REAL QUADRATIC BORCHERDS PRODUCTS

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To Benedict Gross, on his 70th birthday

Abstract. *Rigid meromorphic cocycles* were introduced in [DV21] to formulate a notion of singular moduli for real quadratic fields. The present work further develops their foundations and fleshes out their analogy with meromorphic modular functions with CM divisor by describing a real quadratic analogue of the Borcherds lift mapping certain weakly holomorphic modular forms of weight $1/2$ to the group of rigid meromorphic cocycles with rational RM divisor.

Contents

1. Introduction

Let p be a rational prime and let $\mathcal{H}_p$ denote Drinfeld's p-adic upper half plane, a rigid analytic space whose $\mathbb{C}_p$-points are identified with $\mathbb{C}_p - \mathbb{Q}_p$. The *Ihara group* $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ acts by Möbius transformations on $\mathcal{H}_p$ (with non-discrete orbits), and, by translation, on the multiplicative group $\mathcal{M}^\times$ of non-zero rigid meromorphic functions on $\mathcal{H}_p$. This action preserves the subset $\mathcal{H}_p^{\mathrm{RM}} \subset \mathcal{H}_p$ of *real multiplication*, or RM points, namely, points of $\mathcal{H}_p$ that lie in a real quadratic field. A henceforth fixed choice of embeddings of $\bar{\mathbb{Q}}$ into both $\mathbb{C}_p$ and $\mathbb{C}$ gives an inclusion $\mathcal{H}_p^{\mathrm{RM}} \hookrightarrow \mathbb{R}$ and allows one to view RM points as real numbers, and to make sense of their sign.

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A *rigid meromorphic cocycle* is an element of the first cohomology group $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$. Such a class is said to be *parabolic* if it admits a (necessarily unique, up to torsion) representative cocycle J whose restriction to the group Γ_∞ of upper triangular matrices is $\mathbb{C}_p^\times$-valued. A parabolic rigid meromorphic cocycle is often conflated with this distinguished representative. In this setting, Theorem 1 of [DV21] asserts that the meromorphic function $J(S)$, where $S \in \Gamma$ is the standard matrix of order 4, can be written as a finite product

$$J(S)(z) = \prod_{\tau \in \Gamma \backslash \mathcal{H}_p^{\mathrm{RM}}} \alpha_\tau(z)^{n_\tau}, \qquad n_\tau \in \mathbb{Z}, \tag{1}$$

where $\alpha_\tau(z)$ is a rigid meromorphic function whose zeroes (resp. poles) are concentrated at the negative norm elements in the Γ-orbit of τ which are positive (resp. negative):

$$\mathrm{Divisor}(\alpha_\tau) = \sum_{\substack{w \in \Gamma\tau, \\ ww' < 0}} \mathrm{sign}(w) \cdot (w).$$

While $\mathrm{Divisor}(\alpha_\tau)$ is an infinite sum of points on $\mathcal{H}_p$, its intersection with any affinoid subset of $\mathcal{H}_p$ is a finite divisor.

The *divisor* of the cocycle J satisfying (1) is then defined to be

$$\mathrm{Divisor}(J) \ := \sum_{\tau \in \Gamma \backslash \mathcal{H}_p^{\mathrm{RM}}} n_\tau [\tau] \quad \in \ \mathrm{Div}(\Gamma \backslash \mathcal{H}_p^{\mathrm{RM}}),$$

where $\mathrm{Div}(X)$ denotes the group of finite formal integer linear combinations of elements of a set X. Rigid meromorphic cocycles can be envisaged as real quadratic analogues of meromorphic functions on a modular or Shimura curve whose divisors are concentrated on CM points, such as those arising in the image of Borcherds' singular theta lift. One of the goals of this note is to bring this analogy into sharper focus by showing that the principal parts of certain weakly holomorphic modular forms of weight $1/2$ on $\Gamma_0(4p)$ encode divisors of rigid meromorphic cocycles.

1.1. **The main theorem.** The *discriminant* of $\tau \in \mathcal{H}_p^{\mathrm{RM}}$ is the discriminant of the unique primitive integral binary quadratic form of which τ is a root. The set $\mathcal{H}_p^D$ of RM points of discriminant D is non-empty if and only if p is inert or ramified in the real quadratic field $\mathbb{Q}(\sqrt{D})$. One then has

$$\left(\frac{D}{p}\right) \neq 1, \qquad D \equiv 0, 1 \pmod 4.$$

The set $\mathcal{H}_p^D$ is stable under the action of $\mathrm{SL}_2(\mathbb{Z}) \subset \Gamma$ by Möbius transformations, and the quotient $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}_p^D$ is naturally identified with the narrow ideal class group of the order of discriminant D, with composition given by the classical Gauss composition of binary quadratic forms.

Let $-d_1$ and $-d_2$ be a pair of negative discriminants satisfying the following conditions:

(1) $-d_1$ is fundamental, and prime to p;
(2) $\mathrm{ord}_p(-d_2) \leq 1$;
(3) $\left(\frac{d_1 d_2}{p}\right) \neq 1$.

Under these assumptions, $D := d_1 d_2$ is the discriminant of a real quadratic order which is maximal at p, and $\mathcal{H}_p^D$ is non-empty. As in [Ko85, p. 238], the pair $(-d_1, -d_2)$ gives rise to a character ω_{-d_1} of the narrow class group $\mathrm{Cl}^+(D)$ of discriminant D sending the equivalence class of $Q = ax^2 + bxy + cy^2$ to

$$\omega_{-d_1}(Q) = \begin{cases} 0 & \text{if } \gcd(a,b,c,d_1) > 1; \\ \left(\frac{-d_1}{r}\right) & \text{if } \gcd(a,b,c,d_1) = 1, \quad \text{with } Q \text{ representing } r, \text{ and } \gcd(r,d_1) = 1. \end{cases}$$

We associate to the pair $(-d_1, -d_2)$ the RM divisors

$$\begin{aligned} \mathbb{D}_{-d_1,-d_2} &:= \sum_{Q \in \mathrm{Cl}^+(D)} \omega_{-d_1}(Q) \cdot [\tau_Q], \\ \mathbb{D}^+_{-d_1,-d_2} &:= \sum_{Q \in \mathrm{Cl}^+(D)} \omega_{-d_1}(Q) \cdot ([\tau_Q] + [p\tau_Q]), \\ \mathbb{D}^-_{-d_1,-d_2} &:= \sum_{Q \in \mathrm{Cl}^+(D)} \omega_{-d_1}(Q) \cdot ([\tau_Q] - [p\tau_Q]), \end{aligned} \tag{2}$$

where, for $Q(x,y) = ax^2 + bxy + cy^2$,

$$\tau_Q := \frac{-b + \sqrt{D}}{2a},$$

and $[\tau_Q]$ denotes the class of τ_Q in $\Gamma \backslash \mathcal{H}_p$. These divisors do not depend on the choice of p-adic embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ that was made to define them; i.e., replacing $\sqrt{D}$ by $-\sqrt{D}$ leaves $\mathbb{D}_{-d_1,-d_2}$ unchanged. This follows from the fact that

$$\omega_{-d_1}(-Q) = -\omega_{-d_1}(Q), \qquad \tau_{-Q} = \tau'_Q := \frac{-b - \sqrt{D}}{2a}, \qquad \mathrm{sign}(\tau'_Q) = -\mathrm{sign}(\tau_Q).$$

Definition (2) is extended to arbitrary discriminants $-d_1$ by setting

$$\mathbb{D}^+_{-d_1,-d_2p^{2m}} := \mathbb{D}^+_{-d_1,-d_2}, \qquad \mathbb{D}^-_{-d_1,-d_2p^{2m}} := (-1)^m \cdot \mathbb{D}^-_{-d_1,-d_2}, \qquad \text{for all } m \geq 1. \tag{3}$$

For any integer $N \geq 1$ and a weight $k \in \frac{1}{2}\mathbb{Z}$, let

$$S_k(N) \subset M_k(N) \subset M_k^!(N)$$

denote the usual spaces of weight k cusp forms, modular forms, and weakly holomorphic modular forms on $\Gamma_0(N)$. It will also be convenient to introduce the subspace $M_k^{!!}(N) \subset M_k^!(N)$ consisting of the weakly holomorphic modular forms that are regular at all the cusps except ∞. Any form $\phi \in M_k^!(N)$ admits a Fourier expansion at the cusp ∞ of the form

$$\phi(q) = \sum_{n \gg -\infty} c_\phi(n) q^n, \qquad q := e^{2\pi i z}.$$

The space $M_k^{!!}(N)$ is an infinite-dimensional complex vector space and admits a basis consisting of modular forms with integer Fourier coefficients.

The main result of this paper is

Main Theorem. *Let $\phi := \sum_{n \gg -\infty} c_\phi(n) q^n \in M^{!!}_{1/2}(4p)$ be a weakly holomorphic modular form with integer Fourier coefficients.*

(1) If $\left(\frac{-d_1}{p}\right) = 1$, there exists a rigid meromorphic cocycle $J_{-d_1,\phi} \in \mathrm{H}^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$ with

$$\mathrm{Divisor}(J_{-d_1,\phi}) = \sum_{d_2 \equiv 0,3 \bmod 4} c_\phi(-d_2) \cdot \mathbb{D}^+_{-d_1,-d_2},$$

where the sum runs over the positive integers d_2 satisfying $\left(\frac{-d_2}{p}\right) \neq 1$.

(2) If $\left(\frac{-d_1}{p}\right) = -1$, there exists a rigid meromorphic cocycle $J_{-d_1,\phi} \in \mathrm{H}^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$ with

$$\mathrm{Divisor}(J_{-d_1,\phi}) = \sum_{d_2 \equiv 0,3 \bmod 4} c_\phi(-d_2) \cdot \mathbb{D}^-_{-d_1,-d_2},$$

where the sum runs over the positive integers d_2 satisfying $\left(\frac{-d_2}{p}\right) \neq -1$.

The rigid meromorphic cocycle $J_{-d_1,\phi}$ is called a *real quadratic Borcherds lift* attached to the discriminant $-d_1$ and to the weakly homolorphic modular form ϕ.

1.2. **Outline.** The proof of the main theorem relies on the classification of rigid meromorphic cocycles of [DV21, Theorem 1.23], which rests on the study of the group $\mathrm{H}^1(\Gamma, \mathcal{M}^\times / \mathbb{C}_p^\times)$ of rigid meromorphic cocycles *modulo scalars*. These objects are called *theta-cocycles* because of their analogy with the p-adic theta functions arising in the uniformisation theory of Jacobians of Mumford–Schottky curves, discussed in the Appendix.

In § 2 general rigid theta-cocycles and their RM values are discussed. The subgroup of analytic cocycles is determined in § 3, and it is intimately connected to classical weight two modular forms on $\Gamma_0(p)$ by the *Schneider–Teitelbaum lift*. It is shown in § 4 that the group of rigid *meromorphic* theta-cocycles is of infinite rank; more precisely, that any RM divisor arises as the divisor of a (not necessarily unique) meromorphic theta-cocycle. No essential information is lost in passing from a rigid meromorphic cocycle to its associated theta-cocycle, because the natural map

$$\mathrm{H}^1(\Gamma, \mathcal{M}^\times) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{M}^\times / \mathbb{C}_p^\times)$$

has finite kernel (of exponent dividing 12). However, this map fails to be surjective in general. We analyse the *obstruction* to lifting a rigid meromorphic theta-cocycle with given RM divisor to an actual meromorphic cocycle in § 5, and relate it to the *Stark–Heegner point* attached to that divisor. Such a relation suggests that these lifting obstructions can be packaged into modular generating series of weight $3/2$ for $\Gamma_0(4p)$, which is done in § 6 by building on the Gross–Kohnen–Zagier theorem for Stark–Heegner points proved in [DT08]. The main theorem then follows from a simple application of Serre duality.

2. Generalities on theta cocycles and theta symbols

2.1. **Rational functions on $\mathcal{H}_p$.** Let $\mathbb{C}_p$ be a completion of an algebraic closure of $\mathbb{Q}_p$. Although it is a natural ground field for the theory of rigid analytic spaces, most of our calculations (and all of the interesting ones) take place over the compositum C_p of all the quadratic extensions of $\mathbb{Q}_p$,

which is biquadratic when $p \neq 2$, and may be viewed as a suitable p-adic analogue of $\mathbb{C}$ in the setting of this paper.

Every $z \in \mathbb{P}_1(\mathbb{C}_p)$ can be represented in projective coordinates by a pair $[z_1\colon z_2]$ where (z_1, z_2) belongs to the set $(\mathcal{O}_{\mathbb{C}_p} \times \mathcal{O}_{\mathbb{C}_p})'$ of *primitive vectors* having at least one coordinate in $\mathcal{O}_{\mathbb{C}_p}^\times$. If $(a, b) \in (\mathbb{Z}^2)'$ is a primitive vector, satisfying $\gcd(a, b) = 1$, then $\mathrm{ord}_p(z_1 b - z_2 a) \in \mathbb{Q} \cup \{\infty\}$ depends only on z and not on its representation. It belongs to $\mathbb{Q}$ when $z \notin \mathbb{P}_1(\mathbb{Q}_p)$, and is a half integer when $z \in \mathbb{P}_1(C_p) - \mathbb{P}_1(\mathbb{Q}_p)$.

Let $\mathcal{T}$ be the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose set $\mathcal{V}(\mathcal{T})$ of vertices is in bijection with the homothety clases of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$, two vertices being joined by an (unordered) edge if they admit representative lattices containing each other with index p. Write $\mathcal{E}(\mathcal{T})$ for the set of such unordered edges of $\mathcal{T}$ and identify $\mathcal{T}$ with its combinatorial realisation

$$\mathcal{T} := \mathcal{V}(\mathcal{T}) \cup \mathcal{E}(\mathcal{T}).$$

The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathcal{T}$ via its left multiplication action on $\mathbb{Q}_p^2$ viewed as column vectors. The subgroup Γ acts transitively on $\mathcal{E}(\mathcal{T})$ and breaks up $\mathcal{V}(\mathcal{T})$ into two distinct orbits, in such a way that adjacent vertices lie in distinct Γ-orbits. The vertex $v_\circ$ attached to the standard lattice $\mathbb{Z}_p^2 \subset \mathbb{Q}_p^2$ is called the *standard vertex*. Its stabiliser $\Gamma_{v_\circ}$ in Γ is equal to $\mathrm{SL}_2(\mathbb{Z})$. More generally, the vertex and edge stabilisers in Γ are $\mathrm{GL}_2(\mathbb{Z}[1/p])$-conjugate to $\mathrm{SL}_2(\mathbb{Z})$ and to the Hecke congruence group $\Gamma_0(p)$, respectively. This leads to a description of Γ as an amalgamated product

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{SL}_2(\mathbb{Z})',$$

which is useful in relating the cohomology of Γ with the cohomology of $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0(p)$. If $\mathcal{G}$ is any finite subgraph of $\mathcal{T}$, the stabiliser of $\mathcal{G}$ in Γ is denoted $\Gamma_{\mathcal{G}}$. The groups $\Gamma_{\mathcal{G}}$ are conjugate to finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and act discretely on $\mathcal{H}$.

Let $\mathcal{H}_p$ denote Drinfeld's upper half-plane, a rigid analytic space whose underlying set of $\mathbb{C}_p$-points is identified with $\mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathcal{H}_p$ by Möbius transformations, and there is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant *reduction map*

$$\mathrm{red} : \mathcal{H}_p \longrightarrow \mathcal{T}$$

which satisfies the following properties.

• The *standard affinoid region*

$$\mathcal{H}_p^\circ := \{z = (z_1\colon z_2) \in \mathcal{O}_{\mathbb{C}_p} \text{ such that } \mathrm{ord}_p(z_1 b - z_2 a) = 0 \text{ for all } (a, b) \in (\mathbb{Z}_p^2)'\}, \tag{4}$$

which is the complement in $\mathbb{P}_1(\mathbb{C}_p)$ of $p + 1$ residue discs, maps to the standard vertex $v_\circ$ under the reduction map.

