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Diagonal restrictions of p-adic Eisenstein families

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Abstract

We compute the diagonal restriction of the first derivative with respect to the weight of a p-adic family of Hilbert modular Eisenstein series attached to a general (odd) character of the narrow class group of a real quadratic field, and express the Fourier coefficients of its ordinary projection in terms of the values of a distinguished *rigid analytic cocycle* in the sense of Darmon and Vonk (Duke Math J, to appear, 2020) at appropriate real quadratic points of Drinfeld's p-adic upper half-plane. This can be viewed as the p-adic counterpart of a seminal calculation of Gross and Zagier (J Reine Angew Math 355:191–220, 1985, §7) which arose in their "analytic proof" of the factorisation of differences of singular moduli, and whose inspiration can be traced to Siegel's proof of the rationality of the values at negative integers of the Dedekind zeta function of a totally real field. Our main identity enriches the dictionary between the classical theory of complex multiplication and its extension to real quadratic fields based on RM values of rigid meromorphic cocycles, and leads to an expression for the p-adic logarithms of Gross–Stark units and Stark–Heegner points in terms of the first derivatives of certain twisted Rankin triple product p-adic L-functions.

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Introduction

In their influential work on singular moduli [17, §7], Gross and Zagier consider the diagonal restriction of a family, indexed by a complex parameter s, of non-holomorphic Hilbert modular Eisenstein series of parallel weight one attached to an odd genus character ψ of a real quadratic field. Since this family vanishes identically at $s = 0$, it becomes natural to study its first derivative, a real analytic modular form of weight two on $\mathrm{SL}_2(\mathbb{Z})$. The vanishing of its holomorphic projection is used to calculate the arithmetic intersections of singular moduli attached to the two imaginary quadratic subfields of the biquadratic field cut out by ψ. The derivative of the non-holomorphic Eisenstein family provides a simple but illustrative instance of the *Kudla program*, a framework that seeks similar modular generating series for the topological and arithmetic intersections of a broader class of special cycles on Shimura varieties.

The present work transposes the calculation of Gross and Zagier to a p-adic setting by studying the diagonal restriction of the first derivative with respect to the weight of a p-adic family of Hilbert modular Eisenstein series attached to a general (odd) character of the narrow class group of a real quadratic field. The Fourier coefficients of its *ordinary projection* are expressed in terms of the values of a distinguished *rigid analytic cocycle* at appropriate "real multiplication points" of Drinfeld's p-adic upper half-plane. Such RM values are related to a panoply of invariants defined (conjecturally) over ring class fields of real quadratic fields, notably, the Stark–Heegner points of [4], the Gross–Stark units of [5], and the real quadratic singular moduli of [10]. Our main identity enriches the analogy between the classical theory of complex multiplication and its extension to real quadratic fields based on the RM values of rigid meromorphic cocycles. It also leads to a new expression for the p-adic logarithms of Gross–Stark units and Stark–Heegner points in terms of the first derivatives of certain twisted Rankin triple product p-adic L-functions.

Let F be a real quadratic field of discriminant $D > 0$, and let H denote its Hilbert class field in the narrow sense. The narrow class group $\mathrm{Cl}^+(D) = \mathrm{Gal}\,(H/F)$ of F is endowed with a canonical element c of order 1 or 2 represented by the class of the principal ideal (α), where $\alpha \in F^\times$ is an element of negative norm, which corresponds to the complex conjugation in $\mathrm{Gal}\,(H/F)$. Given $\mathcal{C} \in \mathrm{Cl}^+(D)$, write $\mathcal{C}^* := c \cdot \mathcal{C}$. A function ψ on $\mathrm{Cl}^+(D)$ is said to be *odd* if it satisfies $\psi(\mathcal{C}^*) = -\psi(\mathcal{C})$.

Assume from now on that ψ is such an odd function on $\mathrm{Cl}^+(D)$. For each $k \geq 1$, it gives rise to a holomorphic Eisenstein series of (odd) parallel weight k on the full Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_F)$, whose Fourier expansion for $k > 1$ is given by

$$E_k(1,\psi)(z_1,z_2) := L(F,\psi,1-k) + 4 \sum_{\nu \in \mathfrak{d}_+^{-1}} \sigma_{k-1,\psi}(\nu\mathfrak{d}) \exp(2\pi i(\nu_1 z_1 + \nu_2 z_2)), \tag{1}$$

where

$$L(F,\psi,s) = \sum_{I \triangleleft \mathcal{O}_F} \psi(I)\mathrm{Nm}(I)^{-s}, \quad (\mathrm{Re}(s) > 1) \tag{2}$$

is the zeta-function attached to ψ, the index set $\mathfrak{d}_+^{-1}$ denotes the cone of totally positive elements in the inverse different of F, and $\sigma_{k-1,\psi}$ is the function

$$\sigma_{k-1,\psi}(\alpha) := \sum_{I | (\alpha)} \psi(I)\mathrm{Nm}(I)^{k-1}, \tag{3}$$

the sum being taken over all the integral ideals I of $\mathcal{O}_F$ that divide (α).

Let p be a rational prime. The p-stabilisation of $E_k(1,\psi)$ has Fourier expansion given by

$$E_k^{(p)}(1,\psi) := L^{(p)}(F,\psi,1-k) + 4 \sum_{\nu \in \mathfrak{d}_+^{-1}} \sigma_{k-1,\psi}^{(p)}(\nu\mathfrak{d}) \exp(2\pi i(\nu_1 z_1 + \nu_2 z_2)), \tag{4}$$

where $L^{(p)}(F,\psi,s)$ and $\sigma_{k-1,\psi}^{(p)}(\alpha)$ are obtained from $L(F,\psi,s)$ and $\sigma_{k-1,\psi}(\alpha)$ respectively by restricting the sums arising in their definitions to the ideals whose norm are prime to p. The Eisenstein series $E_k^{(p)}(1,\psi)$ is of parallel weight k on the Hecke congruence group of $\mathrm{SL}_2(\mathcal{O}_F)$ consisting of matrices that are upper triangular modulo p, and hence its restriction to the diagonal $\mathcal{H} \subset \mathcal{H} \times \mathcal{H}$ is a classical modular form of weight $2k$ on $\Gamma_0(p)$:

$$G_k(\psi) := E_k^{(p)}(1,\psi)(\tau,\tau) \in M_{2k}(\Gamma_0(p)), \quad \text{for all} \quad k \in \mathbb{Z}^{\geq 1}. \tag{5}$$

As functions of k, the Fourier coefficients of $G_k(\psi)$ interpolate to analytic (Iwasawa) functions on weight space

$$\mathcal{W} := \hom(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) = \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p.$$

This can be verified directly for the coefficients of q^n with $n \geq 1$, and general principles first described and exploited by Serre in [26] reveal that this property is inherited by the constant term $L^{(p)}(F, \psi, 1-k)$. It can thus be viewed as the value at $s = 1-k$ of an analytic function, the p-adic L-function $L_p(F, \psi, s)$ attached to ψ[1].

Our first theorem—Theorem A below—relates the Fourier expansion of the weight two specialisation $G_1(\psi)$ to Mazur's *winding element*, defined as the image

$$g_w \in H_1(X_0(p); \{0, \infty\}, \mathbb{Z})$$

of the vertical path on the Poincaré upper half plane joining 0 to ∞ in the homology of $X_0(p)$ relative to the cusps.

If p is split in $F/\mathbb{Q}$, then every narrow ideal class of F can be represented by a primitive binary quadratic form $Q(x, y) = Ax^2 + Bxy + Cy^2$ with $p|A$. Such a form is called a *Heegner form* (at p). The set of Heegner forms in a given narrow ideal class $\mathcal{C}$ consists of two $\Gamma_0(p)$-orbits of Heegner forms, depending on a square root s of D modulo p, and determined by the condition $B \equiv s \pmod{p}$. These orbits are denoted $\mathcal{C}_s$ and $\mathcal{C}_{-s}$ respectively. The automorph attached to a Heegner form Q belongs to $\Gamma_0(p)$, and its image in the homology $H_1(Y_0(p), \mathbb{Z})$, denoted g_Q, depends only on the $\Gamma_0(p)$-orbit of Q. The two homology classes of Heegner forms in $\mathcal{C}$ are denoted $g_{\mathcal{C},s}$ and $g_{\mathcal{C},-s}$ respectively. Define

$$g_\psi = \sum_{\mathcal{C} \in \mathrm{Cl}(D)} \psi^{-1}(\mathcal{C})(g_{\mathcal{C},s} + g_{\mathcal{C},-s}) \in H_1(Y_0(p), \mathbb{Z}[\psi]),$$

where $\mathbb{Z}[\psi]$ is the ring generated over $\mathbb{Z}$ by the values of ψ.

Let $\mathbb{T}_k(p)$ be the algebra of Hecke operators acting faithfully on the space $M_k(\Gamma_0(p))$ of modular forms of weight k on $\Gamma_0(p)$. It is generated by the Hecke operators T_n for all $n \geq 1$, where T_p is used here to denote what is sometimes referred to as U_p. These operators are described in the standard way in terms of double cosets, and act naturally on the homology groups $H_1(X_0(p); \{0, \infty\}, \mathbb{Z})$ and $H_1(Y_0(p), \mathbb{Z})$, in a way that is compatible with the intersection pairing

$$\langle \cdot , \cdot \rangle : H_1(X_0(p); \{0, \infty\}, \mathbb{Z}) \times H_1(Y_0(p), \mathbb{Z}) \longrightarrow \mathbb{Z}. \tag{6}$$

These structures extend by linearity to the homology groups with coefficients in more general rings like $\mathbb{Z}[\psi]$. Our first result, which is shown in Sect. 1, is

[1] Its restriction to $\hom(\mathbb{Z}_p^\times, 1 + p\mathbb{Z}_p)$ is customarily denoted $L_p(F, \psi\omega_p, s)$ in the literature.

Theorem A *The Fourier expansion of the weight two specialisation $G_1(\psi)$ is given by*

$$G_1(\psi) = \begin{cases} L_p(F,\psi,0) - 2\sum_{n=1}^{\infty} \langle g_w, T_n g_\psi \rangle \, q^n & \text{if } \left(\frac{D}{p}\right) = 1, \\ 0 & \text{if } \left(\frac{D}{p}\right) = -1. \end{cases}$$

Assume henceforth that $\left(\frac{D}{p}\right) = -1$. Since the family $G_k(\psi)$ vanishes identically at $k = 1$, it can be envisaged as the p-adic counterpart of the families of real analytic modular forms that arise in [17] and in Kudla's theory of incoherent Eisenstein series, as explored, for instance, in [19]. It then becomes natural to consider the first derivative

$$G_1'(\psi) := \frac{d}{dk} G_k(\psi)_{k=1}, \tag{7}$$

which is shown in Sect. 2.1 to be an overconvergent p-adic modular form of weight two and tame level one. Its image

$$G_1'(\psi)_{\text{ord}} := e_{\text{ord}} G_1'(\psi) := \lim_{n\to\infty} U_p^{n!} G_1'(\psi) \;\in\; M_2(\Gamma_0(p)) \tag{8}$$

under Hida's ordinary projector, which plays the same role as the holomorphic projection operator in the work of Gross–Zagier, is a classical form of weight two on $\Gamma_0(p)$. Our second objective is to calculate the Fourier coefficients of $G_1'(\psi)_{\text{ord}}$ and relate them to certain *rigid cocycles*, whose RM values provide a natural framework for extending the theory of complex multiplication to real quadratic fields, and whose definition is now briefly recalled.

Let $\mathcal{H}_p$ denote Drinfeld's p-adic upper half plane and let $\mathcal{M}^\times$ be the multiplicative group of non-zero rigid meromorphic functions on $\mathcal{H}_p$, endowed with the translation action of

$$\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$$

by Möbius transformations. A *rigid meromorphic cocycle* is an $\mathcal{M}^\times$-valued one-cocycle on Γ. It is said to be *rigid analytic* if it takes values in the group $\mathcal{A}^\times$ of non-zero rigid analytic functions on $\mathcal{H}_p$. The groups of rigid meromorphic and analytic cocycles are denoted by $H^1(\Gamma, \mathcal{M}^\times)$ and $H^1(\Gamma, \mathcal{A}^\times)$ respectively.

Because $H^1(\Gamma, \mathbb{C}_p^\times)$ is finite, the natural map $H^1(\Gamma, \mathcal{M}^\times) \longrightarrow H^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ has finite kernel. It is convenient to work with elements of the larger group of *cocycles modulo scalars*, which are called *theta-cocycles*. This terminology is motivated by the analogy with the theta functions that arise in the p-adic uniformisation theory of Mumford curves, which are invariant under the translation action of a p-adic Schottky group, but only up to multiplicative scalars. Although there is a non-trivial obstruction in $H^2(\Gamma, \mathbb{C}_p^\times)$ to lifting a theta-cocycle J to an element of $H^1(\Gamma, \mathcal{M}^\times)$, the restriction of J to $\mathrm{SL}_2(\mathbb{Z})$ lifts to an element

$$J^\circ \in H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}^\times)$$

because of the vanishing of $H^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$, and this lift is essentially unique because the group $H^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$ is finite.

A simple example of an analytic theta cocycle is the *universal theta-cocycle* J_{univ}, defined by fixing a base point ξ in $\mathbb{P}_1(\mathbb{Q}_p)$ and letting $J_{\mathrm{univ}}(\gamma)$ be the (unique, up to multiplicative scalars) rational function having $(\gamma\xi) - (\xi)$ as a divisor. This example is too simple to be of real arithmetic interest, since it takes values in rational functions rather than in the larger group of rigid analytic functions. But there are other instances, and in fact the group $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ is intimately related to the space $M_2(p)$. Namely, the group $H^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$ admits an action of the Hecke operators T_n for all $n \geq 1$, described in the standard way in terms of double cosets. This action preserves the subgroup of analytic theta-cocycles, and its restriction to this subspace factors through the algebra $\mathbb{T}_2(p)$. In fact, there is an explicit Hecke-equivariant map

$$\mathrm{ST}^\times : H^1(\Gamma_0(p), \mathbb{Z}) \longrightarrow H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)/J_{\mathrm{univ}}^{\mathbb{Z}}, \tag{9}$$

described in [11], referred to as the “multiplicative Schneider–Teitelbaum lift”.

A rigid meromorphic cocycle can be *evaluated* at real multiplication points of $\mathcal{H}_p$ following a recipe that is described in [10]. Namely, a point $\tau \in \mathcal{H}_p$ is said to be an RM point if $F(\tau, 1) = 0$ for some primitive integral binary quadratic form $F(x, y) = Ax^2 + Bxy + Cy^2$ of positive discriminant, and the discriminant $B^2 - 4AC$ is also called the discriminant of τ. The RM points are characterised as those in $\mathcal{H}_p$ for which the stabiliser

$$\Gamma[\tau] := \mathrm{Stab}_\Gamma(\tau) \tag{10}$$

of τ in Γ is an infinite group, of rank one modulo torsion. A generator γ_τ of $\Gamma[\tau]$ modulo torsion admits the column vector $(\tau, 1)$ as an eigenvector, with eigenvalue a unit ε of F. It can be chosen in a consistent way by fixing a real embedding of $F = \mathbb{Q}(\tau)$ and insisting that $\varepsilon > 1$, which implies that for all $\xi \in \mathcal{H}$,

$$\lim_{j \longrightarrow -\infty} \gamma_\tau^j \xi = \tau', \quad \lim_{j \longrightarrow \infty} \gamma_\tau^j \xi = \tau. \tag{11}$$

The *value* of a cocycle $J \in H^1(\Gamma, \mathcal{M}^\times)$ at τ is simply

$$J[\tau] := J(\gamma_\tau)(\tau) \quad \in \mathbb{C}_p \cup \{\infty\}. \tag{12}$$

More generally, a theta-cocycle J can also be evaluated at RM points τ whose discriminant is prime to p. Indeed, in this case the automorph γ_τ belongs to $\mathrm{SL}_2(\mathbb{Z})$, and one can simply define

$$J[\tau] := J^\circ(\gamma_\tau)(\tau) \in \mathbb{C}_p \cup \{\infty\}.$$

Let $\mathcal{H}_p^D$ be the set of $\tau \in \mathcal{H}_p^{\mathrm{RM}}$ of discriminant D. The theory of composition of binary quadratic forms identifies the orbit space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_p^D$ with the narrow class group $\mathrm{Cl}(D)$, and hence ψ can be viewed as a function on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_p^D$. Let

$$\Delta_\psi := \sum_{\tau \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_p^D} \psi(\tau)\cdot\tau \in \mathrm{Div}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_p^D), \tag{13}$$

be the associated formal degree zero divisor on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_p$. Because $p \nmid D$, a theta-cocycle J can be evaluated at the points of $\mathcal{H}_p^D$, and we can set

$$J[\Delta_\psi] := \sum_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_p^D} \psi(\tau)J[\tau]. \tag{14}$$

Section 2 introduces the *winding cocycle*, an explicit theta-cocycle

$$J_w \in H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times), \tag{15}$$

that is related to the winding element g_w via

$$J_w := \mathrm{ST}^\times(g_w) \pmod{J_{\mathrm{univ}}^{\mathbb{Z}}},$$

viewing g_w as an element of $H^1(\Gamma_0(p), \mathbb{Z})$ via the intersection pairing. The quantities $J_w[\Delta_\psi]$ belong to $F_p^\times \subset \mathbb{C}_p^\times$, where F_p is the completion of F at p, and we may consider their image under the norm map Nm from $F_p^\times$ to $\mathbb{Q}_p^\times$. From now on, choose Iwasawa's branch of the p-adic logarithm

$$\log_p : \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p \tag{16}$$

which is trivial on the torsion subgroup of $\mathbb{C}_p^\times$ as well as on p.

