one can extend μ to arbitrary continuous, compactly supported functions on $\mathbb{X}$. There is even a *multiplicative refinement* of the integral against μ, defined by

$$\fint_{\mathbb{X}} f(x, y) d\mu(x, y) := \lim_{\{U_\alpha\}} \prod_\alpha f(x_\alpha, y_\alpha)^{\mu(U_\alpha)},$$

where the limit is taken over finer and finer open covers $\{U_\alpha\}$ of the support of f, and (x_α, y_α) is a sample point in U_α. Here $f : \mathbb{X} \longrightarrow \mathbb{C}_p^\times$ is a continuous, compactly supported function on $\mathbb{X}$ (which means that it takes the value 1 outside a compact subset of $\mathbb{X}$).

Let $\mathbb{D}_0(\mathbb{X}_0, \mathbb{Z})$ be the $\mathbb{Z}$-module of distributions on $\mathbb{X}_0$ satisfying

$$\mu(\mathbb{X}_0) = 0.$$

The multiplicative Poisson transform of $\mu \in \mathbb{D}_0(\mathbb{X}_0, \mathbb{Z})$ is the rigid analytic function $J(\mu)$ on $\mathcal{H}_p$ defined by setting

$$J(\mu)(\tau) = \fint_{\mathbb{X}_0} (x\tau + y) d\mu(x, y).$$

This assignment gives rise to an $\mathrm{SL}_2(\mathbb{Z})$-equivariant map

$$J : \mathbb{D}_0(\mathbb{X}_0, \mathbb{Z}) \longrightarrow \mathcal{A}^\times,$$

i.e.,

$$J(\mu|\gamma)(\tau) = J(\mu)|\gamma(\tau) = J(\mu)(\gamma\tau), \qquad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Identifying $\mathbb{D}_0(\mathbb{X}_0, \mathbb{Z})$ with the module $\mathbb{D}_0(\mathbb{X}, \mathbb{Z})$ of distributions on $\mathbb{X}$ satisfying

$$\mu(\mathbb{X}_0) = 0, \qquad \mu(pU) = \mu(U),$$

the same rule J (where one continues to integrate over the compact subset $\mathbb{X}_0 \subset \mathbb{X}$) determines a Γ-equivariant map

(20) $$J : \mathbb{D}_0(\mathbb{X}, \mathbb{Z}) \longrightarrow \mathcal{A}^\times / p^\mathbb{Z}.$$

The reason for this somewhat weaker invariance property is that while $\mathrm{SL}_2(\mathbb{Z})$ preserves the region $\mathbb{X}_0$ of integration defining $J(\mu)$, the full p-arithmetic group Γ does not. Nonetheless, if $\gamma \in \Gamma$, one still can write (following the reasoning in the proof of 1.3)

$$\mathbb{X}_0\,\gamma = p^{m_1} U_1 \sqcup \cdots \sqcup p^{m_t} U_t, \qquad \text{with} \qquad \mathbb{X}_0 = U_1 \sqcup \cdots \sqcup U_t,$$

and the integrand $(x - \tau y)$ arising in the definition of J obeys a simple transformation property under multiplication by p. It follows that $J(\mu|\gamma) = J(\mu)|\gamma \pmod{p^\mathbb{Z}}$, for all $\gamma \in \Gamma$.

Let

$$J_{\mathrm{DR}} := J(\mu_{\mathrm{DR}}) \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times / p^\mathbb{Z})$$

be the image of the measure-valued cocycle μ_{DR} under the multiplicative Poisson transform of (20). It is represented by the one-cochain $J_{\mathrm{DR}} : \Gamma \longrightarrow \mathcal{A}^\times$ (denoted by the same symbol, by an abuse of notation) defined by

$$J_{\mathrm{DR}}(\gamma)(\tau) = J(\mu_{\mathrm{DR}}(\gamma))(\tau),$$

which satisfies the cocycle relation modulo $p^\mathbb{Z}$,

$$J_{\mathrm{DR}}(\gamma_1\gamma_2) = J_{\mathrm{DR}}(\gamma_1) \times J_{\mathrm{DR}}(\gamma_2)|\gamma_1^{-1} \pmod{p^\mathbb{Z}}.$$

Its restriction to $\mathrm{SL}_2(\mathbb{Z})$ also satisfies the *full cocycle relation*, with no $p^\mathbb{Z}$-ambiguity, because of the $\mathrm{SL}_2(\mathbb{Z})$-equivariance of J.

In order to prove Theorem A of the introduction, it now suffices to calculate the image of J_{DR} under the sequence of maps

$$\eta : \mathrm{H}^1(\Gamma, \mathcal{A}^\times / p^\mathbb{Z}) \longrightarrow \mathrm{H}^2(\Gamma, p^\mathbb{Z}) = \mathrm{H}^1(\Gamma_0(p), p^\mathbb{Z}).$$

Theorem 1.6. *The image of J_{DR} under η is*

$$\eta(J_{\mathrm{DR}}) = p^{\varphi_{\mathrm{DR}}}.$$

Proof. The action of Γ on the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$ leads to an expression for Γ as an amalgamated product of the groups

$$\mathrm{SL}_2(\mathbb{Z}), \qquad \mathrm{SL}_2(\mathbb{Z})' = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

whose intersection is $\Gamma_0(p)$. The fact that $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), p^{\mathbb{Z}}) = 0$ and that $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), p^{\mathbb{Z}})$ is of order 12 ensures the existence of unique lifts to $\mathcal{A}^\times$ of the restrictions of J_{DR}^{12} to $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})'$:

$$\mathcal{J}_{\mathrm{DR}} \in \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times), \qquad \mathcal{J}_{\mathrm{DR}}' \in \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z})', \mathcal{A}^\times).$$

One then has, for all $\gamma \in \Gamma_0(p)$,

(21)
$$\eta(J_{\mathrm{DR}}^{12})(\gamma) = \mathcal{J}_{\mathrm{DR}}(\gamma) \div \mathcal{J}_{\mathrm{DR}}'(\gamma).$$

Concretely, $\mathcal{J}_{\mathrm{DR}}$ and $\mathcal{J}_{\mathrm{DR}}'$ may be expressed as multiplicative Poisson transforms of μ_{DR}, by setting

$$\mathcal{J}_{\mathrm{DR}}(\gamma)(\tau) := \fint_{\mathbb{X}_0} (x\tau + y)^{12} d\mu_{\mathrm{DR}}(\gamma)(x, y), \qquad \mathcal{J}_{\mathrm{DR}}'(\gamma)(\tau) := \fint_{\mathbb{X}_0'} (x\tau + y)^{12} d\mu_{\mathrm{DR}}(\gamma)(x, y),$$

where $\mathbb{X}_0' := (p\mathbb{Z}_p \times \mathbb{Z}_p)'$ is the translate of $\mathbb{X}_0$ under the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, a region whose stabiliser in Γ is the group $\mathrm{SL}_2(\mathbb{Z})'$. Observe that

(22)
$$\mathbb{X}_0 \cap \mathbb{X}_0' = p\mathbb{Z}_p \times \mathbb{Z}_p^\times, \qquad \mathbb{X}_0 - \mathbb{X}_0' = \mathbb{Z}_p^\times \times \mathbb{Z}_p, \qquad \mathbb{X}_0' - \mathbb{X}_0 = p(\mathbb{Z}_p^\times \times \mathbb{Z}_p).$$

Hence, for all $\gamma \in \Gamma_0(p)$,

$$
\begin{aligned}
\mathcal{J}_{\mathrm{DR}}(\gamma) \div \mathcal{J}_{\mathrm{DR}}'(\gamma) &= \fint_{\mathbb{X}_0} (x\tau + y)^{12} d\mu_{\mathrm{DR}}(\gamma)(x, y) \div \fint_{\mathbb{X}_0'} (x\tau + y)^{12} d\mu_{\mathrm{DR}}(\gamma)(x, y) \\
&= \fint_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} (x\tau + y)^{12} d\mu_{\mathrm{DR}}(\gamma)(x, y) \div \fint_{p(\mathbb{Z}_p^\times \times \mathbb{Z}_p)} (x\tau + y)^{12} d\mu_{\mathrm{DR}}(\gamma)(x, y) \\
&= \fint_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} p^{-12} d\mu_{\mathrm{DR}}(\gamma)(x, y),
\end{aligned}
$$

where the penultimate equality follows from (22) and the last from the invariance of $\mu_{\mathrm{DR}}(\gamma)$ under multiplication by p. Because $(\mathbb{Z}_p^\times \times \mathbb{Z}_p)$ is the complement of $(p\mathbb{Z}_p \times \mathbb{Z}_p^\times)$ in $\mathbb{X}_0$, and $\mu_{\mathrm{DR}}(\gamma)(\mathbb{X}_0) = 0$, this implies that

$$\mathcal{J}_{\mathrm{DR}}(\gamma) \div \mathcal{J}_{\mathrm{DR}}'(\gamma) = \fint_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} p^{12} d\mu_{\mathrm{DR}}(\gamma)(x, y) = p^{12\mu_{\mathrm{DR}}(\gamma)(p\mathbb{Z}_p \times \mathbb{Z}_p^\times)} = p^{12\varphi_{\mathrm{DR}}(\gamma)},$$

where the last equality follows from Lemma 1.5. Combining this with (21) shows that $\eta(J_{\mathrm{DR}})$ and $p^{\varphi_{\mathrm{DR}}}$ agree, since the group they belong to is torsion-free. This completes the proof of Theorem A. $\qquad\square$

2. The winding cocycle

The goal of this section is to recall the definition and key properties of the winding cocycle introduced in [DPV21, § 2.3] and to reduce Theorem B of the introduction to Theorem C.

2.1. The residue map.

The group $\mathrm{H}^1(\Gamma, \mathcal{A}^\times / \mathbb{C}_p^\times)$ of rigid analytic theta cocycles is finitely generated and closely related to the space of modular forms of weight two on the Hecke congruence group $\Gamma_0(p)$. More precisely, it is a module over the Hecke algebra $\mathbb{T}_0(p)$ of Hecke operators acting faithfully on the weight two modular forms on $\Gamma_0(p)$. To see this, let

$$U := \{z \in \mathbb{P}_1(\mathbb{C}_p) \text{ with } 1 < |z| < p\} \subset \mathcal{H}_p$$

be the standard annulus whose stabiliser in Γ is $\Gamma_0(p)$. The logarithmic annular residue map

(23)
$$\partial_U : \mathcal{A}^\times / \mathbb{C}_p^\times \longrightarrow \mathbb{Z}_p, \qquad \partial_U(f) := \mathrm{Res}_U(\mathrm{dlog}\, f)$$

is equivariant for the action of $\Gamma_0(p)$, and hence composing it with the restriction to $\Gamma_0(p)$ yields a map on cohomology

(24) $$\partial_U : \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) \longrightarrow \mathrm{H}^1(\Gamma_0(p), \mathbb{Z}_p),$$

which is denoted by the same symbol by abuse of notation. This map is compatible with the action of the Hecke operators, and with the involution W_∞ determined by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which lies in the normaliser of both Γ and $\Gamma_0(p)$. Let $\mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)^\pm$ denote the plus and minus eigenspaces for this involution in the space of rigid analytic theta cocycles, and denote by $\mathrm{H}^1(\Gamma_0(p), \mathbb{Z}_p)^\pm$ the corresponding eigenspaces in the cohomology of $\Gamma_0(p)$.

While the map in (23) has an infinite rank kernel, it is notable that the induced map on rigid analytic theta cocycles is essentially an isomorphism:

Lemma 2.1. *Up to torsion kernels and cokernels, the map ∂_U of (24) is surjective, and its kernel is generated by the "trivial" theta-cocycle*

$$J_{\mathrm{triv}} \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times), \qquad J_{\mathrm{triv}}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(z) = cz + d.$$

In particular, the induced map

(25) $$\partial_U^- : \mathbb{Q} \otimes \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)^- \longrightarrow \mathrm{H}^1(\Gamma_0(p), \mathbb{Q})^-$$

is an isomorphism.

Proof. The first assertion is a reformulation of [DPV21, Theorem 3.1]. The last follows from the fact that J_{triv} is fixed by W_∞, as can be checked directly from the definition of J_{triv}. $\qquad\square$

2.2. The winding cocycle.

In [DPV21, §2.3], the so-called *winding cocycle*

$$J_w \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)^-$$

is introduced. Unlike the Dedekind–Rademacher cocycle, it is not an eigenclass for the Hecke operators, although it belongs to the -1 eigenspace for the involution W_∞. The greater complexity of J_w on the spectral side is offset by a gain in simplicity on the geometric side, evidenced by the fact that the rigid analytic functions $J_w(\gamma)$ admit explicit infinite product expansions.

Let $Y_0(p) = \Gamma_0(p)\backslash\mathcal{H}$ be the open modular curve, and let $X_0(p)$ be its standard compactification, obtained by adding the two cusps 0 and ∞. The intersection pairing on homology (Poincaré duality) defines isomorphisms

(26) $$\mathrm{H}_1(X_0(p); \{0, \infty\}, \mathbb{Q})^\pm = \mathrm{H}^1(Y_0(p), \mathbb{Q})^\mp = \mathrm{H}^1(\Gamma_0(p), \mathbb{Q})^\mp.$$

Mazur's *winding element*

$$\varphi_w \in \mathrm{H}^1(\Gamma_0(p), \mathbb{Z})^-$$

is defined to be the class of the path from 0 to ∞ in the homology of the modular curve $X_0(p)$ relative to the cusps, viewed as an element of $\mathrm{H}^1(\Gamma_0(p), \mathbb{Z})$ via (26). By [DPV21, Prop. 3.3] and its proof, the winding cocycle is characterised by the identity

(27) $$\partial_U^-(J_w) = 2\varphi_w.$$

2.3. Theorem C implies Theorem B.

Theorem B of the introduction is reduced to Theorem C by writing the modular form G_τ of this theorem as a linear combination of eigenforms.

To this end, observe that $\mathrm{H}^1(\Gamma_0(p), \bar{\mathbb{Q}})^-$ is generated as a $\bar{\mathbb{Q}}$-vector space by the Dedekind–Rademacher morphism φ_{DR} of (3) encoding the periods of the weight two Eisenstein series $E_2^{(p)}$ defined in the introduction, and the homomorphisms φ_f^- attached to the minus modular symbol for f, where f runs through

a basis of cuspidal Hecke eigenforms in $S_2(\Gamma_0(p))$. A direct calculation of integration pairings in [DPV21, Lemma 3.4] then yields the spectral decomposition of the winding element,

$$\varphi_w \;=\; \frac{1}{p-1} \cdot \varphi_{\mathrm{DR}} \;+\; \sum_f \lambda_f \cdot \varphi_f^-, \tag{28}$$

where the coefficient $\lambda_f \in \bar{\mathbb{Q}}$ is a suitable non-zero multiple of $L(f,1)$ whose exact nature is not germane to the proof of Theorem B. (But see [DPV21, § 3] for more details.)