• The *standard annulus*

$$U := \{z \in \mathcal{O}_{\mathbb{C}_p} \text{ such that } 1 < |z| < p\} \tag{5}$$

maps to the edge of $\mathcal{T}$ attached to the lattice pair $(\mathbb{Z}_p^2 \supset \mathbb{Z}_p \oplus p\,\mathbb{Z}_p)$, which is denoted $e_\circ$ and called the *standard edge* of $\mathcal{T}$. Its stabiliser $\Gamma_{e_\circ}$ is equal to $\Gamma_0(p)$.

These two properties determine the reduction map, because $\mathcal{H}_p$ is the union of the $\mathrm{GL}_2(\mathbb{Q}_p)$-translates of $\mathcal{H}_p^\circ$ and U. The set $\mathcal{H}_p^D$ of RM points of p-discriminant D maps to $\mathcal{V}(\mathcal{T})$ under the reduction map if $p \nmid D$, and to $\mathcal{E}(\mathcal{T})$ if $p|D$.

If v is a vertex of $\mathcal{T}$, let

$$\mathcal{A}_v := \mathrm{red}^{-1}(\{v\}), \qquad \mathcal{W}_v := \mathrm{red}^{-1}(\{v, e_1, \dots, e_{p+1}\}), \tag{6}$$

where $e_1, \dots, e_{p+1}$ are the edges of $\mathcal{T}$ having v as an endpoint. The sets $\mathcal{A}_v$ and $\mathcal{W}_v$ are called the *standard affinoid* and *standard wide open space* attached to v, respectively.

The distance $d(v, w)$ between two vertices v and w of $\mathcal{T}$ is the number of edges in the shortest path joining them. This notion extends to edges by setting, for all edges $e = (v, v')$ and all edges $e' \neq e$:

$$d(w, e) = d(e, w) = \frac{1}{2}(d(v, w) + d(v', w)), \qquad d(e, e') = \frac{1}{2}(d(v, e') + d(v', e')).$$

With this definition, the distance between two vertices or two edges is a positive integer, while the distance between a vertex and an edge is a half of an odd integer. For each integer $n \geq 0$, let $\mathcal{T}^{\leq n}$ and $\mathcal{T}^{<n}$ denote the subgraphs of $\mathcal{T}$ consisting of vertices and edges that are at distance $\leq n$ and $< n$ respectively from $v_\circ$, and let $\mathcal{H}_p^{\leq n}$ and $\mathcal{H}_p^{<n}$ denote their inverse images under the reduction map. The collection of $\mathcal{H}_p^{\leq n}$ and $\mathcal{H}_p^{<n}$ give coverings of $\mathcal{H}_p$ by affinoid subsets and wide open subsets, respectively, that are preserved under the action of $\mathrm{SL}_2(\mathbb{Z})$. They can be described directly as

$$\mathcal{H}_p^{\leq n} = \{[z_1 \colon z_2] \in (\mathcal{O}_{\mathbb{C}_p} \times \mathcal{O}_{\mathbb{C}_p})' \text{ with } \mathrm{ord}_p(z_1 b - z_2 a) \leq n \text{ for all } (a, b) \in (\mathbb{Z}^2)'\}, \tag{7}$$

and likewise for $\mathcal{H}_p^{<n}$.

A rational function t on $\mathbb{P}_1(\mathbb{C}_p)$ can be evaluated on divisors on $\mathbb{P}_1(\mathbb{C}_p)$ in the usual way, by setting

$$t\left(\sum_i n_i \cdot (z_i)\right) := \prod_i t(z_i)^{n_i}.$$

Given a degree zero divisor $\mathscr{D}$ on $\mathbb{P}_1(\mathbb{C}_p)$, there is a unique rational function $t_{\mathscr{D}}$ with that divisor, up to multiplication by a non-zero scalar. In particular, if $\mathscr{D}_1$ and $\mathscr{D}_2$ are degree zero divisors with disjoint supports, the quantity

$$[\mathscr{D}_1; \mathscr{D}_2] := t_{\mathscr{D}_2}(\mathscr{D}_1) \in \mathbb{C}_p^\times \tag{8}$$

does not depend on the choice of $t_{\mathscr{D}_2}$. The quantity $[(z_1) - (z_2); (z_3) - (z_4)]$ is just the familiar cross-ratio of (z_1, z_2, z_3, z_4). The symbol $[\mathscr{D}_1; \mathscr{D}_2]$ is bilinear, symmetric (by Weil reciprocity) and $\mathrm{GL}_2(\mathbb{C}_p)$-equivariant. The following lemma controls the p-adic valuation of $[\mathscr{D}_1; \mathscr{D}_2]$.

Lemma 2.1. *If $\mathscr{D}_1$ is a degree zero divisor supported on $\mathcal{H}_p^{\leq n}$, and $\mathscr{D}_2$ is a degree zero divisor supported on $\mathcal{W}_v$ with $d(v, v_\circ) = m > n$, then $[\mathscr{D}_1; \mathscr{D}_2]$ belongs to $1 + p^{m-n}\,\mathcal{O}_{\mathbb{C}_p}$.*

Proof. This is a special case of a more general result: if $\mathcal{G}_1$ and $\mathcal{G}_2$ are *oriented* subgraphs of $\mathcal{T}$ satisfying

$$\partial\,\mathcal{G}_1 = \mathrm{red}(\mathscr{D}_1), \qquad \partial\,\mathcal{G}_2 = \mathrm{red}(\mathscr{D}_2),$$

then $[\mathscr{D}_1;\mathscr{D}_2]$ is a p-adic unit when $\mathcal{G}_1$ and $\mathcal{G}_2$ can be taken to be disjoint, and it is congruent to 1 modulo p^m, where m is the *distance* between $\mathcal{G}_1$ and $\mathcal{G}_2$, defined in the natural way as the minimum distance between vertices and edges of $\mathcal{G}_1$ and of $\mathcal{G}_2$. □

When $\mathscr{D}_2$ is supported on elements of $\mathbb{P}_1(\mathbb{Q})$, the definition of $[\mathscr{D}_1,\mathscr{D}_2]$ extends to divisors $\mathscr{D}_1$ of arbitrary degree, in a way that continues to be equivariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. To see this, assume for simplicity that $\mathscr{D}_2=(r)-(s)$, and write $r=a/b$ and $s=c/d$ as fractions in lowest terms, adopting the usual convention that $\infty=1/0$. The quantity $[z;(r)-(s)]$ is then defined by setting

$$[z;(r)-(s)]=\frac{bz-a}{dz-c}.$$

The resulting symbol, extended by bilinearity to $\mathrm{Div}(\mathbb{P}_1(\mathbb{C}_p))\times\mathrm{Div}^0(\mathbb{P}_1(\mathbb{Q}))$, coincides with the earlier one on degree zero divisors and satisfies the weaker equivariance properties

$$\begin{array}{rcll}[\gamma z;(\gamma r)-(\gamma s)] &=& [z;(r)-(s)] & \text{for all } \gamma\in\mathrm{SL}_2(\mathbb{Z}),\\ [\gamma z;(\gamma r)-(\gamma s)] &=& [z;(r)-(s)] \pmod{p^{\mathbb{Z}}} & \text{for all } \gamma\in\Gamma,\end{array}\tag{9}$$

which can be checked directly.

It shall be useful to control the p-adic valuation of the restriction of the rational function of $[z;(r)-(s)]$ to the affinoid subsets $\mathcal{H}_p^{\leq n}$. For $r=a/b$ and $s=c/d$ as above, define

$$\det(r,s):=\pm(ad-bc).$$

Lemma 2.2. *Let $0\leq n<m$ be integers. For all $(r,s)\in\mathbb{P}_1(\mathbb{Q})^2$ with $\mathrm{ord}_p(\det(r,s))=m$,*

(1) the restriction of $[z;(r)-(s)]$ to $\mathcal{H}_p^{\leq n}$ takes values in $v+p^{m-n}\,\mathcal{O}_{\mathbb{C}_p}$ for some $v\in\mathbb{Z}_p^\times$;
(2) If $\mathscr{D}$ is a degree zero divisor supported on $\mathcal{H}_p^{\leq n}$, then $[\mathscr{D};(r)-(s)]$ belongs to $1+p^{m-n}\,\mathcal{O}_{\mathbb{C}_p}$.

Proof. If $r=\frac{a}{b}$ and $s=\frac{c}{d}$, the fact that p^m divides $ad-bc$ implies that the primitive vectors (a,b) and (c,d) in $\mathbb{Z}^2$ are proportional to each other modulo p^m. Hence there exists $v\in\mathbb{Z}_p^\times$ for which $(a,b)=v\cdot(c,d)+p^m(e,f)$ for some $(e,f)\in\mathbb{Z}^2$. It follows that

$$[z;(r)-(s)]=v+p^m\frac{fz-e}{dz-c}.$$

But as z ranges over $\mathcal{H}_p^{\leq n}$, the rational function $\frac{fz-e}{dz-c}$ takes values in $p^{-n}\,\mathcal{O}_{\mathbb{C}_p}$ by (7), and the first statement follows. The second follows directly from the first. □

2.2. **Theta cocycles and symbols.** Because all the examples that are relevant to this work are defined over C_p, it will be convenient to slightly modify the set up in the introduction, by letting $\mathcal{A}^\times$ denote the multiplicative group of nowhere vanishing rigid analytic functions on $\mathcal{H}_p$, and $\mathcal{M}^\times$ the group of nonzero rigid meromorphic functions on this domain, which *are defined over C_p*. This point of view shall remain in force throughout this article. The groups $\mathcal{A}^\times$ and $\mathcal{M}^\times$ continue to be equipped with their standard (weight zero) left action by the group $\Gamma=\mathrm{SL}_2(\mathbb{Z}[1/p])$.

Definition 2.3. A rigid analytic theta-cocycle is a class in $\mathrm{H}^1(\Gamma,\mathcal{A}^\times/C_p^\times)$, and a rigid meromorphic theta-cocycle is a class in $\mathrm{H}^1(\Gamma,\mathcal{M}^\times/C_p^\times)$.

It is useful to dispose of a variant of these objects involving Γ-invariant *modular symbols* instead of one-cocycles. The passage from one to the other is akin to working with the cohomology of a modular curve relative to the cusps instead of the cohomology of the associated open modular curve, and hence the two notions can be thought of as being in duality with each other.

If Ω is a Γ-module, the Γ-module of *modular symbols* with values in Ω is the set $\mathrm{MS}(\Omega)$ of functions $m : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \Omega$ satisfying

$$\begin{aligned} m\{r,s\} &= -m\{s,r\}, \\ m\{r,t\} &= m\{r,s\} + m\{s,t\}, \end{aligned}$$

for all $r, s, t \in \mathbb{P}_1(\mathbb{Q})$. If Π is any subgroup of $\mathrm{GL}_2(\mathbb{Q})$, let $\mathrm{MS}^\Pi(\Omega) := \mathrm{H}^0(\Pi, \mathrm{MS}(\Omega))$ denote the group of Π-invariant modular symbols, satisfying

$$m\{\gamma r, \gamma s\} = \gamma m\{r,s\} \quad \text{for all } \gamma \in \Pi, \;\; r, s \in \mathbb{P}_1(\mathbb{Q}).$$

If Π is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, there is a well-known identification between $\mathrm{MS}^\Pi(\Omega)$ and the cohomology of the associated modular curve X_Π relative to the cusps:

$$\mathrm{MS}^\Pi(\Omega) = \mathrm{H}^1(X_\Pi, \mathrm{cusps}; \Omega).$$

The natural map from the relative cohomology to the cohomology of the complete curve corresponds to the map

$$\mathrm{MS}^\Pi(\Omega) \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Pi, \Omega)$$

sending the Π-invariant Ω-valued modular symbol m to the class on the one-cocycle

$$c(\gamma) := m\{\infty, \gamma\infty\}. \tag{10}$$

This map is defined for any subgroup Π of $\mathrm{SL}_2(\mathbb{Q})$ such as the group $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$ (cf. [DV21, Lemma 1.3]), and it fits into the exact sequence

$$\Omega^\Pi \hookrightarrow \bigoplus_{x \in \Pi\backslash\mathbb{P}_1(\mathbb{Q})} \Omega^{\Pi_x} \longrightarrow \mathrm{MS}^\Pi(\Omega) \xrightarrow{\delta} \mathrm{H}^1(\Pi, \Omega) \longrightarrow \bigoplus_{x \in \Pi\backslash\mathbb{P}_1(\mathbb{Q})} \mathrm{H}^1(\Pi_x, \Omega), \tag{11}$$

where the direct sums are taken over a system of representatives for the Π-orbits on $\mathbb{P}_1(\mathbb{Q})$ and Π_x is the stabiliser of x in Π. The image of δ shall be referred to as the *parabolic cohomology* of Π with values in Ω, although the reader is cautioned that for p-arithmetic groups like $\Pi = \Gamma$, the cusp stabilisers do not consist solely of parabolic matrices.

Definition 2.4. A rigid analytic theta-symbol is a class in $\mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$, and a rigid meromorphic theta-symbol is a class in $\mathrm{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$.

Lemma 2.5. *The invariants $(\mathcal{A}^\times/C_p^\times)^{\Gamma_\infty} = (\mathcal{M}^\times/C_p^\times)^{\Gamma_\infty}$ are trivial. In particular, the map δ of* (11) *is injective, and every parabolic theta-cocycle is described by a unique theta-symbol.*

Proof. Let N be the parabolic subgroup of Γ_∞. Any homomorphism in $\mathrm{H}^1(\Gamma_\infty, C_p^\times)$ is trivial on the commutator subgroup of Γ_∞, which contains N^{p^2-1}. It follows that any

$$f \in \mathrm{H}^0(\Gamma_\infty, \mathcal{M}^\times/C_p^\times)$$

lifts to an element of $\mathrm{H}^0(N^{p^2-1}, \mathcal{M}^\times)$. But there are no non-constant rigid meromorphic functions invariant under the translation $z \mapsto z + p^2 - 1$, and the result follows. □

Corollary 2.6. *A parabolic theta-cocycle admits a unique representative one-cocycle, up to torsion, whose restriction to Γ_∞ is trivial.*

Proof. Let $\Omega := \mathcal{A}^\times/C_p^\times$ or $\mathcal{M}^\times/C_p^\times$. A parabolic theta-cocycle J corresponds to a theta-symbol in $\mathrm{MS}^\Gamma(\Omega)$ by Lemma 2.5, and the one-cocycle attached to this modular symbol following the recipe of (10) vanishes identically on Γ_∞. □

It will be convenient henceforth to identify parabolic theta-cocycles with Γ-invariant modular symbols. In this way, for a parabolic theta-cocycle J viewed as both a one cocycle and a modular symbol, we have $J(S) = J\{0,\infty\}^{-1}$. The description of J as a theta-symbol often leads to more transparent calculations.

2.3. **RM values.** Recall that the *value* of a rigid meromorphic cocycle J at an RM point τ is defined by setting

$$J[\tau] := J(\gamma_\tau)(\tau),$$

where γ_τ is a generator of the stabiliser of τ in Γ, normalised so that $\gamma_\tau^n(z)$ tends to τ in the *real topology* as n tends to ∞, and to $\bar\tau$ as n tends to $-\infty$. The element γ_τ is called the *automorph* of τ. If J is parabolic, one also has

$$J[\tau] = J\{r, \gamma_\tau r\}(\tau),$$

where $r \in \mathbb{P}_1(\mathbb{Q})$ is any base point. These RM values are the main subject of [DV21].

It is useful for the considerations of this paper to extend the notion of RM values to theta-cocycles and theta-symbols. If v is any vertex of $\mathcal{T}$, the stabilizer $\Gamma_v \subset \Gamma$ of v in Γ is conjugate to $\mathrm{SL}_2(\mathbb{Z})$ under a matrix in $\mathrm{GL}_2(\mathbb{Z}[1/p])$. The second cohomology group $\mathrm{H}^2(\Gamma_v, C_p^\times)$ is therefore trivial, and hence the restriction of a theta-cocycle J to $\mathrm{H}^1(\Gamma_v, \mathcal{M}^\times/C_p^\times)$ lifts to an element

$$J_v \in \mathrm{H}^1(\Gamma_v, \mathcal{M}^\times).$$

This lift is unique up to 12-torsion, because $\mathrm{H}^1(\Gamma_v, C_p^\times) \simeq \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), C_p^\times)$ is of order ≤ 12.