Our second main result is

Theorem B *For all fundamental discriminants $D > 0$, for all primes p that are inert in $F = \mathbb{Q}(\sqrt{D})$, and for all odd functions ψ on* $\mathrm{Cl}(D)$,

$$G_1'(\psi)_{\mathrm{ord}} = L_p'(F, \psi, 0) - 2\sum_{n=1}^{\infty} \log_p\left(\mathrm{Nm}((T_n J_w)[\Delta_\psi])\right) q^n.$$

Theorems A and B can be used to compute the spectral expansions of the modular forms $G_1(\psi)$ and $G_1'(\psi)_{\mathrm{ord}}$. To this end, a normalised eigenform $f \in S_2(\Gamma_0(p))$ with Fourier coefficients in a ring $\mathcal{O}_f$ gives rise to a modular abelian variety quotient A_f of $J_0(p)$ with endomorphism ring containing $\mathcal{O}_f$, and to a pair of homomorphisms

$$\tilde{\varphi}_f^+, \tilde{\varphi}_f^- \in H^1(\Gamma_0(p), \mathbb{C}), \quad \tilde{\varphi}_f^\pm(\gamma) := \int_\gamma (\omega_f \pm \bar{\omega}_f), \quad \omega_f := 2\pi i f(z)dz.$$

These classes, which encode the real and imaginary periods of f respectively, can be rescaled to take values in the the ring $\mathcal{O}_f$ of Fourier coefficients of f, by choosing appropriate periods $\Omega_f^{\pm} \in \mathbb{C}$ and setting

$$\varphi_f^+ := (\Omega_f^+)^{-1}\tilde{\varphi}_f^+, \quad \varphi_f^- := (\Omega_f^-)^{-1}\tilde{\varphi}_f^-. \tag{17}$$

From the latter one obtains a pair of theta-cocycles

$$J_f^{\pm} := \mathrm{ST}^{\times}(\varphi_f^{\pm}) \in \left(H^1(\Gamma, \mathcal{A}^{\times}/\mathbb{C}_p^{\times})/J_{\mathrm{univ}}^{\mathbb{Z}}\right) \otimes \mathcal{O}_f,$$

which are eigenvectors for the Hecke operators with the same eigenvalues as f. They are the elliptic modular cocycles described in [11, § 3]. The images of the RM values $J_f^{\pm}[\Delta_\psi] \in \mathcal{O}_{F_p}^{\times}$ under the Tate–Morikawa p-adic uniformisation of A_f are the *Stark–Heegner points* in $A_f(F_p) \otimes \mathcal{O}_f$, conjectured to be defined over suitable ring class fields of F.

The elliptic cocycles $J_f^{\pm}$ can be envisaged as the cuspidal counterparts of the *Dedekind–Rademacher cocycle* of [11, §3] attached to the periods of the Eisenstein series of weight two on $\Gamma_0(p)$ (normalised so that is first Fourier coefficient is 1), defined by

$$E_2^{(p)}(q) = \frac{p-1}{24} + \sum_{n\geq 1} \sigma_1^{(p)}(n)q^n, \quad E_2^{(p)}(q)\frac{dq}{q} = \frac{1}{24}\,\mathrm{dlog}(\Delta(q^p)/\Delta(q)).$$

It is given by

$$J_{\mathrm{DR}} := \mathrm{ST}^{\times}(\varphi_{\mathrm{DR}}), \tag{18}$$

where

$$\varphi_{\mathrm{DR}}(\gamma) := 24\int_\gamma E_2^{(p)}(z)dz \tag{19}$$

is the Dedekind–Rademacher homomorphism, whose expression in terms of Dedekind sums can be found in [23] for example. In the theory of rigid meromorphic cocycles, the Dedekind Rademacher cocycle plays the role of the modular unit $\Delta(z)/\Delta(pz)$. The refinement of Gross's p-adic Stark conjecture proposed in [5] predicts that the RM value $J_{\mathrm{DR}}[\Delta_\psi]$ belongs to $(\mathcal{O}_H[1/p])^{\times} \otimes \mathbb{Q}$.

For the next statement, let us choose the periods $\Omega_f^{\pm}$ in (17) in such a way that

$$\Omega_f^+\Omega_f^- = \langle f, f\rangle, \quad \varphi_f^+ \in H^1(\Gamma_0(p), K_f), \quad \varphi_f^- \in H^1(\Gamma_0(p), \mathcal{O}_f).$$

The Manin–Drinfeld theorem implies that the quantity

$$L_{\mathrm{alg}}(f,1) := (\Omega_f^+)^{-1}\int_0^\infty \omega_f$$

belongs to the field K_f, and in particular is algebraic. It is a multiple of the special value $L(f,1)$ by a simple non-zero factor, and can therefore be envisaged as its "algebraic part".

The third main result, discussed in Sect. 3, is readily deduced from Theorems A and B, is

Theorem C *The classical forms $G_1(\psi)$ and $G_1'(\psi)_{\mathrm{ord}}$ obtained in the coherent and incoherent cases respectively may be written as a combination of newforms as follows:*

1. (*Coherent case*). *If p is split in $F/\mathbb{Q}$, then we have*

$$G_1(\psi) \;=\; \lambda_0 \cdot E_2^{(p)} \;+\; \sum_f \lambda_f \cdot f,$$

where f runs over the basis of normalised newforms in $S_2(\Gamma_0(p))$, and

$$\lambda_0 = \frac{-2}{p-1} \cdot \varphi_{\mathrm{DR}}(g_\psi), \quad \lambda_f = -2L_{\mathrm{alg}}(f,1) \cdot \varphi_f^-(g_\psi).$$

2. (*Incoherent case*). *If p is inert in $F/\mathbb{Q}$, then we have*

$$G_1'(\psi)_{\mathrm{ord}} = \lambda_0' \cdot E_2^{(p)} + \sum_f \lambda_f' \cdot f,$$

where the coefficients λ_0' and λ_f' are given by

$$\lambda_0' = \frac{-4}{p-1} \cdot \log_p\left(\mathrm{Nm}(J_{\mathrm{DR}}[\Delta_\psi])\right), \quad \lambda_f' = -4L_{\mathrm{alg}}(f,1) \cdot \log_p\left(\mathrm{Nm}(J_f^-[\Delta_\psi])\right).$$

Table 1 illustrates Theorem C for $p = 11$ and ψ ranging over some odd unramified characters of real quadratic fields. We consider all genus characters of discriminant $D < 100$, corresponding to factorisations $D = D_1 D_2$ of D into a product of two negative fundamental discriminants. The space $M_2(\Gamma_0(11))$ is spanned by the Eisenstein series $E_2^{(11)}$ and the newform f attached to the Weil curve

$$E \;:\; y^2 + y = x^3 - x^2 - 10x - 20$$

of conductor 11, which has rank zero over $\mathbb{Q}$, and $L_{\mathrm{alg}}(f,1) = 1/5$. In the coherent case, the coefficients λ_0 and λ_f of Part 1 of Theorem C are rational numbers, and it was checked that $\lambda_0 = \varphi_{\mathrm{DR}}(g_\psi)$, as claimed. In the incoherent case, it was checked, to 50 significant 11-adic digits, that

(1) the coefficient λ_0' agrees with a rational multiple of the 11-adic logarithm of a global 11-unit in the biquadratic field $\mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$. More precisely, this unit belongs to $\mathbb{Q}(\sqrt{D_1})$, where (D_1, D_2) are ordered in such a way that $(\frac{D_1}{p}) = -(\frac{D_2}{p}) = 1$.

(2) the coefficient λ_f' agrees with a small rational multiple of the formal group logarithm of a global point in $E(\mathbb{Q}(\sqrt{D_1}))$. This is consistent with a theorem of Mok [24] extending the main result of [1] to elliptic curves of prime conductor, which

Table 1 The spectral expansions of $G_1(\psi)$ and $G_1'(\psi)_{\mathrm{ord}}$.

D	$D_1 \cdot D_2$	$\left(\frac{D}{11}\right)$	λ_0	λ_f	λ_0'	λ_f'
12	$(-3)(-4)$	1	$\frac{8}{5}$	$-\frac{8}{5}$		
21	$(-7)(-3)$	-1	0	0	$\frac{16}{5}\log\left(\frac{2+\sqrt{-7}}{11}\right)$	$\frac{8}{5}\log_E\left(\frac{1-\sqrt{-7}}{2}, 1+2\sqrt{-7}\right)$
24	$(-8)(-3)$	-1	0	0	$\frac{16}{5}\log\left(\frac{3+\sqrt{-2}}{11}\right)$	$-\frac{8}{5}\log_E\left(-3-\sqrt{-2}, -4-3\sqrt{-2}\right)$
28	$(-7)(-4)$	-1	0	0	$\frac{24}{5}\log\left(\frac{2+\sqrt{-7}}{11}\right)$	$-\frac{8}{5}\log_E\left(\frac{1-\sqrt{-7}}{2}, 1+2\sqrt{-7}\right)$
56	$(-8)(-7)$	1	0	0	0	0
57	$(-19)(-3)$	-1	0	0	$\frac{16}{5}\log\left(\frac{5+\sqrt{-19}}{22}\right)$	$-\frac{8}{5}\log_E\left(-7+2\sqrt{-19}, -38-2\sqrt{-19}\right)$
69	$(-23)(-3)$	1	$\frac{48}{5}$	$-\frac{8}{5}$		
76	$(-19)(-4)$	-1	0	0	$\frac{24}{5}\log\left(\frac{5+\sqrt{-19}}{22}\right)$	$\frac{8}{5}\log_E\left(-7+2\sqrt{-19}, -38-2\sqrt{-19}\right)$
93	$(-3)(-31)$	1	$\frac{48}{5}$	$-\frac{8}{5}$		

implies that the quantities $J_f^-[\Delta_\psi]$ map to a global point in $E(\mathbb{Q}(\sqrt{D_1})) \otimes \mathbb{Q}$ under Tate's p-adic uniformisation when ψ is a genus character.

Weight one Eisenstein series attached to odd genus characters also play a prominent role in the calculations of [17]. That theorems B and C are not a direct p-adic counterpart of the formulae in loc.cit. is suggested by the fact that they apply to arbitrary (odd) class characters, and not just genus characters. This feature, which accounts for the relevance of Theorem C to explicit class field theory for real quadratic fields, is illustrated in Sect. 3.6, where a numerical illustration is offered in its support.

Remark 1 Part 1 of Theorem C is essentially equation (1.4) of [22] with the genus character replaced by a general odd ideal class character of F, while Part 2 can be viewed as a p-adic "incoherent" counterpart of this result.

Remark 2 When the prime p is split in F, comparing the constant terms for $G_1(\psi)$ given in Theorem A and in Part 1 of Theorem C, we obtain

$$L_p(F, \psi, 0) = \frac{1}{12}\varphi_{\mathrm{DR}}(g_\psi),$$

a classical result that follows from Meyer's formula [30, §4] for the value at $s = 0$ of the L-function of a totally odd ring class character of a real quadratic field in terms of the Dedekind–Rademacher homomorphism. When p is inert in F, comparing the constant terms for $G_1'(\psi)_{\mathrm{ord}}$ given in Theorem B and in Part 2 of Theorem C leads to the identity

$$L_p'(F, \psi, 0) = \frac{1}{12}\log_p(\mathrm{Nm}\, J_{\mathrm{DR}}[\Delta_\psi]),$$

which essentially recovers one of the main theorems of [5]. The proof of the p-adic Gross–Stark conjecture given in [12] shows that $L_p'(F, \psi, 0)$ is a rational multiple of the p-adic logarithm of the norm to $\mathbb{Q}_p$ of a global p-unit—the *Gross–Stark unit* attached to ψ—and leads to theoretical evidence for the algebraicity of the RM values of the Dedekind–Rademacher cocycle. In a forthcoming work [8], the authors will parlay the infinitesimal deformations of $E_k(1, \psi)$ in the *anti-parallel direction* and their diagonal restrictions into a proof of the algebraicity of the *full* invariant $J_{\mathrm{DR}}[\Delta_\psi]$. This gives a new proof of one of the main results of [13], in the setting of real quadratic fields. It is worth noting that the results in [13] apply to general totally real fields, whereas the connection with the theory of rigid cocycles is at present restricted to the quadratic case.

Remark 3 The coefficients λ'_f that occur in the spectral expansion of $G'_1(\psi)_{\mathrm{ord}}$ can be viewed as the first derivatives of certain twisted Rankin p-adic L-functions attached to f and to the diagonal restriction of a family of Hilbert modular Eisenstein series. These quantities can be likened to the "p-adic iterated integrals" of [6] arising from a pair of weight one cusp forms, by viewing such a pair as a "Hilbert modular form of weight (1, 1) for the split quadratic algebra $\mathbb{Q} \times \mathbb{Q}$". The connection between the products of logarithms of pairs of Stark–Heegner points and the *second derivatives* of Rankin triple product L-functions has already been exploited, notably in [9] and [2]. The simpler connection with the *first derivatives* of their twisted variants that is revealed by Theorem C offers the prospect of a more direct geometric approach to Stark–Heegner points via the K-theory of Hilbert modular surfaces, which it would be interesting to flesh out.

1 Diagonal restrictions of Hilbert Eisenstein series

The modular form $G_k(\psi)$ of weight $2k$ on $\Gamma_0(p)$ described in (5) of the introduction has Fourier expansion given by

$$G_k(\psi) := L^{(p)}(F, \psi, 1-k) + 4\sum_{n=1}^{\infty}\left(\sum_{\substack{\nu\in\mathfrak{d}_+^{-1} \\ \mathrm{Tr}(\nu)=n}}\ \sum_{\substack{I\,|\,(\nu)\mathfrak{d}, \\ p\nmid I}} \psi(I)\mathrm{Nm}(I)^{k-1}\right) q^n. \quad (20)$$

The goal of this section is to investigate its weight 2 specialisation and prove Theorem A from the introduction. When p splits in F, which by analogy with common nomenclature in the Kudla programme is referred to as the *coherent* case, the weight two specialisation $G_1(\psi)$ is the generating series for the homological intersection product of certain geodesics, while it vanishes identically when p is inert in F. The latter *incoherent* setting is arithmetically richer: the first derivative $G'_1(\psi)$ that is the object of Theorem B is studied in Sect. 2.

Notation Retaining some of the notations and assumptions of the introduction, F will denote a real quadratic field with discriminant D, ring of integers $\mathcal{O}_F$, set of integral ideals $\mathscr{I}_F$, and different ideal $\mathfrak{d}$. The notation $\mathfrak{d}_+^{-1}$ means the subset of totally positive elements of the inverse different $\mathfrak{d}^{-1}$. Denote by Nm and Tr the norm and trace maps from F to $\mathbb{Q}$.

1.1 The weight two specialisation $G_1(\psi)$.

The weight two specialisation of the family considered above plays a central role. The following lemma is well-known to experts.

Lemma 1.1 *The specialisation $G_1(\psi)$ is a classical modular form of weight two and level $\Gamma_0(p)$. Its constant term is given by*

$$L_p(F,\psi,0) = L(F,\psi,0)\cdot\prod_{\mathfrak{p}|p}(1-\psi(\mathfrak{p})). \tag{21}$$

In particular, the constant term vanishes if p is inert in F.

Proof By [12, Prop. 3.2], the weight one specialisation of the family $E_k^{(p)}(1,\psi)$ is a classical Hilbert modular form of parallel weight one and level

$$\Gamma_0(p\mathcal{O}_F) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathcal{O}_F)\ \mid\ c\in p\mathcal{O}_F,\right\}.$$

Thus, its diagonal restriction is a classical elliptic modular form of weight two and level $\Gamma_0(p)$. The formula for the constant coefficient follows from the fact that $L_p(F,\psi,s)$ interpolates the values of $L^{(p)}(F,\psi,n)$ for every $n\in\mathbb{Z}_{\le 0}$. In particular, if p is inert in F, the p-adic L-function admits a trivial zero at $s=0$, since the conductor of ψ is trivial. □

Remark 1.2 The weight one specialisation of the Eisenstein family can in fact be obtained by p-stabilising the Eisenstein series of level one with Fourier expansion

$$E_1(1,\psi)(z_1,z_2) = 4\sum_{\nu\in\mathfrak{d}_+^{-1}}\sigma_{0,\psi}(\nu\mathfrak{d})\exp(2\pi i(\nu_1 z_1+\nu_2 z_2))$$

in the notation of (3). However, the constant term of the p-stabilisation $E_1^{(p)}(1,\psi)$ may not vanish in the coherent case, due to the contribution of non-zero constant terms at other cusps. For more on the constant terms at various cusps, see Shih [27] and Dasgupta–Kakde [14].