Now consider the rigid analytic theta-cocycle

$$J_f^- \;\in\; \bar{\mathbb{Q}} \otimes \mathrm{H}^1(\Gamma, \mathcal{A}^\times / \mathbb{C}_p^\times)^-$$

characterised by $\partial_U^-(J_f^-) = \varphi_f^-$. By Lemma 2.1 and (27),

$$J_w \;=\; \frac{2}{p-1} \cdot J_{\mathrm{DR}} \;+\; \sum_f 2\lambda_f \cdot J_f^- \qquad \text{in } \bar{\mathbb{Q}} \otimes \mathrm{H}^1(\Gamma, \mathcal{A}^\times / \mathbb{C}_p^\times)^-, \tag{29}$$

where additive notation has been adopted to describe the operations in this group in spite of its multiplicative nature. For each $n \geq 1$, applying the Hecke operator T_n to this identity then gives

$$\begin{aligned} T_n J_w \;&=\; \frac{2}{p-1} \cdot T_n J_{\mathrm{DR}} \;+\; \sum_f 2\lambda_f \cdot T_n J_f^- \\[2mm] &=\; \frac{2}{p-1} \cdot J_{\mathrm{DR}} \cdot \sigma_1^{(p)}(n) \;+\; \sum_f 2\lambda_f \cdot J_f^- \cdot a_n(f) \end{aligned} \tag{30}$$

in $\bar{\mathbb{Q}} \otimes \mathrm{H}^1(\Gamma, \mathcal{A}^\times / \mathbb{C}_p^\times)^-$. After evaluating at the RM point τ and taking p-adic logarithms, it follows that

$$\log_p(T_n J_w[\tau]) \;=\; \frac{2\log_p(J_{\mathrm{DR}}[\tau])}{p-1} \cdot \sigma_1^{(p)}(n) \;+\; \sum_f 2\lambda_f \log_p(J_f^-[\tau]) \cdot a_n(f). \tag{31}$$

Substituting this identity into Theorem C of the introduction yields the spectral expansion

$$G_\tau(q) \;=\; \frac{\log J_{\mathrm{DR}}[\tau]}{12(p-1)} \cdot E_2^{(p)}(q) \;+\; \sum_f \beta_f \cdot f(q), \tag{32}$$

where $E_2^{(p)}$ is the Eisenstein series of (2), and

$$\beta_f = 2\lambda_f \log_p(J_f^-[\tau]). \tag{33}$$

Comparing the zero-th Fourier coefficient of G_τ in (32) with the one in Theorem C shows that

$$\frac{\log J_{\mathrm{DR}}[\tau]}{12} = \log(u_\tau),$$

thereby reducing Theorem B of the introduction to Theorem C.

The remainder of the paper is devoted to the construction of the modular generating series required for the proof of Theorem C.

Remark. The coefficients β_f in (33) are immaterial to the proof of Theorem B but are of independent interest, insofar as they involve the RM values of the elliptic rigid analytic theta-cocycles J_f^-: these values are the formal group logarithms of certain *Stark–Heegner points* in the modular Jacobian $J_0(p)$. Although poorly understood theoretically, these Stark–Heegner points are conjectured to be defined over the narrow ring class field H_τ. The approach to the algebraicity of $J_{\mathrm{DR}}[\tau]$ based on deformations of Galois representation does not seem to shed any immediate light on the algebraicity of these more mysterious invariants.

3. Deformations of Hilbert Eisenstein series

This section studies the derivatives of certain p-adic analytic families of Hilbert modular forms for F parametrised by the weight and specialising to a certain Hilbert Eisenstein series of parallel weight one.

This Eisenstein series has several notable features. Firstly, it is cuspidal when viewed as a p-adic modular form, and admits cuspidal p-adic deformations. Secondly, it vanishes upon diagonal restriction. This implies that the derivatives of both cuspidal and Eisenstein families specialising to f, in spite of not displaying any simple modularity properties themselves, yield p-adic modular forms after taking diagonal restriction. A suitable linear combination of these derivatives is considered in § 4, and the Fourier coefficients of its ordinary projection are related to the RM values of the winding cocycle.

While p-adic Eisenstein families only occur in parallel weight, cuspidal families vary over a larger weight space. The main result of this section is Theorem 3.13, which describes the Fourier coefficients of the derivatives of a cuspidal family in the "anti-parallel" direction of the weight space. Much like in the archimedean settings, the Fourier expansions of p-adic Eisenstein families are entirely explicit; however, no general expression is available for cuspidal families. Our approach to studying cuspidal deformations of a Hilbert Eisenstein series rests on the analysis of the associated Galois deformation problems. Roughly speaking, first order deformations of the Artin representation attached to a Hilbert Eisenstein series of parallel weight one are described in terms of the Galois cohomology of the adjoint representation, which cuts out a finite abelian extension H of F. A class in the Galois cohomology of the adjoint cuts out an abelian p-adic Lie extension of H, and the Frobenius traces on the associated Galois deformation involve p-adic logarithms of global p-units in H, via the reciprocity law of global class field theory for H. This translates into the appearance of the logarithms of Gross–Stark units in the Fourier coefficients of first order deformations of Hilbert Eisenstein series, and accounts for the presence of the same quantities in the constant term of the generating series G_τ of Theorem C.

The Galois deformation arguments are clarified and not substantially lengthened by working in the setting where F is an arbitrary totally real field of degree d in which p is inert. This will be assumed until § 3.5, when the main results will be specialised to the case where F is real quadratic.

3.1. **Hilbert modular forms and Hecke algebras.** Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let F be a totally real field in which p is inert, and denote by $\mathfrak{d}$ the different of its ring of integers $\mathcal{O}_F$. Write $\alpha_1, \ldots, \alpha_d$ for the distinct embeddings of F into $\bar{\mathbb{Q}}_p$, so that α_1 is the embedding given by the restriction to F of the chosen embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Via the choice of an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$, one obtains a corresponding indexing of embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. For $x \in F$, let $(x_1, \ldots, x_d)$ denote the image of x under the embeddings $\alpha_1, \ldots, \alpha_d$, viewed as a d-tuple of either complex or p-adic numbers, depending on the context.
It is assumed throughout that the Leopoldt Conjecture holds for F. (When F is quadratic, this assumption is known to be satisfied.)

Fix a totally odd character ψ of the narrow class group of F. Let E be a finite extension of $\mathbb{Q}_p$ containing the images of F under all embeddings $\alpha_1, \ldots, \alpha_d$ and the values of the character ψ.

We now recall some definitions and conventions related to Hilbert modular forms and their associated Hecke algebras, following the treatment that is given in [Shi78], [Hi88, § 2] and [Hi91, § 3].

Let $\mathbf{k} = (k_j) \in \mathbb{Z}_{\geq 2}^d$ be a d-tuple of integers. Denote $\mathbf{t} = (t_j)$ the vector with $t_j = 1$ for every $1 \leq j \leq d$. Choose a vector $\mathbf{v} \in \mathbb{Z}^d$ of non-negative integers such that $\mathbf{k} + 2\mathbf{v} = m\mathbf{t}$ for some $m \in \mathbb{Z}$, and define $\mathbf{w} = \mathbf{k} + \mathbf{v} - \mathbf{t}$. The space of *Hilbert modular forms* of weight $(\mathbf{k}, \mathbf{w})$ and full level $\mathfrak{N}$, defined as in [Hi88, § 2], is a finite-dimensional complex vector space. Let $\mathscr{H}_{\mathbf{k}, \mathbf{w}}(\mathfrak{N})$ be the algebra of Hecke operators acting faithfully on the subspace of cuspforms. It is free of finite rank as a $\mathbb{Z}$-module.

Fix $(\mathbf{k}, \mathbf{w})$ as above. The p-adic Hecke algebra is defined to be

$$(34) \qquad \mathcal{T} := \varprojlim_{\alpha} \mathscr{H}_{\mathbf{k}, \mathbf{w}}(p^\alpha) \otimes \mathcal{O}_E,$$

where the inverse limit is taken with respect to restriction of increasing full level structure at p. It contains in particular diamond operators $\langle \mathfrak{l} \rangle$ for every integral ideal $\mathfrak{l}$ coprime to p, as well as Hecke operators

$$(35) \qquad \mathbf{T}(y) = \varprojlim_{\alpha} T(y) y_p^{-\mathbf{v}}$$

for any idèle $y \in \widehat{\mathcal{O}_F} \cap \mathbf{A}_F^\times$, whose component at p is denoted y_p. When y_p is a unit, the operator $T(y)$ depends only on the integral ideal $\mathfrak{m}$ defined by y, and we write $T_\mathfrak{m}$ and $\mathbf{T}_\mathfrak{m}$ for $T(y)$ and $\mathbf{T}(y)$. The compact ring $\mathcal{T}$ has a unique decomposition $\mathcal{T} = \mathcal{T}^{\mathrm{n.ord}} \oplus \mathcal{T}^{\mathrm{ss}}$ such that $\mathbf{T}(p)$ is a unit in $\mathcal{T}^{\mathrm{n.ord}}$ and is topologically nilpotent in $\mathcal{T}^{\mathrm{ss}}$. The ring $\mathcal{T}^{\mathrm{n.ord}}$ is called the *nearly ordinary cuspidal Hecke algebra*. It is independent of the choice of $(\mathbf{k}, \mathbf{w})$.

Write $\mathsf{U} = (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$ and let Z be the Galois group of the maximal abelian extension of F unramified outside p and ∞. The Iwasawa algebra

$$\Lambda := \mathcal{O}_E[\![\mathsf{U} \times \mathsf{Z}]\!]$$

is abstractly isomorphic to a ring of power series in several variables with coefficients in a finite group ring over $\mathcal{O}_E$. Denote

$$\kappa^{\mathrm{univ}} \colon \mathsf{U} \times \mathsf{Z} \longrightarrow \Lambda^\times$$

its universal character. Since p is inert in F, the group U is identified with the units of F_p. Denote U° and Z° the torsion free parts of U and Z respectively and let $\Lambda^\circ = \mathcal{O}_E[\![\mathsf{U}^\circ \times \mathsf{Z}^\circ]\!]$. Let χ_p and ω_p be the cyclotomic and the Teichmüller characters of G_F respectively. They factor through the quotient Z. Define $\mathsf{q} = 4$ when $p = 2$, and $\mathsf{q} = p$ otherwise. Then the homomorphism

$$\chi_p \cdot \omega_p^{-1} \colon \mathsf{Z} \to 1 + \mathsf{q}\mathbb{Z}_p$$

induces an isomorphism when restricted to Z° if the Leopoldt conjecture holds for F. For any weight $(\mathbf{k}, \mathbf{w})$ as above, consider the character $\kappa_{\mathbf{k}, \mathbf{w}} \colon \mathsf{G} = \mathsf{U} \times \mathsf{Z} \longrightarrow \bar{\mathbb{Q}}_p^\times$ defined by

$$(36) \qquad (a, z) \longmapsto a^\mathbf{v} \cdot \chi_p^{m-1}(z).$$

With a slight abuse of notation, the corresponding ring homomorphism will also be denoted by $\kappa_{\mathbf{k}, \mathbf{w}}$ and referred to as the weight $(\mathbf{k}, \mathbf{w})$-specialisation. If $\mathbf{v} = 0$, so that $\mathbf{k} = m \cdot \mathbf{t}$, the pair $(\mathbf{k}, \mathbf{w})$ will be called parallel weight m. Parallel weight specialisations are parametrised by the Iwasawa algebra $\mathcal{O}_E[\![\mathsf{Z}]\!]$, which shall be regarded as a quotient of Λ.

The nearly ordinary Hecke algebra $\mathcal{T}^{\mathrm{n.ord}}$ is a Λ-algebra via the action of the diamond operators. The main theorem of [Hi89a] asserts that $\mathcal{T}^{\mathrm{n.ord}}$ is finitely generated and torsion-free as a Λ°-module. In addition, the quotient of $\mathcal{T}^{\mathrm{n.ord}}$ by the ideal generated by the kernel of $\kappa_{\mathbf{k}, \mathbf{w}}$, for $(\mathbf{k}, \mathbf{w})$ as above, is isomorphic to the ordinary part of the classical Hecke algebra of weight $(\mathbf{k}, \mathbf{w})$ and Iwahori level at p. This result is often referred to as *Hida's Control Theorem*.

p-adic families of Hilbert modular forms and weight one Eisenstein series.

The space of Hilbert modular forms of weight $(\mathbf{k}, \mathbf{w})$ and any level is automatically cuspidal unless $(\mathbf{k}, \mathbf{w})$ is parallel [Shi78, (1.8_a)]. However, for parallel weights, non-trivial Eisenstein forms exist and can be interpolated in explicit p-adic families parametrised by $\mathcal{O}_E[\![\mathsf{Z}]\!]$. The study of congruences between cuspidal and Eisenstein families of Hilbert modular forms is at the heart of Wiles' proof of Iwasawa Main Conjecture over totally real fields [Wi90]. In a similar spirit, we consider certain cuspidal and Eisenstein families sharing the same specialisation at parallel weight one, i.e. $\mathbf{k} = \mathbf{t}$ and $\mathbf{w} = 0$.

For any pair (φ, η) of unramified characters of F with $\varphi\eta$ totally odd, there exists a family $\mathcal{E}(\varphi, \eta)$ with Fourier expansions as described in [DDP11, § 3]. The cases where $(\varphi, \eta) = (1, \psi)$ or $(\psi, 1)$ are of particular relevance in the calculations leading to the proof of Theorem C.

Let $E_1(1, \psi)$ be the classical Eisenstein series of weight 1 and trivial level with Fourier expansion:

$$E_1(1, \psi)(z_1, \ldots, z_d) = L(F, \psi, 0) + \psi^{-1}(\mathfrak{d})L(F, \psi^{-1}, 0) + 2^d \sum_{\nu \in \mathfrak{d}_+^{-1}} \sigma_{0,\psi}(\nu\mathfrak{d})e^{2\pi i(\nu_1 z_1 + \cdots + \nu_d z_d)},$$

where $z = (z_1, \ldots z_d) \in \mathcal{H}^d$ and

$$\sigma_{k,\psi}(\alpha) = \sum_{I \lhd \mathcal{O}_F, I|(\alpha)} \psi(I)\mathrm{Nm}(I)^k \qquad \text{and} \qquad L(F, \psi, s) = \sum_{I \lhd \mathcal{O}_F} \psi(I)\mathrm{Nm}(I)^{-s},$$

the latter converging for $\mathrm{Re}(s)$ large enough, analytically continued to all $s \in \mathbb{C}$.