To define $J[\tau]$, it is convenient to distinguish two cases:

Case 1. The RM point τ is *unramified*, i.e., it generates a real quadratic field in which the prime p is inert. In that case, τ maps to a vertex v of the Bruhat-Tits tree under the reduction map, and its automorph γ_τ belongs to Γ_v. We can therefore define

$$J[\tau] := J_v(\gamma_\tau)(\tau) \in C_p \cup \{\infty\}. \tag{12}$$

Case 2. The RM point τ is *ramified*, i.e., it generates a real quadratic field in which p is ramified. The point τ then reduces to an edge $e = (v_1, v_2)$ of the Bruhat-Tits tree, and the automorph γ_τ belongs to the edge stabiliser $\Gamma_e = \Gamma_{v_1} \cap \Gamma_{v_2}$. We then define

$$J[\tau]^2 := J_{v_1}(\gamma_\tau)(\tau) \times J_{v_2}(\gamma_\tau)(\tau) \in C_p \cup \{\infty\}. \tag{13}$$

A direct calculation reveals that the value $J[\tau]$ depends only on the Γ-orbit of τ, i.e., that

$$J[\gamma\tau] = J[\tau], \qquad \text{for all } \gamma \in \Gamma.$$

When J lifts to a rigid meromorphic cocycle, the two notions of RM values are of course compatible, and the conjectures of [DV21] predict that $J[\tau]$ is an algebraic number lying in a compositum

of ring class fields of real quadratic fields. When such a lift does not exist, the quantities $J[\tau]$ are expected to be transcendental in general, but still retain some arithmetic interest, insofar as they are related to Gross–Stark units and Stark–Heegner points, as will be discussed shortly.

3. Analytic theta-cocycles

We now turn to the classification of analytic theta-cocycles, and show they form a finitely generated group of rank $2g+2$, where g is the genus of $X_0(p)$. The techniques used to prove this theorem rely on ideas of Stevens and Schneider–Teitelbaum, (cf. [PS11], [Sch84], [Te90]) and occupy § 3.1–3.6, from which the classification is deduced in § 3.7. This results in a list of three types of generators for the group of theta-cocycles, each with their concomitant idiosyncracies:

- The boundary theta cocycle,
- The Dedekind–Rademacher cocycle,
- The modular cocycles attached to newforms in $S_2(\Gamma_0(p))$.

A discussion of known results about their RM values, which notably include *Gross–Stark units* and *Stark–Heegner points*, can be found in § 3.7. These classes are all proper for the action of the Hecke algebra. In § 3.8 the *toric cocycles*, which are attached to a pair of cusps, are introduced. These theta-cocycles behave less straightforwardly with respect to the action of the Hecke algebra, but are completely explicit, and play an important role in what follows.

3.1. Rigid analytic functions and boundary distributions.

The classification begins with a study of the module of *additive* analytic functions on $\mathcal{H}_p$ endowed with the weight k action of Γ, denoted $\mathcal{A}_k$. It rests on the fact that $\mathcal{A}_k$ is isomorphic to a space of locally analytic distributions on the boundary $\mathbb{P}_1(\mathbb{Q}_p)$ of $\mathcal{H}_p$. Assume henceforth that $k=2$ for simplicity, although the results described below can certainly be extended to more general positive even weights.

The dual of the space of locally analytic functions on $\mathbb{P}_1(\mathbb{Q}_p)$, equipped with the strong topology of uniform convergence on compact open subsets, is called the space of *locally analytic distributions* on $\mathbb{P}_1(\mathbb{Q}_p)$, and is denoted $\mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$. Given $\mu \in \mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$, the notation

$$\mu(h) =: \int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t) d\mu(t)$$

shall be adopted. More generally, if U is a compact open subset of $\mathbb{P}_1(\mathbb{Q}_p)$ and 1_U is its characteristic function, we define

$$\int_U h(t) d\mu(t) := \int_{\mathbb{P}_1(\mathbb{Q}_p)} 1_U(t) h(t) d\mu(t).$$

A distribution $\mu \in \mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ satisfying $\mu(1)=0$, where 1 denotes the constant function 1 on $\mathbb{P}_1(\mathbb{Q}_p)$, is said to be *of total volume zero*, and the space of such locally analytic distributions is denoted $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$.

The group $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts naturally on $\mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ and on $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$ via the weight zero action on locally analytic functions on $\mathbb{P}_1(\mathbb{Q}_p)$. More precisely,

$$\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t)d(\mu|\gamma)(t) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} h(\gamma t)d\mu(t), \qquad \text{where } \gamma t := \frac{at+b}{ct+d}.$$

To any rigid analytic function $f \in \mathcal{A}_2$, we attach a locally analytic distribution μ_f on $\mathbb{P}_1(\mathbb{Q}_p)$ by setting, for all analytic functions $h(t)$ on a compact open $U_e \subset \mathbb{P}_1(\mathbb{Q}_p)$ corresponding to an oriented edge $e \in \mathcal{T}_1^*$,

$$\int_{U_e} h(t)d\mu_f(t) := \mathrm{res}_e(f(z)h(z)dz).$$

Here, res_e denotes the p-adic annular residue along the oriented annulus $\mathcal{W}_e$. The distribution μ_f is called the *boundary distribution* attached to f. It is a direct consequence of the residue theorem that μ_f belongs to $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$.

Proposition 3.1. *The map $f \mapsto \mu_f$ induces a topological isomorphism*

$$\mathrm{BD} : \mathcal{A}_2 \xrightarrow{\sim} \mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$$

that is compatible with the $\mathrm{PGL}_2(\mathbb{Q}_p)$-actions on both sides.

Proof. Setting $k = 2$ in the statement of Theorem 2.2.1 of [DaTe], the dual of the map denoted I_2 in loc.cit. induces an isomorphism

$$\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)) \longrightarrow \mathcal{A}_2,$$

in light of the fact that $\mathcal{A}_2$ is a reflexive Frechet space and hence is identified with its double dual. The "boundary distribution map" BD is just the inverse of this isomorphism. □

The map BD induces an isomorphism on the parabolic cohomology groups, denoted by the same symbol by a slight abuse of notation:

$$\mathrm{BD} : \mathrm{MS}^\Gamma(\mathcal{A}_2) \xrightarrow{\sim} \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))). \tag{14}$$

This reduces the problem of understanding $\mathrm{MS}^\Gamma(\mathcal{A}_2)$ to that of classifying the Γ-invariant modular symbols with values in $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$. An element μ of the latter is simply a collection of distributions $\mu\{r,s\}$ on $\mathbb{P}_1(\mathbb{Q}_p)$, indexed by elements $r, s \in \mathbb{P}_1(\mathbb{Q})$, that satisfy the usual modular symbol relations

$$\mu\{r,s\} = -\mu\{s,r\}, \qquad \mu\{r,s\} + \mu\{s,t\} = \mu\{r,t\},$$

together with the equivariance property

$$\int_{\gamma B} h(t)d\mu\{r,s\}(t) = \int_B h(\gamma t)d\mu\{\gamma^{-1}r, \gamma^{-1}s\}(t), \qquad \text{for all } \gamma \in \Gamma. \tag{15}$$

Let w_∞ and w_p be elements of determinant -1 and p respectively in $\mathrm{GL}_2(\mathbb{Z}[1/p])$. Their classes modulo Γ generate the normaliser of Γ, and thus they act as involutions on various cohomology groups for Γ. If μ is an eigensymbol for these involutions, then the invariance property of (15) even holds for all $\gamma \in \mathrm{PGL}_2(\mathbb{Z}[1/p])$:

$$\int_{\gamma B} h(t)d\mu\{r,s\}(t) = \pm \int_B h(\gamma t)d\mu\{\gamma^{-1}r, \gamma^{-1}s\}(t), \qquad \text{for all } \gamma \in \mathrm{PGL}_2(\mathbb{Z}[1/p]), \tag{16}$$

where the sign is given by

$$\mathrm{sgn}_\infty(\mu)^{\mathrm{sgn}(\det(\gamma))} \times \mathrm{sgn}_p(\mu)^{v_p(\det(\gamma))}, \qquad \text{with } \begin{cases} \mu \mid w_\infty &= \mathrm{sgn}_\infty(\mu)\mu, \\ \mu \mid w_p &= \mathrm{sgn}_p(\mu)\mu. \end{cases}$$

3.2. **Restriction to $\mathbb{Z}_p$.** The compact open subset $\mathbb{Z}_p \subset \mathbb{P}_1(\mathbb{Q}_p)$ is a ball whose stabiliser in Γ is the usual congruence group $\Gamma_0(p)$. The restriction map $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)) \longrightarrow \mathcal{D}(\mathbb{Z}_p)$ to the space of distributions on $\mathbb{Z}_p$ therefore induces a map on modular symbols:

$$\mathrm{res}_{\mathbb{Z}_p} : \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)).$$

The target of this map is called the space of *overconvergent modular symbols* of weight two and level p.

Lemma 3.2. *The map* $\mathrm{res}_{\mathbb{Z}_p}$ *is injective.*

Proof. The matrix $\iota_p := \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ interchanges $\mathbb{Z}_p$ and its complement $\mathbb{Z}_p' := \mathbb{P}_1(\mathbb{Q}_p) - \mathbb{Z}_p$, and normalises $\Gamma_0(p)$. It therefore induces mutually inverse isomorphisms

$$\mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) \overset{\iota_p}{\longleftrightarrow} \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p'))$$

for which the diagram

$$\begin{array}{ccc} \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) & \xrightarrow{w_p} & \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \\ \downarrow{\scriptstyle \mathrm{res}_{\mathbb{Z}_p}} & & \downarrow{\scriptstyle \mathrm{res}_{\mathbb{Z}_p'}} \\ \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) & \xrightarrow{\iota_p} & \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p')) \end{array}$$

commutes. In particular, the involution w_p interchanges the kernels of $\mathrm{res}_{\mathbb{Z}_p}$ and of $\mathrm{res}_{\mathbb{Z}_p'}$, and it suffices to show that $\mathrm{res}_{\mathbb{Z}_p'}$ is injective. If μ is in the kernel of $\mathrm{res}_{\mathbb{Z}_p'}$, then

$$\mu\{r,s\}|_{\mathbb{Z}_p'} = 0, \quad \text{for all } r,s \in \mathbb{P}_1(\mathbb{Q}). \tag{17}$$

The domain $\mathbb{P}_1(\mathbb{Q}_p)$ admits a decomposition as a disjoint union of $p+1$ open balls,

$$\mathbb{P}_1(\mathbb{Q}_p) = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_{p-1} \sqcup \mathbb{Z}_p', \tag{18}$$

where $B_j \subset \mathbb{Z}_p$ is the mod p residue disc of $-j$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the collection $\{B_0, B_1, \ldots, \mathbb{Z}_p'\}$. Let $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix satisfying $\mathbb{Z}_p' = \gamma_j B_j$. Then for all $j = 0, \ldots, p-1$, and for all $r, s \in \mathbb{P}_1(\mathbb{Q})$,

$$\mu\{r,s\}|_{B_j} = \mu\{r,s\}|_{\gamma_j^{-1}\mathbb{Z}_p'} = (\mu\{\gamma_j r, \gamma_j s\}|_{\mathbb{Z}_p'})|\gamma_j = 0,$$

where the last equality follows from (17). It now follows from (18) that $\mu\{r,s\} = 0$ as a distribution on $\mathbb{P}_1(\mathbb{Q}_p)$, for all $r, s \in \mathbb{P}_1(\mathbb{Q})$. The lemma follows. □

The space of overconvergent modular symbols is equipped with a Hecke operator U_p, defined explicitly by

$$\int_{\mathbb{Z}_p} h(t) d(U_p\mu)\{r,s\}(t) := \sum_{j=0}^{p-1} \int_{\mathbb{Z}_p} h(\alpha_j^{-1}t) d\mu\{\alpha_j r, \alpha_j s\}(t), \qquad \text{where } \alpha_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}. \tag{19}$$

The space $\mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$ decomposes as a direct sum

$$\mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)) = \mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))^{+} \oplus \mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))^{-},$$

where $\mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))^{\epsilon}$ denotes, for $\epsilon \in \{+,-\}$, the ϵ-eigenspace for the action of the involution w_p. Let $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p=\epsilon}$ denote the space of overconvergent modular symbols on which U_p acts as multiplication by ϵ.

Proposition 3.3. *The map* $\mathrm{res}_{\mathbb{Z}_p}$ *induces Hecke-equivariant inclusions*

$$\mathrm{res}_{\mathbb{Z}_p} : \mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))^{\epsilon} \hookrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p=\epsilon}.$$

Proof. For $j = 0, 1, \ldots, p-1$, let $B_j \subset \mathbb{Z}_p$ denote, as in the proof of Lemma 3.2, the residue class of $-j$ modulo p, so that

$$\mathbb{Z}_p = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_{p-1}, \qquad \alpha_j B_j = \mathbb{Z}_p,$$

with α_j as in (19). By the additivity of the distribution $\mu\{r,s\} \in \mathcal{D}(\mathbb{Z}_p)$, we have, for any locally analytic function h on $\mathbb{Z}_p$:

$$\begin{aligned}\int_{\mathbb{Z}_p} h(t)d\mu\{r,s\}(t) &= \sum_{j=0}^{p-1}\int_{B_j} h(t)d\mu\{r,s\}(t) = \sum_{j=0}^{p-1}\int_{\alpha_j^{-1}\mathbb{Z}_p} h(t)d\mu\{r,s\}(t)\\ &= \epsilon\sum_{j=0}^{p-1}\int_{\mathbb{Z}_p} h(\alpha_j^{-1}t)d\mu\{\alpha_j r,\alpha_j s\}(t),\end{aligned}$$

where the last equality follow from (16) in light of the fact that the matrices $\alpha_j \in \mathrm{PGL}_2(\mathbb{Z}[1/p])$ have determinant p. The proposition now follows from the definition of the U_p operator given in (19). □

Proposition 3.3 shows that U_p preserves the image of $\mathrm{res}_{\mathbb{Z}_p}$ and that the minimal polynomial of its restriction to this space divides $x^2 - 1$. Composing the map BD of (14) with the restriction map $\mathrm{res}_{\mathbb{Z}_p}$ thus gives an injection

$$\mathrm{MS}^{\Gamma}(\mathcal{A}_2) \hookrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p^2=1}. \tag{20}$$

3.3. **Stevens' control theorem.** The "total measure" map $\mathcal{D}(\mathbb{Z}_p) \longrightarrow C_p$ that sends μ to $\mu(1)$ induces a "weight two specialisation map"

$$\rho : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p) \tag{21}$$

that is compatible with the actions of the Hecke operators on both sides.

Theorem 3.4 (Stevens)**.** *The weight two specialisation map* ρ *induces an isomorphism*

$$\rho : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p^2=1} \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p). \tag{22}$$

Proof. The *ordinary subspace* of a Hecke module M is the direct summand of it on which the U_p operator acts with slope zero, and is denoted M^{ord}. The *control theorem for overconvergent modular symbols* (cf. the case $k = 0$ of Theorem 1.1 of [PS11]) asserts that ρ induces an isomorphism

$$\rho^{\mathrm{ord}} : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{\mathrm{ord}} \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p)^{\mathrm{ord}}.$$

But $\mathrm{MS}^{\Gamma_0(p)}(C_p)$ is isomorphic as a Hecke module to the direct sum of an "Eisenstein line" with two copies of the space of modular forms of weight two on $\Gamma_0(p)$. Since all such modular forms are new at p, it follows that U_p^2 acts as the identity on this space, and that

$$\mathrm{MS}^{\Gamma_0(p)}(C_p)^{\mathrm{ord}} = \mathrm{MS}^{\Gamma_0(p)}(C_p)^{U_p^2=1} = \mathrm{MS}^{\Gamma_0(p)}(C_p).$$

The theorem follows. □

Remark 3.5. The statement of Theorem 3.4 relies crucially on the fact that there are no old-forms of weight two for $\Gamma_0(p)$, and would have to be suitably adapted if $\Gamma_0(p)$ was replaced by a congruence subgroup obtained by imposing (say) an additional $\Gamma_1(M)$-level structure with $p \nmid M$. Note however that in general, if Γ is then replaced by the corresponding finite index congruence subgroup Γ' of Γ, [Dar01, part 3 of lemma 1.3] implies that the Hecke algebra acts on $\mathrm{H}^1_f(\Gamma', \mathcal{A}^\times/C_p^\times)$ through its image in the endomorphism ring of the *p-new subspace* of the space of weight two modular forms for $\Gamma_1(M) \cap \Gamma_0(p)$.