1.2 Ideals and RM points

The Fourier coefficient a_n of the diagonal restriction (20) may be written as

$$a_n = 4\sum_{\mathcal{C}\in\mathrm{Cl}^+(D)}\psi(\mathcal{C})\sum_{(I,\nu)\,\in\,\mathbb{I}(n,\mathcal{C})-\mathbb{I}(n,\mathcal{C})_p}\mathrm{Nm}(I)^{k-1} \tag{22}$$

where the index sets are given by

$$\begin{aligned}\mathbb{I}(n,\mathcal{C}) &:= \left\{(I,\nu)\in\mathscr{I}_F\times\mathfrak{d}_+^{-1}\ :\ \mathrm{Tr}(\nu)=n,\quad I\mid(\nu)\mathfrak{d},\quad [I]=\mathcal{C}\right\},\\ \mathbb{I}(n,\mathcal{C})_p &:= \{(I,\nu)\in\mathbb{I}(n,\mathcal{C})\ :\ p\mid\mathrm{Nm}(I)\}.\end{aligned}$$

The finite index sets $\mathbb{I}(n, \mathcal{C})$ and $\mathbb{I}(n, \mathcal{C})_p$ will be placed in an explicit bijection with certain sets of RM points. To ease the exposition, the case $n = 1$, where this set of RM points may be described in a particularly simple way, is treated separately. The calculations for the case $n > 1$ are more involved, and are dealt with in the remainder of the section.

An RM point is defined to be a real quadratic irrationality, and is said to be of discriminant D if it is the root of a primitive binary quadratic form of discriminant D. The set of RM points of discriminant D, denoted $\mathbb{RM}(D)$, is preserved by the action of $\mathrm{SL}_2(\mathbb{Z})$.

Extending the definition in (10), write $G[\tau] \subset G$ for the stabiliser of τ in G, for G any congruence subgroup of Γ and τ an element of $\mathcal{H}_p$. If τ is an RM point, then $G[\tau]$ is always of rank one, i.e., it is of the form $G[\tau] = \pm\langle\gamma_\tau\rangle$ for the generator γ_τ that is uniquely determined by the property that τ is it *stable* fixed point in the sense of (11). As in the introduction, the (open) hyperbolic geodesic in $\mathcal{H}$ between two RM points τ_1, τ_2 will be denoted by (τ_1, τ_2), whereas the (closed) hyperbolic geodesic segment between two points ξ_1, ξ_2 of the extended upper-half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}_1(\mathbb{Q})$ is denoted by the symbol $[\xi_1, \xi_2]$. The intersection number between two geodesics in $\mathcal{H}$, which is always ± 1 or 0, is defined in the natural way after fixing a standard orientation on $\mathcal{H}$, and is denoted by the symbol "$\cdot$" as above.

Ideals and RM points are related by the canonical bijection

$$\begin{aligned} \mathrm{Cl}^+(D) &\longrightarrow \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{RM}(D), \\ [I] &\longmapsto \omega_1/\omega_2, \end{aligned} \tag{23}$$

where (ω_1, ω_2) is any positive $\mathbb{Z}$-basis of I, i.e. a basis satisfying $\omega_1\omega_2' - \omega_1'\omega_2 > 0$. This is well-defined, and defines a bijection with inverse given by:

$$\mathrm{cl} : \mathbb{RM}(D) \longrightarrow \mathrm{Cl}^+(D), \qquad \mathrm{cl}(\tau) = \begin{cases} \mathbb{Z}\tau \oplus \mathbb{Z} & \text{if } \tau - \tau' > 0, \\ \sqrt{D}(\mathbb{Z}\tau \oplus \mathbb{Z}) & \text{if } \tau - \tau' < 0, \end{cases}$$

which is constant on $\mathrm{SL}_2(\mathbb{Z})$-orbits. Given a narrow ideal class $\mathcal{C}$ in $\mathrm{Cl}^+(D)$, let

$$\mathbb{RM}(\mathcal{C}) := \mathrm{cl}^{-1}(\mathcal{C}) = \mathrm{SL}_2(\mathbb{Z})\tau,$$

where τ is any preimage of $\mathcal{C}$. The group $\mathrm{GL}_2(\mathbb{Z})$ acts transitively on $\mathbb{RM}(\mathcal{C}) \cup \mathbb{RM}(\mathcal{C}^*)$, and any matrix of determinant -1 interchanges $\mathbb{RM}(\mathcal{C})$ and $\mathbb{RM}(\mathcal{C}^*)$. In particular

$$\mathrm{cl}(-\tau) = \mathrm{cl}(\tau)^*. \tag{24}$$

Definition 1.3 An RM point τ is said to be *reduced*[2] if $\tau\tau' < 0$. A reduced RM point is called *positive* if $\tau' < 0 < \tau$, and *negative* if $\tau < 0 < \tau'$. Denote by $\mathbb{RM}_+$ and $\mathbb{RM}_-$ the sets of positive and negative (reduced) RM points, and set

$$\mathbb{RM}_\pm(D) = \mathbb{RM}(D) \cap \mathbb{RM}_\pm, \quad \mathbb{RM}_\pm(\mathcal{C}) = \mathbb{RM}(\mathcal{C}) \cap \mathbb{RM}_\pm.$$

[2] Note that this differs from the notion of reducedness defined by Gauß in his *Disquisitiones Arithmeticae* [15]. Reduced forms in his sense are always reduced in our sense, but the converse is not true.

Lemma 1.4 *The sets* $\mathbb{RM}_\pm(\mathcal{C})$ *are finite. The assignment* $\tau \mapsto -\tau$ *induces a bijection from* $\mathbb{RM}_+(\mathcal{C}^*)$ *to* $\mathbb{RM}_-(\mathcal{C})$.

Proof Any τ in $\mathbb{RM}_\pm(D)$ is the root of a primitive binary quadratic form $Ax^2+Bxy+Cy^2$ of discriminant D in which $AC < 0$. There are finitely many such forms, so the first assertion follows. The second follows from (24) given that $\tau \mapsto -\tau$ interchanges $\mathbb{RM}_+(D)$ and $\mathbb{RM}_-(D)$. □

Definition 1.5 Let p be a prime that does not divide D. An RM point in $\mathbb{RM}(D)$ is said to be *Heegner* (relative to p) if $p\tau$ also lies in $\mathbb{RM}(D)$. Equivalently, τ is a Heegner RM point if it is the root of a binary quadratic form of discriminant D that is Heegner at p in the sense of the introduction. The set of Heegner RM points in $\mathbb{RM}(\mathcal{C})$ is denoted $\mathbb{RM}(\mathcal{C})_p$, and likewise $\mathbb{RM}_\pm(\mathcal{C})_p$ denotes the set of Heegner RM points in $\mathbb{RM}_\pm(\mathcal{C})$.

Note that if p is inert in F, then $\mathbb{RM}(D)_p$ is empty. If p splits in F, it is nonempty and stable under the action of $\Gamma_0(p)$, with two distinct orbits, as described in the introduction.

Lemma 1.6 *The sets* $\mathbb{I}(1, \mathcal{C})$ *and* $\mathbb{RM}_+(\mathcal{C})$ *are in bijection via the map*

$$(I, \nu) \longmapsto \frac{\nu\sqrt{D}}{\mathrm{Nm}(I)}.$$

Proof Any totally positive $\nu \in \mathfrak{d}^{-1}$ of trace 1 can be uniquely expressed as

$$\nu = \frac{-b+\sqrt{D}}{2\sqrt{D}}, \quad \text{where} \quad b \in \mathbb{Z}, \quad b^2 - D < 0, \quad b \equiv D \pmod 2.$$

Given an ideal $I \mid (\nu)\mathfrak{d}$, its norm $a := \mathrm{Nm}(I)$ is a positive divisor of the negative integer $\mathrm{Nm}(\nu\sqrt{D}) = (b^2 - D)/4$, hence its quotient by a is equal to a negative integer c. Define

$$\tau := \frac{-b+\sqrt{D}}{2a}$$

which is a root of $ax^2 + bxy + cy^2$ and therefore contained in $\mathbb{RM}_+(D)$. If (I, ν) belongs to $\mathbb{I}(1, \mathcal{C})$ it is readily checked that τ belongs to $\mathbb{RM}_+(\mathcal{C}) \subset \mathbb{RM}_+(D)$.

Conversely, if $\tau = (-b + \sqrt{D})/2a$ belongs to $\mathbb{RM}_+(\mathcal{C})$, then an element (I, ν) of $\mathbb{I}(1, \mathcal{C})$ in the preimage of τ can be constructed by setting

$$I := (a, \nu\sqrt{D}), \quad \nu := \frac{-b+\sqrt{D}}{2\sqrt{D}}.$$

□

The bijection in Lemma 1.6 will now be extended to general n. Due to the non-constancy of discriminants in the set of RM points corresponding to $\mathbb{I}(n, \mathcal{C})$, greater care becomes necessary in introducing notation for certain double cosets that feature in the target of the desired bijection.

As before, let $\tau \in \mathbb{RM}(\mathcal{C})$. Let

$$\mathrm{Mat}^{(n)}_{2\times 2}(\mathbb{Z}) := \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \ : \ \det(A) = n\} \tag{25}$$

and let M_n and $M_n(\tau)$ be a system of representatives for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{Mat}^{(n)}_{2\times 2}(\mathbb{Z})$ and for the double coset space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{Mat}^{(n)}_{2\times 2}(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})[\tau]$ respectively, where as above we write $\mathrm{SL}_2(\mathbb{Z})[\tau]$ for the stabiliser of τ in $\mathrm{SL}_2(\mathbb{Z})$, a group of rank 1. In other words,

$$\mathrm{Mat}^{(n)}_{2\times 2}(\mathbb{Z}) = \bigsqcup_{\gamma_n \in M_n} \mathrm{SL}_2(\mathbb{Z}) \cdot \gamma_n \tag{26}$$

$$= \bigsqcup_{\delta_n \in M_n(\tau)} \mathrm{SL}_2(\mathbb{Z}) \cdot \delta_n \cdot \mathrm{SL}_2(\mathbb{Z})[\tau]. \tag{27}$$

It will be convenient to choose the standard set of representatives of the Hecke operator T_n

$$M_n := \left\{ \begin{pmatrix} n/d & j \\ 0 & d \end{pmatrix} : \quad d|n, \quad 0 \le j \le d-1 \right\}, \tag{28}$$

and to assume without loss of generality that $M_n(\tau)$ is contained in M_n.

Definition 1.7 For any choice of sign $\pm$ define the set

$$\mathbb{RM}_\pm(n, \mathcal{C}) := \{(w, \delta_n) \in \mathbb{RM}_\pm \times M_n(\tau) \ : w \in \mathrm{SL}_2(\mathbb{Z})\delta_n\tau\} \tag{29}$$

Let $w \in \mathbb{RM}_\pm(n, \mathcal{C})$, and let $ax^2 + bxy + cy^2$ be the unique quadratic form[3] of discriminant n^2D which has w as its stable root. Using the first coefficient of this quadratic form gives a well-defined map

$$\mathbb{RM}_\pm(n, \mathcal{C}) \longrightarrow \mathbb{Z} \ : \ w \mapsto a(w) := a. \tag{30}$$

Write $\mathbb{RM}_\pm(n, \mathcal{C})_p$ for the subsets of those (w, δ_n) for which $p \mid a(w)$. Note that the sets $\mathbb{RM}_\pm(\mathcal{C})$ defined in Sect. 1.2 are canonically identified with $\mathbb{RM}_\pm(1, \mathcal{C})$.

The following is a generalisation of Lemma 1.4 for all $n \ge 1$:

Lemma 1.8 *The sets $\mathbb{RM}_\pm(n, \mathcal{C})$ are finite. The map*

$$\mathbb{RM}_+(n, \mathcal{C}^*) \ \longrightarrow \ \mathbb{RM}_-(n, \mathcal{C}) \ : \ (w, \delta_n) \ \longmapsto \ (-w, \delta_n^*),$$

is a bijection, where δ_n^* is the representative of the conjugate of δ_n by the matrix

$$W_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[3] Note that this form may fail to be primitive.

Proof The finiteness of $\mathbb{RM}_{\pm}(n, \mathcal{C})$ follows from the fact that the discriminant of $\delta_n\tau$ divides n^2D, and that the set of reduced RM points of a fixed discriminant is finite. The second statement follows easily from the observation that $-w = W_\infty w$. □

The following more general version of Lemma 1.6 establishes a bijection between the index set appearing in the expression (20) for the Fourier coefficient a_n of the diagonal restriction, and an explicit set of "augmented RM points" of the above form.

Lemma 1.9 *There exists a bijection*

$$\mathbb{I}(n, \mathcal{C}) \longrightarrow \mathbb{RM}_{+}(n, \mathcal{C})$$

such that if (I, ν) *corresponds to* (w, δ_n)*, then* $\mathrm{Nm}(I) = a(w)$.

Proof Define the ideal $\mathfrak{a} = (A, (-B + \sqrt{D})/2)$ where $Ax^2 + Bxy + Cy^2$ is the unique quadratic form of discriminant D whose stable root is τ, whose narrow ideal class is $\mathcal{C}$.

First, let $(I, \nu) \in \mathbb{I}(n, \mathcal{C})$ and define a triple of integers a, b, c by

$$\begin{cases} a = & \mathrm{Nm}(I) \\ b = & \text{unique integer such that } \nu = \frac{-b+n\sqrt{D}}{2\sqrt{D}} \\ c = & -\mathrm{Nm}(J), \text{ where } IJ = (\nu)\mathfrak{d}. \end{cases}$$

One readily sees that $c < 0 < a$ and $b^2 - 4ac = n^2D$. Now define

$$w = \frac{-b + n\sqrt{D}}{2a} \in \mathbb{RM}_{+}.$$

If $(\cdot)'$ is the non-trivial automorphism of F, then I' is in the narrow ideal class of $\mathfrak{a}^{-1}$. Define λ to be a totally positive generator of the principal ideal $I'\mathfrak{a}$. Then the lattice

$$\Lambda = \mathbb{Z}\lambda + \mathbb{Z}w\lambda$$

is well defined up to multiplication by a totally positive unit in $\mathcal{O}_F^\times$. We claim that Λ is a lattice in $\mathfrak{a}$ of index n. Indeed, λ belongs to $\mathfrak{a}$, and on the other hand $w\lambda$ also belongs to $\mathfrak{a}$ since

$$\begin{aligned}(w\lambda) &= (\nu\sqrt{D}/\mathrm{Nm}(I))I'\mathfrak{a} \\ &= J\mathfrak{a}.\end{aligned}$$

The resulting containment $\Lambda \subseteq \mathfrak{a}$ is of index n, since the quadratic form

$$\mathrm{Nm}(\lambda x - \mu y)/\mathrm{Nm}(\mathfrak{a})$$

is equal to $ax^2 + bxy + cy^2$, which is of discriminant $n^2 D$. Therefore

$$\begin{pmatrix} \lambda w \\ \lambda \end{pmatrix} = N \begin{pmatrix} A\tau \\ A \end{pmatrix}, \quad \det N = n,$$

and hence there is a unique $\delta_n \in M_n(\tau)$ such that

$$N \in \mathrm{SL}_2(\mathbb{Z}) \cdot \delta_n \cdot \mathrm{SL}_2(\mathbb{Z})[\tau].$$

Note that δ_n is well-defined, since multiplication of λ by a totally positive unit in $\mathcal{O}_F^\times$ changes N by right multiplication by an element of $\mathrm{SL}_2(\mathbb{Z})[\tau]$, and hence does not change δ_n. Since w belongs to $\mathrm{SL}_2(\mathbb{Z})\delta_n\tau$, it follows that w lies in $\mathbb{RM}_+(n, \mathcal{C})$.

To check that this defines a bijection, we construct an explicit inverse. For an element $(w, \delta_n) \in \mathbb{RM}_+(n, \mathcal{C})$, let $ax^2 + bxy + cy^2$ be the unique quadratic form of discriminant $n^2 D$ whose stable root is w. Define

$$\nu = \frac{-b + n\sqrt{D}}{2\sqrt{D}},$$

which is a totally positive element of $\mathfrak{d}^{-1}$ of trace n. Write $w = \gamma\delta_n\tau$, and define λ by

$$\begin{pmatrix} \lambda w \\ \lambda \end{pmatrix} = \gamma\delta_n \begin{pmatrix} A\tau \\ A \end{pmatrix}.$$

Note that $\gamma\delta_n$ is only well-defined up to left multiplication by elements in $\mathrm{SL}_2(\mathbb{Z})[w]$, and up to right multiplication by elements in $\mathrm{SL}_2(\mathbb{Z})[\tau]$, which makes λ well-defined up to totally positive units. This makes the integral ideals

$$I = (\lambda')/\mathfrak{a}', \quad J = (\lambda w)\mathfrak{a}^{-1}$$

well-defined, and one checks directly that $IJ = (\nu)\mathfrak{d}$. Therefore (I, ν) belongs to $\mathbb{I}(n, \mathcal{C})$, and this assignment defines the desired inverse to the map defined above. □

1.3 An unfolding lemma for geodesics

This section presents an unfolding identity between certain sums of intersection numbers of geodesics that will appear multiple times in subsequent calculations. The results below are stated for a congruence subgroup $\Gamma_0(N)$ for a general N, but only the cases $N = 1$ or $N = p$ will be used.