In the case where p is inert in F, one has $\psi(p) = 1$, and the p-adic L-functions $L_p(\psi, s)$ and $L_p(\psi^{-1}, s)$ have exceptional zeros at $s = 0$. The Eisenstein series $E_1(1, \psi)$ then admits a unique p-stabilisation

(37) $$f(z) := E_1^{(p)}(1, \psi)(z) := E_1(1, \psi)(z) - E_1(1, \psi)(pz)$$

which is the weight one specialisation of the Eisenstein families $\mathcal{E}(1, \psi)$ and $\mathcal{E}(\psi, 1)$. The derivatives of the Fourier coefficients of $\mathcal{E}(1, \psi)$ and $\mathcal{E}(\psi, 1)$ at weight 1 will be exploited in §4. Let

$$\kappa_{1+\varepsilon} \colon \mathcal{O}_E[\![\mathsf{Z}]\!] \to E[\varepsilon]/(\varepsilon^2)$$

be the algebra homomorphism whose restriction to Z is given by

$$\kappa_{1+\varepsilon}(u, z) = \chi_p(z)^{-1}(1 + \log_p(\chi_p(z))\varepsilon),$$

and let $E_{1+\varepsilon}^{(p)}(\eta, \varphi)$ be the image of $\mathcal{E}(\eta, \varphi)$ under $\kappa_{1+\varepsilon}$. The Fourier expansion of $E_{1+\varepsilon}^{(p)}(\eta, \varphi)$ can be written as

(38) $$E_{1+\varepsilon}^{(p)}(\eta, \varphi) = a_0(\eta, \varphi) + \sum_\nu a_\nu(\eta, \varphi)q^\nu.$$

where the coefficients can be read off from [DDP11, Prop. 2.1, 3.2], as summarised in the following lemma:

Lemma 3.1. *The Fourier coefficients of $E_{1+\varepsilon}^{(p)}(\eta, \varphi)$ are given by*

(39)
$$a_0(\eta, \varphi) = \frac{L_p'(\eta^{-1}\varphi, 0)}{4\eta(\mathfrak{d})} \cdot \varepsilon$$
$$a_\nu(\eta, \varphi) = \sum_{p \nmid I|(\nu)\mathfrak{d}} \eta\left(\frac{(\nu)\mathfrak{d}}{I}\right) \varphi(I)\left(1 + \varepsilon \log_p \mathrm{Nm}(I)\right).$$

The article [DDP11] constructs an explicit *cuspidal* family parametrised by $\mathcal{O}_E[\![\mathsf{Z}]\!]$ specialising to f at weight 1. The family is not an eigenform over $\mathcal{O}_E[\![\mathsf{Z}]\!]$. Nevertheless, since f itself is an eigenform, one can deduce the existence of a morphism

(40) $$\pi_f \colon \mathcal{T}^{\mathrm{n.ord}} \to \mathcal{O}_E$$

encoding the eigenvalues of Hecke operators acting on f. The composition with the morphism $\Lambda \to \mathcal{T}^{\mathrm{n.ord}}$ will be denoted by

(41) $$\pi_1 \colon \Lambda \to \mathcal{O}_E$$

and corresponds to the character $\kappa_{\mathbf{t},\mathbf{0}} \cdot \psi$.

The remainder of § 3 will be dedicated to studying lifts of the morphism π_f to $E[\varepsilon]/(\varepsilon^2)$. Geometrically, this corresponds to studying the geometry of $\mathrm{Spec}(\mathcal{T}^{\mathrm{n.ord}})$ in an infinitesimal neighborhood of the prime ideal defined by π_f.

Remark 1. The cuspidal family appearing in [DDP11] was used to obtain the explicit formula for the derivative of the p-adic L-function $L_p(\psi, s)$ at $s = 0$ conjectured by Gross, asserting that

$$(42) \qquad L_p'(\psi, 0) = \mathscr{L}(\psi)L(\psi, 0),$$

where $\mathscr{L}(\psi)$ is the $\mathscr{L}$-invariant described in § 3.2. In recent work of Betina, Dimitrov and Shih [BDS] Gross' formula is linked to the study of the geometry of eigenvarieties from a Galois theoretic perspective. The approach of [BDS] informs the present work, and is carried out in a broader setting.

3.2. **Galois cohomology and $\mathscr{L}$-invariants.** This section develops some results on Galois cohomology, which will be used later, notably in § 3.4, to describe the tangent space of certain Galois deformation functors. These preliminary results are well known to experts.

Although it will not be used, it is worth noting that most of the arguments below are quite general and also work for general number fields F. Let H be a Galois extension of F with Galois group G. The $E[G]$-module $\mathrm{Hom}(G_H^{\mathrm{ab}}, E)$ can be described explicitly via class field theory. This will be used to show that certain global Galois cohomology classes for the totally odd character ψ are determined by their images in local cohomology.

A finite place v of H determines a prime ideal of H whose decomposition group is isomorphic to the absolute Galois group of H_v, denoted by G_v. For each v, there is an isomorphism between the completion of $H_v^\times$ and G_v^{ab} induced by the local Artin reciprocity map

$$\mathrm{rec}_v \colon H_v^\times \longrightarrow G_v^{\mathrm{ab}}$$

for which the geometric normalisation is adopted. Note that the image of G_v in G_H^{ab} is canonical. Thus, there is a restriction map res_v defined by

$$(43) \qquad \begin{array}{rccc} \mathrm{res}_v \colon & \mathrm{Hom}(G_H^{\mathrm{ab}}, E) & \longrightarrow & \mathrm{Hom}(H_v^\times, E) \\ & f & \longmapsto & \mathrm{rec}_v \circ f|_{G_v^{\mathrm{ab}}}. \end{array}$$

The following lemma is well-known to experts, but its proof is sketched below for completeness.

Lemma 3.2. *There is an exact sequence of $E[G]$-modules*

$$(44) \qquad 0 \to \mathrm{Hom}(G_H^{\mathrm{ab}}, E) \xrightarrow{(\mathrm{res}_v)_{v|p}} \prod_{v|p} \mathrm{Hom}(H_v^\times, E) \longrightarrow \mathrm{Hom}(\mathcal{O}_H[1/p]^\times, E).$$

In addition, the rightmost map is surjective if and only if Leopoldt's Conjecture holds for H.

Proof. Let $\mathbb{A}_H^\times/H^\times$ be the idèle class group of H. The global Artin reciprocity map $\mathrm{rec}_H \colon \mathbb{A}_H^\times/H^\times \to G_H^{\mathrm{ab}}$ is compatible with its local versions via the restriction maps, and gives a sequence

$$(45) \qquad 0 \to \mathrm{Hom}(G_H^{\mathrm{ab}}, E) \to \mathrm{Hom}(\mathbb{A}_H^\times, E) \to \mathrm{Hom}(H^\times, E)$$

of continuous group homomorphisms and the topology on $H^\times$ is discrete. This sequence must be exact for topological reasons, and the two terms on the right are understood via their restrictions to the places above p, by the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{A}_H^\times, E) & \xrightarrow{\Delta^\vee} & \mathrm{Hom}(H^\times, E) \\ \downarrow & & \downarrow \\ \prod_{v|p} \mathrm{Hom}(H_v^\times, E) & \xrightarrow{\Delta_p^\vee} & \mathrm{Hom}(\mathcal{O}_H[1/p]^\times, E) \end{array}$$

where $\Delta^\vee$, $\Delta_p^\vee$ are induced by the diagonal embeddings. Note that any element in $\mathrm{Hom}(\mathbb{A}_H^\times, E)$ must be trivial on the units $\mathcal{O}_v^\times$ of H_v for any $v \nmid p\infty$, and standard continuity arguments then show that the resulting map $\ker \Delta^\vee \to \ker \Delta_p^\vee$ is an isomorphism. Finally, by Dirichlet's Unit Theorem,

$$\mathrm{rk}_\mathbb{Z} \, \mathcal{O}_H[1/p]^\times = \mathrm{rk}_\mathbb{Z} \, \mathcal{O}_H^\times + |\{v \mid p\}|,$$

so the rightmost map of (44) is surjective if and only if the $\mathbb{Z}_p$-rank of the closure of the image of $\mathcal{O}_H^\times$ in $\prod_{v|p} \mathcal{O}_v^\times$ is equal to the $\mathbb{Z}$-rank of $\mathcal{O}_H^\times$, that is, if Leopoldt's Conjecture holds for H. $\quad\square$

Let $\varphi \colon G \to E^\times$ be any character, viewed as a character of G_F, and consider the global Galois cohomology group $\mathrm{H}^1(F, E(\varphi))$. The inflation-restriction sequence for continuous group cohomology of $E(\varphi)$ leads to the identification

$$(46) \qquad \mathrm{H}^1(F, E(\varphi)) \simeq \mathrm{H}^1(H, E)^{\varphi^{-1}} = \mathrm{Hom}(G_H^{\mathrm{ab}}, E)^{\varphi^{-1}}.$$

The cohomology group $H^1(G, E(\varphi))$ is related to certain units in H. Let S be a finite set of places of F, containing all infinite places and let $\mathcal{O}_{H,S}^\times$ be the S-units of H, and write

$$U_\varphi := (\mathcal{O}_{H,S}^\times \otimes E)^{\varphi^{-1}}.$$

Then, the Galois-equivariant version of Dirichlet's Unit Theorem yields

$$(47) \qquad \dim_E U_\varphi = |\{w \in S \mid \varphi(w) = 1\}| - \dim_E E(\varphi)^G.$$

The above discussion will now be specialised to the case where H is the narrow Hilbert class field of F. In what follows, the character φ will be taken to be either the trivial character or the unramified totally odd character ψ, viewed as a character of $G = \mathrm{Gal}(H/F)$. Let S be the set of places containing the place corresponding to the prime (p) of F and all infinite places of F. It follows from (47) that $\dim_E U_\mathbb{1} = d$, since F is totally real and $\dim_E U_\psi = 1$, because (p) splits completely in H/F and ψ is totally odd.

Recall that we wrote

$$\alpha_1, \ldots, \alpha_d : F \hookrightarrow \bar{\mathbb{Q}}_p$$

for the distinct p-adic embeddings of F. The prime ideal $p\mathcal{O}_F$ splits completely in H/F, and the choice of a prime $\mathfrak{p}$ of H above p determines an identification $H_\mathfrak{p} = F_p$. Fix the choice of $\mathfrak{p}$ corresponding to the chosen embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ once and for all, and write

$$\tilde{\alpha}_1, \ldots, \tilde{\alpha}_d : H \hookrightarrow \bar{\mathbb{Q}}_p$$

for the p-adic embeddings of H extending $\alpha_1, \ldots, \alpha_d$ respectively, i.e.,

$$\tilde{\alpha}_j|_F = \alpha_j \qquad \text{and} \qquad \tilde{\alpha}_j^{-1}(\mathfrak{m}_{\bar{\mathbb{Z}}_p}) = \mathfrak{p}.$$

Lemma 3.3. *If the Leopoldt Conjecture holds for H, then:*

$$(48) \qquad \dim_E \mathrm{H}^1(F, E) = 1 \qquad \text{and} \qquad \dim_E \mathrm{H}^1(F, E(\psi)) = d.$$

Proof. Combining (46) with Lemma 3.2, for any character φ of G, the cohomology group $\mathrm{H}^1(F, E(\varphi))$ is equal to the φ^{-1}-eigenspace of the kernel of

$$(49) \qquad \prod_{\mathfrak{p}_i|p} \mathrm{Hom}(H_{\mathfrak{p}_i}^\times, E) \longrightarrow \mathrm{Hom}(\mathcal{O}_H[1/p]^\times, E).$$

Let $\mathfrak{p}_1 = \mathfrak{p}$ be the chosen prime. Then $\mathrm{Hom}(H_{\mathfrak{p}_1}^\times, E)$ is spanned by $(\log_p \circ \tilde{\alpha}_j)_j$ and $\mathrm{ord}_{\mathfrak{p}_1}$. Since p splits completely in H/F, the E-vector space $\mathrm{Hom}(H_{\mathfrak{p}_1}^\times, E)$ generates the source of (49) as an $E[G]$-module. The $E[G]$-span of each basis element of $\mathrm{Hom}(H_{\mathfrak{p}_1}^\times, E)$ is isomorphic to the right regular representation of G. In particular, the multiplicity of every one-dimensional representation of G in the source of (49) is equal to $\dim_E(\mathrm{Hom}(H_{\mathfrak{p}_1}^\times, E)) = (d+1)$.

Thus, (47) leads to the inequality (which is an equality if Leopoldt's Conjecture holds for H)

$$\dim_E \mathrm{H}^1(F, E(\varphi)) \geq (d+1) - \dim_E U_{\varphi^{-1}} = \begin{cases} 1, & \text{if } \varphi = \mathbb{1}, \\ d, & \text{if } \varphi \text{ is totally odd.} \end{cases}$$

$\square$

When φ is either totally odd or trivial, we wish to describe the restriction to the decomposition group at the prime p of a basis of $\mathrm{H}^1(F, E(\varphi))$. Denote

$$\mathrm{res}_p \colon \mathrm{H}^1(F, E(\varphi)) \to \mathrm{H}^1(F_p, E(\varphi))$$

for the restriction to the decomposition group at p, which is characterised by the choice of embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Since $\varphi|_{G_{F_p}} = 1$, by local class field theory, we can choose a basis of the target given by the p-adic valuation in $F_p^\times$, denoted by o_p, together with the homomorphisms

$$\ell_{p,j} = \log_p \circ \alpha_j, \qquad 1 \leq j \leq d.$$

Choose an auxiliary prime $\mathfrak{p}_i$, for some $1 \leq i \leq n$. Then, under the running assumptions, U_ψ is a one-dimensional E-vector space. Choose a generator u_ψ of U_ψ, and note that $\mathrm{ord}_{\mathfrak{p}_i}(u_\psi) \neq 0$, since the ψ^{-1}-eigenspace of $\mathcal{O}_H^\times \otimes E$ is trivial by (47).

Definition 3.4. *The quantity*

$$(50) \qquad\qquad \mathscr{L}_j(\psi) = -\frac{(\log_p \circ \tilde{\alpha}_j)(u_\psi)}{(\mathrm{ord}_p \circ \tilde{\alpha}_j)(u_\psi)}, \qquad (1 \leq j \leq d)$$

is called the partial $\mathscr{L}$-invariant *of ψ with respect to the j-th embedding of F into $\bar{\mathbb{Q}}_p$.*

Note that this definition is independent of the choice of the generator u_ψ of U_ψ and of the auxiliary choice of the prime $\mathfrak{p}$ of H above p. However, it depends on the choice of the p-adic embedding α_j of F, thus justifying the notation.