Corollary 3.6. *The map*

$$\eta := \rho \circ \mathrm{res}_{\mathbb{Z}_p} \circ \mathrm{BD} : \mathrm{MS}^{\Gamma}(\mathcal{A}_2) \hookrightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p)$$

is injective.

Proof. This follows from Propositions 3.1 and 3.3 combined with Theorem 3.4. □

Our goal in what follows is to show that the map η is surjective as well.

3.4. **The residue map.** Let Ω be a Γ-module.

Definition 3.7. A function $c : \mathcal{T}_1^* \longrightarrow \Omega$ is said to be *harmonic* if it satisfies

$$c(\bar{e}) = -c(e), \text{ for all } e \in \mathcal{T}_1^*, \quad \text{and} \quad \sum_{s(e)=v} c(e) = 0, \text{ for all } v \in \mathcal{T}_0\,.$$

The space of harmonic functions on the set $\mathcal{T}_1^*$ of oriented edges with values in Ω is denoted $C_{\mathrm{har}}(\Omega)$. The action of Γ on $\mathcal{T}$ induces a natural right action of Γ on the space $C_{\mathrm{har}}(\Omega)$. In what follows we will be primarily interested in the case where $\Omega = \mathbb{Z}$ or a bounded subgroup of C_p, equipped with the trivial action of Γ.

Remark 3.8. Elsewhere in the literature (e.g., in [Te90]) it is cusomary to refer to harmonic functions as *harmonic cocycles*. Since the noun "cocycle" already being used in its more standard form in this article, a more transparent terminology was chosen for Definition 3.7.

One can associate to any $f \in \mathcal{A}_2$ a harmonic function $c_f \in C_{\mathrm{har}}(C_p)$ by the rule

$$c_f(e) = \partial_e f(z)dz, \qquad \text{for } e \in \mathcal{T}_1^*,$$

where

$$\partial_e : \mathcal{A}(\mathcal{W}_e) \longrightarrow C_p$$

is the p-adic annular residue on the space of rigid differentials on the oriented annulus $\mathcal{W}_e$. The $\mathrm{PGL}_2(\mathbb{Q}_p)$-equivariant map

$$\partial : \mathcal{A}_2 \longrightarrow C_{\mathrm{har}}(C_p)$$

sending f to c_f is called the *residue map*. The same notation and terminology is used to describe the induced map

$$\partial : \mathrm{MS}^{\Gamma}(\mathcal{A}_2) \longrightarrow \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(C_p)) \tag{23}$$

on modular symbols. Let e_0 denote the standard edge of $\mathcal{T}^*$, whose stabiliser is $\Gamma_0(p)$ and whose associated open ball in $\mathbb{P}_1(\mathbb{Q}_p)$ is $\mathbb{Z}_p$. The evaluation at e_0

$$\mathrm{ev}_{e_0} : C_{\mathrm{har}}(C_p) \longrightarrow C_p$$

is $\Gamma_0(p)$-equivariant, and hence induces a map

$$\mathrm{ev}_{e_0} : \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(C_p)) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p). \tag{24}$$

Our strategy to show the surjectivity of η will be to prove the surjectivity of the maps ev_{e_0} and ∂ that fit into the following commutative diagram:

$$\begin{array}{ccc} \mathrm{MS}^{\Gamma}(\mathcal{A}_2) & \xrightarrow{\partial} & \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(C_p)) \\ {\scriptstyle \mathrm{BD}}\downarrow & \overset{\eta}{\searrow} & \downarrow{\scriptstyle \mathrm{ev}_{e_0}} \\ \mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) & & \\ {\scriptstyle \mathrm{res}_{\mathbb{Z}_p}}\downarrow & & \\ \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) & \xrightarrow{\rho} & \mathrm{MS}^{\Gamma_0(p)}(C_p). \end{array} \tag{25}$$

The surjectivity of ev_{e_0} is elementary:

Lemma 3.9. *The map* $\mathrm{ev}_{e_0} : \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(C_p)) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p)$ *is an isomorphism.*

Proof. The injectivity of ev_{e_0} follows from much the same argument as in the proof of the injectivity of the map $\mathrm{res}_{\mathbb{Z}_p}$ given in Lemma 3.2. Namely, an element c of its kernel satisfies

$$c\{r,s\}(e_0) = 0 \qquad \text{for all } r,s \in \mathbb{P}_1(\mathbb{Q}).$$

Since Γ acts transitively on $\mathcal{T}_1^+$, it then follows from the Γ-equivariance of c that $c\{r,s\}(e) = 0$ for all $e \in \mathcal{T}_1^+$, and hence, for all $e \in \mathcal{T}_1^*$ by the harmonicity of $c\{r,s\}$. To check surjectivity, given $c_0 \in \mathrm{MS}^{\Gamma_0(p)}(C_p)$, define $c \in \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(C_p))$ by setting, for all $e = \gamma^{-1}e_0 \in \mathcal{T}_1^+$,

$$\begin{array}{lcll} c\{r,s\}(e) & := & c_0\{\gamma r, \gamma s\} & \text{for all } r,s \in \mathbb{P}_1(\mathbb{Q}), \\ c\{r,s\}(\bar{e}) & := & -c\{r,s\}(e) & \end{array}$$

Although γ is only well-defined up to left multiplication by elements of $\Gamma_0(p)$, the $\Gamma_0(p)$-invariance of c_0 ensures that the value of $c\{r,s\}(e)$ does not depend on the choice of γ. Since for any vertex $v \in \mathcal{T}_0$ the quantities

$$S_v\{r,s\} := \sum_{s(e)=v} c\{r,s\}$$

define an $\mathrm{SL}_2(\mathbb{Z})$-invariant modular symbol valued in C_p, they must be trivial, whence we see that $c\{r,s\} \in C_{\mathrm{har}}(C_p)$. It follows from the construction that $\mathrm{ev}_{e_0}(c) = c_0$. □

3.5. **The Schneider–Teitelbaum transform.** We now show that the residue map ∂ in (23) is surjective. The main ingredient for achieving this is the integral "p-adic Poisson transform" of Schneider and Teitelbaum which allows one to recover certain elements of $\mathcal{A}_2$ from their associated boundary distributions.

Let $C^b_{\mathrm{har}}(C_p) \subset C_{\mathrm{har}}(C_p)$ denote the subspace of *bounded* harmonic functions, i.e., those whose values lie in a bounded subset of C_p. A element $c \in C^b_{\mathrm{har}}(C_p)$ can be parlayed into a bounded linear functional μ_c on the space of locally constant functions on $\mathbb{P}_1(\mathbb{Q}_p)$, by setting

$$\int_{U_e} 1 d\mu_c := c(e).$$

The boundedness of μ_c implies that it extends uniquely to a *measure* on $\mathbb{P}_1(\mathbb{Q}_p)$, i.e., a continuous functional on the space of continuous functions on $\mathbb{P}_1(\mathbb{Q}_p)$ endowed with the sup norm. This extension exploits the fact that every continuous function $h(t)$ on $\mathbb{P}_1(\mathbb{Q}_p)$ is a uniform limit of locally constant functions to express $\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t) d\mu_c(t)$ as a limit of (finite) Riemann sums.

Proposition 3.10 (Schneider, Teitelbaum)**.** *There is a unique Γ-equivariant splitting of the residue map ∂ on $C^b_{\mathrm{har}}(C_p)$, i.e., a map $\mathrm{ST} : C^b_{\mathrm{har}}(C_p) \longrightarrow \mathcal{A}_2$ for which the diagram*

$$\begin{array}{ccccc} \mathcal{A}_2 & \xrightarrow{\quad\partial\quad} & C_{\mathrm{har}}(C_p) & \longrightarrow & 0 \\ & \overset{\mathrm{ST}}{\searrow} & \cup & & \\ & & C^b_{\mathrm{har}}(C_p) & & \end{array} \tag{26}$$

commutes.

Proof. The map ST is constructed by integrating a "Poisson kernel" against this measure, as in [Te90], namely, one sets

$$\mathrm{ST}(c)(z) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_c(t), \qquad \text{for all } z \in \mathcal{H}_p\,. \tag{27}$$

See [Te90] for more details. □

Lemma 3.11. *The natural inclusion $\mathrm{MS}^\Gamma(C^b_{\mathrm{har}}(C_p)) \hookrightarrow \mathrm{MS}^\Gamma(C_{\mathrm{har}}(C_p))$ is an isomorphism.*

Proof. Given $c \in \mathrm{MS}^\Gamma(C_{\mathrm{har}}(C_p))$, consider its image in $\mathrm{H}^1(\Gamma_0(p), C_p)$ under the map ev_{e_0}. Since $\Gamma_0(p)$ is finitely generated, there is a bounded subset $\Omega \subset C_p$ for which $\mathrm{ev}_{e_0}(c) \in \mathrm{H}^1(\Gamma_0(p), \Omega)$. But then the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\Omega)) & \xrightarrow{\mathrm{ev}_{e_0}} & \mathrm{MS}^{\Gamma_0(p)}(\Omega) \\ \downarrow & & \downarrow \\ \mathrm{MS}^\Gamma(C_{\mathrm{har}}(C_p)) & \xrightarrow{\mathrm{ev}_{e_0}} & \mathrm{MS}^{\Gamma_0(p)}(C_p), \end{array}$$

in which the horizontal arrows are isomorphisms by Lemma 3.9, implies that c belongs to $\mathrm{MS}^\Gamma(C_{\mathrm{har}}(\Omega)) \subset \mathrm{MS}^\Gamma(C_{\mathrm{har}}(C_p))$. □

Corollary 3.12. *The residue map*

$$\partial : \mathrm{MS}^\Gamma(\mathcal{A}_2) \longrightarrow \mathrm{MS}^\Gamma(C_{\mathrm{har}}(C_p))$$

of (23) *is an isomorphism.*

Proof. The injectivity of ∂ is apparent from the fact that the injective map η in the commutative diagram (25) factors through it. Given $c \in \mathrm{MS}^\Gamma(C_{\mathrm{har}}(C_p))$, the harmonic functions $c\{r,s\}$ belong to $C^b_{\mathrm{har}}(C_p)$ for all $r, s \in \mathbb{P}_1(\mathbb{Q})$, by Lemma 3.11. We may therefore set

$$f\{r,s\} := \mathrm{ST}(c\{r,s\}) \in \mathcal{A}_2\,.$$

The assignment $(r,s) \mapsto f\{r,s\}$ is an element of $\mathrm{MS}^\Gamma(\mathcal{A}_2)$ satisfying $\partial(f) = c$, and the result follows. □

Theorem 3.13. *The map η of* (25) *gives a Hecke-equivariant isomorphism*

$$\eta : \mathrm{MS}^\Gamma(\mathcal{A}_2) \xrightarrow{\sim} \mathrm{MS}^{\Gamma_0(p)}(C_p).$$

Proof. This follows immediately from Lemma 3.9 and Corollary 3.12. □

Definition 3.14. The inverse of the isomorphism η, denoted

$$L^{\mathrm{ST}} : \mathrm{MS}^{\Gamma_0(p)}(C_p) \xrightarrow{\sim} \mathrm{MS}^\Gamma(\mathcal{A}_2)$$

is called the *Schneider–Teitelbaum lift.*

3.6. **The multiplicative Schneider–Teitelbaum lift.** The logarithmic derivative gives a natural injection

$$\mathrm{dlog} : \mathcal{A}^\times / C_p^\times \longrightarrow \mathcal{A}_2$$

sending the local section f to f'/f. It induces a map on the space of Γ-invariant modular symbols:

$$\mathrm{dlog} : \mathrm{MS}^\Gamma(\mathcal{A}^\times / C_p^\times) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{A}_2) \tag{28}$$

The space $\mathrm{dlog}(\mathcal{A}^\times) \subset \mathcal{A}_2$ is called the space of *rigid differentials of the third kind* on $\mathcal{H}_p$, and consists of differentials whose image under ∂ are $\mathbb{Z}$-valued harmonic functions on $\mathcal{T}_1^*$. The image of (28) is likewise called the space of rigid analytic modular symbols of the third kind.

Proposition 3.15. *There is a Hecke equivariant morphism*

$$L^\times_{\mathrm{ST}} : \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z}) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{A}^\times / C_p^\times)$$

for which the following diagram commutes:

$$\begin{array}{ccc} \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z}) & \xrightarrow{L^\times_{\mathrm{ST}}} & \mathrm{MS}^\Gamma(\mathcal{A}^\times / C_p^\times) \\ \big\downarrow & & \big\downarrow \mathrm{dlog} \\ \mathrm{MS}^{\Gamma_0(p)}(C_p) & \xrightarrow{L_{\mathrm{ST}}} & \mathrm{MS}^\Gamma(\mathcal{A}_2) \end{array}$$

The map $L^\times_{\mathrm{ST}}$ is called the *multiplicative Schneider–Teitelbaum lift.* It is constructed, following [Dar01, §3.3], as a multiplicative refinement of (27), as we now describe. First note that a class

$$\phi \in \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$$

gives rise to a unique element $\mu_\phi \in \mathrm{MS}^\Gamma(\mathcal{D})$, where $\mathcal{D}$ is the module of $\mathbb{Z}$-valued distributions on $\mathbb{P}_1(\mathbb{Q}_p)$. This element is characterised by the properties

$$\begin{aligned} \mu_\phi\{r,s\}(\mathbb{Z}_p) &= \phi\{r,s\}, \qquad \text{for all } r,s \in \mathbb{P}_1(\mathbb{Q}), \\ \mu_\phi\{\gamma r, \gamma s\}(\gamma U) &= \mu_\phi\{r,s\}(U), \qquad \text{for all compact open } U \subset \mathbb{P}_1(\mathbb{Q}_p), \text{ and } \gamma \in \Gamma. \end{aligned}$$

The existence and uniqueness of μ_ϕ exploits the fact that Γ acts transitively on the compact open balls in $\mathbb{P}_1(\mathbb{Q}_p)$ and that the stabiliser of $\mathbb{Z}_p$ is $\Gamma_0(p)$. Given $n \geq 1$ and $a := [a_1\colon a_2] \in \mathbb{P}_1(\mathbb{Z}/p^n\mathbb{Z})$, let

$$B(a,n) := \{(z_1\colon z_2) \in \mathbb{P}_1(\mathbb{Q}_p) \qquad \text{such that } |a_1 z_2 - a_2 z_1| \leq p^{-n}\} \subset \mathbb{P}_1(\mathbb{Q}_p)$$

be the "mod p^n residue disc centered at a", and write

$$\mathscr{D}_{\phi,n}\{r,s\} = \sum_{a \in \mathbb{P}_1(\mathbb{Z}/p^n\mathbb{Z})} \mu_\phi\{r,s\}(B(a,n)) \cdot \tilde{a} \quad \in \quad \mathrm{Div}^0(\mathbb{P}_1(\mathbb{Q}_p)),$$

where $\tilde{a}$ is any sample point in $B(a,n)$. Then we have

Lemma 3.16. *Let $J := \mathrm{ST}^\times(\phi)$. Then*

$$J_{v_\circ}\{r,s\} = \lim_{n \longrightarrow \infty} [z; \mathscr{D}_{\phi,n}\{r,s\}].$$

Proof. The convergence, modular symbol and $\mathrm{SL}_2(\mathbb{Z})$-equivariance properties of the right hand side follow immediately from the similar properties for the divisors $\mathscr{D}_{\phi,n}\{r,s\}$, using Lemma 2.2. The lemma follows from a comparison of divisors. □

This definition of the RM values of theta-symbols may have seemed a bit contrived, but finds a modicum of justification in the following proposition.