Lemma 1.10 *Suppose τ is an RM point and let γ_τ be the normalised generator for the stabiliser subgroup $\Gamma_0(N)[\tau]$ modulo torsion. Let $n \geq 1$ be an integer that is relatively* *prime to N. Then for any point $\eta \in \mathcal{H}$,*

$$\sum_{\substack{\delta_n \in M_n(\tau) \\ \gamma \in \Gamma_0(N)/\Gamma_0(N)[\delta_n\tau]}} [0,\infty]\cdot(\gamma\delta_n\tau', \gamma\delta_n\tau) = \frac{1}{2}\sum_{\substack{\gamma_n \in M_n \\ \gamma \in \Gamma_0(N)}} [0,\infty]\cdot[\gamma\gamma_n\eta,\ \gamma\gamma_n\gamma_\tau\eta]. \tag{31}$$

Proof For any RM point ρ, let γ_ρ be its automorph in $\Gamma_0(N)$, i.e. the generator of $\Gamma_0(N)[\rho]$ whose stable fixed point is ρ. Equation (11) implies that, for any $\xi \in \mathcal{H}$,

$$[0,\infty]\cdot(\gamma\rho', \gamma\rho) = \sum_{j=-\infty}^{\infty} [0,\infty]\cdot[\gamma\gamma_\rho^j\xi, \gamma\gamma_\rho^{j+1}\xi].$$

Setting $\rho = \delta_n\tau$, this allows us to unfold the left hand side of (31) into the expression

$$\frac{1}{2}\sum_{\delta_n \in M_n(\tau)}\ \sum_{\gamma \in \Gamma_0(N)} [0,\infty]\cdot[\gamma\xi, \gamma\gamma_\rho\xi] \tag{32}$$

where the factor $1/2$ accounts for the torsion subgroup $\pm I$ of $\Gamma_0(N)[\delta_n\tau]$. Now note that $\gamma_\rho = \delta_n\gamma_\tau^f\delta_n^{-1}$ for some $f \geq 1$. Setting $\eta = \delta_n^{-1}\xi$ we can rewrite

$$\begin{aligned} [0,\infty]\cdot[\gamma\xi, \gamma\gamma_\rho\xi] &= [0,\infty]\cdot \quad [\gamma\delta_n\eta,\ \gamma\delta_n\gamma_\tau^f\eta] \\ &= [0,\infty]\cdot\big([\gamma\delta_n\eta,\ \gamma\delta_n\gamma_\tau\eta] \\ &\qquad + \ldots + [\gamma(\delta_n\gamma_\tau^{f-1})\eta,\ \gamma(\delta_n\gamma_\tau^{f-1})\gamma_\tau\eta]\big) \end{aligned}$$

Note that the sum on the right hand side of (31) can be identified with the intersection product in homology of two homology classes on the open modular curve of level N, the first being the geodesic between the cusps 0 and ∞ viewed as a class in the homology of $X_0(N)$ relative to the cusps, and the second being the T_n-translate of the geodesic joining the images of η and $\gamma_\tau\eta$, whose class in $H_1(Y_0(N), \mathbb{Z})$ is independent of η. Since

$$\bigsqcup_{\delta_n \in M_n(\tau)} \Gamma_0(N)\cdot\delta_n\cdot\Gamma_0(N)[\tau] = \bigsqcup_{\delta_n \in M_n(\tau)}\ \bigsqcup_{i=0}^{f-1} \Gamma_0(N)\cdot\delta_n\gamma_\tau^i,$$

the elements $\delta_n\gamma_\tau^i$ form a complete set of coset representatives for the action of the Hecke operator T_n, so the sum (32) agrees with the right hand side of (31), as claimed. □

1.4 The Fourier expansion of $G_1(\psi)$

This section is devoted to the *coherent* case, where it is assumed that the prime p splits in F. Its goal is to prove Theorem A of the introduction, which asserts that the form $G_1(\psi)$ is the generating series for certain intersection products of the classes of geodesics in the homology of the modular curve $X_0(p)$.

Lemma 1.11 *For every $n \geq 1$, the sum $S := \sum_{\mathcal{C} \in \mathrm{Cl}^+(D)} \psi(\mathcal{C})|\mathbb{I}(n, \mathcal{C})|$ vanishes.*

Proof It suffices to show this when ψ is the odd indicator function of a narrow class $\mathcal{C}$, i.e.

$$\psi(I) = \begin{cases} 1 & \text{if } I \in \mathcal{C}, \\ -1 & \text{if } I \in \mathcal{C}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Combining Lemma 1.9 and Lemma 1.8 gives

$$\begin{aligned} S &= |\mathbb{RM}_+(n, \mathcal{C})| - |\mathbb{RM}_+(n, \mathcal{C}^*)| \\ &= |\mathbb{RM}_+(n, \mathcal{C})| - |\mathbb{RM}_-(n, \mathcal{C})|. \end{aligned}$$

Let τ be an RM point such that $\mathrm{cl}(\tau) = \mathcal{C}$. For any RM point ρ,

$$[0, \infty] \cdot (\rho, \rho') = \begin{cases} 1 & \text{if } \rho \in \mathbb{RM}_+, \\ -1 & \text{if } \rho \in \mathbb{RM}_-, \\ 0 & \text{otherwise} \end{cases}$$

by definition, where the oriented intersections are taking place on $\mathcal{H}$. The set S can therefore be rewritten as

$$\begin{aligned} S &= \sum_{(w, \delta_n) \in \mathbb{RM}_+(n, \mathcal{C})} 1 \quad - \sum_{(w, \delta_n) \in \mathbb{RM}_-(n, \mathcal{C})} 1 \\ &= \sum_{\substack{\delta_n \in M_n(\tau) \\ \gamma \in \mathrm{SL}_2(\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z})[\delta_n \tau]}} [0, \infty] \cdot (\gamma \delta_n \tau', \gamma \delta_n \tau). \end{aligned}$$

By Lemma 1.10,

$$S = \frac{1}{2} \sum_{\gamma_n \in M_n} \langle g_w, \gamma_n g_\psi \rangle = \frac{1}{2} \langle g_w, T_n g_\psi \rangle$$

where $\langle \cdot\,, \cdot \rangle$ denotes the intersection pairing between the homology of $X_0(1)$ relative to the cusps and the homology of the open modular curve $Y_0(1)$, and g_w and g_ψ are as defined in the introduction. Since these homology groups are trivial, it follows that $S = 0$ as claimed. □

We now come to Theorem A, asserting (in the coherent case) that $G_1(\psi)$ is a generating series for intersection products of geodesics on $X_0(p)$. The proof uses the previous lemma, and consists of a rearrangement of a sum over $\mathrm{SL}_2(\mathbb{Z})$ with an additional restriction to sums over $\Gamma_0(p)$ without any restriction, identifying the latter with the intersection number in homology.

Theorem 1.12 *If p splits in F, the Fourier expansion of $G_1(\psi)$ is given by*

$$G_1(\psi) = L_p(F, \psi, 0) - 2\sum_{n=1}^{\infty} \langle g_w, T_n g_\psi \rangle_p \, q^n.$$

Proof It suffices to show this when ψ is the odd indicator function of a narrow ideal class $\mathcal{C}$. Equation (22) and Lemma 1.9 imply the following explicit expression for the n-th Fourier coefficient of $G_1(\psi)$:

$$a_n = 4\,|\mathbb{RM}_+(n, \mathcal{C}) \setminus \mathbb{RM}_+(n, \mathcal{C})_p| - 4\,|\mathbb{RM}_+(n, \mathcal{C}^*) \setminus \mathbb{RM}_+(n, \mathcal{C}^*)_p|. \quad (33)$$

Using Lemmas 1.11 and 1.8, this may be rewritten as

$$\begin{aligned} a_n &= -4|\mathbb{RM}_+(n, \mathcal{C})_p| + 4|\mathbb{RM}_+(n, \mathcal{C}^*)_p| \\ &= -4|\mathbb{RM}_+(n, \mathcal{C})_p| + 4|\mathbb{RM}_-(n, \mathcal{C})_p| \end{aligned}$$

As in the proof of Lemma 1.11, this can be further rewritten as

$$\begin{aligned} a_n &= -4 \sum_{(w,\delta_n) \in \mathbb{RM}_+(n,\mathcal{C})_p} 1 + 4 \sum_{(w,\delta_n) \in \mathbb{RM}_-(n,\mathcal{C})_p} 1 \quad (34) \\ &= -4 \sum_{\delta_n \in M_n(\tau)} \sum_{\substack{w \in \mathrm{SL}_2(\mathbb{Z})\delta_n\tau \\ p \mid a(w)}} [0, \infty] \cdot (w', w), \quad (35) \end{aligned}$$

This is almost in the correct form for the unfolding of Lemma 1.10, except for the condition $p \mid a(w)$, which will be removed by passing from $\mathrm{SL}_2(\mathbb{Z})$-orbits to $\Gamma_0(p)$-orbits.

Suppose first that n is coprime to p. Fix an RM point $\tau \in \mathbb{RM}(\mathcal{C})$, and choose representatives for the two $\Gamma_0(p)$-orbits in $\mathbb{RM}(\mathcal{C})_p$

$$\begin{cases} \tau_s = A_s\tau, & A_s \in \mathrm{SL}_2(\mathbb{Z}) \\ \tau_{-s} = A_{-s}\tau, & A_{-s} \in \mathrm{SL}_2(\mathbb{Z}) \end{cases}$$

characterised by the property that all elements in the orbit are the stable roots of quadratic forms of discriminant D whose middle coefficient is congruent to s and $-s$ respectively, for a fixed choice s of square root of D modulo p. The set of Heegner forms $ax^2 + bxy + cy^2$ in an $\mathrm{SL}_2(\mathbb{Z})$-orbit of discriminant $n^2 D$ is likewise the disjoint union of two $\Gamma_0(p)$-orbits, distinguished by the congruences $b \equiv ns \pmod p$ and $b \equiv -ns \pmod p$. We will first identify two explicit representatives for these orbits.

Choose subsets $N_n^{(s)}$ and $N_n^{(-s)}$ of M_n such that

$$\mathrm{Mat}_{2\times 2}^{(n)}(\mathbb{Z}) = \bigsqcup_{\delta_n^{(s)} \in N_n^{(s)}} \mathrm{SL}_2(\mathbb{Z}) \cdot \delta_n^{(s)} \cdot \mathrm{SL}_2(\mathbb{Z})[\tau_s] \tag{36}$$

$$= \bigsqcup_{\delta_n^{(-s)} \in N_n^{(-s)}} \mathrm{SL}_2(\mathbb{Z}) \cdot \delta_n^{(-s)} \cdot \mathrm{SL}_2(\mathbb{Z})[\tau_{-s}] \tag{37}$$

For any $\delta_n \in M_n(\tau)$, the two matrices $\delta_n^{(s)} \in N_n^{(s)}$ and $\delta_n^{(-s)} \in N_n^{(-s)}$ are defined to be the double coset representatives of $A_s \delta_n A_s^{-1}$ and $A_{-s} \delta_n A_{-s}^{-1}$. Then

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) \cdot \delta_n^{(s)} \cdot \mathrm{SL}_2(\mathbb{Z})[\tau_s] &= \mathrm{SL}_2(\mathbb{Z}) \cdot A_s \delta_n A_s^{-1} \cdot \mathrm{SL}_2(\mathbb{Z})[\tau_s] \\ &= \mathrm{SL}_2(\mathbb{Z}) \cdot \delta_n \cdot \mathrm{SL}_2(\mathbb{Z})[\tau] \cdot A_s^{-1} \end{aligned}$$

and likewise for $\delta_n^{(-s)}$, from which one may conclude that the maps

$$\begin{cases} M_n(\tau) \longrightarrow N_n^{(s)} : \delta_n \longmapsto \delta_n^{(s)} \\ M_n(\tau) \longrightarrow N_n^{(-s)} : \delta_n \longmapsto \delta_n^{(-s)} \end{cases}$$

are bijections, and therefore that

$$\begin{aligned} \{w \in \mathrm{SL}_2(\mathbb{Z})\delta_n \tau\} &= \{w \in \mathrm{SL}_2(\mathbb{Z})\delta_n^{(s)} \tau_s\} \\ &= \{w \in \mathrm{SL}_2(\mathbb{Z})\delta_n^{(-s)} \tau_{-s}\} \end{aligned}$$

Now observe that the action of matrices in M_n on quadratic forms is via

$$(ax^2 + bxy + cy^2) \cdot \begin{pmatrix} d & j \\ 0 & n/d \end{pmatrix} = (ad^2)x^2 + (nb + 2dja)xy + (\cdots)\, y^2$$

Inspection of the first two coefficients reveals that for any $\delta_n \in M_n(\tau)$,

- $\delta_n^{(s)} \tau_s$ and $\delta_n^{(-s)} \tau_{-s}$ are the stable roots of Heegner forms, and are hence in $\mathbb{RM}(n, \mathcal{C})_p$.
- $\delta_n^{(s)} \tau_s$ and $\delta_n^{(-s)} \tau_{-s}$ are the stable roots of quadratic forms whose middle coefficients are respectively congruent to ns and $-ns$ modulo p, and hence are not equivalent under $\Gamma_0(p)$.

It follows from these two observations that

$$\{w \in \mathrm{SL}_2(\mathbb{Z})\delta_n \tau \;:\; p \mid a(w)\} \;=\; \Gamma_0(p)\delta_n^{(s)} \tau_s \;\bigsqcup\; \Gamma_0(p)\delta_n^{(-s)} \tau_{-s}$$

Equation (35) can now be rewritten as

$$a_n = -4 \sum_{\delta_n^{(s)} \in N_n^{(s)}} \sum_{\gamma \in \Gamma_0(p)/\Gamma_0(p)[\delta_n^{(s)}\tau_s]} [0,\infty] \cdot \left(\gamma\delta_n^{(s)}\tau_s',\ \gamma\delta_n^{(s)}\tau_s\right)$$
$$-4 \sum_{\delta_n^{(-s)} \in N_n^{(-s)}} \sum_{\gamma \in \Gamma_0(p)/\Gamma_0(p)[\delta_n^{(-s)}\tau_{-s}]} [0,\infty] \cdot \left(\gamma\delta_n^{(-s)}\tau_{-s}',\ \gamma\delta_n^{(-s)}\tau_{-s}\right). \tag{38}$$

It remains to show that both of these double sums are in the required form for Lemma 1.10, in the case $\Gamma_0(N) = \Gamma_0(p)$. Note that τ_s is the root of a Heegner form, so that

$$\mathrm{SL}_2(\mathbb{Z})[\tau_s] = \Gamma_0(p)[\tau_s],$$

and hence for any $\gamma_n, \gamma_n' \in M_n$ (which are upper triangular) and $M \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_n\gamma_{\tau_s}^i = M\gamma_n'$, it must be that M belongs to $\Gamma_0(p)$. It follows that

$$\bigsqcup_{\gamma_n \in M_n} \Gamma_0(p)\cdot\gamma_n \ = \bigsqcup_{\delta_n^{(s)} \in N_n^{(s)}} \Gamma_0(p)\cdot\delta_n^{(s)}\cdot\Gamma_0(p)[\tau_s]$$

and likewise for τ_{-s}. Since n is coprime to p, the left hand side is equal to the union of double cosets defining T_n for the congruence subgroup $\Gamma_0(p)$. Lemma 1.10 can now be applied for $\Gamma_0(p)$ in order to rewrite (38) as

$$a_n = -2 \sum_{\substack{\gamma_n \in M_n \\ \gamma \in \Gamma_0(p)}} \left([0,\infty]\cdot[\gamma\gamma_n\xi, \gamma\gamma_n\gamma_{\tau_s}\xi] + [0,\infty]\cdot[\gamma\gamma_n\xi, \gamma\gamma_n\gamma_{\tau_{-s}}\xi]\right).$$

This expression is equal to the topological intersection between g_w and $T_n g_\psi$ on the modular curve $X_0(p)$. This shows the proposition for the Fourier coefficients away from p.

Since the higher coefficients of the Fourier expansion on the right hand side is obtained from a linear function on the Hecke algebra for $\Gamma_0(p)$, they must agree with the Fourier expansion of some modular form f in $M_2(\Gamma_0(p))$. The difference of $G_1(\psi)$ and f has vanishing Fourier coefficients away from p and must therefore be an oldform. Since there are no non-trivial oldforms, the statement follows for all Fourier coefficients. □

Remark 1.13 Even though the above argument relies on the triviality of the space $M_2(\mathrm{SL}_2(\mathbb{Z}))$, we expect it to go through with minimal changes for more general congruence subgroups of Γ, where this triviality fails. Our reliance on this fact merely simplifies the argument.

We now complete the proof of Theorem A, by showing that, in the *incoherent* setting when p is inert in F, the weight 2 specialisation $G_1(\psi)$ of (20) vanishes identically.