The following lemma related the partial $\mathscr{L}$-invariants of Definition 3.4 with the Gross–Stark unit u_τ of the introduction. In order to state it precisely, fix a choice of the unit u_ψ by fixing an RM point τ of discriminant D and setting

$$(51) \qquad\qquad u_\psi = \prod_{\sigma \in \mathrm{Gal}(H/F)} (\sigma u_\tau)^{\psi(\sigma^{-1})}.$$

Lemma 3.5. *For all odd characters ψ of $\mathrm{Gal}(H/F)$, and all $1 \leq j \leq d$,*

$$\mathscr{L}_j(\psi) L(\psi, 0) = \log_p(\tilde{\alpha}_j(u_\psi)).$$

Proof. This follows after noting that, by definition of the Gross–Stark unit u_ψ,

$$\mathrm{ord}_p(\tilde{\alpha}_j(u_\psi)) = -L(\psi, 0).$$

$\square$

The *full $\mathscr{L}$-invariant* of ψ is the quantity

$$\mathscr{L}(\psi) = \sum_{j=1}^d \mathscr{L}_j(u_\psi).$$

It can alternatively be defined as

$$\mathscr{L}(\psi) = -\frac{(\log_p \circ \mathrm{Nm}_{\mathbb{Q}_p}^{F_p} \circ \tilde{\alpha}_j)(u_\psi)}{(\mathrm{ord}_p \circ \tilde{\alpha}_j)(u_\psi)}.$$

Remark 2. Of primary interest is the case where F is quadratic and ψ is an odd narrow class character. Under theses assumptions, the character ψ satisfies $\psi^{-1}(\sigma) = \psi(\tau\sigma\tau^{-1})$ for any $\tau \in G_{\mathbb{Q}} \smallsetminus G_F$. This implies that the partial $\mathscr{L}$-invariants satisfy the relations

$$\mathscr{L}_1(\psi) = \mathscr{L}_2(\psi^{-1}), \qquad \mathscr{L}_2(\psi) = \mathscr{L}_1(\psi^{-1}).$$

Proposition 3.6. *Let φ be any character of G. Denote $\chi_p \colon G_F \to \mathbb{Z}_p^{\times}$ the p-adic cyclotomic character. If Leopoldt's Conjecture holds for H, then:*

(1) *If φ is trivial, $\eta_{\mathbb{1}} := -\log_p \circ \chi_p$ generates $\mathrm{H}^1(F, E(\varphi))$. Its restriction to the decomposition group at p satisfies*

$$\mathrm{res}_p(\eta_{\mathbb{1}}) = \sum_{j=1}^{d} \ell_{p,j};$$

(2) *If φ is totally odd, the cohomology group $\mathrm{H}^1(F, E(\varphi))$ has a basis $\{\eta_{\varphi,j}\}_{1 \le j \le d}$ such that*

$$\mathrm{res}_p(\eta_{\varphi,j}) = \ell_{p,j} + \mathscr{L}_j(\varphi^{-1})o_p.$$

Proof. By Proposition 3.6, $\mathrm{H}^1(F, E)$ is one dimensional; thus, it is generated by the (non-zero element) class of $\eta_{\mathbb{1}} = -\log_p(\chi_p)$. The restriction to the decomposition group at p can be calculated by observing that, since χ_p is obtained by restriction to G_F of a character of $G_{\mathbb{Q}}$, the same applies to the local characters at p. This implies that $\mathrm{res}_p(\eta_{\mathbb{1}})$ factors through the norm map from F_p to $\mathbb{Q}_p$.

Since $\mathfrak{p}_1$ is the prime of H determined by the fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, the diagram

$$
\begin{array}{ccc}
\mathrm{H}^1(F, E(\varphi)) & \xrightarrow{\mathrm{res}_p} & \mathrm{H}^1(F_p, E(\varphi)) \\
\downarrow{\scriptstyle \mathrm{res}_H} & & \downarrow{\scriptstyle \mathrm{res}_{H_{\mathfrak{p}_1}}} \\
\mathrm{H}^1(H, E(\varphi)) & \xrightarrow{\mathrm{res}_{\mathfrak{p}_1}} & \mathrm{H}^1(H_{\mathfrak{p}_1}, E(\varphi))
\end{array}
$$

commutes, where all the maps are given by restriction. In addition, $\mathrm{res}_{H_{\mathfrak{p}_1}}$ is an isomorphism, because (p) splits completely in H/F; more precisely $\mathrm{res}_{H_{\mathfrak{p}_1}}$ satisfies

(52) $$\mathrm{res}_{H_{\mathfrak{p}_1}}(o_p) = \mathrm{ord}_p \circ \tilde{\alpha}_j \quad \text{and} \quad \mathrm{res}_{H_{\mathfrak{p}_1}}(\ell_{j,p}) = \log_p \circ \tilde{\alpha}_j$$

for every $1 \le j \le d$. It is worth noting at this stage that $\mathrm{ord}_p \circ \tilde{\alpha}_j$ is independent of j, while $\log_p \circ \tilde{\alpha}_j$ depends very much on j. After identifying $\mathrm{H}^1(F, E(\varphi))$ with the φ^{-1} eigenspace of the kernel of (49), the image of res_p is isomorphic to the image of this subspace via the projection

$$\bigoplus_{i=1}^{n} \mathrm{Hom}(H_{\mathfrak{p}_i}^{\times}, E) \longrightarrow \mathrm{Hom}(H_{\mathfrak{p}_1}^{\times}, E)$$

on the first component (which is, of course, not Galois equivariant). Let $\{1, 2, \ldots, n\}$ be the G-set characterised by $\sigma\mathfrak{p}_i = \mathfrak{p}_{\sigma i}$ for every $\sigma \in G$. Let $(f_i)_{i \in I}$ an element of $\bigoplus_{i=1}^{n} \mathrm{Hom}(H_{\mathfrak{p}_i}^{\times}, E)$. The action of $\sigma \in G$ on $(f_i)_i$ is given by

$$\sigma(f_i)_i = (f_{\sigma^{-1}i} \circ \sigma^{-1})_i.$$

In particular, let $(f_i)_i$ belong to the φ^{-1}-eigenspace. The action of G on primes above p is simply transitive. Let $i = \sigma^{-1}1$ for $\sigma \in G$. Then

$$f_i = \varphi(\sigma)^{-1}(f_1 \circ \sigma)$$

Let $u_{\varphi^{-1}}$ be a generator of the φ-eigenspace of $\mathcal{O}_H^{\times} \otimes E$. Then

$$(f_i)_{i \in I}(\Delta_p(u_{\varphi^{-1}})) = \sum_{i=1}^{n} f_i(u_{\varphi}) = \sum_{i=1}^{n} \varphi(\sigma)^{-1}(f_1(\sigma u_{\varphi})) = n f_1(u_{\varphi}).$$

Thus, $(f_i)_i$ belongs to the φ^{-1}-component of the kernel of (49) if and only if $f_1(u_{\varphi^{-1}}) = 0$. Let

$$f_1 = \sum_{j=1}^{d} x_j \log_p \circ \tilde{\alpha}_j + y \mathrm{ord}_p \circ \tilde{\alpha}_j.$$

The condition $f_1(u_{\varphi^{-1}}) = 0$ cuts out a d-dimensional subspace, since $\mathrm{ord}_p \circ \tilde{\alpha}_j(u_{\varphi^{-1}}) \neq 0$. After re-writing this condition in terms of the $\mathscr{L}$-invariants $\{\mathscr{L}_j(\varphi^{-1})\}$ and comparing with (52), the proposition follows. $\qquad\square$

3.3. Λ-adic Galois representations.

A general result of Hida establishes the connection between the nearly ordinary Hecke algebra introduced in §3.1 and Galois representations. More precisely, in [Hi89b], certain Galois representations are constructed which interpolate the representation corresponding to classical specialisations of Hida families for the Hecke algebra $\mathcal{T}^{\mathrm{n.ord}}$. Exploiting the properties of these Galois representations, the study of the Hecke algebra $\mathcal{T}^{\mathrm{n.ord}}$ infinitesimally at the prime ideal corresponding to the system of eigenvalues of f can be reduced to that of a deformation ring that will be introduced in §3.4.

The ultimate goal is to leverage the properties of the Galois representation to extract explicit formulae for the derivatives of the cuspidal family specialising to f, in the spirit of [DLR15]. For this purpose, it suffices to consider the completed local ring $\mathcal{T}_f$ obtained as the nilreduction of the completion of the localisation of $\mathcal{T}^{\mathrm{n.ord}}$ at the prime ideal $\mathfrak{q}_f$ given by the kernel of the morphism π_f defined in (40). (Although this is not crucial for this application, it can be showed as in [C05, Prop. 6.4] that the completion of the localisation of $\mathcal{T}^{\mathrm{n.ord}}$ at the point corresponding to $\mathfrak{q}_f$ is automatically reduced.)

It is natural to view $\mathcal{T}_f$ as an algebra over Λ_1, the completion of the localisation of Λ at the prime ideal $\mathfrak{p}_1 = \ker \pi_1$; the latter is isomorphic to a ring of power series in $d+1$ variables over E.

In this section, Hida's results are slightly refined in order to obtain a two-dimensional representation with coefficients in $\mathcal{T}_f$ (Prop. 3.8), satisfying certain additional properties. The proof follows Mazur and Wiles' approach to the (somewhat delicate) study of deformations of residually reducible representations. The treatment of Bellaiche and Chenevier [BC06], which is well-suited to working over reduced henselian local rings such as $\mathcal{T}_f$, will be followed.

Write K_f for the total ring of fractions of $\mathcal{T}_f$; it is isomorphic to a product of fields $\prod_i K_{f,i}$, each corresponding to a minimal local component at of $\mathrm{Spec}(\mathcal{T}^{\mathrm{n.ord}})$ at the point corresponding to $\mathfrak{q}_f$.

Thus $\mathcal{T}_f$ can be viewed as a subring of K_f. In this context, the main result of [Hi89b] can be phrased as follows.

Theorem 3.7 (Hida). *There exists a totally odd, continuous Galois representation*

$$\rho_{K_f} \colon G_F \to \mathrm{GL}_2(K_f)$$

satisfying the following properties:

(1) *the pushforward $\rho_{K_{f_i}}$ of ρ_{K_f} to $K_{f,i}$ is absolutely irreducible, for every i;*

(2) *ρ_{K_f} is unramified outside p;*

(3) *For every $\mathfrak{l}$ prime ideal of F such that $\mathfrak{l} \nmid p$, let $\mathrm{Frob}_{\mathfrak{l}}$ be a Frobenius element. Then*

$$\det(1 - X\rho(\mathrm{Frob}_{\mathfrak{l}})) = 1 - \mathbf{T}_{\mathfrak{l}}X + \langle\mathfrak{l}\rangle\mathrm{Nm}(\mathfrak{l})X^2;$$

(4) *The restriction of ρ_{K_f} to G_{F_p} is nearly ordinary, i.e., it satisfies*

$$\rho_{K_f}|_{G_{F_p}} \simeq \begin{bmatrix} \epsilon & * \\ 0 & \delta \end{bmatrix},$$

where

$$\delta \circ \mathrm{rec}_p(p) = \mathbf{T}(p),$$

$$\delta \circ \mathrm{rec}_p(u) = (\kappa^{\mathrm{univ}})((u, 1)) \quad \textit{for all } u \in \mathsf{U},$$

where $\mathrm{rec}_p \colon F_p^\times \to G_{F_p}$ *is the local Artin reciprocity map.*

In order to relate the ring $\mathcal{T}_f$ to a deformation ring, it is important to refine the Galois representation ρ_{K_f} to an integral version with coefficients in $\mathcal{T}_f$. By the Čebotarev density theorem, the trace of ρ_{K_f}, as well as the characters δ and ϵ, take values in $\mathcal{T}_f \subset K_f$. Following Bellaïche–Chenevier [BC06], the existence of a free rank two $\mathcal{T}_f$-module stable under the action of G_F can be related to a certain global cohomology group. In addition, the condition that the Galois representation ρ_{K_f} is nearly ordinary imposes some local conditions on the global cohomology classes, that allow to show that ρ_{K_f} is conjugate to a representation with coefficients in $\mathcal{T}_f$, following an argument which will now be described.

Let $M_{K_f} \simeq K_f^2$ be the two-dimensional Galois representation of G_F provided by Theorem 3.7. Relative to a basis (e^+, e^-) of M_{K_f} on which a choice of complex conjugation for F acts diagonally as $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right]$, the representation ρ_{K_f} is given by

$$\rho_{K_f} = \begin{bmatrix} a_f & b_f \\ c_f & d_f \end{bmatrix} : G_F \to \mathrm{GL}_2(K_f).$$

The fact that the traces of ρ_{K_f} lie in $\mathcal{T}_f$ implies that $a_f(\sigma) \pm d_f(\sigma)$ belong to $\mathcal{T}_f$, and hence, that

$$a_f(\sigma), d_f(\sigma) \in \mathcal{T}_f, \quad \text{for all } \sigma \in G_F.$$

It follows that

$$b_f(\sigma) \cdot c_f(\tau) = a_f(\sigma\tau) - a_f(\sigma)a_f(\tau) \in \mathcal{T}_f, \quad \text{for all } \sigma, \tau \in G_F.$$

Let B_f and C_f be the $\mathcal{T}_f$-submodules of K_f generated by the values of the functions b_f and c_f respectively. The *reducibility ideal* I_f^{red} is the (proper) integral ideal of $\mathcal{T}_f$ generated by the products $b_f(\sigma)c_f(\tau)$ for all $\sigma, \tau \in G_F$.

Fix a G_{F_p}-stable free one-dimensional submodule L_{K_f} of $M_{K_f} = K_f^2$, and denote by ϵ_f and δ_f the local characters of G_{F_p} acting on L_{K_f} and M_{K_f}/L_{K_f} respectively.

Proposition 3.8. *There exists a free $\mathcal{T}_f[G_F]$-submodule $M_{\mathcal{T}_f} \subset M_{K_f}$ of rank two over $\mathcal{T}_f$, whose associated Galois representation $\rho_{\mathcal{T}_f} : G_F \to \mathrm{GL}(M_{\mathcal{T}_f})$ satisfies the following properties:*

(1) The residual representation $M_E := M_{\mathcal{T}_f} \otimes E$ is semisimple;
(2) There exists a free rank one summand $L_{\mathcal{T}_f}$ of $M_{\mathcal{T}_f}$ such that
- *$L_{\mathcal{T}_f}$ is G_{F_p}-stable and G_{F_p} acts on $M_{\mathcal{T}_f}/L_{\mathcal{T}_f}$ via δ_f;*
- *The subspace $L_E := L_{\mathcal{T}_f} \otimes E$ of M_E is not G_F-stable.*

Proof. Let $B_{p,f} \subset B_f$ be the $\mathcal{T}_f$-module generated by $b(G_{F_p})$, and likewise for $C_{f,p} \subset C_f$. We claim that the natural inclusion $B_{p,f} \hookrightarrow B_f$ is surjective. The $\mathcal{T}_f$-module B_f is finitely generated by continuity of the representation ρ_{K_f}, and hence, by Nakayama's lemma, it suffices to show that the induced map

$$i_B : B_{f,p}/\mathfrak{m}_{\mathcal{T}_f} B_{f,p} \to B_f/\mathfrak{m}_{\mathcal{T}_f} B_f$$

is surjective. Consider the commutative diagram

(55)
$$\begin{array}{ccc}
\operatorname{Hom}(B_f/\mathfrak{m}_{\mathcal{T}_f}B_f, E) & \xrightarrow{\ \Gamma\ } & \operatorname{H}^1(F, E(\psi)) \ , \\
\downarrow{\scriptstyle i_B^\vee} & & \downarrow{\scriptstyle \operatorname{res}_p} \\
\operatorname{Hom}(B_{p,f}/\mathfrak{m}_{\mathcal{T}_f}B_{p.f}, E) & \xrightarrow{\ \Gamma_p\ } & \operatorname{H}^1(F_p, E)
\end{array}$$

where the top horizontal map Γ maps $\theta \in \operatorname{Hom}(B_f/\mathfrak{m}_{\mathcal{T}_f}B_f, E)$ to the class of the cocycle

$$\sigma \longmapsto \theta(b(\sigma))$$

for every $\sigma \in G_F$, and Γ_p is the corresponding map on local cohomology. By [BC06, Lemma 3], the map Γ is injective, and Proposition 3.6 implies that res_p is also injective. The commutativity of diagram (55) implies that $i_B^\vee$ is injective, and therefore i_B is surjective, as claimed, so that $B_f = B_{p,f}$. The same argument shows that $C_f = C_{p,f}$.