Proposition 3.17. *If $J = \mathrm{ST}^\times(\phi)$, and τ is an RM point, then $J[\tau]$ belongs to $\mathcal{O}_{C_p}^\times$.*

Proof. If τ is unramified and even, assume without loss of generality that τ reduces to $v_\circ$, so that

$$J[\tau] = J_{v_\circ}\{\infty, \gamma_\tau \infty\}(\tau).$$

But since τ belongs to $\mathcal{H}_p^\circ$, the quantities

$$[\tau; \mathscr{D}_{\phi,n}\{\infty, \gamma_\tau\infty\}]$$

are p-adic units, and therefore so is their limit $J[\tau]$ as $n \longrightarrow \infty$. If τ is unramified and odd, a similar argument applies after replacing the even vertex $v_\circ$ by some fixed odd vertex. Finally, when τ is ramified, assume without loss of generality that it reduces to the standard edge $e_\circ = (v_\circ, v_1)$. A direct calculation shows that

$$\mathrm{ord}_p(J_{v_\circ}\{r,s\}(\tau)) = \frac{1}{2}\phi\{r,s\}, \qquad \mathrm{ord}_p(J_{v_1}\{r,s\}(\tau)) = -\frac{1}{2}\phi\{r,s\},$$

and the proposition follows from the definition of $J[\tau]$ for ramified τ. □

3.7. **Classification of analytic theta-cocycles.** Using the multiplicative Schneider–Teitelbaum lift, one may now classify the space of rigid analytic theta-cocycles $\mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times)\otimes\mathbb{Q}$.

There are precisely two conjugacy classes of parabolic subgroups of $\Gamma_0(p)$, the group P_∞ consisting of the upper triangular matrices, stabilising the cusp ∞, and the group P_0 consisting of lower triangular matrices, stabilising the cusp 0. The $\Gamma_0(p)$-module $\mathcal{F}(\mathbb{P}_1(\mathbb{Q}),\mathbb{Q})$ of $\mathbb{Q}$-valued functions on $\mathbb{P}_1(\mathbb{Q})$ therefore decomposes as a direct sum of the two induced modules

$$\mathcal{F}(\mathbb{P}_1(\mathbb{Q}),\mathbb{Q}) \;=\; \mathrm{Ind}_{P_\infty}^{\Gamma_0(p)}\mathbb{Q} \,\oplus\, \mathrm{Ind}_{P_0}^{\Gamma_0(p)}\mathbb{Q},$$

and the $\Gamma_0(p)$-cohomology of the exact sequence (11) with $\Omega=\mathbb{Q}$ leads to a short exact sequence

$$0\longrightarrow\mathbb{Q}\longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Q})\longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma_0(p),\mathbb{Q})\longrightarrow 0. \tag{29}$$

The one-dimensional kernel is spanned by the boundary symbol $m_\sharp\in\mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$ defined by

$$m_\sharp\{r,s\}=\begin{cases} 0 & \text{if } \Gamma_0(p)r=\Gamma_0(p)s,\\ 1 & \text{if } r\in\Gamma_0(p)0, \quad s\in\Gamma_0(p)\infty,\\ -1 & \text{if } r\in\Gamma_0(p)\infty, \quad s\in\Gamma_0(p)0.\end{cases}$$

It accounts for the discrepancy between modular symbols and parabolic cohomology classes. Using the multiplicative Schneider–Teitelbaum transform above, we therefore see that the space of modular theta symbols $\mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)\otimes\mathbb{Q}$ is generated by

- The boundary theta cocycle attached to the boundary symbol $m_\sharp$ defined above
- The modular cocycles attached to the cohomology classes of newforms in $S_2(\Gamma_0(p))$.

To obtain non-parabolic classes, note that the residue map induces an exact sequence

$$0\longrightarrow\mathbb{Q}\longrightarrow\mathbb{Q}\otimes\mathrm{H}^1(\Gamma,\mathcal{A}^\times/C_p^\times)\longrightarrow\mathrm{H}^1(\Gamma_0(p),\mathbb{Q})\longrightarrow 0,$$

It is classical that the space $\mathrm{H}^1(\Gamma_0(p),\mathbb{Q})$ is of dimension $2g+1$, where g is the genus of $X_0(p)$, and has a $2g$-dimensional parabolic subspace. Thus the space of analytic theta-cocycles

$$\mathbb{Q}\otimes\mathrm{H}^1(\Gamma,\mathcal{A}^\times/C_p^\times)$$

is of dimension $2g+2$, and the quotient by the parabolic classes is a line, accounted for by the Dedekind–Rademacher cocycle below. We now discuss all these classes in more detail.

3.7.1. *The boundary theta-symbol.* The simplest example of a theta-symbol is the so-called *boundary theta-symbol* described by fixing a base point $\eta\in\mathbb{P}_1(C_p)$ and setting

$$J_\sharp\{r,s\}(z):=[(z)-(\eta);(r)-(s)]. \tag{30}$$

The boundary theta-symbol $J_\sharp$ is the multiplicative Schneider–Teitelbaum lift of the boundary symbol $m_\sharp$. When viewed as a parabolic theta-cocycle, it lies in the kernel of the residue map to $\mathrm{H}^1(\Gamma_0(p),\mathbb{Z})$. Note that it is easily seen to lift to an element of $\mathrm{H}^1(\Gamma,\mathcal{A}^\times)$ defined by

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\;\longmapsto\;(cz+d)^{-1}$$

Because $J_\sharp$ is very simple, taking values in the group of rational functions on $\mathbb{P}_1$, its RM values are not very interesting. If $F(x,s) = Ax^2+Bxy+Cy^2$ is a binary quadratic form of discriminant $D = B^2 - 4AC$, then its root is $\tau_F = (-B+\sqrt{D})/2A$, while its stabiliser is generated by

$$\gamma_F = \begin{pmatrix} u - Bv & -2Cv \\ 2Av & u + Bv \end{pmatrix}, \qquad u^2 - Dv^2 = 1,$$

where $u + v\sqrt{D}$ is a fundamental solution to Pell's equation. A straightforward calculation shows that for any $r \in \mathbb{P}_1(\mathbb{Q})$ we have

$$J_\sharp[\tau_F] = J_\sharp\{r, \gamma_\tau r\}(\tau_F) = u \pm v\sqrt{D} \pmod{\mathbb{Z}[1/p]^\times}.$$

It follows that the cocycle $J_\sharp$ has algebraic RM values, albeit somewhat uninteresting ones, since they always belong to the field of "real multiplication". To obtain more interesting class invariants we now consider the RM values of analytic theta-cocycles arising from the multiplicative Schneider–Teitelbaum lifts of richer elements of $\mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$.

3.7.2. *Modular theta-symbols.* Let $f \in S_2(\Gamma_0(p))$ be a normalised cuspidal newform with Fourier coefficients in a field $K_f \subset \mathbb{R}$, and let $\omega_f := 2\pi i f(z)dz$ be the associated regular differential on $X_0(p)$. The real analytic differentials

$$\omega_f^+ := \frac{1}{2}(\omega_f + \bar{\omega}_f), \qquad \omega_f^- := \frac{1}{2}(\omega_f - \bar{\omega}_f)$$

give rise to modular symbols φ_f^+ and $\varphi_f^- \in \mathrm{MS}^{\Gamma_0(p)}(K_f)$, defined by

$$\varphi_f^+\{r,s\} := (\Omega_f^+)^{-1}\int_r^s \omega_f^+, \qquad \varphi_f^-\{r,s\} = (\Omega_f^-)^{-1}\int_r^s \omega_f^-, \tag{31}$$

where Ω_f^+ and Ω_f^- are so-called *real and imaginary periods* attached to f, which can be chosen in such a way that $\varphi_f^\pm$ is K_f-valued. This only determines these modular symbols, and the periods $\Omega_f^\pm$, up to multiplication by $K_f^\times$, and we always have

$$\Omega_f^+\Omega_f^- = \Omega_f := \langle f, f\rangle = \langle \omega_f^+, \omega_f^-\rangle \pmod{K_f^\times}.$$

We can further insist that $\varphi_f^\pm$ takes values in $\mathcal{O}_{K_f}$, and maps surjectively to $\mathbb{Z}$ when f has integer Fourier coefficients, which then determines Ω_f^- and Ω_f^+ up to a sign.

The multiplicative Schneider–Teitelbaum lifts of these morphisms are denoted

$$J_f^+, J_f^- \in \mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \otimes \mathcal{O}_{K_f}.$$

They are called the *modular theta-symbols* attached to f. Recall again that we can exploit the natural injection

$$\mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times)$$

to view $J_f^\pm$ as elements of the latter group when this is convenient. Assume now for simplicity that f has rational Fourier coefficients and hence corresponds to an elliptic curve E_f of conductor p, and let $\Phi_{\mathrm{Tate}} : C_p^\times \longrightarrow E_f(C_p)$ be the Tate p-adic uniformistion of E_f. One of the main conjectures of [Dar01] is

Conjecture 3.18. *After eventually replacing the theta-symbols $J_f^{\pm}$ by suitable powers, the local points $\Phi_{\mathrm{Tate}}(J_f^{\pm}[\tau]) \in E(C_p)$ are defined over H_τ, for all $\tau \in \mathcal{H}_p^{\mathrm{RM},\circ}$.*

The algebraicity of Stark–Heegner points attached to genus characters was proved in [BD09], and later extended in [Mo11]. The general case remains open.

3.7.3. *The Dedekind–Rademacher theta-cocycle.* Let

$$E_2^{(p)}(z) = \frac{p-1}{12} + 2\sum_{n\geq 1} \sigma_1^{(p)}(n) e^{2\pi i n z}, \qquad \text{where } \sigma_1^{(p)}(n) = \sum_{p\nmid d \mid n} d$$

be the weight two Eisenstein series on $\Gamma_0(p)$, and let $\omega_{\mathrm{Eis}} := 2\pi i E_2^{(p)}(z)dz$ be the associated regular differential on the open modular curve $Y_0(p)$. The periods of ω_{Eis} are encoded in the *Dedekind–Rademacher homomorphism* $\varphi_{\mathrm{DR}} : \Gamma_0(p) \longrightarrow \mathbb{Z}$ defined by

$$\varphi_{\mathrm{DR}} := (2\pi i)^{-1} \int_{z_0}^{\gamma z_0} \omega_{\mathrm{Eis}}. \tag{32}$$

The *Dedekind–Rademacher cocycle* $J_{\mathrm{DR}} \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times)$ was defined in [DPV2, § 1]. Its image under the residue map is equal to φ_{DR}. It is a prototypical instance of a non-parabolic theta-cocycle, and in fact generates (up to finite index) the cokernel of the natural inclusion

$$\mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times).$$

Concerning the RM values of J_{DR}, one has the following

Theorem 3.19. *For all $\tau \in \mathcal{H}_p^{\mathrm{RM},\circ}$, the value $J_{\mathrm{DR}}[\tau]$ is a p-unit in the class field H_τ attached to τ, and generates it if the order attached to τ does not admit a unit of norm -1.*

This algebraicity result follows from the proof of the p-adic Gross-Stark conjecture given in [DDP11] and from the proof of Gross's "tower of norms" conjecture in [DK]. An independent and somewhat more direct approach based on the antiparallel deformations of p-irregular Hilbert modular Eisenstein series and modeled on the "analytic proof" of the factorisation of norms of singular moduli [GZ85] is described in [DPV2].

3.8. **Toric theta-cocycles.** If $(r,s) \in \mathbb{P}_1(\mathbb{Q})^2$ is an ordered pair, its stabiliser in $\mathrm{GL}_2(\mathbb{Q})$ is a split torus whose $\mathbb{Z}[1/p]$-points are a group of rank one, generated up to finite index by a loxodromic transformation with eigenvalues of the form $p^{\pm e}$. This section asssociates to such an ordered pair a rigid analytic theta-cocycle, referred to as a *toric cocycle*. Unlike the Hecke eigensymbols described in previous subsections, $J_{r,s}$ is not an eigenvector for the action of the Hecke operators, but admits an explicit description which shall be useful in later calculations.

If ξ_1 and ξ_2 are two points of the extended upper-half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}_1(\mathbb{Q})$, the symbol $[\xi_1, \xi_2]$ is used to denote the hyperbolic geodesic segment on $\mathcal{H}$ going from ξ_1 to ξ_2. The intersection of two (open or closed) geodesic segments on $\mathcal{H}$ is defined in the natural way. A point $\xi \in \mathcal{H}^*$ is said to be (r,s)-*admissible* if it does not lie on any geodesic in $\Gamma[r,s]$. Clearly, all p-admissible ξ belong to $\mathcal{H}$, and the set of p-admissible base points is preserved by the action of Γ. Since the non-admissible points are contained in a countable union of sets of measure zero, the existence of admissible base points is clear.

Let $\Sigma_{r,s}$ denote the Γ-orbit of the pair (r,s), and let $\Sigma_{r,s}^{(m)} \subset \Sigma_{r,s}$ be the subset of pairs (u,v) with $\mathrm{ord}_p(\det(u,v)) = m$. It is not hard to see that $\Sigma_{r,s}^{(m)}$ is non-empty for all $m \geq 0$ and that

$$\Sigma_{r,s} = \bigcup_{m=0}^{\infty} \Sigma_{r,s}^{(m)}.$$

Fix a base point $\eta \in \mathcal{H}_p$ and an (r,s)-admissible base point $\xi \in \mathcal{H}$, and set

$$J_{r,s}(\gamma)(z) = \prod_{(u,v)\in\Sigma_{r,s}} [(z)-(\eta);(u)-(v)]^{[u,v]\cdot[\xi,\gamma\xi]} := \prod_{m=0}^{\infty} J_{r,s}^{(m)}(\gamma)(z),$$

where

$$J_{r,s}^{(m)}(\gamma)(z) := \prod_{(u,v)\in\Sigma_{r,s}^{(m)}} [(z)-(\eta);(u)-(v)]^{[u,v]\cdot[\xi,\gamma\xi]}.$$

Lemma 3.20. *For each $\gamma \in \Gamma$, the infinite product defining $J_{r,s}(\gamma)$ converges to a rigid analytic function on $\mathcal{H}_p$ and its image in $\mathcal{A}^\times/C_p^\times$ satisfies a cocycle condition modulo scalars, namely*

$$J_{r,s}(\gamma_1\gamma_2) = J_{r,s}(\gamma_1) \times \gamma_1 \cdot J_{r,s}(\gamma_2).$$

Proof. Observe first that $\Gamma_\circ := \mathrm{SL}_2(\mathbb{Z})$ acts on the set $\Sigma_{r,s}^{(m)}$ by Möbius transformations, and that there are finitely many orbits for this action:

$$\Sigma_{r,s}^{(m)} \quad = \quad \Gamma_\circ\cdot(r_1,s_1) \ \sqcup\ \Gamma_\circ\cdot(r_2,s_2) \ \sqcup\ \cdots\ \sqcup\ \Gamma_\circ\cdot(r_\ell,m_\ell).$$

But the cardinality of the set

$$\{(u,v)\in\Gamma_\circ(r_j,s_j) \text{ such that } [u,v]\cdot[\xi,\gamma\xi] = \pm 1\}$$

represents the number of intersection points between the images of the geodesics $[r_j,s_j]$ and $[\xi,\gamma\xi]$ in the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$. Since this number is finite, it follows that the product defining $J_{r,s}^{(m)}(\gamma)(z)$ has finitely many factors that are $\neq 1$, and hence is a rational function of z. To prove convergence of

$$J_{r,s}(\gamma)(z) := \prod_{m=0}^{\infty} J_{r,s}^{(m)}(\gamma)(z)$$

as a rigid meromorphic function of $z \in \mathcal{H}_p^{\leq n}$, it suffices to show that the restriction of $J_{r,s}^{(m)}(\gamma)$ to $\mathcal{H}_p^{\leq n}$ converges uniformly to 1 as $m \longrightarrow \infty$. But this follows directly from Lemma 2.2. We have hence showed that the infinite product defining $J_{r,s}(z)$ converges absolutely and uniformly on affinoid subsets of $\mathcal{H}_p$. The cocycle condition for $J_{r,s}$ modulo scalars also follows by a direct calculation. □