Proposition 1.14 *If p is inert in F, the weight two specialisation $G_1(\psi)$ vanishes.*

Proof Let $n \geq 1$, and suppose (I, ν) is an element of $\mathbb{I}(n, \mathcal{C})$ such that $p \nmid \mathrm{Nm}(I)$. Let J be the minimal ideal that is coprime to (p) and such that IJ divides $(\nu)\mathfrak{d}$. The map

$$(I, \nu) \longmapsto (J', \nu')$$

defines an involution, and $\psi(J') = -\psi(I)$. Since ν and ν' both have trace n, it follows from the expression (22) that the n-th Fourier coefficient of $G_1(\psi)$ vanishes, and the proposition follows. □

Remark 1.15 Note that the proof of Proposition 1.14 only used the fact that $\psi((p)) = 1$, and shows for example also that the series $G_1(\psi)$ vanishes when $p = \mathfrak{p}\mathfrak{p}'$ and $\psi(\mathfrak{p}) = 1$ (see Sect. 3.6). When ψ is unramified and p is inert in F, one can alternatively observe that the operation of p-stabilisation commutes with the diagonal restriction, and therefore $G_1(\psi)$ is the p-stabilisation of a weight two modular form on $\mathrm{SL}_2(\mathbb{Z})$. The proposition then follows from the fact that there are no non-zero modular form of weight two and level one.

2 The incoherent Eisenstein series and its diagonal restriction

The goal of this section is to prove Theorem B of the introduction, showing that the overconvergent form $G_1'(\psi)_{\mathrm{ord}}$ discussed in the introduction is a generating series for the RM values of an appropriate rigid analytic theta cocycle. Assume for the remainder of this section that p is inert in F.

2.1 The overconvergence of $G_1'(\psi)$

Arguments similar to those of Buzzard–Calegari [3, §8] will be used to show that the first derivative $G_1'(\psi)$ is an overconvergent p-adic modular form of level 1. The following general lemma considers the first derivative of an "overconvergent family" at a point where it vanishes identically:

Lemma 2.1 *Suppose $\mathcal{G}(t)$ is a family of overconvergent forms of weight $\kappa(t)$, indexed by a parameter t on a closed rigid analytic disk D. If $\mathcal{G}(0) = 0$, and $k = \kappa(0) \in \mathbb{Z}$, then*

$$\left(\frac{\partial}{\partial t}\mathcal{G}(t)\right)\Big|_{t=0}$$

is an overconvergent modular form of weight k.

Proof Let E be the level one modular form

$$\mathrm{E} = \begin{cases} E_{p-1} & \text{if } p \geq 5 \\ E_6 & \text{if } p = 3 \\ E_4 & \text{if } p = 2 \end{cases} \tag{39}$$

where E_k is the unique level one Eisenstein series of weight k with constant term 1 at the cusp ∞. Since the weight of E is a multiple of $(p-1)$, and its q-expansion reduces to 1 modulo p, it must be a lift of a power of the Hasse invariant, and therefore $|\mathrm{E}-1|<1$ on a strict neighbourhood of the ordinary locus of $X_0(p)$. In particular, by shrinking D if necessary, there is a power series $e(t)$ such that $\mathrm{E}^{e(t)}$ converges for all $t \in D$ to an overconvergent form of weight $\kappa(t)\kappa(0)^{-1}$.

Let $A_k^\dagger$ be the space of overconvergent modular forms of weight k, and A_k^{ord} the space of p-adic modular forms of weight k, so that $A_k^\dagger \subset A_k^{\mathrm{ord}}$. The rigid analytic functions on the closed disk D and its boundary $B=\{t \in D : |t|=1\}$ are given by the Tate algebras

$$\mathbb{C}_p\langle t\rangle = \left\{\sum_{n\geq 0} c_n t^n \;:\; \lim_{n\to\infty} |c_n| = 0\right\} \quad \mathbb{C}_p\langle t, t^{-1}\rangle = \left\{\sum_{n\in\mathbb{Z}} c_n t^n \;:\; \lim_{|n|\to\infty} |c_n| = 0\right\}$$

Now consider the family

$$t^{-1}\cdot \mathcal{G}(t)/\mathrm{E}^{e(t)}.$$

Since B is an affinoid, this defines a family of overconvergent forms over B, and therefore an element of $A_k^\dagger \widehat{\otimes}\, \mathbb{C}_p\langle t, t^{-1}\rangle$. On the other hand, since $\mathcal{G}(0)=0$, its q-expansion is integral, and therefore it is an element of $A_k^{\mathrm{ord}} \widehat{\otimes}\, \mathbb{C}_p\langle t\rangle$. Since

$$A_k^{\mathrm{ord}} \widehat{\otimes}\, \mathbb{C}_p\langle t\rangle \bigcap A_k^\dagger \widehat{\otimes}\, \mathbb{C}_p\langle t, t^{-1}\rangle = A_k^\dagger \widehat{\otimes}\, \mathbb{C}_p\langle t\rangle. \tag{40}$$

it follows that it is a family of overconvergent forms of weight k. Multiplying out $\mathrm{E}^{e(t)}$, shows that $t^{-1}\mathcal{G}(t)$ is an overconvergent family over the disk D, so that in particular

$$\left(\frac{\partial}{\partial t}\mathcal{G}(t)\right)\bigg|_{t=0}$$

which is its value at $t=0$, is an overconvergent modular form, of weight k. □

Lemma 2.2 *The modular form $G_1'(\psi)$ is overconvergent.*

Proof Lemma 2.1 applies to the family $G_k(\psi)$, which is overconvergent because it is the diagonal restriction of the (overconvergent) Hilbert Eisenstein family. It follows from this lemma that $G_1'(\psi)$ is also overconvergent. □

Remark 2.3 For numerical computations, it is useful to quantify the rate of overconvergence of $G_1'(\psi)$. The ideas above can be refined to show that $G_1'(\psi)$ is r-overconvergent for any $r < p/(p+1)$. Since this finer result is not needed in this paper, its proof shall merely be sketched. The work of Goren–Kassaei [16, Theorem A] shows that the family $E_k(1,\psi)$ analytically continues to the *canonical region* $\mathcal{V}_{\mathrm{can}}$. The diagonal embedding on moduli stacks is given by $E/S \mapsto E\otimes_{\mathbb{Z}}\mathcal{O}_F \simeq E\times_S E$, endowed with the natural pieces of extra structure, and it can be checked directly that the valuations of the lifts of the partial Hasse invariants appearing in *loc. cit.* all coincide with the valuation of the lift of the Hasse invariant on E. It follows that the diagonal embedding induces an embedding $X_r \hookrightarrow \mathcal{V}_{\mathrm{can}}$ for any r with $|r| < p/(p+1)$, and hence the family of diagonal restrictions is r-overconvergent for any such r. By adapting the proofs of Lemma 2.1, one shows that $G_1'(\psi)$ inherits the same rate of overconvergence. See also Buzzard–Calegari [3, §8].

2.2 The Bruhat–Tits tree and the Drinfeld upper half-plane

We first establish some notation related to the Drinfeld upper half plane $\mathcal{H}_p$. Let $v_\circ$ be the standard vertex of the Bruhat–Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose stabiliser in Γ is $\mathrm{SL}_2(\mathbb{Z})$. For each integer $n \geq 0$, let $\mathcal{T}^{<n}$ and $\mathcal{T}^{\leq n}$ denote the subgraph of $\mathcal{T}$ consisting of vertices and edges that are at distance $< n$ and $\leq n$ from $v_\circ$, respectively. Let $\mathcal{H}_p^{<n} \subset \mathcal{H}_p^{\leq n} \subset \mathcal{H}_p$ denote the inverse images of $\mathcal{T}^{<n}$ and $\mathcal{T}^{\leq n}$ under the reduction map. The collection of $\mathcal{H}_p^{<n}$ and $\mathcal{H}_p^{\leq n}$ gives an admissible covering of $\mathcal{H}_p$ by wide open subsets and affinoid subsets respectively, which are stable under the action of $\mathrm{SL}_2(\mathbb{Z})$.

A pair $(x, y) \in \mathcal{O}_{\mathbb{C}_p}^2$ is said to be *primitive* if $\gcd(x, y) = 1$. Any $\tau \in \mathbb{P}_1(\mathbb{C}_p)$ can be written in projective coordinates as $\tau = (\tau_1 : \tau_2)$, where $(\tau_1, \tau_2) \in \mathcal{O}_{\mathbb{C}_p}^2$ is primitive. With this convention, the sets $\mathcal{H}_p^{<n}$ and $\mathcal{H}_p^{\leq n}$ can be described as

$$\begin{aligned} \mathcal{H}_p^{<n} &= \{(\tau_1 : \tau_2) \text{ such that } \operatorname{ord}_p(a\tau_1 - b\tau_2) < n, \text{ for all primitive } (a, b) \in \mathbb{Z}_p^2\}, \\ \mathcal{H}_p^{\leq n} &= \{(\tau_1 : \tau_2) \text{ such that } \operatorname{ord}_p(a\tau_1 - b\tau_2) \leq n, \text{ for all primitive } (a, b) \in \mathbb{Z}_p^2\}. \end{aligned} \tag{41}$$

2.3 The winding cocycle

We now define the *winding cocycle*, which gives a class

$$J_w \in H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times),$$

appearing in Theorem B. The cocycle J_w is obtained by taking suitable infinite products of *cross-ratios*. Recall that for any four points p_1, p_2, p_3, p_4 in $\mathbb{P}^1(\mathbb{C}_p)$, using the usual convention when some of the points are ∞, the cross-ratio is defined by

$$(p_1, p_2; p_3, p_4) := \frac{p_3 - p_1}{p_3 - p_2} \cdot \frac{p_4 - p_2}{p_4 - p_1}, \tag{42}$$

and is invariant under the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on all four points simultaneously.

The definition of J_w depends on a choice of *admissible* base points $\xi = (\xi_p, \xi_\infty) \in \mathcal{H}_p \times \mathcal{H}$, whose class in $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ will turn out to be independent of this choice (cf. Lemma 2.6). The pair $\xi = (\xi_p, \xi_\infty)$ is said to be *admissible* if:

- $\xi_\infty \in \mathcal{H}_\infty$ does not lie on any geodesic in the Γ-orbit of $[0, \infty]$,
- $\xi_p \in \mathcal{H}_p$ lies in the affinoid $\mathcal{H}_p^{\leq 0}$ of (41).

Since the non-admissible points have ξ_∞ contained in a countable union of sets of measure zero, the existence of admissible base points is apparent.

Remark 2.4 For computational purposes, it may be desirable to dispose of explicit choices for ξ_∞. For instance, let ξ be the root of a primitive integral binary quadratic form $[a, b, c]$ of discriminant $\Delta := b^2 - 4ac < 0$ for which

(1) the prime p is inert in the imaginary quadratic order of discriminant Δ,
(2) the class of $[a, b, c]$ is of order > 2 in the class group attached to Δ.

Letting $\xi_\infty \in \mathcal{H}_\infty$ and $\xi_p \in \mathcal{H}_p$ be the complex and p-adic root of the same binary quadratic form obtained by choosing embeddings of $\mathbb{Q}(\xi)$ in $\mathbb{C}_p$ and $\mathbb{C}$ respectively, it can be shown that the pair (ξ_p, ξ_∞) is admissible. Since no use will be made of this fact in this paper, its proof is omitted.

The *determinant* of a pair (r, s) of distinct elements of $\mathbb{P}_1(\mathbb{Q})$ is $ad - bc$, where $r = a/b$ and $s = c/d$ are expressions for r and s as fractions in lowest terms, adopting the usual convention that $\infty = 1/0$. It is an integer that is well-defined up to sign, hence shall always be normalised to be positive. If (r, s) and (r', s') are Γ-equivalent, then their determinants equal up to multiplication by a power of p. Let Σ denote the Γ-orbit of the pair $(0, \infty)$, and let $\Sigma^{(m)} \subset \Sigma$ be the subset of pairs (r, s) with $\mathrm{ord}_p(\det(r, s)) = m$. It is not hard to see that $\Sigma^{(m)}$ is non-empty for all $m \geq 0$ and that

$$\Sigma = \bigcup_{m=0}^{\infty} \Sigma^{(m)}. \tag{43}$$

Choose an admissible base point $\xi = (\xi_p, \xi_\infty)$, and define

$$J_w^\xi(\gamma)(z) = \prod_{(r,s) \in \Sigma} \left(r, s; \xi_p, z\right)^{[r,s] \cdot [\xi_\infty, \gamma\xi_\infty]}, \tag{44}$$

where the exponent $[r, s] \cdot [\xi_\infty, \gamma\xi_\infty]$ denotes the topological intersection of these two hyperbolic geodesic segments on the Poincaré upper half-plane.

Proposition 2.5 *For each $\gamma \in \Gamma$, the infinite product defining $J_w^\xi(\gamma)$ converges to a rigid analytic function on $\mathcal{H}_p$ and it satisfies a cocycle condition modulo scalars, namely*

$$J_w^\xi(\gamma_1\gamma_2) = J_w^\xi(\gamma_1) \times \gamma_1 \cdot J_w^\xi(\gamma_2) \pmod{\mathbb{C}_p^\times}. \tag{45}$$

Proof Observe first that $\Gamma_\circ := \mathrm{SL}_2(\mathbb{Z})$ acts on the set $\Sigma^{(m)}$ by Möbius transformations, and that there are finitely many orbits for this action:

$$\Sigma^{(m)} = \Gamma_\circ \cdot (r_1, s_1) \sqcup \Gamma_\circ \cdot (r_2, s_2) \sqcup \cdots \sqcup \Gamma_\circ \cdot (r_\ell, m_\ell).$$

But the cardinality of the set

$$\{\alpha \in \Gamma_\circ \text{ such that } [\alpha r, \alpha s] \cdot [\xi_\infty, \gamma\xi_\infty] = \pm 1\}$$

is equal to the number of intersection points between the images of the geodesics $[r, s]$ and $[\xi_\infty, \gamma\xi_\infty]$ in the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_\infty$. Since this number is finite, it follows that the product

$$J^{\xi}_{w,m}(\gamma) := \prod_{(r,s)\in\Sigma^{(m)}} \big(r, s; \xi_p, z\big)^{[r,s]\cdot[\xi_\infty,\gamma\xi_\infty]}$$

has finitely many factors $\neq 1$, so it is a rational function of z. To prove convergence of

$$J^{\xi}_{w}(\gamma)(z) := \prod_{m=0}^{\infty} J^{\xi}_{w,m}(\gamma)(z)$$

as a rigid meromorphic function of $z \in \mathcal{H}_p^{\leq n}$ it suffices to show that the restriction of $J^{\xi}_{w,m}(\gamma)$ to $\mathcal{H}_p^{\leq n}$ converges uniformly to 1 as $m \longrightarrow \infty$. To see this, write $r = a/b$ and $s = c/d$ in lowest terms as above, let $z := (z_0 : z_1)$ and $\xi_p := (\xi_0 : \xi_1)$ be primitive homogenous coordinates in $\mathcal{O}_{\mathbb{C}_p}$ for z and ξ_p, and note that

$$\begin{aligned}\big(r, s; \xi_p, z\big) &= 1 - \big(r, \xi_p; s, z\big)\\ &= 1 - \frac{(ad - bc)}{(\xi_1 c - \xi_0 d)} \cdot \frac{(\xi_1 z_0 - \xi_0 z_1)}{(b z_0 - a z_1)}\end{aligned}$$

It follows from the definitions of $\mathcal{H}_p^{\leq 0}$ and $\mathcal{H}_p^{\leq n}$ in (41) that

$$|(r, s; \xi_p, z) - 1| \leq p^{n-m}$$

when $z \in \mathcal{H}_p^{\leq n}$ and $(r, s) \in \Sigma^{(m)}$. Therefore, the infinite product defining $J^{\xi}_{w}(z)$ converges absolutely and uniformly on affinoid subsets of $\mathcal{H}_p$. The cocycle condition for J^{ξ}_{w} modulo scalars now follows from a direct calculation. □

The following proposition asserts that the choice of admissible base point ξ that went into the definition of this cocycle does not affect its class in cohomology.

Proposition 2.6 *The class of J^{ξ}_{w} in $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$, denoted J_w, does not depend on the choice of admissible base point ξ that was made to define it.*

Proof The requirement that $\xi_p \in \mathcal{H}_p^{\leq 0}$ implies that

$$(r, s; \xi_p, z) = (r, s; \xi'_p, z) \pmod{\mathcal{O}_{\mathbb{C}_p}^\times}$$

for any other choice of $\xi'_p \in \mathcal{H}_p^{\leq 0}$, and hence changing ξ_p to ξ'_p does not affect the cocycle in $Z^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$. As for replacing ξ_∞ by ξ'_∞, a direct calculation reveals that the associated cocycles differ by the coboundary dF, where $F \in \mathcal{A}^\times$ is defined by

$$F(z) = \prod_{(r,s)\in\Sigma} (r, s;\ \xi'_p, z)^{[r,s]\cdot[\xi_\infty,\xi'_\infty]}.$$

□

2.4 Hecke operators on rigid cocycles

Our goal is to investigate generating series constructed from the sequence $T_n J_w$ for all $n \geq 1$ of Hecke translates of the winding cocycle J_w constructed above. We now briefly recall the definition of the Hecke operators T_n.