From the fact that $\rho_{K_{f_i}}$ is absolutely irreducible for every i, it follows that the the annihilator of the module B_f (respectively C_f) is 0. One can deduce that, without loss of generality, the vectors (e^+, e^-) can be rescaled by a pair of elements of $K_f^\times$ so that

$$L_{K_f} = \langle e^+ + e^- \rangle.$$

Note that this basis is unique up to scaling, and hence, the resulting matrix representation of ρ_{K_f} is uniquely determined.

Changing the basis (e^+, e^-) to $(e^- + e^+, e^+)$, the representation ρ_{K_f} is given in matrix form by

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_f & b_f \\ c_f & d_f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a_f + b_f & b_f \\ -b_f + (d_f - a_f) + c_f & -b_f + d_f \end{bmatrix}.$$

In particular, for every $\sigma \in G_{F_p}$,

$$a_f(\sigma) + b_f(\sigma) = \epsilon_f(\sigma) \qquad \text{and} \qquad -b_f(\sigma) + d_f(\sigma) - a_f(\sigma) + c_f(\sigma) = 0.$$

Since a_f and d_f are valued in $\mathcal{T}_f$ and $a_f(\sigma) = d_f(\sigma) = 1 \pmod{\mathfrak{m}_{\mathcal{T}_f}}$, the first equation implies that b_f takes values in $\mathfrak{m}_{\mathcal{T}_f}$. Similarly, because $d_f(\sigma) - 1 \in \mathfrak{m}_{\mathcal{T}_f}$, from the second equation it also follows that $c_f(\sigma) \in \mathfrak{m}_{\mathcal{T}_f}$. Thus, $B_f = B_{p,f} \subset \mathfrak{m}_{\mathcal{T}_f}$ and $C_p = C_{p,f} \subset \mathfrak{m}_{\mathcal{T}_f}$. It follows that ρ_{K_f} has coefficients in $\mathcal{T}_f$ relative to the basis (e^+, e^-), giving rise to a Galois stable $\mathcal{T}_f$-lattice $M_{\mathcal{T}_f} := \mathcal{T}_f e^+ + \mathcal{T}_f e^-$ which satisfies all the claims of Proposition 3.8. $\qquad\qquad\square$

3.4. **A deformation ring for residually reducible representations.** This section describes an abstract deformation ring $\mathcal{R}_{\rho,L}^{\mathrm{n.ord}}$ relevant to the Eisenstein series f defined in (37). This deformation ring is equipped with a natural Λ_1-algebra structure

(56)
$$\Phi_{\mathcal{R}} \colon \Lambda_1 \longrightarrow \mathcal{R}_{\rho,L}^{\mathrm{n.ord}}.$$

The construction of $\mathcal{R}_{\rho,L}^{\mathrm{n.ord}}$ is complicated by the residual reducibility, and we follow the approach of Calegari–Emerton [CE05] to overcome this. The main result of this section is Proposition 3.10, which computes the map induced by $\Phi_{\mathcal{R}}$ on tangent spaces and shows it is an isomorphism.

3.4.1. *The deformation functor.* Let $\mathfrak{C}_E$ be the category of local complete noetherian rings with residue field E. Consider the functor

$$\mathcal{D}^{\mathrm{det}} \times \mathcal{D}^{\mathrm{loc}} \colon \mathfrak{C}_E \longrightarrow \mathbf{Sets}$$

which takes any $(A, \mathfrak{m}_A) \in \operatorname{Ob}(\mathfrak{C}_E)$ to the pairs of continuous characters
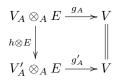
$$(\upsilon_A, \vartheta_A) \colon \mathsf{U} \times \mathsf{Z} \to A^\times, \quad \text{with} \quad (\upsilon_A, \vartheta_A) = (1, 1) \pmod{\mathfrak{m}_A}.$$

This functor is represented by the completed local ring Λ_1 of the Iwasawa algebra introduced in § 3.3. The ring Λ_1 is isomorphic to a ring of formal power series in $(d+1)$ variables over E.

We first define a deformation ring $\mathcal{R}_{\rho,L}$. Let $\rho = \psi \oplus 1$ with the standard basis (v_1, v_2) of $V = E^2$. This is a semisimple reducible representation, and as such it admits non-scalar endomorphisms. To obtain a representable functor, the deformation problem needs to be suitably rigidified. This is done, following Calegari–Emerton [CE05], by setting $L = \langle v_1 + v_2 \rangle$, which is a line that is *not* stable under the action of G_F. A *strict deformation* of (V, L, ρ) over an E-algebra A is a quadruple (V_A, L_A, ρ_A, g_A) where

- V_A is a free A-module of rank 2;
- L_A is a free rank 1 summand of V_A;
- $\rho_A : G_F \to \mathrm{GL}(V_A)$ is a continuous representation;
- $g_A : V_A \otimes_A E \simeq V$ is an isomorphism of $E[G_\mathbb{Q}]$-modules sending $L_A \otimes E$ to L.

Two strict deformations (V_A, L_A, ρ_A, g_A) and $(V'_A, L'_A, \rho'_A, g'_A)$ are said to be *equivalent* if there is an $A[G_\mathbb{Q}]$-module isomorphism $h : V_A \longrightarrow V'_A$ sending L_A to L'_A and for which the diagram

$$
\begin{array}{ccc}
V_A \otimes_A E & \xrightarrow{g_A} & V \\
{\scriptstyle h \otimes E}\downarrow & & \Vert \\
V'_A \otimes_A E & \xrightarrow{g'_A} & V
\end{array}
$$

commutes. A *deformation* of (V, L, ρ) over an E-algebra A is an equivalence class of strict deformations over A.

Consider the functor

$$\mathcal{D}_{\rho,L} : \mathfrak{C}_E \longrightarrow \mathbf{Sets}$$

which associates to an object A in $\mathfrak{C}_E$ the set of deformations of (V, L, ρ) over A. The functor $\mathcal{D}_{\rho,L}$ is representable by a complete local Noetherian ring $\mathcal{R}_{\rho,L}$ with residue field E. The representability can be verified as in [Ma89, Prop. 1]. It is ensured by the additional datum of a line lifting L, which can be viewed a "partial framing" of the functor parametrising deformations, forcing automorphisms of a deformation to consist only of scalars. Thus rigidified, the deformation functor presents the advantage of being fine enough to be representable, while still being conceivably comparable with a Hecke algebra, as in [CE05].

Finally, consider the functor $\mathcal{D}_{\rho,L}^{\mathrm{n.ord}}$ classifying quintuples $(V_A, L_A, \rho_A, g_A, \iota_A)$ such that:

- the equivalence class of the quadruple (V_A, L_A, ρ_A, g_A) belongs to $\mathcal{D}_{\rho,L}(A)$;
- the free rank one summand L_A is G_{F_p}-stable;
- $\iota_A : F_p^\times \to A^\times$ is the character satisfying

$$((\rho_A \circ \mathrm{rec}_p)y)v = \iota_A(y)v \mod L_A$$

for every $y \in F_p^\times$ and $v \in V_A$, where rec_p denotes the local Artin reciprocity map.

Remark 3. Of course the datum of a character $\iota_A : F_p^\times \to A^\times$ is redundant in the previous definition; it is nonetheless useful to keep track of it, since it plays a key role in the calculations.

The deformation functor $\mathcal{D}_{\rho,L}^{\mathrm{n.ord}}$ is representable by a quotient of $\mathcal{R}_{\rho,L}$, denoted $\mathcal{R}_{\rho,L}^{\mathrm{n.ord}}$. Indeed, choose an $\mathcal{R}_{\rho,L}$-basis $(\tilde{v}_1, \tilde{v}_2)$ for the universal representation, lifting (v_1, v_2), in such a way that the universal free rank one summand is given by

$$L_{\mathcal{R}_{\rho,L}} = \langle \tilde{v}_1 + \tilde{v}_2 \rangle.$$

Then the ring $\mathcal{R}_{\rho,L}^{\mathrm{n.ord}}$ is the quotient of $\mathcal{R}_{\rho,L}$ by the ideal

$$J^{\mathrm{n.ord}} = \langle \alpha(\sigma) + \beta(\sigma) - \gamma(\sigma) - \delta(\sigma) \mid \forall \sigma \in G_{F_p} \rangle,$$

where $\alpha, \beta, \gamma, \delta$ are the entries of $\rho_{\mathcal{R}_{\rho,L}} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with respect to the chosen basis.

We now describe the Λ_1-algebra structure of $\mathcal{R}_{\rho,L}^{\mathrm{n.ord}}$. Given a quintuple $(V_A, L_A, \rho_A, g_A, \iota_A) \in \mathcal{D}_{\rho,L}^{\mathrm{n.ord}}(A)$, the pair $(\det \rho_A \cdot \psi^{-1}, \iota|_{\mathcal{O}_{F_p}^{\times}})$ belongs to $\mathcal{D}^{\det} \times \mathcal{D}^{\mathrm{loc}}(A)$. In particular, taking (a representative of) the universal object of $\mathcal{D}_{\rho,L}^{\mathrm{n.ord}}$ yields a morphism $\Phi_{\mathcal{R}}$. As a consequence of Proposition 3.8, the universal property of $\mathcal{R}_{\rho,L}^{\mathrm{n.ord}}$ yields a morphism Υ from the deformation ring to the Hecke algebra, which makes the following diagram commute:

(57)
$$
\begin{array}{ccc}
\mathcal{R}_{\rho,L}^{\mathrm{n.ord}} & \xrightarrow{\ \ \Upsilon\ \ } & \mathcal{T}_f \\
& \searrow^{\Phi_{\mathcal{R}}} \qquad \nearrow^{\Phi_{\mathcal{T}}} & \\
& \Lambda_1 &
\end{array}
$$

3.4.2. *Tangent spaces.* We now come to the main results of this subsection, and describe the map on tangent spaces induced by Φ_R. Let $E[\varepsilon]$ be the ring of dual numbers over E. Let

$$t^{\mathrm{det,loc}} = (\mathcal{D}^{\det} \times \mathcal{D}^{\mathrm{loc}})(E[\varepsilon]), \qquad t_{\rho,L} = \mathcal{D}_{\rho,L}(E[\varepsilon]) \qquad \text{and} \qquad t_{\rho,L}^{\mathrm{n.ord}} = \mathcal{D}_{\rho,L}^{\mathrm{n.ord}}(E[\varepsilon])$$

be the tangent spaces of the three functors introduced above. The following lemma describes them explicitly.

Lemma 3.9. *(1) There is an isomorphism $G : \mathrm{H}^1(F, \mathrm{ad}(\rho)) \to t_{\rho,L}$ sending the cohomology class of $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ in $\mathrm{H}^1(F, \mathrm{ad}(\rho))$ to the equivalence class of*

(58)
$$\left(E[\varepsilon]^2, \left(1 + \varepsilon \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right) \rho, \langle [\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}] \rangle, g_\varepsilon \right),$$

where g_ε sends the standard basis of $E[\varepsilon]^2$ to (v_1, v_2).

(2) Let $\mathrm{H}^1(F, \mathrm{ad}(\rho))^{\mathrm{n.ord}}$ be the subspace of $\mathrm{H}^1(F, \mathrm{ad}(\rho))$ consisting of cocycles for which

$$\alpha(\sigma) + \beta(\sigma) - \gamma(\sigma) - \delta(\sigma) = 0, \qquad \forall \sigma \in G_{F_p}.$$

There is an isomorphism $G' : \mathrm{H}^1(F, \mathrm{ad}(\rho))^{\mathrm{n.ord}} \to t_{\rho,L}^{\mathrm{n.ord}}$ given by

$$G' \left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right) = \left(G \left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right), 1 + \varepsilon(\delta - \beta) \circ \mathrm{rec}_p \right).$$

Proof. (1) For any equivalence class in $t_{\rho,L} = \mathcal{D}_{\rho,L}(E[\varepsilon])$, we can choose a representative of the form (58) for some cocycle in $\mathrm{Z}^1(F, \mathrm{ad}(\rho))$. It suffices to verify that G is well-defined; in other words, it is enough to show that two lifts if $\rho_\varepsilon, \rho'_\varepsilon$ of ρ are conjugate by a matrix

$$g \in \ker(\mathrm{GL}_2(E[\varepsilon]) \to \mathrm{GL}_2(E)),$$

then they are conjugate by a matrix stabilizing the line $\langle [\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}] \rangle$. This follows from the fact that the space of coboundaries for the adjoint representation of the form

$$\sigma \mapsto \begin{bmatrix} \psi(\sigma) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} \begin{bmatrix} \psi(\sigma)^{-1} & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} 0 & (\psi(\sigma) - 1)s \\ (\psi(\sigma^{-1}) - 1)t & 0 \end{bmatrix}$$

for $\sigma \in G_F$ and $[\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}] \in \mathrm{M}_2(E)$ is spanned by coboundaries of matrices fixing the line $\langle [\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}] \rangle$.