Lemma 3.21. *The class of $J_{r,s}$ in $\mathrm{H}^1(\Gamma,\mathcal{A}^\times/C_p^\times)$ does not depend on the choice of base point $\eta\in\mathcal{H}_p$ and of admissible base point $\xi\in\mathcal{H}$ that were made to define it.*

Proof. Changing the base point η to η' merely multiplies the functions $J_{r,s}(\gamma)$ by a non-zero scalar, and hence does not affect the cocycle $J_{r,s} \in Z^1(\Gamma,\mathcal{A}^\times/C_p^\times)$. As for replacing ξ by ξ'

in the definition of $J_{r,s}$, a direct calculation reveals that the associated cocycles differ by the coboundary dF, where

$$F(z) = \prod_{(u,v)\in\Sigma_{r,s}} [(z)-(\eta);(u)-(v)]^{[u,v]\cdot[\xi,\xi']} \in \mathcal{A}^\times .$$

□

We now ask whether the theta-cocycle $J_{r,s} \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times)$ lifts to a genuine analytic cocycle

$$\tilde{J}_{r,s} \overset{?}{\in} \mathrm{H}^1(\Gamma, \mathcal{A}^\times).$$

As mentioned before, the cocycle $J_{r,s}$ need not admit such a lift, but its restriction to $\Gamma_\circ := \mathrm{SL}_2(\mathbb{Z})$ does, by the essential triviality of $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), C_p^\times)$. To describe such a lift, we first restrict $J_{r,s}$ to $\mathrm{H}^1(\Gamma_\circ, \mathcal{A}^\times/C_p^\times)$ and construct an explicit lift of it to a class

$$J^\circ_{r,s} \in \mathrm{H}^1(\Gamma_\circ, \mathcal{A}^\times).$$

The first part of Lemma 2.2 suggests that replacing $[(z)-(\eta);(r)-(s)]$ by $[z;(r)-(s)]$ in the definition of $J_{r,s}$ leads to an infinite product which need not converge in general. However, we have

Proposition 3.22. *For all $\gamma \in \Gamma_\circ$, the infinite product*

$$J^\circ_{r,s}(\gamma)(z) := \prod_{m=0}^{\infty} J^{\circ,(m)}_{r,s}(\gamma)(z), \qquad \text{where } J^{\circ,(m)}_{r,s}(\gamma)(z) := \prod_{(u,v)\in\Sigma^{(m)}_{r,s}} ([z;(u)-(v)])^{[u,v]\cdot[\xi,\gamma\xi]}$$

converges to a rigid analytic function on $\mathcal{H}_p$, up to 12-th roots of unity, and gives rise to an element of $\mathrm{H}^1(\Gamma_\circ, \mathcal{A}^\times/\mu_{12})$.

Proof. For integers $m > n \geq 0$, consider the restriction of $J^{\circ,(m)}_{r,s}(\gamma)(z)$ to the affinoid $\mathcal{H}_p^{\leq n}$. By Lemma 2.2, this restriction is constant modulo p^{m-n} and hence its mod p^{m-n} reduction defines a cocycle in $\mathrm{H}^1(\Gamma_\circ, (\mathbb{Z}/p^{m-n}\mathbb{Z})^\times)$. Since the abelianisation of $\Gamma_\circ$ is of order 12, it follows that

$$J^{\circ,(m)}_{r,s}(\gamma)(z)|_{\mathcal{H}_p^{\leq n}} \in \mu_{12} \pmod{p^{m-n}}.$$

The convergence of the infinite product defining $J^\circ_{r,s}$ (up to 12-th roots of unity) follows. The fact that it is a cocycle for $\Gamma_\circ$ follows from the $\Gamma_\circ$-equivariance of the expression $[z;(u)-v)]$. □

This proposition gives some information about the RM values of the theta-cocycle $J_{r,s}$:

Corollary 3.23. *For all RM points $\tau \in \mathcal{H}_p$, the value $J_{r,s}[\tau]$ belongs to $\mathcal{O}_{C_p}^\times$.*

Proof. If τ is an unramified point, we can assume without loss of generality, by translating it by an appropriate element of $\mathrm{GL}_2(\mathbb{Z}[1/p])$, that it belongs to the standard affinoid $\mathcal{H}_p^\circ$. The definition of this affinoid implies that

$$[\tau;(r)-(s)] \in \mathcal{O}_{C_p}^\times,$$

and hence that $J^{\circ,(m)}_{r,s}(\gamma_\tau)(\tau)$ is a p-adic unit, for all m. The corollary follows in this case. The proof for ramified τ proceeds along similar lines. □

Let us now compute the image of the theta-cocycle $J_{r,s}$ in $\mathrm{H}^1(\Gamma_0(p),\mathbb{Z})$ under the residue map. Let $\Sigma_{r,s}^{(00)}$ be the subset of $\Sigma_{r,s}^{(0)}$ consisting of pairs (u,v) satisfying

$$u \notin \mathbb{Z}_p, \qquad v \in \mathbb{Z}_p\,.$$

This subset is nonempty, stable under the action of $\Gamma_0(p)$, and breaks up as a finite union of $\Gamma_0(p)$-orbits:

$$\Sigma_{r,s}^{(00)} = \Gamma_0(p)(r_1,s_1) \cup \cdots \cup \Gamma_0(p)(r_t,s_t).$$

Let $[r_j,s_j]_{X_0(p)}$ be the image of the geodesic path $[r_j,s_j]$ on $\mathcal{H}^*$ in the relative homology of $X_0(p)$ relative to the cusps. It does not depend on the choice of orbit representatives, and their sum depends only on (r,s). Let $\varphi_{r,s} : \Gamma_0(p) \longrightarrow \mathbb{Z}$ be the homomorphism defined by

$$\varphi_{r,s}(\gamma) = ([r_1,s_1]_{X_0(p)} + \cdots + [r_t,s_t]_{X_0(p)}) \cdot \gamma,$$

where $\cdot$ denotes the intersection pairing

$$\mathrm{H}_1(X_0(p);\{0,\infty\},\mathbb{Z}) \;\times\; \mathrm{H}_1(Y_0(p),\mathbb{Z}) \;\longrightarrow\; \mathbb{Z}\,.$$

Proposition 3.24. *The image of $J_{r,s}$ under the residue map is equal to $\varphi_{r,s}$.*

Proof. Recall the standard annulus U of (5) having $\Gamma_0(p)$ as its stabiliser in Γ. The residue map takes a cocycle $J \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times)$ to the homomorphism

$$\phi_J : \Gamma_0(p) \longrightarrow \mathbb{Z}, \qquad \phi_J(\gamma) := \mathrm{res}_U(\mathrm{dlog}\, J(\gamma)),$$

where res_U is the p-adic annular residue attached to U. Consider the infinite product expression of Proposition 3.22 for $J_{r,s}$ and observe that the terms $\mathrm{dlog}\, J_{r,s}^{(m)}(\gamma)$ for $m \geq 1$ contribute nothing to the annular residue at U: indeed, two cusps u,v for which $p|\det(u,v)$ belong to the same component of the complement of U, and hence $\mathrm{res}_U(\mathrm{dlog}\, t_{u,v}(z)) = 0$ for such pairs. On the other hand, if (u,v) belongs to $\Sigma_{r,s}^{(0)}$, then

$$\mathrm{res}_U(\mathrm{dlog}[z;(u)-(v)]) = \begin{cases} 1 & \text{if } u \notin \mathbb{Z}_p, v \in \mathbb{Z}_p, \\ -1 & \text{if } u \in \mathbb{Z}_p, v \notin \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, any pair (u,v) for which the residue of $\mathrm{dlog}[z;(u)-(v)]$ is equal to 1 (resp. -1) is of the form $(\alpha r_j, \alpha s_j)$, (resp. $(\alpha s_j, \alpha r_j)$) for some $\alpha \in \Gamma_0(p)$. It follows that

$$\begin{aligned} \mathrm{res}_U(\mathrm{dlog}\, J_{r,s}(\gamma)) &= \sum_{j=1}^{t} \sum_{\alpha \in \Gamma_0(p)} (+1)[\alpha r_j, \alpha s_j] \cdot [\xi, \gamma\xi] \;+\; \sum_{\alpha \in \Gamma_0(p)} (-1)[\alpha s_j, \alpha r_j] \cdot [\xi, \gamma\xi] \\ &= 2 \sum_{j=1}^{t} \sum_{\alpha \in \Gamma_0(p)} [\alpha r_j, \alpha s_j] \cdot [\xi, \gamma\xi]. \end{aligned}$$

In this last expression one can recognise the intersection product of the relative homology class attached to $[r_1,s_1] + \cdots [r_j,s_j]$ with the class of γ in $\mathrm{H}_1(Y_0(p),\mathbb{Z})$. The proposition follows. □

4. Meromorphic theta-symbols

Following [DV21], this section recalls the construction of certain *meromorphic* theta-symbols

$$J_\tau \in \mathrm{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$$

which are indexed by orbits of RM points $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$ and generate, along with the space of analytic theta symbols $\mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$, the group of meromorphic theta-symbols up to torsion.

The RM values of meromorphic theta symbols appear to be of great arithmetic significance, and yield in particular the singular moduli for real quadratic fields that were introduced in [DV21]. In the current paper, we investigate instead their *toric values*, defined in § 5, which are of central importance to the Gross–Kohnen–Zagier theorem obtained in § 6.

4.1. **Divisors.** Let $\tau \in \mathcal{H}_p(C_p)$ be an RM point of p-discriminant D, and let F be the real quadratic field that it generates. The element τ is the root of a unique (up to sign) primitive integral binary quadratic form, whose discriminant is of the form Dp^{2n} with $n \geq 0$. Our fixed complex and p-adic embedding of $\bar{\mathbb{Q}}$ allows us to view F simultaneously as a subfield of $\mathbb{R}$ and of C_p.

For $w \in F$, let (w, w') denote the geodesic on $\mathcal{H}$ joining w to its Galois conjugate w', which maps to a countable union of copies of the same basic closed geodesic on $\Pi\backslash\mathcal{H}$ for any subgroup $\Pi \subset \Gamma$. Likewise, if r, s are elements of $\mathbb{P}_1(\mathbb{Q})$, recall that $[r, s]$ denotes the hyperbolic geodesic on $\mathcal{H}$ joining these two elements, which maps to a compact (but not necessarily closed) geodesic on any quotient $\Pi\backslash\mathcal{H}$. The geodesics (w, w') and $[r, s]$ always intersect properly, and we set

$$\delta_{r,s}(w) := (w, w') \cdot [r, s] = \begin{cases} 1 & \text{if the two geodesics intersect positively;} \\ -1 & \text{if they intersect negatively;} \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

The infinite formal sum

$$\Delta_\tau\{r, s\} := \sum_{w \in \Gamma\tau} \delta_{r,s}(w) \cdot (w) \tag{34}$$

defines a Γ-invariant modular symbol with values in the Γ-module $\mathbb{Z}\langle\Gamma\tau\rangle$ of (possibly infinite) $\mathbb{Z}$-linear combinations of points of $\Gamma\tau$. Set

$$\Delta_\tau^{\leq n}\{r, s\} := \sum_{w \in (\Gamma\tau) \cap \mathcal{H}_p^{\leq n}} \delta_{r,s}(w) \cdot (w),$$

and define $\Delta_\tau^{<n}\{r, s\}$ analogously, replacing $\mathcal{H}_p^{\leq n}$ by $\mathcal{H}_p^{<n}$.

Lemma 4.1. *The expressions $\Delta_\tau^{\leq n}\{r, s\}$ and $\Delta_\tau^{<n}\{r, s\}$ are degree zero divisors on $\mathcal{H}_p$.*

Proof. The RM points w in the support of $\Delta_\tau^{\leq n}\{r, s\}$ consist of zeroes of binary quadratic forms of discriminant Dp^{2m} with $1/2\,\mathrm{ord}_p(D) + m \leq n$ for which $\delta_{r,s}(w) \neq 0$, i.e., for which the geodesic (w, w') intersects $[r, s]$ non-trivially. There are finitely many binary quadratic forms of a given discriminant satisfying this condition, and hence $\Delta_\tau^{\leq n}\{r, s\}$ belongs to $\mathrm{Div}(\mathcal{H}_p^{\mathrm{RM}})$ for all $(r, s) \in \mathbb{P}_1(\mathbb{Q})^2$. Since $\mathcal{H}_p^{\leq n}$ is preserved by the action of $\mathrm{SL}_2(\mathbb{Z})$, the function

$$(r, s) \mapsto \deg \Delta_\tau^{\leq n}\{r, s\}$$

defines an element in $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathbb{Z})$. Since this space of modular symbols is trivial, it follows that $\deg \Delta_\tau^{\leq n}\{r,s\}$ is identically zero for all n, τ and (r,s). The proof for $\Delta_\tau^{<n}\{r,s\}$ is the same. □

If v is a vertex of $\mathcal{T}$, recall the wide open subset $\mathcal{W}_v$ of $\mathcal{H}_p$ associated to v, consisting of the points whose image under the reduction map is either v or an edge containing v. The same reasoning as in the proof of Lemma 4.1 shows that the expression

$$\Delta_\tau^v\{r,s\} := \sum_{w\in(\Gamma\tau)\cap\mathcal{W}_v} \delta_{r,s}(w)\cdot(w)$$

is a degree zero divisor. If τ is unramified, then for all $n\geq 0$ we have

$$\Delta_\tau^{\leq n}\{r,s\} = \sum_{v\in\mathcal{V}(\mathcal{T}^{\leq n})} \Delta_\tau^v\{r,s\}.$$

When τ is a ramified RM point, and hence reduces to an edge of $\mathcal{T}$, the situation is a bit more subtle. Namely, we can write $\Delta_\tau^{\leq n}\{r,s\} = \Delta_\tau^{<n}\{r,s\}$ as

$$\Delta_\tau^{<n}\{r,s\} = \sum_{v\in\mathcal{V}(\mathcal{T}^{<n})} \Delta_\tau^v\{r,s\},$$

with the righthand sum ranging only over the *even* vertices of $\mathcal{T}$ when n is odd, and over the odd vertices of $\mathcal{T}$, when n is even.

4.2. **Definitions and convergence.** To define the rigid meromorphic theta symbols J_τ, one starts from the association $(r,s)\mapsto\Delta_\tau^{\leq n}\{r,s\}$ which yields, by Lemma 4.1, an element of the group of $\mathrm{SL}_2(\mathbb{Z})$-invariant modular symbols valued in divisors:

$$\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathrm{Div}^0(\mathcal{H}_p^{\leq n})).$$

These modular symbols are now upgraded to meromorphic theta symbols. Since the combinatorial structures are different, the cases where τ is unramified or ramified are treated separately.

Suppose first that the RM point τ in $\mathcal{H}_p$ is *unramified*. Recall the symbols $[\mathscr{D}_1,\mathscr{D}_2]$ defined in (8) for a pair of divisors $\mathscr{D}_1,\mathscr{D}_2$. Fix an auxiliary base point $\eta\in\mathcal{H}_p$.

Lemma 4.2. *Assume that τ is unramified. The rational functions*

$$J_\tau^{\leq n}\{r,s\}(z) := [(z)-(\eta);\Delta_\tau^{\leq n}\{r,s\}]$$

converge uniformly on affinoid subsets of $\mathcal{H}_p$ to a rigid analytic function $J_\tau\{r,s\}(z)$ on $\mathcal{H}_p$.