These Hecke operators are defined in terms of relevant coset representatives. For all $n \geq 1$, choose a finite set Γ_n such that

$$\bigcup_{\substack{\alpha\in \mathrm{M}_2(\mathbb{Z})\\ \det(\alpha)=n}} \Gamma\alpha\Gamma \ = \ \bigsqcup_{\gamma_n\in\Gamma_n} \Gamma\cdot\gamma_n.$$

For $p \nmid n$, one may choose the usual set of representatives $\Gamma_n = M_n$ defined in (26). On the other hand, when $n = p^m$ we may take

$$\Gamma_{p^m} = \left\{ \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Following Shimura [28, § 8.3] we describe the action of the Hecke operators T_n on $H^1(\Gamma, A)$ for any multiplicative Γ-module A. Let $\gamma \in \Gamma$, then for any $\gamma_n \in \Gamma_n$,

$$\gamma_n\gamma = \gamma'\gamma'_n, \quad \text{for some } \gamma' \in \Gamma,\ \gamma'_n \in \Gamma_n.$$

Suppose J is in $Z^1(\Gamma, A)$, then one defines

$$(T_n J)(\gamma) = \prod_{\gamma_n\in\Gamma_n} \gamma_n^\iota \cdot J(\gamma')$$

where the involution $(-)^\iota$ is defined by $\alpha^\iota = \det(\alpha)\cdot\alpha^{-1}$. It can be checked that with these definitions, $T_n J$ defines an element in $Z^1(\Gamma, A)$, whose equivalence class in group cohomology does not depend on the choice of coset representatives Γ_n.

There are also two involutions W_∞ and W_p determined by the matrices

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

which lie in the normaliser of Γ in $\mathrm{GL}_2(\mathbb{Q})$. The action of these involutions on the cohomology class of the winding cocycle are easily described, as in the following lemma.

Lemma 2.7 *The cohomology class defined by the winding cocycle J_w has eigenvalue -1 for the involution W_∞, and eigenvalue $+1$ for W_p.*

Proof The action of W_∞ on the winding cocycle J_w^ξ with respect to some choice of base point $\xi = (\xi_\infty, \xi_p)$ is defined by

$$
\begin{aligned}
(W_\infty J_w^\xi)(\gamma) &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot J_w^\xi(\gamma^\iota) \\
&= \prod_{(r,s)\in\Sigma} (r, s; \xi_p, -z)^{[r,s]\cdot[\xi_\infty, \gamma^\iota \xi_\infty]} && (46) \\
&= \prod_{(r,s)\in\Sigma} (-r, -s; -\xi_p, z)^{-[-r,-s]\cdot[-\overline{\xi_\infty}, -\overline{\gamma^\iota \xi_\infty}]} && (47) \\
&= \prod_{(r,s)\in\Sigma} (-r, -s; -\xi_p, z)^{-[-r,-s]\cdot[-\overline{\xi_\infty}, \gamma\cdot(-\overline{\xi_\infty})]} && (48)
\end{aligned}
$$

The second equality is justified by the fact that the map $a \mapsto -\overline{a}$ defines an orientation reversing diffeomorphism from the upper half plane to itself, causing the sign of the intersection in the exponent to change. To obtain our conclusion, note that it is clear that $(r, s) \mapsto (-r, -s)$ defines a bijection on Σ, so we obtain the equality

$$W_\infty J_w^\xi = (J_w^{\xi'})^{-1}$$

where $\xi' = (-\overline{\xi_\infty}, -\xi_p)$ is a different choice of base point. By Proposition 2.6 the cohomology class of the winding cocycle is independent of the choice of base point ξ, so that the result follows. The statement about W_p is proved similarly. □

2.5 Lifting the winding cocycle

As a preamble to the explicit determination of the RM values of the cocycles $T_n J_w$, we first discuss how to lift their restrictions to $\mathrm{SL}_2(\mathbb{Z})$. Consider the natural diagram

$$
\begin{array}{ccccccc}
 & & & & H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) & & \\
 & & & & \downarrow \scriptstyle{\mathrm{res}} & & \\
H^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times) & \longrightarrow & H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times) & \longrightarrow & H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times/\mathbb{C}_p^\times) & \longrightarrow & H^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)
\end{array}
$$

where the vertical arrow is restriction to the natural subgroup $\Gamma_\circ := \mathrm{SL}_2(\mathbb{Z})$ of Γ, which is the stabiliser of the standard vertex $v_\circ$ in the Bruhat–Tits tree. Whereas the cocycles $T_n J_w$ need not admit a lift to $H^1(\Gamma, \mathcal{A}^\times)$, their restrictions to $\Gamma_\circ$ do admit a lift

$$(T_n J_w)^\circ \in H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times),$$

by the triviality of $H^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$. Since $H^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}_p^\times)$ is a finite group of order at most 12, this lift is unique up to torsion. We start by giving an explicit description of $(T_n J_w)^\circ$.

For each pair (r, s) define functions $t_{r,s}(z)$ by expressing $r = a/b$ and $s = c/d$ as fractions in lowest terms, in such a way that $ad - bc > 0$, and setting

$$t_{r,s}(z) = \frac{bz - a}{dz - c}. \tag{49}$$

The function $t_{r,s}(z)$ depends only on the pair (r, s) and its divisor is equal to $(r) - (s)$. Hence

$$t_{r,s}(z)/t_{r,s}(\xi_p) = (r, s; \xi_p, z) \tag{50}$$

Observe that the constant $t_{r,s}(\xi_p)$ of proportionality lies in $\mathcal{O}_{\mathbb{C}_p}^\times$, since ξ_p lies in $\mathcal{H}_p^{\leq 0}$.

Lemma 2.8 *For all* $\gamma \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ *with* $\det(\gamma) > 0$ *and all* $(r, s) \in \mathbb{P}_1(\mathbb{Q})^2$,

$$t_{\gamma r,\gamma s}(\gamma z) = \frac{d_2}{d_1} \cdot t_{r,s}(z),$$

for some positive divisors d_1, d_2 *of* $\det(\gamma)$. *In particular,* $t_{\gamma r,\gamma s}(\gamma z) = t_{r,s}(z)$ *when* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Proof Let $r = a/b$ in lowest terms, so that $au + bt = 1$ for some $u, v \in \mathbb{Z}$. We have

$$\gamma r = \frac{Aa + Bb}{Ca + Db}, \quad \text{where} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{51}$$

and furthermore

$$\begin{aligned} &\gcd(Aa + Bb, Ca + Db) \mid (Aa + Bb)(Dt - Cu) + (Ca + Db)(Au - Bt) \\ &\quad = AD - BC = \det(\gamma). \end{aligned}$$

This implies that up to some divisor $\pm d_1$ of $\det(\gamma)$, the fraction in (51) is in lowest terms, and analogously we find a divisor $\pm d_2$ for γs. Furthermore,

$$(Aa + Bb)(Cc + Dd) - (Ca + Db)(Ac + Bd) = (AD - BC)(ad - bc) > 0$$

so that the quantity d_2/d_1 is positive. □

We now give an explicit description of the lift $(T_n J_w)^\circ$, where T_n is the Hecke operator defined in Sect. 2.4. Since $(T_n J_w)^\circ$ does not depend on the choice of ξ_p, we will simplify our notation and simply write ξ for ξ_∞.

Proposition 2.9 *For all* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, *the infinite product*

$$(T_n J_w)^\circ(\gamma) := \prod_{m=0}^{\infty} (T_n J_w)_m^\circ(\gamma),$$

where the factors are defined by

$$(T_n J_w)^\circ_m(\gamma) := \prod_{\gamma_n \in \Gamma_n} \prod_{(r,s)\in\Sigma^{(m)}} (t_{r,s}(\gamma_n z))^{[r,s]\cdot[\xi,\, \gamma_n\gamma(\gamma'_n)^{-1}\xi]}$$

converges to a rigid analytic function on $\mathcal{H}_p$, up to 12-th roots of unity, and defines an element of $H^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{A}^\times/\mu_{12})$, which is the unique lift of the restriction of $T_n J_w$ to $\Gamma_\circ = \mathrm{SL}_2(\mathbb{Z})$.

Proof For integers $m > N + \mathrm{ord}_p(n) \geq 0$, consider the restriction of $(T_n J_w)^\circ_m(\gamma)$ to the affinoid $\mathcal{H}_p^{\leq N}$. Suppose $(r,s) \in \Sigma^{(m)}$ with $r = a/b$ and $s = c/d$ in lowest terms. The fact that $\mathrm{ord}_p(ad - bc) = m$ implies that the primitive vectors (a, b) and (c, d) in $\mathbb{Z}^2$ are proportional to each other modulo p^m. Hence there exists $v \in \mathbb{Z}_p^\times$ for which

$$(a, b) = v \cdot (c, d) + p^m(e, f)$$

for some $(e, f) \in \mathbb{Z}^2$. It follows that

$$t_{r,s}(z) = v + p^m \frac{fz - e}{dz - c}.$$

If $\gamma_n \in M_n$ and $z \in \mathcal{H}_p^{\leq N}$, then $\gamma_n z \in \mathcal{H}_p^{\leq N'}$ with $N' = N + \mathrm{ord}_p(n)$. The description of the latter set given in (41) shows that

$$\frac{f\gamma_n z - e}{d\gamma_n z - c} \in p^{-N'}\mathcal{O}_{\mathbb{C}_p},$$

so that $(T_n J_w)^\circ_m(\gamma)$ is constant modulo $p^{m-N'}$, and its reduction defines a cocycle of $\mathrm{SL}_2(\mathbb{Z})$ valued in the trivial module $(\mathbb{Z}/p^{m-N'}\mathbb{Z})^\times$. Since the abelianisation of $\mathrm{SL}_2(\mathbb{Z})$ is of order 12, it follows that

$$(T_n J_w)^\circ_m(\gamma)(z)|_{\mathcal{H}_p^{\leq N}} \in \mu_{12} \pmod{p^{m-N'}}.$$

The convergence of the infinite product (up to 12th roots of unity) follows. The rest of the statement follows by definition of the Hecke action on cohomology. □

2.6 RM values of the winding cocycle

The main interest in the winding cocycle and its Hecke translates $T_n J_w$ lies in their RM values, which we now investigate. Recall that if $\tau \in \mathcal{H}_p$ is an RM point, then we defined

$$T_n J_w[\tau] := (T_n J_w)^\circ(\gamma_\tau)(\tau) \in \mathbb{C}_p \cup \{\infty\}.$$

We will now obtain an explicit formula for this quantity, using Proposition 2.9.

First, we fix some notation. Suppose n is coprime to p, and choose $M_n(\tau) \subseteq M_n$ as in (26). Note that the condition $(n, p) = 1$ implies that

$$\bigsqcup_{\gamma_n \in M_n} \Gamma \cdot \gamma_n = \bigsqcup_{\delta_n \in M_n(\tau)} \Gamma \cdot \delta_n \cdot \Gamma[\tau]. \tag{52}$$

We also use the notation $\tilde{\Gamma} := \mathrm{GL}_2^+(\mathbb{Z}[1/p])$ for the group of invertible matrices with entries in $\mathbb{Z}[1/p]$ and positive determinant, and let D and $\tilde{D}$ be the subgroups of diagonal matrices in Γ and $\tilde{\Gamma}$ respectively. We have the following explicit formula for $T_n J_w[\tau]$:

Theorem 2.10 *For all $\tau \in \mathcal{H}_p^D$, and for all $n \geq 1$ such that $(n, p) = 1$,*

$$T_n J_w[\tau] = \prod_{\delta_n \in M_n(\tau)} \prod_{\substack{w \in \tilde{\Gamma}\delta_n\tau \\ v_p(w)=0}} w^{[0,\infty]\cdot(w',w)}. \tag{53}$$

Proof We start by choosing a set of coset representatives Γ_n for the Hecke operator T_n that is more convenient for our purposes than the standard choice M_n: For every δ_n in $M_n(\tau)$, there is an integer f such that the stabiliser subgroup $\Gamma[\delta_n\tau]$ is generated modulo torsion by the matrix $\delta_n\gamma_\tau^f\delta_n^{-1}$. Now choose

$$\Gamma_n = \bigsqcup_{\delta_n \in M_n(\tau)} \left\{\delta_n,\ \delta_n\gamma_\tau,\ \ldots,\ \delta_n\gamma_\tau^{f-1}\right\}, \tag{54}$$

which is a set of coset representatives for T_n. The Hecke action is independent of the choice of representatives, and with this choice, we find

$$\begin{cases} \gamma_n\gamma_\tau(\gamma_n')^{-1} = \gamma_{\delta_n\tau} & \text{if } \gamma_n = \delta_n \in M_n(\tau) \\ \gamma_n\gamma_\tau(\gamma_n')^{-1} = 1 & \text{otherwise,} \end{cases}$$

where we recall that $\gamma_{\delta_n\tau} = \delta_n\gamma_\tau^f\delta_n^{-1}$ is the automorph of the RM point $\delta_n\tau$. It now follows from Proposition 2.9 that

$$T_n J_w[\tau] = \prod_{\delta_n \in M_n(\tau)} \prod_{\gamma \in \tilde{\Gamma}/\tilde{D}} t_{\gamma 0,\gamma\infty}(\delta_n\tau)^{[\gamma 0,\gamma\infty]\cdot[\xi,\gamma_{\delta_n\tau}\xi]}$$

where we used the fact that the map $\gamma \mapsto (\gamma 0, \gamma\infty)$ gives a natural identification

$$\tilde{\Gamma}/\tilde{D} = \bigcup_{m=0}^{\infty} \Sigma^{(m)},$$

where as before $\Sigma^{(m)} \subset \Sigma$ is the subset of pairs (r, s) with $\mathrm{ord}_p(\det(r, s)) = m$.

Since $(n, p) = 1$, the stabiliser subgroup $\Gamma[\delta_n\tau]$ is contained in $\mathrm{SL}_2(\mathbb{Z})$, and it follows from Lemma 2.8 that the quantity $t_{\gamma 0,\gamma\infty}(\delta_n\tau)$ only depends on the double coset of γ in $\gamma_{\delta_n\tau}^{\mathbb{Z}}\backslash\tilde{\Gamma}/\tilde{D}$. As a consequence, a similar unfolding argument as in Lemma 1.10 implies that

$$\begin{aligned} T_n J_w[\tau] &= \prod_{\delta_n \in M_n(\tau)} \prod_{\gamma \in \Gamma[\delta_n\tau]\backslash\tilde{\Gamma}/\tilde{D}} t_{\gamma 0,\gamma\infty}(\delta_n\tau)^{\sum_{j\in\mathbb{Z}}[\gamma 0,\gamma\infty]\cdot[\gamma_{\delta_n\tau}^{j}\xi,\gamma_{\delta_n\tau}^{j+1}\xi]} \\ &= \prod_{\delta_n \in M_n(\tau)} \prod_{\gamma \in \Gamma[\delta_n\tau]\backslash\tilde{\Gamma}/\tilde{D}} t_{\gamma 0,\gamma\infty}(\delta_n\tau)^{[\gamma 0,\gamma\infty]\cdot(\delta_n\tau',\delta_n\tau)} \end{aligned}$$

Consider $\tilde{\Gamma}_{\text{prim}} \subset \tilde{\Gamma}$, consisting of the elements whose two columns are primitive vectors in $\mathbb{Z}^2$. Clearly, each coset in $\tilde{\Gamma}/\tilde{D}$ has a unique primitive representative, and hence the natural inclusion $\tilde{\Gamma}_{\text{prim}}/\pm 1 \subset \tilde{\Gamma}/\tilde{D}$ is a bijection. Furthermore, if γ is primitive, then

$$t_{\gamma 0,\gamma\infty}(\delta_n\tau) = \gamma^{-1}\delta_n\tau.$$

Now observe the equality of sets

$$\{\gamma^{-1}\delta_n\tau \;:\; \gamma \in \tilde{\Gamma}_{\text{prim}}/\pm 1\} = \{w \in \tilde{\Gamma}\delta_n\tau \;:\; v_p(w) = 0\},$$

which allows us to rewrite the above expression as

$$T_n J_w[\tau] = \prod_{\delta_n \in M_n(\tau)} \prod_{\gamma \in \Gamma[\delta_n\tau]\backslash\tilde{\Gamma}_{\text{prim}}} (\gamma^{-1}\delta_n\tau)^{[0,\infty]\cdot(\gamma^{-1}\delta_n\tau',\gamma^{-1}\delta_n\tau)} \tag{55}$$

$$= \prod_{\delta_n \in M_n(\tau)} \prod_{\substack{w \in \tilde{\Gamma}\delta_n\tau \\ v_p(w)=0}} w^{[0,\infty]\cdot(w',w)}. \tag{56}$$

□

2.7 Diagonal restrictions: the incoherent case

Finally, we return to the incoherent case of the diagonal restriction of the p-adic family of Hilbert Eisenstein series. Recall that we showed that when p is inert in F, the diagonal restriction $G_1(\psi)$ vanishes identically, and the first order derivative $G_1'(\psi)$ is an overconvergent form of weight 2 and tame level 1. We are now ready to prove Theorem B from the introduction:

Theorem 2.11 *For any odd function ψ on* $\mathrm{Cl}(D)$*,*

$$G_1'(\psi)_{\text{ord}} = L_p'(F,\psi,0) - 2\sum_{n=1}^{\infty} \log_p\left(\mathrm{Nm}((T_n J_w)[\Delta_\psi])\right)\; q^n. \tag{57}$$

Proof Note that all Fourier coefficients are linear in the character ψ, so it suffices to prove this for the odd indicator function of the class $\mathcal{C}$ attached to an RM point τ, which takes values 1 and -1 on $\mathcal{C}$ and $\mathcal{C}^*$, and 0 elsewhere. Let τ be an RM point in $\mathbb{RM}(\mathcal{C})$.