(2) Let X be the cohomology class of $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{Z}^1(F, \mathrm{ad}(\rho))$. Denote $\rho_\varepsilon = (1 + \varepsilon \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix})\rho$. The image of X under G belongs to the $t_{\rho,L}^{\mathrm{n.ord}}$ if the line $\langle[\begin{smallmatrix}1\\1\end{smallmatrix}]\rangle$ is stable for the action of G_{F_p}. In the basis $(\langle[\begin{smallmatrix}1\\1\end{smallmatrix}]\rangle, \langle[\begin{smallmatrix}0\\1\end{smallmatrix}]\rangle)$ the matrix of ρ_ε is given by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \psi + \varepsilon(\psi\alpha + \beta) & \beta\varepsilon \\ (1 - \psi) + \varepsilon(-\alpha\psi - \beta + \gamma\psi + \delta) & 1 + \delta\varepsilon - \beta\varepsilon \end{bmatrix}.$$

The claim follows by observing that the restriction of ψ to G_{F_p} is trivial. $\qquad\square$

With these identifications, explicit bases for the relevant tangent spaces can be obtained by using the results in § 3.2. Identify $t^{\mathrm{det,loc}}$ with $\mathrm{Hom}(\mathsf{Z}, E) \oplus \mathrm{Hom}(\mathsf{U}, E)$, and choose the basis

$$(59) \qquad\qquad e_0 := (\eta_{\mathbb{1}}, 0), \qquad e_j = (0, \ell_{p,j}), \qquad 1 \le j \le d.$$

Given this choice of basis of the tangent space, an identification between $\Lambda_1 = E[\![X_0, X_1, \ldots, X_d]\!]$ can be chosen so that the universal characters are given by the pair $(\upsilon^{\mathrm{univ}}, \vartheta^{\mathrm{univ}})$ satisfying

$$\vartheta^{\mathrm{univ}} = 1 + X_0 e_0 \pmod{\mathfrak{m}_{\Lambda_1}^2} \qquad\qquad \upsilon^{\mathrm{univ}} = 1 + \sum_{j=1}^d X_j e_j \pmod{\mathfrak{m}_{\Lambda_1}^2}.$$

For the tangent space $t_{\rho,L}$, note that since $\rho = \psi \oplus \mathbb{1}$, it follows that

$$\mathrm{ad}(\rho) = \mathbb{1}^2 \oplus \psi \oplus \psi^{-1}.$$

i.e. the adjoint representation of ρ splits completely. Hence, by Lemma 3.3, the cohomology group $\mathrm{H}^1(F, \mathrm{ad}(\rho))$ has dimension $2d + 2$, and we may choose the E-basis consisting of

$$(60) \qquad A = \begin{bmatrix} \eta_{\mathbb{1}} & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & \eta_{\mathbb{1}} \end{bmatrix}, \quad B_j = \begin{bmatrix} 0 & \eta_{\psi,j} \\ 0 & 0 \end{bmatrix}, \quad C_j = \begin{bmatrix} 0 & 0 \\ \eta_{\psi^{-1},j} & 0 \end{bmatrix}, \qquad 1 \le j \le d.$$

where the entries are described by Proposition 3.6. With respect to these choices of bases, we now explicitly describe the map on tangent spaces induced by $\Phi_{\mathcal{R}}$, denoted

$$(61) \qquad\qquad\qquad \Theta\colon t_{\rho,L}^{\mathrm{n.ord}} \longrightarrow t^{\mathrm{det,loc}}.$$

Proposition 3.10. *If $\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1}) \ne 0$, the map Θ is an isomorphism and its inverse satisfies*

$$(62) \qquad \Theta^{-1}(e_0) = \frac{\mathscr{L}(\psi^{-1})}{\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1})}\left(A + \sum_{k=1}^d C_k\right) + \frac{\mathscr{L}(\psi)}{\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1})}\left(D + \sum_{k=1}^d B_k\right)$$

$$(63) \qquad \Theta^{-1}(e_j) = \frac{\mathscr{L}_j(\psi) - \mathscr{L}_j(\psi^{-1})}{\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1})}\left(A - D + \sum_{k=1}^d C_k - \sum_{k=1}^d B_k\right) - B_j - C_j$$

for $1 \le j \le d$.

Proof. By Lemma 3.9, up to composing with G', the tangent space $t_{\rho,L}^{\mathrm{n.ord}}$ is identified with a subspace of $\mathrm{H}^1(F, \mathrm{ad}(\rho))$ given by the kernel of

$$P_1\colon \mathrm{H}^1(F, \mathrm{ad}(\rho)) \to \mathrm{H}^1(F_p, E), \qquad\qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \mathrm{res}_p(\alpha + \beta - \gamma - \delta).$$

where res_p denotes the restriction to G_{F_p} (note that this is well defined, because $\psi(G_{F_p}) = 1$). Similarly, the map $\Theta\colon t_{\rho,L}^{\mathrm{n.ord}} \to t^{\mathrm{det,loc}}$ sends a quintuple $(V_\varepsilon, L_\varepsilon, \rho_\varepsilon, g_\varepsilon, \iota_\varepsilon)$ to the pair $(\det \rho_\varepsilon \cdot \psi^{-1}, \iota_\varepsilon|_{\mathcal{O}_{F_p}^\times})$. Again, by Lemma 3.9, it can be interpreted in terms of Galois cohomology as the restriction to the kernel of P_1 of

$$P_2\colon \mathrm{H}^1(F, \mathrm{ad}(\rho)) \to \mathrm{Hom}(\mathsf{Z}^\circ, E) \oplus \mathrm{Hom}(\mathsf{U}^\circ, E), \qquad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \left(\alpha + \delta, (\delta - \beta) \circ \mathrm{rec}_p|_{\mathcal{O}_{F_p}^\times}\right).$$

Thus, in order to show that Θ is an isomorphism, it suffices to show that $P := (P_1, P_2)$ is an isomorphism. Choose the bases $(A, D, B_1, \ldots, B_d, C_1, \ldots, C_d)$ for $\mathrm{H}^1(F, \mathrm{ad}(\rho))$ and

$$(o_p, 0), \ (\ell_{p,1}, 0), \ldots, (\ell_{p,d}, 0), \ (0, e_0), \ldots, (0, e_d)$$

for the target of P. Let $(x_A, x_D, x_{B_1}, \ldots, x_{B_d}, x_{C_1}, \ldots, x_{C_d})$ be the coordinates of a class in $\mathrm{H}^1(F, \mathrm{ad}(\rho))$ with respect to the basis above. The map P yields a system of $(2d+2)$ linear equations in $(2d+2)$-variables:

$$\sum_{j=1}^d (\mathscr{L}_j(\psi) x_{C_j} - \mathscr{L}_j(\psi^{-1}) x_{B_j}) = 0$$
$$x_A - x_D + x_{B_j} - x_{C_j} = 0$$
$$x_A + x_D = 0$$
$$x_D - x_{B_j} = 0$$

where $1 \le j \le d$. The corresponding matrix has determinant $\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1})$. The expressions for the inverse of Θ can be obtained by the inverting the matrix of P with respect to the above bases. $\square$

Remark 4. In the case of interest to this paper where $[F : \mathbb{Q}] = 2$, the non-vanishing of $\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1})$ is clear since $\mathscr{L}(\psi) = \mathscr{L}(\psi^{-1}) \ne 0$. In fact, this non-vanishing holds in general. For details, compare with [BDS].

Recall the commutative diagram (57) arising from Proposition 3.8.

Theorem 3.11. *The maps* $\Phi_{\mathcal{R}}, \Phi_{\mathcal{T}}$ *and* Υ *are isomorphisms.*

Proof. Since $\Phi_{\mathcal{R}}$ is a morphism of complete local noetherian rings with residue field E, the injectivity of (61) implies that $\Phi_{\mathcal{R}}$ is surjective. The top row is surjective because all Hecke operators are in the image; thus it follows that $\Phi_{\mathcal{T}}$ is surjective. But $\mathcal{T}_f$ is a torsion free Λ_1-algebra; in particular $\Phi_{\mathcal{T}}$ is an injective, hence an isomorphism. It follows that $\Phi_{\mathcal{R}}$ and Υ are isomorphisms as well. $\square$

Proposition 3.12. *The inverse of* $\Phi_{\mathcal{T}}$ *satisfies*

(64)
$$\begin{array}{rcll}
\Phi_{\mathcal{T}}^{-1}(\mathbf{T}_{\mathfrak{l}}) & = & 1 + \psi(\mathfrak{l}) + \log_p(\mathrm{Nm}(\mathfrak{l})) \cdot (\lambda + \mu\psi(\mathfrak{l})) & (\mathrm{mod}\ \mathfrak{m}_{\Lambda_1}^2) \\
\Phi_{\mathcal{T}}^{-1}(\langle \mathfrak{l} \rangle)\, \mathrm{Nm}(\mathfrak{l}) & = & \psi(\mathfrak{l})(1 + (\lambda + \mu)\log_p(\mathrm{Nm}(\mathfrak{l}))) & (\mathrm{mod}\ \mathfrak{m}_{\Lambda_1}^2) \\
\Phi_{\mathcal{T}}^{-1}(\mathbf{T}(p)) & = & 1 + \xi & (\mathrm{mod}\ \mathfrak{m}_{\Lambda_1}^2),
\end{array}$$

where $\mathfrak{l}$ *is a prime ideal of* F *such that* $p \nmid \mathrm{Nm}(\mathfrak{l})$ *and* $\lambda, \mu, \xi \in \mathfrak{m}_{\Lambda_1}$ *are given by*

$$\lambda = (\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1}))^{-1}\left(\mathscr{L}(\psi^{-1})X_0 + \sum_{j=1}^d (\mathscr{L}_j(\psi) - \mathscr{L}(\psi^{-1}))X_j\right)$$

$$\mu = (\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1}))^{-1}\left(\mathscr{L}(\psi)X_0 - \sum_{j=1}^d (\mathscr{L}_j(\psi) - \mathscr{L}(\psi^{-1}))X_j\right)$$

$$\xi = (\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1}))^{-1}\left(-\mathscr{L}(\psi)\mathscr{L}(\psi^{-1})X_0 + \sum_{j=1}^d (\mathscr{L}_j(\psi)\mathscr{L}(\psi^{-1}) + \mathscr{L}_j(\psi^{-1})\mathscr{L}(\psi))X_j\right).$$

Proof. Let $(V^{\mathrm{univ}}, L^{\mathrm{univ}}, \rho^{\mathrm{univ}}, g^{\mathrm{univ}}, \iota^{\mathrm{univ}})$ be a representative of the universal object of the functor $\mathcal{D}_{\rho,L}^{\mathrm{n.ord}}$ over the deformation ring $\mathcal{R}_{\rho,L}^{\mathrm{n.ord}}$. Then

$$\Upsilon^{-1}(\mathbf{T}_{\mathfrak{l}}) = \mathrm{Tr}(\rho^{\mathrm{univ}})(\mathrm{Frob}_{\mathfrak{l}}) \qquad \text{and} \qquad \Upsilon^{-1}(\mathbf{T}(p)) = \iota^{\mathrm{univ}}(\mathrm{Frob}_p).$$

Denoting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi_{\mathcal{R}}^{-1} \circ \rho^{\mathrm{univ}} \colon G_F \longrightarrow \mathrm{GL}_2(\Lambda_1),$$

it follows from Proposition 3.10 that modulo $\mathfrak{m}_{\Lambda_1}^2$,

$$a = \psi + \psi\eta_{\mathbb{1}}(\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1}))^{-1}\left(\mathscr{L}(\psi^{-1})X_0 + \sum_{j=1}^d (\mathscr{L}_j(\psi) - \mathscr{L}(\psi^{-1}))X_j\right)$$

$$d = 1 + \eta_{\mathbb{1}}(\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1}))^{-1}\left(\mathscr{L}(\psi)X_0 - \sum_{j=1}^d (\mathscr{L}_j(\psi) - \mathscr{L}(\psi^{-1}))X_j\right)$$

from which the expression for $\Upsilon^{-1}(\mathbf{T}_{\mathfrak{l}})$ is obtained. Similarly, $\Phi_{\mathcal{R}}^{-1} \circ \Phi^{\mathrm{univ}} = d - b$. Since $\eta_{\mathbb{1}}(\mathrm{Frob}_p) = 0$, it can be seen that $(d-b)(\mathrm{Frob}_p) = 1 - b(\mathrm{Frob}_p)$, which is equal to

$$1 + \left(\frac{\mathscr{L}(\psi)(\sum_{k=1}^d \eta_{\psi,k}(\mathrm{Frob}_p))}{\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1})} X_0 + \sum_{j=1}^d \frac{(\mathscr{L}_j(\psi^{-1}) - \mathscr{L}_j(\psi))\sum_{k=1}^d \eta_{\psi,k}(\mathrm{Frob}_p)}{\mathscr{L}(\psi) + \mathscr{L}(\psi^{-1})} - \eta_{\psi,j}(\mathrm{Frob}_p) \right) X_j$$

modulo $\mathfrak{m}_{\Lambda_1}^2$ by Proposition 3.10. The equality (64) then follows from Lemma 3.6. $\qquad\square$

3.5. Fourier coefficients.

The above results will now be specialised to the case where F is a real quadratic field and ψ is an unramified totally odd character of F. In this case, we compute the Fourier coefficients of the *anti-parallel* family through the Eisenstein series f of parallel weight 1 discussed above.

The *anti-parallel* weight direction in the tangent space of the Iwasawa algebra is the direction corresponding to the morphism $\Lambda_1 \longrightarrow E[\varepsilon]/(\varepsilon^2)$ given in terms of generators as follows:

$$(65) \qquad \begin{cases} X_0, X_1 & \longmapsto \quad \varepsilon, \\ \quad X_2 & \longmapsto \quad 0. \end{cases}$$

Since, by Theorem 3.11, the structural map $\Phi_{\mathcal{T}} \colon \Lambda_1 \to \mathcal{T}_f$ is an isomorphism, the map (65) gives rise to a morphism from the nearly ordinary Hecke algebra to the ring of dual numbers

$$(66) \qquad \tilde{\pi}_f : \mathcal{T}^{\mathrm{n.ord}} \longrightarrow E[\varepsilon]/(\varepsilon^2).$$

lifting the morphism π_f defined in (40). This corresponds to the system of Hecke eigenvalues of a first order eigenfamily of Hilbert modular forms $\mathcal{F}$, whose Fourier coefficients, which can be recovered from $\tilde{\pi}_f$, play a central role in what follows.

In the anti-parallel weight direction, one immediately checks that the quantities λ, μ, and ξ appearing in the description of the tangent space of $\mathcal{T}^{\mathrm{n.ord}}$ in Proposition 3.12 specialise to

$$(67) \qquad \lambda = \frac{\mathscr{L}_1(\psi)}{\mathscr{L}(\psi)} \qquad\qquad \mu = \frac{\mathscr{L}_2(\psi)}{\mathscr{L}(\psi)} \qquad\qquad \xi = 0.$$

Using the results in § 3.4, the image under the anti-parallel weight morphism $\tilde{\pi}_f$ of the operators $\langle \mathfrak{l} \rangle$ and $\mathbf{T}_{\mathfrak{l}^n}$ for $\mathfrak{l} \neq (p)$, as well as $\mathbf{T}(p^n)$, can now be computed in terms of these quantities:

- Let $\mathfrak{l} \neq (p)$ be a prime ideal. Proposition 3.12 immediately implies that

$$\begin{aligned} \tilde{\pi}_f(\mathbf{T}_{\mathfrak{l}}) &= 1 + \psi(\mathfrak{l}) &+& \varepsilon \log_p(\mathrm{Nm}(\mathfrak{l})) \cdot (\lambda\psi(\mathfrak{l}) + \mu) \\ \tilde{\pi}_f(\langle \mathfrak{l} \rangle)\,\mathrm{Nm}(\mathfrak{l}) &= \psi(\mathfrak{l}) &+& \varepsilon \log_p(\mathrm{Nm}(\mathfrak{l})) \cdot \psi(\mathfrak{l}). \end{aligned}$$

  The recursion relation proved in [Hi88, Corollary 4.2], which states

$$\mathbf{T}_{\mathfrak{l}^n}\mathbf{T}_{\mathfrak{l}} = \mathbf{T}_{\mathfrak{l}^{n+1}} + \langle \mathfrak{l} \rangle\,\mathrm{Nm}(\mathfrak{l})\mathbf{T}_{\mathfrak{l}^{n-1}}$$

  can be used to determine the image of the Hecke operators attached to powers of $\mathfrak{l}$. A straightforward inductive argument now shows that

$$\begin{aligned} \tilde{\pi}_f(\mathbf{T}_{\mathfrak{l}^n}) &= \sum_{j=0}^n \psi(\mathfrak{l})^j \left(1 + \varepsilon \log_p \mathrm{Nm}(\mathfrak{l})\,(j \cdot \lambda + (n-j) \cdot \mu) \right) \\ &= \sum_{I | \mathfrak{l}^n} \psi(I) \left(1 + \varepsilon \left(\lambda \log_p \mathrm{Nm}(I) + \mu \log_p \mathrm{Nm}\left(\frac{\mathfrak{l}^n}{I} \right) \right) \right) \end{aligned}$$