Proof. Let $\mathcal{A}\subset\mathcal{H}_p^{\leq N}$ be some affinoid subset of $\mathcal{H}_p$, and assume without loss of generality (by enlarging $\mathcal{A}$ if necessary) that $\eta\in\mathcal{A}$. For all $N<n<m$,

$$\begin{aligned} J_\tau^{\leq m}\{r,s\}(z) \div J_\tau^{\leq n}\{r,s\}(z) &= [(z)-(\eta);\Delta_\tau^{\leq m}\{r,s\}-\Delta_\tau^{\leq n}\{r,s\}] \\ &= \prod_{n<d(v,v_\circ)\leq m} [(z)-(\eta);\Delta_\tau^v\{r,s\}]. \end{aligned} \tag{35}$$

By Lemma 2.1, each factor in this product belongs to $1+p^{n-N}\,\mathcal{O}_{C_p}$ as z ranges over $\mathcal{H}_p^{\leq N}\supset\mathcal{A}$. It follows that the sequence $(J_\tau^{\leq n}\{r,s\})_{n\geq 1}$ of partial products is uniformly Cauchy when restricted to $\mathcal{A}$, and hence converges to a rigid meromorphic function on this domain. □

The proof of Lemma 4.2 yields the following corollary giving a more flexible expression for the rigid meromorphic function $J_\tau\{r,s\}$ when τ is unramified. We recall that a *rational affinoid* is an affinoid subset of $\mathcal{H}_p$ which can be expressed as the inverse image under the reduction map of a (finite) subgraph of $\mathcal{T}$. For instance, any Γ-translate of $\mathcal{H}_p^{\leq n}$ is a rational affinoid.

Corollary 4.3. *Assume that τ is unramified. Suppose that $\mathcal{H}_p^{[n]}$ is an increasing sequence of rational affinoids satisfying $\mathcal{H}_p = \cup_{n\geq 1}\mathcal{H}_p^{[n]}$, and let*

$$\Delta_\tau^{[n]}\{r,s\} := \sum_{w\in(\Gamma\tau)\cap\mathcal{H}_p^{[n]}} \delta_{r,s}(w)\cdot(w).$$

Then $\Delta_\tau^{[n]}\{r,s\}$ is a degree zero divisor and

$$J_\tau\{r,s\} = \lim_{n\longrightarrow\infty}[(z)-(\eta);\Delta_\tau^{[n]}\{r,s\}].$$

Suppose now that the RM point τ in $\mathcal{H}_p$ is *ramified*. In this case, Lemma 4.2 and Corollary 4.3 cease to be true, since the sequence $(J_\tau^{\leq n}\{r,s\})_{n\geq 0}$ need not converge uniformly on affinoid subsets. The next lemma shows that it is made up of two a priori distinct convergent subsequences

$$\{J_\tau^{\leq n}\{r,s\}(z)\}_{n \text{ even}}, \qquad \{J_\tau^{\leq n}\{r,s\}(z)\}_{n \text{ odd}}.$$

Lemma 4.4. *Assume that τ is a ramified RM point. The rational functions*

$$J_\tau^{\leq n}\{r,s\}(z) := [(z)-(\eta);\Delta_\tau^{\leq n}\{r,s\}]$$

as n ranges over the odd (resp. even) integers converge uniformly on affinoid subsets of $\mathcal{H}_p$ to rigid analytic functions $J_\tau^+\{r,s\}(z)$ (resp. $J_\tau^-\{r,s\}(z)$) on $\mathcal{H}_p$.

Proof. The uniform convergence of each subsequence follows from the identity, generalising (35)

$$J_\tau^{\leq n+2}\{r,s\}(z) \div J_\tau^{\leq n}\{r,s\}(z) = \prod_{d(v,v_\circ)=n+1}[(z)-(\eta);\Delta_\tau^v\{r,s\}]. \tag{36}$$

□

One can give a more intrinsic expression for the rigid meromorphic functions $J_\tau^\pm\{r,s\}$ in terms of more general affinoid coverings of $\mathcal{H}_p$. Namely, a closed, connected, finite subgraph $\mathcal{G}$ of $\mathcal{T}$ is said to be *full* if every vertex v of $\mathcal{G}$ is either of degree 1 or $p+1$. The vertices that are of degree 1 are called the *boundary vertices* of $\mathcal{G}$. The graph $\mathcal{G}$ is then said to be *even* (resp. *odd*) if every boundary vertex of $\mathcal{G}$ is odd (resp. even). An affinoid region in $\mathcal{H}$ is said to be even or odd if it is the inverse image under the reduction map red of a full subgraph of $\mathcal{T}$ of the same parity. The set of even and odd affinoid subsets is preserved by the action of Γ, while elements of $\mathrm{GL}_2(\mathbb{Z}[1/p])$ of determinant p interchange these two types of affinoids. When τ is ramified, and $\mathcal{A}$ is an even (resp. odd) affinoid subset, the divisor $\Delta_\tau^{\mathcal{A}}\{r,s\}$ can be uniquely expressed as a sum of the divisors $\Delta_\tau^v\{r,s\}$ as v ranges over the even (resp. odd) vertices of $\mathcal{T}$ that lie in $\mathrm{red}(\mathcal{A})$.

The following lemma is the counterpart of Corollary 4.3 when τ is ramified:

Lemma 4.5. *Assume that τ is a ramified RM point. Let $\mathcal{H}_p^{[n]}$ be any increasing sequence of even rational affinoids satisfying $\mathcal{H}_p = \cup_{n\geq 1} \mathcal{H}_p^{[n]}$, and let*

$$\Delta_\tau^{[n]}\{r,s\} := \sum_{w\in(\Gamma\tau)\cap\mathcal{H}_p^{[n]}} \delta_{r,s}(w)\cdot(w).$$

Then $\Delta_\tau^{[n]}\{r,s\}$ is a degree zero divisor and

$$J_\tau^+\{r,s\} = \lim_{n\longrightarrow\infty}[(z)-(\eta);\Delta_\tau^{[n]}\{r,s\}].$$

A similar statement holds for J_τ^-, with even affinoids replaced by odd ones.

The proof of this lemma proceeds along the same lines as that of Cor. 4.3, and is therefore omitted.

4.3. **Basic properties.** A number of formal and foundational properties on the collections of rigid meromorphic functions $J_\tau\{r,s\}$ will now be established.

Lemma 4.6. *Suppose τ is unramified. The functions $J_\tau\{r,s\}(z)$ satisfy the the following properties:*

- *(a) The assignment $(r,s)\mapsto J_\tau\{r,s\}$ is a modular symbol with values in $\mathcal{M}^\times$.*
- *(b) The rigid meromorphic function $J_\tau\{r,s\}$ is independent, up to multiplication by a non-zero scalar, of the choice of base point η.*
- *(c) The modular symbol J_τ satisfies the Γ-invariance property*

$$J_\tau\{\gamma r,\gamma s\}(\gamma z) = J_\tau\{r,s\}(z), \quad (\mathrm{mod}\ C_p^\times), \qquad \text{for all } \gamma\in\Gamma.$$

Proof. Property (a) of $J_\tau\{r,s\}$ follows from the similar property of the divisors $\Delta_\tau^{\leq n}\{r,s\}$, while (b) follows from the fact that the functions $J_\tau^\eta\{r,s\}$ and $J_\tau^{\eta'}\{r,s\}$ defined using different base points η and η' satisfy

$$J_\tau^\eta\{r,s\} = \lambda J_\tau^{\eta'}\{r,s\}, \qquad \text{where } \lambda = \lim_{n\longrightarrow\infty}[(\eta')-(\eta);\Delta_\tau^{\leq n}\{r,s\}].$$

Property (c) holds because

$$J_\tau\{\gamma r,\gamma s\}(\gamma z) = \lim_{n\longrightarrow\infty}[(\gamma z)-(\eta);\Delta_\tau^{\leq n}\{\gamma r,\gamma s\}] \tag{37}$$

$$= \lim_{n\longrightarrow\infty}[\gamma((z)-(\gamma^{-1}\eta));\gamma\Delta_\tau^{\mathcal{A}_n}\{r,s\}], \tag{38}$$

where $\mathcal{A}_n := \gamma^{-1}(\mathcal{H}_p^{\leq n})$ and

$$\Delta_\tau^{\mathcal{A}_n}\{r,s\} := \sum_{w\in(\Gamma\tau)\cap\mathcal{A}_n} \delta_{r,s}(w)(w).$$

The Γ-equivariance of the symbol $[\mathscr{D}_1;\mathscr{D}_2]$ implies that

$$J_\tau\{\gamma r,\gamma s\}(\gamma z) = \lim_{n\longrightarrow\infty}[(z)-(\gamma^{-1}\eta);\Delta_\tau^{\mathcal{A}_n}\{r,s\}].$$

Property (c) now follows from Corollary 4.3, in light of the fact that $\mathcal{H}_p = \cup_{n\geq 1}\mathcal{A}_n$, and from part (b) of Lemma 4.6. □

Likewise, one establishes with the same proofs the following ramified counterpart:

Lemma 4.7. *Suppose $\tau \in \mathcal{H}_p$ is a ramified RM point. The elements J_τ^+ and J_τ^- belong to $\mathrm{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$.*

Proof. The argument proceeds exactly as in the proof of Lemma 4.6. □

It is instructive to examine the difference between the two rigid meromorphic theta-symbols J_τ^+ and J_τ^- attached to a ramified RM point τ. To this end, we associate to τ a $\Gamma_0(p)$-invariant modular symbol $m_\tau \in \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$ by setting

$$m_\tau\{r,s\} := \deg(\Delta_\tau^U\{r,s\}) := \sum_{w\in\Gamma\tau\cap U} \delta_{r,s}(w),$$

where U is the standard annulus whose stabiliser in Γ is equal to $\Gamma_0(p)$. The following lemma is an easy consequence of the definitions.

Lemma 4.8. *For all ramified RM points $\tau \in \mathcal{H}_p$, the difference between J_τ^+ and J_τ^- is the rigid analytic theta-symbol given by*

$$J_\tau^+ \div J_\tau^- = \mathrm{ST}^\times(m_\tau).$$

When τ is ramified, it is natural to define the theta-symbol J_τ by averaging over J_τ^+ and J_τ^-.

Definition 4.9. Let τ be a ramified RM point in $\mathcal{H}_p$. The theta-symbol attached to τ is defined by

$$J_\tau\{r,s\} := (J_\tau^+\{r,s\} \times J_\tau^-\{r,s\})^{1/2} = J_\tau^\pm\{r,s\} \times \mathrm{ST}^\times(m_\tau)^{\mp 1/2}.$$

Lemmas 4.6 and 4.7 imply that J_τ is a Γ-invariant modular symbol with values in $\mathcal{M}^\times/C_p^\times$. It is called the *meromorphic theta-symbol* associated to $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$.

Proposition 4.10. *Up to elements of $\mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$, every class in $\mathrm{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$ can be expressed as a finite product of cocycles of the form J_τ, as τ ranges over $\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$.*

Proof. It is shown in [DV21] that the full collection of theta symbols J_τ, as τ ranges over the distinct Γ-orbits of both unramified and ramified RM points, generates $\mathrm{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$ up to analytic theta-symbols. See [DV21, Lemma 2.11, Theorem 2.12]. □

5. Lifting obstructions

5.1. **Definition and basic properties.** Applying the exact functor $\Omega \mapsto \mathrm{MS}(\Omega)$ to the short exact sequences

$$1 \longrightarrow C_p^\times \longrightarrow \mathcal{A}^\times \longrightarrow \mathcal{A}^\times/C_p^\times \longrightarrow 1, \qquad 1 \longrightarrow C_p^\times \longrightarrow \mathcal{M}^\times \longrightarrow \mathcal{M}^\times/C_p^\times \longrightarrow 1$$

and then taking their Γ-cohomology leads to the commutative diagram in which the rows are exact up to finite subgroups, and the vertical maps are injective:

$$\begin{array}{ccccccc} & & & & \mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) & \xrightarrow{\mathrm{Obs}} & \Pi \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{MS}^\Gamma(\mathcal{M}^\times) & \longrightarrow & \mathrm{MS}^\Gamma(\mathcal{M}^\times/C_p^\times) & \xrightarrow{\mathrm{Obs}} & T(C_p), \end{array} \tag{39}$$

where T is the torus over $\mathbb{Q}_p$ whose L-valued points are given by

$$T(L) := \mathrm{H}^1(\Gamma, \mathrm{MS}(L^\times)),$$

and Π is the image of $\mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times)$ under the connecting homomorphism to $T(C_p)$.

Definition 5.1. If J belongs to $\mathrm{MS}^\Gamma(\mathcal{M}^\times/C_p^\times)$, its image under Obs is called the *parabolic lifting obstruction* attached to J.

The terminology is justified by the fact that the theta-symbol J lifts to a genuine rigid meromorphic modular symbol if and only if its lifting obstruction vanishes.

The next lemma analyses the rightmost column of diagram (39).

Lemma 5.2. *There is a natural map*

$$\mathrm{MS}^{\Gamma_0(p)}(C_p^\times) \longrightarrow T(C_p)$$

with finite kernel and co-kernel. In particular, the character group of T is isomorphic to $\mathbb{Z}^{2g+1}$, where g is the genus of the modular curve $X_0(p)$. The subgroup Π is a lattice in $T(C_p)$ (i.e., a discrete subgroup of $T(C_p)$ of rank $2g+1$).

Proof. The asserted map arises from the long exact cohomology sequence

$$1 \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p^\times)) \longrightarrow \mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times)) \longrightarrow \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathrm{MS}(C_p^\times))^2 \longrightarrow \mathrm{H}^1(\Gamma_0(p), \mathrm{MS}(C_p^\times)),$$

which arises from the second exact sequence of [Se80, II. §2.8, Prop. 13] applied to the Γ-module $\mathrm{MS}(C_p^\times)$ and to the action of Γ on the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose edge and vertex stabilisers are isomorphic to $\Gamma_0(p)$ and $\mathrm{SL}_2(\mathbb{Z})$ respectively, and whose fundamental region consists of a single closed edge. The fact that Π is a lattice in $T(C_p)$ follows from the fact that the composition

$$\mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z}) \xrightarrow{\mathrm{ST}^\times} \mathrm{MS}^\Gamma(\mathcal{A}^\times/C_p^\times) \xrightarrow{\mathrm{Obs}} \mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times)) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(C_p^\times) \xrightarrow{\mathrm{ord}_p} \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$$

is the identity. □

Corollary 5.3. *The character group $X(T) := \hom(T, \mathbb{G}_m)$, and the lattice Π, are isomorphic to $\mathbb{Z}^{2g+1}$ up to torsion, and their tensor products with $\mathbb{Q}$ are isomorphic as Hecke modules.*

The lattice Π and the torus T are closely related to the invariants defined and studied in [Das05] in a more general setting.

Lemma 5.4. *If $R \in \mathrm{Div}(\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}})$ is an RM divisor, there is a unique theta-symbol J_R satisfying*

(i) $\mathrm{Divisor}(J_R) = R$;
(ii) $\mathrm{Obs}(J_R) \in \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathcal{O}_{C_p}^\times))$.

Proof. For $\tau \in \mathcal{H}_p^{\mathrm{RM}}$, the theta symbol J_τ has divisor equal to (τ), and hence, it is possible to construct a J_R^0 with RM divisor equal to R. The theta-symbol J_R can then be defined by setting

$$J_R = J_R^0 \div \mathrm{ST}^\times(\mathrm{ord}_p(\mathrm{Obs}(J_R^0))).$$

The uniqueness of J_R satisfying properties (i) and (ii) follows from the fact that Π is a lattice and hence that there are no analytic theta cocycles J with $\mathrm{Obs}(J) \in \mathrm{H}^1(\Gamma, \mathrm{MS}(\mathcal{O}_{C_p}^\times))$. □

The parabolic lifting obstruction attached to J is calculated by lifting J to a (not necessarily Γ-invariant) $\mathcal{M}^\times$-valued modular symbol $\tilde{J}$, and setting

$$\mathrm{Obs}(J)(\gamma)\{r,s\} = \frac{\tilde{J}\{\gamma r, \gamma s\}(\gamma z)}{\tilde{J}\{r,s\}(z)}. \tag{40}$$

Note that the ratio on the right belongs to $C_p^\times$ since the modular symbol $\tilde{J}$ is Γ-invariant modulo multiplicative scalars.