Suppose first that n is not divisible by p. Define the map

$$\tilde{\Gamma}\delta_n\tau \longrightarrow \mathbb{C}_p \,:\, w \longmapsto \log_p(a(w))$$

sending an RM point w to the p-adic logarithm of the leading coefficient of any quadratic form whose prime-to-p discriminant is n^2D, and whose stable root is w. The integer $a(w)$ is only defined up to powers of p, but its logarithm is well-defined. By Lemma 1.8 and Theorem 2.10, we obtain the identities

$$\log_p\left(\mathrm{Nm}((T_nJ_w)[\Delta_\psi])\right) = \sum_{\delta_n\in M_n(\tau)} \sum_{\substack{w\in\tilde{\Gamma}\delta_n\tau\\ v_p(w)=0}} [0,\infty]\cdot(w',w)\cdot\log_p(\mathrm{Nm}(w)) \tag{58}$$

$$= -2\sum_{\delta_n\in M_n(\tau)} \sum_{\substack{w\in\tilde{\Gamma}\delta_n\tau\\ v_p(w)=0}} [0,\infty]\cdot(w',w)\cdot\log_p(a(w)) \tag{59}$$

where the last equality is justified by the obvious relations

$$\begin{aligned} &[0,\infty]\cdot(w',w) = -[0,\infty]\cdot(-1/w',-1/w)\\ &\log_p(\mathrm{Nm}(w)) \;= -\log_p(a(w)) + \log_p(a(-1/w)). \end{aligned} \tag{60}$$

The next step is to rewrite the inner sum of (59). First observe that its index set is

$$\lim_{m\to\infty} X_m(\delta_n), \quad \text{where} \quad X_m(\delta_n) := \{w\in\tilde{\Gamma}\delta_n\tau \,:\, v_p(w)=0,\;\; v_p(\mathrm{disc}(w))\le 2m\}.$$

For any $w\in X_m(\delta_n)$ we let a,b,c be the the unique integers such that

- the stable root of $ax^2+bx+c=0$ is w,
- $b^2-4ac = n^2p^{2m-2k}D$, where $v_p(\mathrm{disc}(w)) = 2m-2k$,

Define $\tilde{w} = p^kw$, then $\tilde{w}$ is the stable root of the equation $ax^2+bp^kx+cp^{2k}=0$. Since $v_p(a)=0$ there exists a matrix $M\in M_2(\mathbb{Z})$ of determinant p^{2m} such that $\tilde{w} = M\delta_n\tau$. This means there is a unique $\delta_{p^m}\in M_{p^m}(\tau)$ such that M belongs to the double coset

$$\mathrm{SL}_2(\mathbb{Z})\cdot\delta_{p^m}\cdot\mathrm{SL}_2(\mathbb{Z})[\delta_n\tau].$$

We claim that the map $w\longmapsto(\tilde{w},\delta_{p^n})$ defines a bijection

$$X_m(\delta_n) \longleftrightarrow \mathbb{RM}(p^m,\delta_n\tau)\setminus\mathbb{RM}(p^m,\delta_n\tau)_p. \tag{61}$$

To prove this claim, note that the image is contained in $\mathbb{RM}(p^m,\delta_n\tau)$, and $p\nmid a(\tilde{w}) = a(w)$. To see that it is a bijection, note that the inverse map is given by

$$(\tilde{w},\delta_{p^m})\mapsto \tilde{w}\cdot p^{-v_p(\tilde{w})}.$$

In conclusion, (61) shows that

$$
\begin{aligned}
2\log_p\left(\mathrm{Nm}((T_n J_w)[\Delta_\psi])\right) &= -4 \lim_{m\to\infty} \sum_{\delta_n \in M_n(\tau)} \sum_{w\in X_m(\delta_n)} \\
&\qquad [0,\infty]\cdot(w',w)\cdot \log_p a(w) \\
&= -4 \lim_{m\to\infty} \sum_{(\tilde{w},\delta_{np^m})\,\in\,\mathbb{RM}(np^m,\mathcal{C})\setminus \mathbb{RM}(np^m,\mathcal{C})_p} \\
&\qquad [0,\infty]\cdot(\tilde{w}',\tilde{w})\cdot \log_p a(\tilde{w})
\end{aligned}
$$

where the inner sum is the np^m-th Fourier coefficient of $G_1'(\psi)$ by (22) and Lemma 1.9.

It now follows that for all n that are prime to p, the nth Fourier coefficient of $G_1'(\psi)_{\mathrm{ord}}$ agrees with the corresponding coefficient on the right-hand side of (57). Because the Hecke action on the space $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ of analytic theta-cocycles factors through the Hecke algebra $\mathbb{T}_2(p) \subset \mathrm{End}(M_2(\Gamma_0(p))$ acting faithfully on $M_2(\Gamma_0(p))$, the formal q-series

$$
G_w := L^? - 2\sum_{n=1}^{\infty} \log_p\left(\mathrm{Nm}((T_n J_w)[\Delta_\psi])\right)\; q^n
$$

is a classical weight two modular form on $\Gamma_0(p)$, for a uniquely determined constant $L^? \in \mathbb{C}_p$. Therefore the difference $G_1'(\psi)_{\mathrm{ord}} - G_w$ is an oldform in $M_2(\Gamma_0(p))$, and therefore zero. It follows that $L^? = L_p'(F,\psi,0)$, and hence that both sides of (57) coincide. □

3 The twisted triple product *p*-adic *L*-function

We now turn to the proof of Theorem C of the introduction, which rests on a careful analysis of the winding element g_w and winding cocycle J_w, and on the decomposition of the latter as a linear combination of the Dedekind–Rademacher cocycle J_{DR} and the elliptic modular cocycles $J_f^\pm$.

3.1 The Schneider–Teitelbaum lift

The logarithmic derivative map embeds the multiplicative group $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ into the $\mathbb{C}_p$-vector space $H^1(\Gamma, \mathcal{A}_2)$, where $\mathcal{A}_2$ denotes the rigid analytic functions on $\mathcal{H}_p$ equipped with the "weight two action" of Γ. Let

$$
U := \{z \in \mathbb{C}_p \text{ with } 1 < |z| < p\} \subset \mathcal{H}_p \tag{62}
$$

denote the standard annulus, whose stabiliser in Γ is equal to $\Gamma_0(p)$. The p-adic annular residue $\omega \mapsto \mathrm{res}_U(\omega)$, as described for instance in [25, §II] or [29], determines a $\Gamma_0(p)$-equivariant map

$$\mathrm{res}_U : \mathcal{A}_2 \longrightarrow \mathbb{C}_p,$$

with $\Gamma_0(p)$ acting trivially on the target.

Theorem 3.1 *The linear map*

$$\mathrm{res}_U : H^1(\Gamma, \mathcal{A}_2) \longrightarrow H^1(\Gamma_0(p), \mathbb{C}_p)$$

induced by the p-adic annular residue is a surjection of $\mathbb{C}_p$-vector spaces. Its kernel is one-dimensional and generated by the cocycle $\mathrm{dlog}\, J_{\mathrm{univ}}$.

The proof of this assertion is given in [11, § 3]. It rests on the construction of an explicit inverse to the residue map, referred to as the *Schneider–Teitelbaum lift*:

$$\mathrm{ST} : H^1(\Gamma_0(p), \mathbb{C}_p) \longrightarrow H^1(\Gamma, \mathcal{A}_2)/\mathbb{C}_p \cdot \mathrm{dlog}\, J_{\mathrm{univ}}.$$

There is also a multiplicative variant, the so-called *multiplicative Schneider–Teitelbaum lift*

$$\mathrm{ST}^\times : H^1(\Gamma_0(p), \mathbb{Z}) \longrightarrow H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)/J_{\mathrm{univ}}^{\mathbb{Z}} \tag{63}$$

of [11, § 3], which fits into the commutative square

$$\begin{array}{ccc} H^1(\Gamma_0(p), \mathbb{Z}) & \xrightarrow{\mathrm{ST}^\times} & H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)/J_{\mathrm{univ}}^{\mathbb{Z}} \\ \big\downarrow & & \big\downarrow \mathrm{dlog} \\ H^1(\Gamma_0(p), \mathbb{C}_p) & \xrightarrow{\mathrm{ST}} & H^1(\Gamma, \mathcal{A}_2)/\mathbb{C}_p \cdot \mathrm{dlog}\, J_{\mathrm{univ}}. \end{array}$$

The multiplicative Schneider–Teitelbaum lift leads to the construction of various explicit rigid analytic theta-cocycles, as described in the introduction and in [11, § 3], namely the Dedekind–Rademacher cocycle $J_{\mathrm{DR}} := \mathrm{ST}^\times(\varphi_{\mathrm{DR}})$ attached to the Dedekind–Rademacher homomorphism, and the elliptic modular cocycles $J_f^\pm := \mathrm{ST}^\times(\varphi_f^\pm)$ attached to the real and imaginary periods of weight two cusp forms on $\Gamma_0(p)$.

Remark 3.2 Although the theta-cocycles J_{DR} and $J_f^\pm$ are only defined up to multiples of J_{univ}, the RM values of the latter are given by

$$J_{\mathrm{univ}}[\tau] := \varepsilon_\tau, \tag{64}$$

where ε_τ is the fundamental unit of the order attached to τ (cf. [11, §3]). Since this quantity depends only on the discriminant of τ rather than on τ itself, it follows that

$$J_{\mathrm{univ}}[\Delta_\psi] = 1 \tag{65}$$

for any odd function ψ, and hence that the RM values $J_{\mathrm{DR}}[\Delta_\psi]$ and $J_f^{\pm}[\Delta_\psi]$ are well-defined.

3.2 The winding element and the winding cocycle

Recall that the *winding element* g_w is the class of the geodesic path from 0 to ∞ in the homology of $X_0(p)$ relative to the cusps, and define

$$\varphi_w : \Gamma_0(p) \longrightarrow \mathbb{Z}, \quad \gamma \longmapsto \langle g_w, \gamma \rangle, \tag{66}$$

where $\langle\ ,\ \rangle$ denotes the intersection pairing of (6)

$$H_1(X_0(p); \{0, \infty\}, \mathbb{Z}) \times H_1(Y_0(p), \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Proposition 3.3 *The winding cocycle J_w is the image of the homomorphism φ_w under the multiplicative Schneider–Teitelbaum lift of* (63)*:*

$$J_w = L_{\mathrm{ST}}^{\times}(2\varphi_w).$$

Proof Recall

the standard annulus U of (62) having $\Gamma_0(p)$ as its stabiliser in Γ. The inverse of the Schneider–Teitelbaum lift takes a cocycle $J \in H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ to the homomorphism

$$\phi_J : \Gamma_0(p) \longrightarrow \mathbb{Z}, \quad \phi_J(\gamma) := \mathrm{res}_U(\mathrm{dlog}\, J(\gamma)),$$

where res_U is the p-adic annular residue attached to U. Consider the infinite product expression of Proposition 2.9 for J_w° and observe that the terms $\mathrm{dlog}\, J_{w,m}^\circ(\gamma)$ for $m \geq 1$ contribute nothing to the annular residue at U: indeed, two cusps r, s for which $\det(r, s) = p^m$ with $m \geq 1$ necessarily belong to the same connected affinoid component of the complement of U, and hence $\mathrm{res}_U(\mathrm{dlog}\, t_{r,s}(z)) = 0$ for such pairs. On the other hand,

$$\mathrm{res}_U(\mathrm{dlog}\, t_{r,s}(z)) = \begin{cases} 1 & \text{if } r \notin \mathbb{Z}_p, s \in \mathbb{Z}_p, \\ -1 & \text{if } r \in \mathbb{Z}_p, s \notin \mathbb{Z}_p. \end{cases}$$

Hence, any pair (r, s) for which the residue of $\mathrm{dlog}\, t_{r,s}(z)$ is equal to 1 is of the form $(\alpha 0, \alpha\infty)$. It follows that

$$\begin{aligned} \mathrm{res}_U(\mathrm{dlog}\, J_{r,s}(\gamma)) &= \sum_{\alpha \in \Gamma_0(p)} (+1)[\alpha 0, \alpha\infty] \cdot [\xi, \gamma\xi] + \sum_{\alpha \in \Gamma_0(p)} (-1)[\alpha\infty, \alpha 0] \cdot [\xi, \gamma\xi] \\ &= 2 \sum_{\alpha \in \Gamma_0(p)} [\alpha 0, \alpha\infty] \cdot [\xi, \gamma\xi]. \end{aligned}$$

This last expression equals twice the intersection product of the relative homology class g_w with the class of γ in $H_1(Y_0(p), \mathbb{Z})$. The proposition follows. □

3.3 Spectral expansion of the winding element

The following lemma describes the decomposition of the cohomology class φ_w relative to the $\bar{\mathbb{Q}}$- basis $(\varphi_{\mathrm{DR}}, \varphi_f^{\pm})$ for $H^1(\Gamma_0(p), \bar{\mathbb{Q}})$ described in (19) and (17).

Lemma 3.4 *The homomorphism φ_w is equal to*

$$\varphi_w = \frac{1}{p-1} \cdot \varphi_{\mathrm{DR}} + \sum_f L_{\mathrm{alg}}(f, 1) \cdot \varphi_f^-,$$

where the sum runs over a basis of normalised eigenforms for $S_2(\Gamma_0(p))$,

$$L_{\mathrm{alg}}(f, 1) := \frac{1}{\Omega_f^+} \int_0^\infty \omega_f^+ \in K_f$$

is the "algebraic part" of the special value $L(f, 1)$, and

$$\varphi_f^-(\gamma) := \frac{1}{\Omega_f^-} \int_{z_0}^{\gamma z_0} \omega_f^- \in \mathcal{O}_f$$

is the minus class in $H^1(\Gamma_0(p), \mathcal{O}_f)$ attached to f, normalised by the periods $\Omega_f^{\pm}$ chosen in (17).

Proof Recall the canonical identifications

$$H_1(Y_0(p); \{0, \infty\}, \mathbb{C}) \longrightarrow H_c^1(Y_0(p))^\vee \longrightarrow H_{\mathrm{dR}}^1(Y_0(p)),$$

where H_c^1 denotes the de Rham cohomology with compact support and the superscript $\vee$ denotes the $\mathbb{C}$-linear dual. The first identification arises from the integration pairing and the second from Poincaré duality. Let G_w be the class in $H_{\mathrm{dR}}^1(Y_0(p))$ corresponding to φ_w under this identification, which is characterised by the equivalent conditions

$$\int_\gamma G_w = \langle \gamma, g_w \rangle, \quad \text{for all } \gamma \in H_1(Y_0(p), \mathbb{Z}), \tag{67}$$

$$\langle G_w, \omega \rangle = \int_0^\infty \omega, \quad \text{for all } \omega \in H_c^1(Y_0(p)). \tag{68}$$

Let α_0 and $\alpha_f^{\pm} \in \mathbb{C}$ be the coordinates of G_w relative to the basis of $H_{\mathrm{dR}}^1(Y_0(p))$ consisting of ω_{Eis} and of the classes ω_f^+ and ω_f^- as f ranges over the normalised weight two eigenforms on $\Gamma_0(p)$:

$$G_w = \alpha_0 \omega_{\mathrm{Eis}} + \sum_f (\lambda_f^+ \omega_f^+ + \lambda_f^- \omega_f^-). \tag{69}$$

Let $\gamma \in H_1(Y_0(p), \mathbb{Z})$ be the class attached to the standard (upper-triangular) parabolic element of $\Gamma_0(p)$, which is orthogonal to the cuspidal classes ω_f^+ and ω_f^-. Applying (67) to this class and substituting for the expansion (69) of G_w, one obtains

$$2\pi i(p-1)\cdot \alpha_0 = 1 \quad \text{and hence} \quad \alpha_0 = \frac{1}{2\pi i(p-1)}. \tag{70}$$

The class $G_w - \alpha_0\omega_{\mathrm{Eis}}$ belongs to $H^1_{\mathrm{dR}}(X_0(p))$ and can therefore be paired against any element of the de Rham cohomology of $X_0(p)$. Applying (68) with $\omega = \omega_f^-$ and substituting for (69) once again, yields

$$-\,\Omega_f \alpha_f^+ = \int_0^\infty \omega_f^- = 0, \quad \text{and hence} \quad \alpha_f^+ = 0. \tag{71}$$

The same calculation with $\omega = \omega_f^+$ reveals that

$$\Omega_f \alpha_f^- = \int_0^\infty \omega_f^+, \quad \text{and hence} \quad \alpha_f^- = (\Omega_f)^{-1}\int_0^\infty \omega_f^+ = L_{\mathrm{alg}}(f,1)(\Omega_f^-)^{-1}. \tag{72}$$

We have thus obtained

$$G_w = \frac{1}{2\pi i(p-1)}\cdot \omega_{\mathrm{Eis}} + \sum_f L_{\mathrm{alg}}(f,1)\cdot(\Omega_f^-)^{-1}\omega_f^-, \tag{73}$$

where the sum is taken over a basis of eigenforms for f. The lemma now follows from (67) and the definitions in (19) and (17). □

3.4 Spectral decomposition: the coherent case

We now turn to the proof of Part 1 of Theorem C of the introduction, concerning the expansion of the modular form $G_1(\psi)$ as a linear combination of eigenforms in $M_2(\Gamma_0(p))$.