- For the nearly ordinary Hecke operators at p, it follows from [Hi89b, Proposition 2.3] that

$$\tilde{\pi}_f(\mathbf{T}(p^n)) = \tilde{\pi}_f(\mathbf{T}(p))^n = 1.$$

We are now ready to compute the Fourier coefficients of the anti-parallel deformation $\mathcal{F}$. Recall that the algebraic notion of q-expansions (cf. Hida [Hi04, Chapter 4]) gives a tuple of power series $\mathcal{F}_i(q)$ indexed by a set $\mathfrak{t}_i$ of integral ideals representing the classes in the narrow ideal class group of F:

$$(68) \qquad \mathcal{F}_i(q) \;=\; a_0(\mathfrak{t}_i) \;+\; \sum_{\nu \in (\mathfrak{t}_i)_+} a_\nu \, q^\nu, \qquad\qquad a_\nu \in E[\varepsilon]/(\varepsilon^2),$$

where as usual $(\mathfrak{t}_i)_+$ denotes the subset of totally positive elements of $\mathfrak{t}_i$. Abbreviate the q-expansion corresponding to the inverse different $\mathfrak{d}^{-1}$ by $\mathcal{F}(q)$. The main result of this section is:

Theorem 3.13. *The anti-parallel family*

$$(69) \qquad \mathcal{F}(q) \;=\; \sum_{\nu \in \mathfrak{d}_+^{-1}} a_\nu \, q^\nu, \qquad a_\nu \in E[\varepsilon]/(\varepsilon^2),$$

has Fourier coefficients given, to first order, by

$$(70) \qquad a_\nu = \sum_{I | (\nu)\mathfrak{d}} \psi(I) \left(1 + \varepsilon \left(-\log_p(\nu) + \frac{\mathscr{L}_1(\psi)}{\mathscr{L}(\psi)} \log_p \mathrm{Nm}(I) + \frac{\mathscr{L}_2(\psi)}{\mathscr{L}(\psi)} \log_p \mathrm{Nm}\left(\frac{(\nu)\mathfrak{d}}{I} \right) \right) \right),$$

for all ν that are relatively prime to p. Furthermore $a_{p^m \nu} = a_\nu$ for all $m \geq 1$.

Proof. The family $\mathcal{F}$ is p-adically cuspidal, so the constant term vanishes. To compute the higher Fourier coefficients, we compute that, for any $\nu \in \mathfrak{d}_+^{-1}$ such that $p \nmid \nu$,

$$\begin{aligned}
\tilde{\pi}_f(\mathbf{T}_{(\nu)\mathfrak{d}}) \;&=\; \prod_{\mathfrak{l}^n \| (\nu)\mathfrak{d}} \tilde{\pi}_f\left(\mathbf{T}_{\mathfrak{l}^n}\right) \\
&=\; \sum_{I | (\nu)\mathfrak{d}} \psi(I) \left(1 + \varepsilon \left(\frac{\mathscr{L}_1(\psi)}{\mathscr{L}(\psi)} \log_p \mathrm{Nm}(I) + \frac{\mathscr{L}_2(\psi)}{\mathscr{L}(\psi)} \log_p \mathrm{Nm}\left(\frac{(\nu)\mathfrak{d}}{I} \right) \right) \right)
\end{aligned}$$

to first order. To determine the Fourier coefficients from this value, the p-adic interpolation properties of the coefficients a_ν, and the density of classical forms, are used to reduce to the relations between Fourier coefficients and the Hecke algebra proved for classical forms in [Hi91].

Consider the rigid analytic fiber of the formal scheme attached to $\mathcal{T}^{\mathrm{n.ord}}$. For a sufficiently small affinoid neighbourhood $V = \mathrm{Spm}(A_V)$ of the point corresponding to the morphism π_f, there is a rigid analytic family $\mathcal{F}_V = \sum_\nu a_{V,\nu} q^\nu$ with normalised Fourier coefficients in $a_{V,\nu} \in A_V$, specialising to $\mathcal{F}$ in the anti-parallel direction. By Hida's Control Theorem, there is a Zariski-dense set of points in V corresponding to systems of Hecke eigenvalues $\pi_g \colon A_V \to \bar{\mathbb{Q}}_p$ of classical modular forms g of weight $(\mathbf{k}_g, \mathbf{w}_g)$ and with Fourier coefficients $a_{g,\nu}$ given by the image of $a_{V,\nu}$ under π_g. Combining the relations for classical forms proved in [Hi91, Eqn. (2.3) et seq./Eqn. (1.5)], one obtains

$$\begin{aligned}
\pi_g(\mathbf{T}_{(\nu)\mathfrak{d}}) \;&=\; a_{g,\nu} \cdot \nu^{\mathbf{v}_g} \\
\pi_g(\mathbf{T}_{p^m}) \;&=\; a_{g,p^m}
\end{aligned}$$

where as before $\mathbf{v}_g = \mathbf{w}_g - \mathbf{k}_g + \mathbf{t}$. The quantity $\nu^{\mathbf{v}_g}$ may be identified with the weight $(\mathbf{k}_g, \mathbf{w}_g)$-specialisation of the universal character κ^{univ} evaluated at $(\nu, 1)$. The image of $\kappa^{\mathrm{univ}}(\nu, 1)$ under the morphism (65) defining the anti-parallel direction is given by

$$1 + \varepsilon \log_p(\nu)$$

so that the density of classical points in V implies that

$$a_\nu = (1 - \varepsilon \log_p(\nu))\tilde{\pi}_f(\mathbf{T}_{(\nu)\mathfrak{d}}).$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Diagonal restrictions and RM values

This section explains how to parlay Theorem 3.13 of § 3 into a proof of Theorem C. In a nutshell, the generating series of Theorem C is obtained from the ordinary projection of the diagonal restriction of a modification of the anti-parallel cuspidal deformation $\mathcal{F}$ described in Theorem 3.13.

Retain the setup of § 3.5. Namely, F denotes a real quadratic field, and ψ is a totally odd unramified character. Let D be the discriminant of F, with ring of integers $\mathcal{O}_F$, set of integral ideals $\mathscr{I}_F$, and different ideal $\mathfrak{d}$. The notation $\mathfrak{d}_+^{-1}$ is used for the subset of totally positive elements of the inverse different $\mathfrak{d}^{-1}$. Write Nm and Tr for the norm and trace functions from F to $\mathbb{Q}$. If $\tau \in \mathcal{H}_p^D$ is an RM point of discriminant D, denote by $\mathfrak{a}_\tau \in \mathrm{Cl}(D)$ the narrow ideal class attached to τ. If J is a rigid cocycle, then

$$J[\psi] := \prod_{\tau \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}_p^{\circ, D}} J[\tau]^{\psi(\mathfrak{a}_\tau)} \in \mathbb{C}_p^\times \otimes \mathbb{Q}(\psi).$$

Remark 5. The character ψ was assumed to be unramified for simplicity, and it would be interesting to generalise the arguments to the case of an arbitrary totally odd ring class character

$$\psi : \mathrm{Cl}(D) \longrightarrow \mathbb{C}_p^\times \tag{71}$$

of discriminant $D = f^2 D_0$ with D_0 fundamental, and $(p, f) = 1$. The deformations studied in § 3 are not sensitive in an essential way to this additional ramification. Moreover, a version of Lemma 4.2 for non-trivial conductors can be found in [LV], and the explicit formula (74) continues to hold. One may therefore expect that Proposition 4.7 is amenable to this generalisation via the strategy of this paper, provided that the left hand side of the equality is replaced by the series obtained by taking the trace to level $\Gamma_0(p)$:

$$\mathrm{Tr}_{\Gamma_0(p)}^{\Gamma_0(fp)} \left(e^{\mathrm{ord}}(\partial_\varepsilon f^+) \right) \quad \in M_2(\Gamma_0(p)). \tag{72}$$

4.1. **The RM values of the winding cocycle.** In contrast the approach of [DD06], the calculations below build on the viewpoint of rigid (theta) cocycles introduced in [DV, §3], by making essential use of the winding cocycle J_w of the prequel [DPV21], some of whose properties were already recalled in § 2.2. This section describes some further results from [DPV21] concerning its RM values. The first key result is an explicit formula for $T_n J_w[\tau]$, which was established in [DPV21, Theorem 2.9].

In order to state it, choose, for any integer $n \geq 1$ and any RM point τ in $\mathcal{H}_p$, a finite set $M_n(\tau)$ of representatives for the double coset space $\mathrm{SL}_2(\mathbb{Z}) \backslash M_2(\mathbb{Z})_n / \Gamma_\tau$, where

$$M_2(\mathbb{Z})_n := \{\alpha \in M_2(\mathbb{Z}) \text{ with } \det(\alpha) = n\}, \qquad \Gamma_\tau = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\tau).$$

In other words

$$M_2(\mathbb{Z})_n = \bigsqcup_{\delta \in M_n(\tau)} \mathrm{SL}_2(\mathbb{Z}) \cdot \delta \cdot \Gamma_\tau. \tag{73}$$

Let $\tilde{\Gamma} := \mathrm{GL}_2^+(\mathbb{Z}[1/p])$ be the group of invertible matrices over $\mathbb{Z}[1/p]$ with positive determinant.

Theorem 4.1. *Let $n \geq 1$ be an integer coprime to p. Then*

$$T_n J_w[\tau] = \prod_{\delta \in M_n(\tau)} \prod_{\substack{w \in \tilde{\Gamma}\delta\tau \\ v_p(w) = 0}} w^{[0, \infty] \cdot (w', w)}. \tag{74}$$

Proof. See [DPV21, Theorem 2.9]. $\qquad\square$

Lemma 4.2 below recalls the existence of a bijection between "level n" sets of RM points and ideals that was constructed in [DPV21, § 1]. Define the (multi)set

$$(75) \qquad \mathrm{RM}_n^+(\tau) := \bigsqcup_{\delta \in M_n(\tau)} \left\{ w \in \tilde{\Gamma}\delta\tau \; : \; \begin{array}{c} w > 0 > w' \\ v_p(w) = 0, \; v_p(\mathrm{disc}(w)) \leq v_p(n) \end{array} \right\}$$

where $\mathrm{disc}(w)$ is defined to be the discriminant of a primitive integral quadratic form that has w as a root. Similarly, define $\mathrm{RM}_n^-(\tau)$ as above, with the condition $w > 0 > w'$ replaced by $w' > 0 > w$.

Remark 6. Note that an RM point w may appear several times in the set $\mathrm{RM}_n^+(\tau)$, and the multiplicity with which it does is a subtle actor in the bijections discussed below. It is therefore important to use a disjoint union in this definition. The nature of the matrices δ, which index the multiplicity with which an RM point w arises, was made clearer in the proof of [DPV21, Lemma 1.9].

The sets $\mathrm{RM}_n^\pm(\tau)$ play a crucial role in the explicit formulae for the Fourier coefficients of the diagonal restrictions of the Eisenstein family $\mathcal{E}$ investigated in [DPV21]. It will be observed below that they appear again in the analysis of the diagonal restriction of the anti-parallel family $\mathcal{F}$ studied in § 3.

Lemma 4.2. *There exist two bijections*

$$\varphi_1 \; : \qquad\qquad \mathrm{RM}_n^-(-\tau) \qquad\qquad \longrightarrow \quad \mathrm{RM}_n^+(\tau)$$

$$\varphi_2 \; : \quad \left\{ (I, \nu) : \begin{array}{cc} \nu \in \mathfrak{d}_+^{-1}, & p \nmid I \,|\, (\nu)\mathfrak{d} \\ \mathrm{Tr}(\nu) = n, & I \sim (1, \tau) \end{array} \right\} \quad \longrightarrow \quad \mathrm{RM}_n^+(\tau)$$

such that, after writing $\nu = p^m \nu_0$, we have

$$\begin{array}{rcl} \varphi_1(w) & = & -w, \\ \varphi_2(I, \nu) & = & \nu_0 \sqrt{\Delta} / \mathrm{Nm}(I). \end{array}$$

Proof. A bijection φ_1 as required may be constructed by letting W_∞ be a diagonal matrix with eigenvalues 1 and -1. If $w \in \tilde{\Gamma}\delta\tau$, then

$$-w = W_\infty w \; \in \; \tilde{\Gamma}\delta'(-\tau)$$

where $\delta' \in M_n(\tau)$ is the double coset representative of $W_\infty \delta W_\infty$. To obtain a bijection φ_2 as above, one first uses a bijection

$$\Phi \; : \quad \left\{ (I, \nu) : \begin{array}{cc} \nu \in \mathfrak{d}_+^{-1}, & I \,|\, (\nu)\mathfrak{d} \\ \mathrm{Tr}(\nu) = n, & I \sim (1, \tau) \end{array} \right\} \quad \longrightarrow \quad \bigsqcup_{\delta \in M_n(\tau)} \{w \in \mathrm{SL}_2(\mathbb{Z})\delta\tau \; : \; w > 0 > w'\}$$

which satisfies $\Phi(I, \nu) = \nu\sqrt{\Delta} / \mathrm{Nm}(I)$. Such a bijection was constructed in [DPV21, Lemma 1.9]. Note that the source of Φ is almost equal to the source of the desired bijection, minus the condition $p \nmid I$. Under the bijection Φ, the condition $p \nmid I$ is equivalent to the condition that $w = w_0 p^m$ for some $m \geq 0$ and $p \nmid w_0$. The map $w \mapsto w_0$ then defines a bijection between

$$\left\{ w \in \mathrm{SL}_2(\mathbb{Z})\delta\tau \; : \; \begin{array}{c} w > 0 > w' \\ w = w_0 p^m, \; p \nmid w_0, \; m \geq 0 \end{array} \right\},$$

and the set

$$\left\{ w \in \tilde{\Gamma}\delta\tau \; : \; \begin{array}{c} w > 0 > w' \\ v_p(w) = 0, \; v_p(\mathrm{disc}(w)) \leq v_p(n) \end{array} \right\},$$

so that the result follows by definition of $\mathrm{RM}_n^+(\tau)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. **Derivatives of diagonal restrictions.** The modular generating series for the RM values of the winding cocycle that is the subject of Theorem C will be constructed from three different analytic families that specialise to the Eisenstein series of parallel weight one. More specifically, the anti-parallel cuspidal family from § 3.5, and the two Eisenstein families of Lemma 3.1:

$$E_{1+\varepsilon}^{(p)}(1, \psi) \quad \text{and} \quad E_{1+\varepsilon}^{(p)}(\psi, 1)$$

The *modularity* of the generating series of Theorem C will follow from two simple results:

(1) The vanishing of the diagonal restriction of $E_1^{(p)}(1, \psi)$, the p-stabilisation of the parallel weight one Eisenstein series (for which the shorthand f was used in § 3),

(2) For any analytic family of p-adic modular forms whose specialisation vanishes, the specialisation of its *derivative* is also a p-adic modular form.