Theorem 3.5 *The modular form $G_1(\psi)$ is equal to*

$$G_1(\psi) \;=\; \lambda_0 \cdot E_2^{(p)} \;+\; \sum_f \lambda_f \cdot f,$$

where the sum runs over the basis of normalised eigenforms f in $S_2(\Gamma_0(p))$, and

$$\lambda_0 = \frac{-2}{p-1}\cdot \varphi_{\mathrm{DR}}(g_\psi), \quad \lambda_f = -2L_{\mathrm{alg}}(f,1)\cdot \varphi_f^-(g_\psi).$$

Proof By Theorem 1.12, the generating series $G_1(\psi)$ is equal to

$$G_1(\psi) = L_p(F, \psi, 0) - 2\sum_{n=1}^{\infty} \varphi_w(T_n g_\psi) q^n.$$

Lemma 3.4 implies the n-th Fourier coefficient in this expression is equal to

$$\begin{aligned}\varphi_w(T_n g_\psi) &= \frac{1}{p-1} \cdot \varphi_{\mathrm{DR}}(T_n g_\psi) + \sum_f L_{\mathrm{alg}}(f, 1) \cdot \varphi_f^-(T_n g_\psi) \\ &= \frac{1}{p-1} \cdot \varphi_{\mathrm{DR}}(g_\psi) a_n(E_2^{(p)}) + \sum_f L_{\mathrm{alg}}(f, 1) \varphi_f^-(g_\psi) a_n(f).\end{aligned}$$

The theorem follows by substituting this into the q-expansion formula for $G_1(\psi)$. □

Remark: The coefficient λ_f in the above decomposition can be understood as the special value of a twisted triple product L-function attached to f and the family of Hilbert modular Eisenstein series $E_k(1, \psi)$.

As an illustration of this result, consider the unique unramified odd character ψ of discriminant 12, which is the odd genus character attached to the factorisation $12 = (-3)(-4)$. Let $p = 23$, which is split in $\mathbb{Q}(\sqrt{12})$, then we compute

$$G_1(\psi) = a_0 + 8q^3 + 8q^4 + 8q^6 + 16q^8 + 8q^9 + 16q^{10} + \dots$$

As before, we express $G_1(\psi)$ in a basis of normalised eigenforms, and obtain

$$G_1(\psi) = \frac{8}{11} \cdot E_2^{(23)} - \frac{8}{11}\left(\frac{7\beta - 4}{5} \cdot f_1 + \frac{7\beta' - 4}{5} \cdot f_2\right)$$

where $\beta = (1 + \sqrt{5})/2$ is the golden ratio, and $f_1 = q - \beta q^2 + \dots$ and its conjugate f_2 are the newforms of weight 2 and level 23. In light of the above result, we note that the algebraic part of the L-value of the modular surface attached to the pair $\{f_1, f_2\}$ is equal to $1/11$, which is consistent with the fact that the trace of $(7\beta - 4)/5$ is -3.

3.5 Spectral decomposition: the incoherent case

The following direct corollary of Lemma 3.4 expresses the theta-cocycle J_w as a linear combination of Hecke eigenvectors.

Lemma 3.6 *The rigid analytic theta cocycle J_w satisfies*

$$J_w = \frac{2}{p-1} \cdot J_{\mathrm{DR}} + \sum_f 2L_{\mathrm{alg}}(f, 1) \cdot J_f^- \pmod{J_{\mathrm{univ}}^{\mathbb{Z}}},$$

where the sum runs over a basis of normalised eigenforms for $S_2(\Gamma_0(p))$, and additive notation is used to denote the group operation in $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times) \otimes K_f$.

Proof This follows by applying the Schneider–Teitelbaum lift to Lemma 3.4. □

We are now ready to prove part (2) of Theorem C.

Theorem 3.7 *The modular form $G_1'(\psi)_{\mathrm{ord}}$ is equal to*

$$G_1'(\psi)_{\mathrm{ord}} \;=\; \lambda_0' \cdot E_2^{(p)} \;+\; \sum_f \lambda_f' \cdot f,$$

where the coefficients λ_0' and λ_f' are given by

$$\lambda_0' = \frac{-4}{p-1} \cdot \log_p \left(\mathrm{Nm}(J_{\mathrm{DR}}[\Delta_\psi])\right), \quad \lambda_f' = -4L_{\mathrm{alg}}(f,1) \cdot \log_p \left(\mathrm{Nm}(J_f^-[\Delta_\psi])\right).$$

Proof By Theorem 2.11, the generating series $G_1'(\psi)_{\mathrm{ord}}$ is equal to

$$G_1'(\psi)_{\mathrm{ord}} = L_p'(F, \psi_\tau^-, 0) - 2\sum_{n=1}^{\infty} \log_p(\mathrm{Nm}((T_n J_w)[\Delta_\psi]))q^n.$$

By Lemma 3.6, the n-th Fourier coefficient $A_n := -2\log_p(\mathrm{Nm}((T_n J_w)[\Delta_\psi]))$ in this q-expansion is equal to

$$\begin{aligned} A_n &= \frac{-4}{p-1} \cdot \log_p(\mathrm{Nm}((T_n J_{\mathrm{DR}})[\Delta_\psi])) \\ &\quad -4 \sum_f L_{\mathrm{alg}}(f,1) \cdot \log_p(\mathrm{Nm}((T_n J_f^-)[\Delta_\psi])) \\ &= \frac{-4a_n(E_2^{(p)})}{p-1} \cdot \log_p(\mathrm{Nm}(J_{\mathrm{DR}}[\Delta_\psi])) \\ &\quad -4 \sum_f L_{\mathrm{alg}}(f,1) \cdot \log_p(\mathrm{Nm}(J_f^-[\Delta_\psi]))a_n(f), \end{aligned}$$

where the sum runs over a basis of normalised eigenforms for $S_2(\Gamma_0(p))$. The result follows in exactly the same way as in the proof of Theorem 3.5. □

Remark 3.8 Recall from [11, § 3] that

$$J_{\mathrm{DR}}[\Delta_\psi] \overset{?}{\in} (\mathcal{O}_H[1/p])^\times \otimes \mathbb{Q}$$

is conjectured to be the Gross–Stark unit attached to the RM divisor Δ_ψ. If f is a normalised eigenform on $\Gamma_0(p)$ having integer Fourier coefficients, so that it corresponds to a modular elliptic curve E_f via the Eichler–Shimura construction, [11, § 3] likewise predicts that the *Stark–Heegner points*

$$J_f^+[\Delta_\psi] \overset{?}{\in} E_f(H), \quad J_f^-[\Delta_\psi] \overset{?}{\in} E_f(H)$$

are global points on E_f. The global point $J_f^+[\Delta_\psi]$ is conjecturally fixed by complex conjugation, i.e., is defined over the class field in the wide sense, while $J_f^-[\Delta_\psi]$ is expected to be in the minus eigenspace for complex conjugation. The coefficient λ_f in the above decomposition can be understood as the special value of a twisted triple product L-function attached to f and the family of Hilbert modular Eisenstein series $E_k(1, \psi)$. It is notable that these coefficients involve the logarithms of the Stark–Heegner points attached to odd modular symbols, which are conjecturally in the minus part for complex conjugation.

3.6 Examples

Using Lemma 1.9, we may efficiently compute the diagonal restrictions $G_1(\psi)$ and the first derivative $G_1'(\psi)$. This will be described in a more general setting in [21]. The algorithms of Lauder [20] can then be used to compute the ordinary projection $G_1'(\psi)_{\mathrm{ord}}$. Expressing these classical modular forms of weight two and level p as linear combinations of eigenforms leads to the following numerical illustrations of Theorem C.

Example 1 When $p = 17$, the space $M_2(\Gamma_0(p))$ is two-dimensional and is spanned by the Eisenstein series $E_2^{(17)}$ and the normalised newform f attached to the elliptic curve

$$E \;:\; y^2 + xy + y = x^3 - x^2 - x - 14$$

of rank 0 over $\mathbb{Q}$, whose associated central L-value is $L_{\mathrm{alg}}(f, 1) = 1/4$.

Table 2 presents the coefficients of the spectral decompositions of $G_1(\psi)$ and $G_1'(\psi)_{\mathrm{ord}}$ for all genus characters associated to a factorisation $D = D_1 \cdot D_2$ with $D < 100$, where the labelling is chosen such that $\left(\frac{D_1}{17}\right) = -\left(\frac{D_2}{17}\right) = 1$ in the incoherent case. The coefficients λ_0 and λ_f are rational numbers, and were computed exactly. The coefficients λ_0' and λ_f' were computed numerically up to 30 digits of 17-adic precision. We note that the exceptional vanishing for $D = 76$ is explained by Remark 1.15.

Example 2 We now turn to the attractive case of elliptic curves of conductor 37, where there are two isogeny classes with different ranks:

$$\begin{array}{ll} \texttt{37a} & E^+ : y^2 + y = x^3 - x \\ \texttt{37b} & E^- : y^2 + y = x^3 + x^2 - 23x - 50 \end{array}$$

We denote f^+ and f^- for the associated modular forms, which span $S_2(\Gamma_0(37))$. The elliptic curve E^+ has non-split multiplicative reduction at 37, and rank 1 over $\mathbb{Q}$, whereas E^- has split multiplicative reduction, and rank 0 over $\mathbb{Q}$. We also have $L_{\mathrm{alg}}(f^-, 1) = 1/3$.

Table 2 The spectral decompositions of $G_1(\psi)$ and $G_1'(\psi)_{\mathrm{ord}}$ when $p = 17$.

D	$D_1 \cdot D_2$	$\left(\frac{D}{17}\right)$	λ_0	λ_f	λ_0'	λ_f'
12	$(-4)(-3)$	-1	0	0	$\log\left(\frac{4-\sqrt{-1}}{17}\right)$	$\log_E\left(3+\sqrt{-1}, -5-4\sqrt{-1}\right)$
21	$(-3)(-7)$	1	2	-2		
24	$(-8)(-3)$	-1	0	0	$-2\log\left(\frac{3-2\sqrt{-2}}{17}\right)$	$2\log_E\left(\frac{-22+17\sqrt{-2}}{9}, \frac{181+34\sqrt{-2}}{27}\right)$
28	$(-4)(-7)$	-1	0	0	$3\log\left(\frac{4-\sqrt{-1}}{17}\right)$	$\log_E\left(3+\sqrt{-1}, -5-4\sqrt{-1}\right)$
33	$(-3)(-11)$	1	2	-2		
44	$(-4)(-11)$	-1	0	0	$3\log\left(\frac{4-\sqrt{-1}}{17}\right)$	$-\log_E\left(3+\sqrt{-1}, -5-4\sqrt{-1}\right)$
56	$(-8)(-7)$	-1	0	0	$6\log\left(\frac{3-2\sqrt{-2}}{17}\right)$	$2\log_E\left(\frac{-22+17\sqrt{-2}}{9}, \frac{181+34\sqrt{-2}}{27}\right)$
57	$(-19)(-3)$	-1	0	0	$2\log\left(\frac{7-\sqrt{-19}}{2\cdot 17}\right)$	$2\log_E\left(\sqrt{-19}-3, -\sqrt{-19}-11\right)$
69	$(-23)(-3)$	1	6	2		
76	$(-4)(-19)$	1	0	0	0	0
77	$(-7)(-11)$	1	6	2		
88	$(-8)(-11)$	-1	0	0	$6\log\left(\frac{3-2\sqrt{-2}}{17}\right)$	$-2\log_E\left(\frac{-22+17\sqrt{-2}}{9}, \frac{181+34\sqrt{-2}}{27}\right)$
93	$(-3)(-31)$	1	6	2		

Table 3 The spectral decompositions of $G_1'(\psi)_{\mathrm{ord}}$ when $p = 37$.

D	$D_1 \cdot D_2$	λ_0'	λ_{f+}'	λ_{f-}'
24	$(-3)(-8)$	$\frac{8}{9}\log\left(\frac{1+7\sqrt{-3}}{2\cdot 37}\right)$	0	$\frac{8}{9}\log_{E^-}\left(-\frac{7}{3}, \frac{-9+19\sqrt{-3}}{18}\right)$
56	$(-7)(-8)$	$\frac{8}{3}\log\left(\frac{3+2\sqrt{-7}}{37}\right)$	0	$\frac{8}{3}\log_{E^-}\left(\frac{-27+\sqrt{-7}}{8}, \frac{15+3\sqrt{-7}}{16}\right)$
57	$(-3)(-19)$	$\frac{8}{9}\log\left(\frac{1+7\sqrt{-3}}{2\cdot 37}\right)$	0	$\frac{8}{9}\log_{E^-}\left(-\frac{7}{3}, \frac{-9+19\sqrt{-3}}{18}\right)$
69	$(-3)(-23)$	$\frac{8}{3}\log\left(\frac{1+7\sqrt{-3}}{2\cdot 37}\right)$	0	0
76	$(-4)(-19)$	$\frac{4}{3}\log\left(\frac{1+6\sqrt{-1}}{37}\right)$	0	$\frac{8}{3}\log_{E^-}\left(-\frac{5}{4}, \frac{-4+37\sqrt{-1}}{8}\right)$
88	$(-11)(-8)$	$\frac{8}{3}\log\left(\frac{7+3\sqrt{-11}}{2\cdot 37}\right)$	0	$\frac{8}{3}\log_{E^-}\left(\frac{-65-5\sqrt{-11}}{18}, \frac{68-10\sqrt{-11}}{27}\right)$
93	$(-3)(-31)$	$\frac{8}{3}\log\left(\frac{1+7\sqrt{-3}}{2\cdot 37}\right)$	0	0

It turns out that the modular form $G_1(\psi)$ vanishes systematically when ψ is a genus character, in the coherent as well as in the incoherent cases. In the coherent setting, this "exceptional vanishing" of $G_1(\psi) = 0$ can be explained by the presence of an exceptional zero of the associated p-adic L-function, as described in Remark 1.15. It follows from Proposition 2.1 that $G_1'(\psi)$ is also overconvergent in the coherent setting. Our numerical experiments found in all these cases that $G_1'(\psi)_{\mathrm{ord}}$ is a nonzero multiple of f^+. We expect the constant of proportionality to be a rational multiple of the p-adic height of a Mordell–Weil generator of $E^+(\mathbb{Q})$, but have not verified this.

Table 3 shows the coefficients of the spectral decompositions of $G_1'(\psi)_{\mathrm{ord}}$ for all genus characters associated to the factorisation $D = D_1 \cdot D_2$ with $D < 100$ and $(D/p) = -1$, with the ordering of D_1 and D_2 as in the previous example. The coefficients λ_0' and λ_f' were computed numerically up to 20 digits of 37-adic precision.

We note that the vanishing of λ_{f-}' for $D = 69$ and 93 in Table 3 can be accounted for by the fact that the twists of E^- by the odd quadratic characters of conductor 23 and 31 have analytic rank equal to 2.

Example 3 Finally, we illustrate how Theorems B and C do not only apply to genus characters, by considering $D = 316$ which has narrow class number 6. Let ψ be the odd character which takes value 1 on the trivial class, value -1 on the class of $\mathfrak{d}$, and zero elsewhere. Setting $p = 11$, we compute that $G_1(\psi) = 0$ and that

$$G_1(\psi)'_{\mathrm{ord}} = \lambda_0' E_2^{(11)} + \lambda_f' f,$$

where f is the modular form attached to the elliptic curve $X_0(11)$ considered also in the introduction. The coefficients λ_0' and λ_f' were calculated to 200 digits of 11-adic precision, and we found that

$$\lambda_0' = -\frac{12}{5}\log_{11}(u)$$

where u is the root of a sextic polynomial $a_6 x^6 + \cdots + a_1 x + a_0$ which generates the narrow Hilbert class field of $\mathbb{Q}(\sqrt{316})$, and whose coefficients are given by

$$\begin{aligned} a_0 &= 11^2 \\ a_1 &= 11^0 \times -23684126 \\ a_2 &= 11^4 \times 38858607 \\ a_3 &= 11^8 \times 1575649852 \\ a_4 &= 11^{14} \times 38858607 \\ a_5 &= 11^{20} \times -23684126 \\ a_6 &= 11^{32} \end{aligned}$$

The constant λ'_f was slightly more difficult to identify because of the large height of the polynomials involved. Using the efficient implementation in Sage by Guitart–Masdeu [18] of the polynomial time algorithm of [7] for computing Stark–Heegner points on elliptic curves, Marc Masdeu verified that

$$\lambda'_f = \frac{1}{100} \log_{11}\left(P_\psi\right) \qquad (\text{mod } 11^{200}),$$

where $P_\psi = (x, y)$ is a global point on $X_0(11)$ defined over the narrow Hilbert class field of $\mathbb{Q}(\sqrt{316})$, whose x-coordinate satisfies the polynomial

$$\begin{aligned} &72456194397209968278659172637696 x^3 - 175475962538109348211894597561280 x^2 \\ &-183621530533243510414048237467536 x + 103446014224118434016969398063313 = 0 \end{aligned}$$

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