These results were also used in [DPV21], where full proofs may be found. Since they play an important role in the argument, they will be briefly reviewed here.

Lemma 4.3. *Suppose p is inert in F, and ψ is an odd unramified character of F. Then*

$$E_1^{(p)}(1, \psi)(z, z) = 0.$$

Proof. Recall that the diagonal restriction of any Hilbert modular form with Fourier coefficients a_ν has the following q-expansion:

(76)
$$a_0 + \sum_{n \geq 1} \sum_{\substack{\nu \in \mathfrak{d}_+^{-1} \\ \mathrm{Tr}(\nu) = n}} a_\nu \, q^n.$$

For the Eisenstein series $E_1^{(p)}(1, \psi)$, the Fourier coefficient a_ν is equal to

$$4 \sum_{p \nmid I \mid (\nu)\mathfrak{d}} \psi(I)$$

For any ideal I in the index set of this summation, we may write $IJ(p^e) = (\nu)\mathfrak{d}$ for some uniquely determined ideal J coprime to p, since p is inert in F. The conjugate J' then defines an ideal coprime to p, dividing $(\nu')\mathfrak{d}$. Observe that, since ψ is odd, we have

$$\psi(J') = \psi(J)^{-1} = \psi(I)\psi(\mathfrak{d})^{-1} = -\psi(I),$$

and it follows that $a_\nu = -a_{\nu'}$. Therefore the diagonal restriction must vanish. □

The three analytic families that specialise to $E_1^{(p)}(1, \psi)$ therefore give families of diagonal restrictions that specialise to zero. It is easy to see that the specialisation of the *derivative* of each of these families of diagonal restrictions is a p-adic modular form of weight two. The following result, appearing as Lemma 2.1 in [DPV21], ascertains that it is even overconvergent, though this is not used in what follows.

Lemma 4.4. *Suppose $\mathcal{F}(t)$ is a family of overconvergent forms of weight $\kappa(t)$, indexed by a parameter t on a closed rigid analytic disk D in weight space. Suppose that*

- *the disk D is centred at an integer $k = \kappa(0) \in \mathbb{Z}$,*
- *the specialisation vanishes $\mathcal{F}(0) = 0$.*

Then the derivative $\partial_t \mathcal{F}(0)$ is an overconvergent modular form of weight k.

4.3. **Proof of Theorem C.** Theorem 3.13 will now be used to construct the modular generating series G_τ of Theorem C, and calculate its constant term. The argument involves three main steps:

(1) The definition of the power series $\partial \mathcal{F}_\psi^+$, a combination of the q-expansions of the first derivatives of the anti-parallel cuspidal family $\mathcal{F}_\psi$ of Theorem 3.13 and a parallel Eisenstein family $\mathcal{E}_\psi$;

(2) The computation of its diagonal restriction ∂f_ψ^+;

(3) The computation of its ordinary projection $e^{\mathrm{ord}}(\partial f_\psi^+)$.

The forms constructed in these three steps lie in increasingly structured spaces: ∂f_ψ^+ is a p-adic modular form of weight two and tame level one, and $e^{\mathrm{ord}}(\partial f_\psi^+)$ is a classical modular form on $\Gamma_0(p)$. The power series $\partial \mathcal{F}_\psi^+$ however lacks the modularity properties of a traditional (classical or p-adic) Hilbert modular form, and is perhaps best envisaged as an instance of a "p-adic mock modular form", of the kind that make an appearance in [DT08, DLR15] for instance.

The series $\mathcal{F}_\psi^+$ is a combination of first order families of modular forms passing through the same Eisenstein series of parallel weight one in *different* weight directions. Its definition was dictated by the algebraic shape of the Fourier coefficients of the anti-parallel family $\mathcal{F}_\psi$ arising from Theorem 3.13, as it causes the desired algebraic cancellation. Precisely, define

$$\mathcal{F}_\psi^+ := \mathcal{F}_\psi + \mathcal{E}_\psi = a_0(\mathcal{F}_\psi^+) + \sum a_\nu(\mathcal{F}_\psi^+) q^\nu, \qquad a_\nu(\mathcal{F}_\psi^+) \in E[\varepsilon]/(\varepsilon^2),$$

where the first term $\mathcal{F}_\psi$ is the anti-parallel weight cuspidal deformation of Theorem 3.13. The second term $\mathcal{E}_\psi$ is the following explicit combination of parallel weight Eisenstein families

$$(77) \qquad\qquad \mathcal{E}_\psi := \frac{\mathscr{L}_2(\psi)}{\mathscr{L}(\psi)} \left(E_{1+\varepsilon}^{(p)}(1, \psi) - E_{1+\varepsilon}^{(p)}(\psi, 1) \right).$$

Recall the Gross–Stark unit u_ψ attached to the odd character ψ, defined in (51). Henceforth, the unit u_ψ is identified with its image under the p-adic embedding $\tilde{\alpha}_2$ in order to lighten the notations and view it as an element of $F_p^\times \otimes \mathbb{Q}(\psi)$, to which the p-adic logarithm $\log_p$ may be unambiguously applied.

Proposition 4.5. *The Fourier coefficients of $\mathcal{F}_\psi^+$ are given by*

$$a_0(\mathcal{F}_\psi^+) = \frac{\varepsilon}{2} \cdot \log_p(u_\psi).$$

$$a_\nu(\mathcal{F}_\psi^+) = \sum_{I | (\nu_0)\mathfrak{d}} \psi(I) \left(1 - \varepsilon \log_p \left(\frac{\nu_0}{\mathrm{Nm}(I)} \right) \right)$$

Proof. Since ψ is odd, we have $\psi(\mathfrak{d}) = -1$. The constant term of $\mathcal{E}_\psi$, given by (39), is therefore

$$a_0(\mathcal{E}_\psi) = \frac{\mathscr{L}_2(\psi)}{\mathscr{L}(\psi)} (a_0(1, \psi) - a_0(\psi, 1)) = \frac{\varepsilon}{4} \cdot \frac{\mathscr{L}_2(\psi)}{\mathscr{L}(\psi)} (L_p'(\psi, 0) + L_p'(\psi^{-1}, 0)).$$

By the Gross–Stark theorem (42)

$$(78) \qquad\qquad L_p'(\psi, 0) = \mathscr{L}(\psi) L(\psi, 0) = \mathscr{L}(\psi^{-1}) L(\psi^{-1}, 0) = L_p'(\psi^{-1}, 0),$$

and hence, using Lemma 3.5, we obtain

$$a_0(\mathcal{E}_\psi) = \frac{\varepsilon}{2} \cdot \frac{\mathscr{L}_2(\psi)}{\mathscr{L}(\psi)} L_p'(\psi, 0) = \frac{\varepsilon}{2} \cdot \mathscr{L}_2(\psi) L(\psi, 0) = \frac{\varepsilon}{2} \cdot \log_p(u_\psi).$$

At $\nu \neq 0$, the Fourier coefficient of $\mathcal{E}_\psi$ is given by

$$a_\nu(\mathcal{E}_\psi) = \frac{\mathscr{L}_2(\psi)}{\mathscr{L}(\psi)} \sum_{I | (\nu_0)\mathfrak{d}} \psi(I) \varepsilon \left(\log_p(\mathrm{Nm}(I) - \log_p \left(\frac{\nu_0}{\mathrm{Nm}(I)} \right) \right).$$

Combining this with the formula for the Fourier coefficients of the anti-parallel deformation $\mathcal{F}_\psi$ given in Theorem 3.13, gives the required identity

$$(79) \qquad a_\nu(\mathcal{F}_\psi^+) \;=\; \sum_{I|(\nu_0)\mathfrak{d}} \psi(I)\left(1 - \varepsilon \log_p\left(\frac{\nu_0}{\mathrm{Nm}(I)}\right)\right)$$

$\square$

Next, we consider the *diagonal restriction* f_ψ^+ of the series $\mathcal{F}_\psi^+$, defined as the sum of the diagonal restrictions of the families $\mathcal{F}_\psi$ and $\mathcal{E}_\psi$. Its derivative with respect to ε is modular. More specifically:

Proposition 4.6. *The power series*

$$(80) \qquad \partial f_\psi^+(q) \;=\; \frac{1}{2}\log_p(u_\psi) \;-\; \sum_{n\geq 1}\;\sum_{\substack{\nu\in\mathfrak{d}_+^{-1}\\ \mathrm{Tr}(\nu)=n}}\;\sum_{I|(\nu_0)\mathfrak{d}} \psi(I)\log_p\left(\frac{\nu_0\sqrt{D}}{\mathrm{Nm}(I)}\right) q^n.$$

is the q-expansion of a p-adic modular form of weight two and tame level one.

Proof. Lemma 4.3 implies that the diagonal restriction f_ψ^+ vanishes at $\varepsilon = 0$, so that the derivative ∂f_ψ^+ is a p-adic modular form (by Lemma 4.4 it is even overconvergent). The statement about its q-expansion follows by (76) from the observation that $\partial f_\psi^+(q)$ differs from the desired result by

$$\sum_{\substack{\nu\in\mathfrak{d}_+^{-1}\\ \mathrm{Tr}(\nu)=n}} \psi(I)\log_p(\sqrt{D}),$$

which is proportional to the n-th Fourier coefficient of the diagonal restriction of the Hilbert Eisenstein series $E_1^{(p)}(1,\psi)$, and is therefore identically zero by Lemma 4.3. $\square$

Finally, we explicitly compute the *ordinary projection* of the p-adic modular form ∂f_ψ^+. This ordinary projection is a classical modular form in $M_2(\Gamma_0(p))$, and its Fourier coefficients can be related to the RM values of the winding cocycle J_w, using the explicit formula for the latter stated in § 4.1.

Proposition 4.7. *The ordinary projection of the p-adic modular form ∂f_ψ^+ is a classical modular form in the space $M_2(\Gamma_0(p))$. Its q-expansion is given by:*

$$(81) \qquad 2e^{\mathrm{ord}}(\partial f_\psi^+) \;=\; \log_p(u_\psi) \;-\; \sum_{n\geq 1}\log_p\left(T_n J_w[\psi]\right) q^n.$$

Proof. Note that the ordinary projection is classical of level $\Gamma_0(p)$, by Coleman's classicality theorem. The statement about the constant term follows from (79). For any $n \geq 1$, the bijection φ_2 of Lemma 4.2 allows us to rewrite the n-th Fourier coefficient of $2\partial f_\psi^+$ appearing in (80) in terms of the level n sets of RM points $\mathrm{RM}_n^\pm(\tau)$. Since $\psi(\tau) = -\psi(-\tau)$, this Fourier coefficient is given by

$$2a_n \;=\; \sum_{\tau\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_p^{\circ,D}} \psi(\tau)\left(\sum_{w\,\in\,\mathrm{RM}_n^+(\tau)} \log_p(w) \;-\; \sum_{w\,\in\,\mathrm{RM}_n^+(-\tau)} \log_p(w)\right)$$

$$=\; \sum_{\tau\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_p^{\circ,D}} \psi(\tau)\left(\sum_{w\,\in\,\mathrm{RM}_n^+(\tau)} \log_p(w) \;-\; \sum_{w\,\in\,\mathrm{RM}_n^-(\tau)} \log_p(w)\right)$$

where the second equality follows from the existence of a bijection φ_1 as in Lemma 4.2. Let $n \geq 1$ coprime to p, then the n-th coefficient of the ordinary projection of $2\partial f_\psi^+$ is given by

$$
2a_n^{\mathrm{ord}} = 2 \lim_{m \to \infty} a_{np^{2m}} \quad = \sum_{\tau \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}_p^{\circ, D}} \psi(\tau) \sum_{\delta \in M_n(\tau)} \sum_{\substack{w \in \bar{\Gamma}\delta\tau \\ v_p(w) = 0}} \big([0, \infty] \cdot (w', w)\big) \log_p(w)
$$

$$
= \log_p(T_n J_w[\psi])
$$

where the last equality uses the explicit formula (74).

To obtain the statement for all $n \geq 1$, note that

$$
T_n \longmapsto \log_p(T_n J_w[\psi]),
$$

is a linear function from the weight two Hecke algebra of level $\Gamma_0(p)$, since the Hecke action on $\mathrm{H}^1(\Gamma, \mathcal{A}^\times / \mathbb{C}_p^\times)$ factors through it, and hence there exists $f \in M_2(\Gamma_0(p))$ with higher Fourier coefficients as in the statement. By what we showed, the n-th Fourier coefficients of f and $2e^{\mathrm{ord}}(\partial f_\psi^+)$ agree when n is coprime to p. Their difference must therefore be an oldform, and hence zero. $\qquad\square$

We are now ready to prove Theorem C of the introduction:

Theorem 4.8. *Let D be a fundamental discriminant and let $\tau \in \mathcal{H}_p$ be an RM point of discriminant D. There is a classical modular form G_τ of weight two on $\Gamma_0(p)$ with p-adic Fourier coefficients, whose q-expansion is given by*

$$
G_\tau(q) = \log(u_\tau) + \sum_{n=1}^{\infty} \log((T_n J_w)[\tau])q^n,
$$

where $\log \, : \, \mathcal{O}_{\mathbb{C}_p}^\times \longrightarrow \mathbb{C}_p$ *is the p-adic logarithm. The modular form G_τ is non-trivial if and only if $\mathbb{Q}(\sqrt{D})$ does not admit a unit of norm -1.*

Proof. Let H be the narrow class field of $\mathbb{Q}(\sqrt{D})$. Proposition 4.7 produces, for each odd character ψ of $\mathrm{Gal}(H/F)$, a classical modular form in $M_2(\Gamma_0(p))$ with q-expansion in $C_p[[q]]$ given by

$$
G_\psi(q) = \log_p(u_\psi) \; - \; \sum_{n \geq 1} \log_p(T_n J_w[\psi]) \, q^n.
$$

The assignment $\psi \mapsto G_\psi(q)$ extends by linearity to a map on the linear span of the odd characters, which is the space of odd functions on $\mathrm{Gal}(H/F)$. Let ψ be the odd indicator function on the class of τ, which is equal to 1 on $[\tau]$, to -1 on $[-\tau] = [\sigma_\infty \tau]$, where $\sigma_\infty \in \mathrm{Gal}(H/F)$ is complex conjugation, and vanishes on all the other $\mathrm{Pic}^+(\mathcal{O}_F)$-translates of $\tau \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}_p^D$. With this choice of ψ, we have

$$
\log_p(u_\psi) = \log_p(u_\tau) - \log_p(\sigma_\infty u_\tau) = 2 \log_p(u_\tau),
$$

and

$$
\log_p(T_n J_w[\psi]) = \log_p(T_n J_w[\tau]) - \log_p(T_n J_w[-\tau]) = 2 \log_p(T_n J_w[\tau]).
$$

The modular form G_τ of Theorem C is obtained by setting

$$
G_\tau = \frac{1}{2} G_\psi.
$$

$\qquad\square$

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