5.2. **Toric values.** The quantities on the right of (40) depend on the choice of lift $\tilde{J}$, and indeed classes in $\mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times))$ need not admit a canonical representative 1-cocycle. It is useful to associate to such a class some well-defined numerical invariants arising from $\mathbb{Q}$-split tori in Γ. More precisely, let (r,s) be an ordered pair of elements of $\mathbb{P}_1(\mathbb{Q})^2$. The stabiliser $\Gamma_{r,s}$ of (r,s) in Γ is the set of $\mathbb{Z}[1/p]$valued points of a global split torus in $\mathrm{GL}_2(\mathbb{Q})$, and is generated by a – unique, up to torsion – loxodromic element γ_{rs} having r as an attractive and s as a repulsive fixed point. Given a 1-cocycle $\kappa \in Z^1(\Gamma, \mathrm{MS}(C_p^\times))$, the quantity

$$\kappa[r,s] := \kappa(\gamma_{rs})\{r,s\} \in C_p^\times$$

is trivial on coboundaries, and therefore leads to a well-defined numerical invariant.

Definition 5.5. The quantity $\kappa[r,s]$ is called the *toric value* of $\kappa \in \mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times))$ at (r,s).

The following lemma gives a simple formula for $\mathrm{Obs}(J)[r,s]$.

Lemma 5.6. *For all* $(r,s) \in \mathbb{P}_1(\mathbb{Q})^2$,

$$\mathrm{Obs}(J)[r,s] = \tilde{J}\{r,s\}(\gamma_{rs} z) \div \tilde{J}\{r,s\}(z), \tag{41}$$

where $\tilde{J}\{r,s\}$ *is* any *rigid meromorphic function lifting* $J\{r,s\}(z)$.

Proof. This is an immediate consequence of the definitions. It is worth reiterating the useful fact that the right hand side in (41) does not depend on the choice of lift $\tilde{J}\{r,s\}$, and hence that any choice would do. □

The following proposition asserts that the non-triviality of a class $\kappa \in \mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times))$ can be detected from its toric values.

Proposition 5.7. *Let* κ *be a class in* $\mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times))$. *If* $\mathrm{ord}_p\, \kappa[r,s] = \log_p \kappa[r,s] = 0$ *for all* $(r,s) \in \mathbb{P}_1(\mathbb{Q})^2$, *then* κ *is a torsion class.*

Proof. If T is a Hecke operator and $J \in \mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times))$, then the toric values of $T(J)$ can be expressed as linear combinations of other toric values, and hence the kernel of the map toric evaluation map composed with ord_p and $\log_p$,

$$\mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times)) \longrightarrow \prod_{(r,s)} \mathbb{Z} \oplus C_p \tag{42}$$

is Hecke stable. Since the Hecke algebra acts semi-simply, the kernel is non-trivial if and only if it contains a Hecke eigenclass. This eigenclass cannot be a power of the lifting obstruction of the boundary symbol of (30), since

$$\mathrm{Obs}(J_\sharp)[0,\infty] = p^2.$$

Hence, it must be associated to a weight two cusp form f on $\Gamma_0(p)$. Denote by κ_f this eigenclass with trivial toric values. Let c be an integer prime to p, and let $\nu \in \mathbb{Z}/c\mathbb{Z}$. Proposition 2.16 of [Dar01] shows that the toric value $\kappa_f[\infty, \nu/c]$, which in *loc.cit.* is denoted W_{Ψ_ν}, depends only on the image of ν in the group $(\mathbb{Z}/c\mathbb{Z})^\times/\langle p^{2\,\mathbb{Z}}\rangle$, and that, for any Dirichlet character χ of conductor c satisfying $\chi(p) = \pm 1$, we have

$$\sum_{\nu\in(\mathbb{Z}/c\mathbb{Z})^\times} \chi(\nu)\,\mathrm{ord}_p\,\kappa_f[\infty,\nu/c] \sim L(f,\chi,1), \qquad \sum_{\nu\in(\mathbb{Z}/c\mathbb{Z})^\times} \chi(\nu)\log_p\,\kappa_f[\infty,\nu/c] \sim L(f,\chi,1),$$

where $\sim$ denotes equality up to multiplication by a simple non-zero scalar depending on κ_f. Since there are infinitely many such Dirichlet characters (in fact, infinitely many quadratic ones) for which $L(f,\chi,1) \neq 0$, it follows that there is no such κ_f, and hence that the map (42) is injective. □

5.3. **The toric values of the lifting obstruction.** An explicit formula for $\mathrm{Obs}(J_\tau)[r,s]$ will now be obtained, for unramified τ in Proposition 5.8, and for ramified τ in Proposition 5.10. It will play a central role in the analysis to follow, as well as in the numerical computation of toric values of the lifting obstruction, which give a computational pathway to the Gross–Stark units and Stark–Heegner points discussed in § 3.7, as we will see in § 5.4.

Recall the degree zero divisors $\Delta_\tau^{\leq n}\{r,s\}$ supported on $\mathcal{H}_p^{\leq n}$ that were used to define $J_\tau\{r,s\}(z)$ as a limit of rational functions. For the calculation that follows, it will be useful to replace the affinoid cover $\{\mathcal{H}_p^{\leq n}\}_{n\geq 0}$ by another cover involving affinoid domains that are better adapted to studying the action of the matrix γ_{rs}. To this end, consider the infinite path $\Upsilon(r,s)$ on the Bruhat-Tits-tree going from r to s, viewed as ends of the tree. When $p \nmid \det(r,s)$, this path contains the standard vertex $v_\circ$, and consecutive vertices of $\Upsilon(r,s)$ can be numbered as $\ldots, v_{-1}, v_0, v_1, \ldots$, in such a way that $v_0 = v_\circ$. The connected subtree $\mathcal{T}_j$ of $\mathcal{T}$ having v_j as its root vertex, and containing no edge of $\Upsilon(r,s)$, is characterised by the property

$$\mathrm{ord}_p([z;(r)-(s)]) = j \text{ if and only if } \mathrm{red}(z) \in \mathcal{T}_j\,.$$

Let $\Upsilon[v_i, v_j]$ be the subgraph of $\Upsilon(r,s)$ consisting of the path from v_i to v_j, with all the vertices between v_i and v_j included, and let $\Upsilon[v_i, v_j) := \Upsilon[v_i, v_j] - \{v_j\}$. The Bruhat–Tits tree can be expressed as the disjoint union

$$\mathcal{T} = \bigcup_{j=-\infty}^{\infty} \Big(\mathcal{T}_j \cup \Upsilon[v_j, v_{j+1}) \Big).$$

The matrix γ_{rs} preserves the path $\Upsilon(r,s)$, sending the vertex v_j to v_{j+2e} where $p^{\pm e}$ are the eigenvalues of γ_{rs}. It also maps the subtree $\mathcal{T}_j$ to $\mathcal{T}_{j+2e}$. Hence the connected subgraph

$$\mathcal{T}_{[v_0,v_{2e})} := \Upsilon[v_0, v_{2e}) \cup \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{2e-1}$$

is a fundamental domain for the $\Gamma_{r,s}$-action on $\mathcal{T}$, and

$$\Omega(r,s) := \mathrm{red}^{-1}(\mathcal{T}_{[v_0,v_{2e})}) = \{z \in \mathcal{H}_p \text{ with } 0 \leq \mathrm{ord}_p([z;(r)-(s)]) < 2e\}$$

is a fundamental region in $\mathcal{H}_p$ for this action. Let $\mathcal{T}_{\widetilde{j}}^{\leq n}$ denote the finite subtree of $\mathcal{T}_j$ of depth n with v_j as root vertex, and let

$$\begin{aligned}\mathcal{T}^{\leq n}_{[v_0,v_{2e})} &:= \Upsilon[v_0,v_{2e}) \cup \mathcal{T}_{\widetilde{0}}^{\leq n} \cup \mathcal{T}_{\widetilde{1}}^{\leq n-1} \cup \mathcal{T}_{\widetilde{2}}^{\leq n} \cup \cdots \cup \mathcal{T}_{\widetilde{2e-1}}^{\leq n-1},\\ \Omega^{[n]}(r,s) &:= \mathrm{red}^{-1}\left(\mathcal{T}^{\leq n}_{[v_0,v_{2e})}\right) \subset \mathcal{H}_p\,.\end{aligned}$$

The sets $\Omega^{[n]}(r,s)$ form a nested sequence whose union is $\Omega(r,s)$, and likewise

$$\mathcal{H}_p^{[n]} := \mathrm{red}^{-1}\left(\bigcup_{j=-n}^{n} \gamma_{rs}^j\, \mathcal{T}^{\leq n}_{[v_0,v_{2e})}\right) = \bigcup_{j=-n}^{n} \gamma_{rs}^j \Omega^{[n]}(r,s), \tag{43}$$

satisfies

$$\mathcal{H}_p = \cup_{n\geq 1}\, \mathcal{H}_p^{[n]}\,. \tag{44}$$

We associate divisors $\Delta_\tau^{[n]}\{r,s\}$ and $\Pi_\tau^{[n]}\{r,s\}$ to the sets $\mathcal{H}_p^{[n]}$ and $\Omega^{[n]}(r,s)$ by setting

$$\Delta_\tau^{[n]}\{r,s\} := \sum_{w\in(\Gamma\tau)\cap\mathcal{H}_p^{[n]}} \delta_{r,s}(w)\cdot(w), \qquad \Pi_\tau^{[n]}\{r,s\} := \sum_{w\in(\Gamma\tau)\cap\Omega^{[n]}(r,s)} \delta_{r,s}(w)\cdot(w). \tag{45}$$

Note that, by (43),

$$\Delta_\tau^{[n]}\{r,s\} = \sum_{j=-n}^{n} \gamma_{rs}^j \Pi_\tau^{[n]}\{r,s\}. \tag{46}$$

Proposition 5.8. *For all unramified $\tau \in \Gamma\backslash\,\mathcal{H}_p^{\mathrm{RM}}$,*

$$\mathrm{Obs}(J_\tau)[r,s] = \lim_{n\longrightarrow\infty}[\Pi_\tau^{[n]}(r,s);(r)-(s)].$$

Proof. Choose a base point η, and assume for simplicity that its reduction is equal to the vertex $v_\circ$. This basepoint determines a lift of $J_\tau\{r,s\}(z)$ to $\mathcal{M}^\times$ by setting

$$\tilde{J}_\tau\{r,s\}(z) := \lim_{n\longrightarrow\infty}[(z)-(\eta);\Delta_\tau^{\leq n}\{r,s\}],$$

following the definition given in Lemma 4.6. In light of (44) and Corollary 4.3, the definition of $\tilde{J}_\tau\{r,s\}(z)$ remains unchanged after replacing $\Delta_\tau^{\leq n}\{r,s\}$ by the divisors $\Delta_\tau^{[n]}\{r,s\}$ of (45), i.e.,

$$\tilde{J}_\tau\{r,s\}(z) = \lim_{n\longrightarrow\infty}[(z)-(\eta);\Delta_\tau^{[n]}\{r,s\}].$$

By definition of the lifting obstruction,

$$\begin{aligned}\mathrm{Obs}(J_\tau)[r,s] &:= \tilde{J}_\tau\{r,s\}(\gamma_{rs}z)/\tilde{J}_\tau\{r,s\}(z)\\ &= \lim_{n\longrightarrow\infty}[(\gamma_{rs}z)-(\eta);\Delta_\tau^{[n]}\{r,s\}] \div [(z)-(\eta);\Delta_\tau^{[n]}\{r,s\}]\\ &= \lim_{n\longrightarrow\infty}[(\gamma_{rs}z)-(z);\Delta_\tau^{[n]}\{r,s\}].\end{aligned}$$

By (46), we may rewrite this as

$$\begin{aligned}\mathrm{Obs}(J_\tau)[r,s] &= \lim_{n\longrightarrow\infty}\prod_{j=-n}^{n}[(\gamma_{rs}z)-(z);\gamma_{rs}^{j}\Pi_\tau^{[n]}\{r,s\}]\\ &= \lim_{n\longrightarrow\infty}\prod_{j=-n}^{n}[(\gamma_{rs}^{-j+1}z)-(\gamma_{rs}^{-j}z);\Pi_\tau^{[n]}\{r,s\}]\\ &= \lim_{n\longrightarrow\infty}[(\gamma_{rs}^{n+1}z)-(\gamma_{rs}^{-n}z);\Pi_\tau^{[n]}\{r,s\}]\\ &= \lim_{n\longrightarrow\infty}[(r)-(s);\Pi_\tau^{[n]}\{r,s\}].\end{aligned}$$

The proposition now follows from Weil reciprocity. □

Corollary 5.9. *If τ is unramified and $p\nmid\det(r,s)$, then $\mathrm{Obs}(J_\tau)[r,s]$ belongs to $\mathcal{O}_{C_p}^\times$.*

Proof. The expression $[\Pi_\tau^{[n]}\{r,s\};(r)-(s)]$ occuring in Proposition 5.8 can be written as a finite product of $[\Delta_\tau^v\{r,s\};(r)-(s)]$ for various vertices v, which are p-adic units since $\Delta_\tau^v\{r,s\}$ is a degree zero divisor supported on a single basic affinoid. □

To deal with the ramified case, the following ramified version of Proposition 5.8 will be needed. Since its proof is essentially identical to that of the unramified version, we leave it to the reader.

Proposition 5.10. *For all ramified $\tau\in\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$,*

$$\begin{aligned}\mathrm{Obs}(J_\tau^+)[r,s] &= \lim_{n\longrightarrow\infty}[\Pi_\tau^{[n]}(r,s);(r)-(s)],\\ \mathrm{Obs}(J_\tau^-)[r,s] &= \lim_{n\longrightarrow\infty}[\Pi_\tau^{[n]}(r,s);(r)-(s)],\end{aligned}$$

where the limits are taken over the odd and even values of n, respectively.

Unlike the lifting obstructions attached to J_τ when τ is unramified, the lifting obstructions attached to J_τ^+ and J_τ^- need not be p-adic units. The following result computes their p-adic valuations. To state it, one associates to τ and (r,s) the unique harmonic cocycle $c_\tau\{r,s\}$ on the Bruhat–Tits tree $\mathcal{T}$ satisfying, for all *even oriented* edges e

$$c_\tau\{r,s\}(e)=\mathrm{sgn}(e)\cdot\deg(\Delta_\tau^{U_e}\{r,s\}),$$

where U_e is the oriented annulus mapping to e under the reduction map.

Proposition 5.11. *For all ramified $\tau\in\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$,*

$$\mathrm{ord}_p(\mathrm{Obs}(J_\tau^+)[r,s])=-\mathrm{ord}_p(\mathrm{Obs}(J_\tau^-)[r,s])=\frac{1}{2}\sum_{e\in\Upsilon[v_0,v_{2e}]}c_\tau\{r,s\}(e). \tag{47}$$

In particular, $\mathrm{ord}_p(\mathrm{Obs}(J_\tau)[r,s])=0$.

Proof. By Proposition 5.10, $\mathrm{Obs}(J_\tau^+)[r,s]$ can be written as the limit of $[\Pi_\tau^{[n]}(r,s);(r)-(s)]$ for n odd, each of which is a finite product of factors of the form

$$[\Delta_\tau^v\{r,s\}\cap\Omega^{[n]}(r,s)\ ;\ (r)-(s)] \tag{48}$$

for various vertices v of *even* distance to v_0. There are two cases to consider: When v is a vertex that does not lie on the path $\Upsilon(r,s)$, we have

$$\Delta_\tau^v\{r,s\} \cap \Omega^{[n]}(r,s) = \Delta_\tau^v\{r,s\},$$

and all points w in the support of $\Delta_\tau^v\{r,s\}$ satisfy $\mathrm{ord}_p[w;(r)-(s)] = j$ for the same $j \in \mathbb{Z}$. Since the divisor $\Delta_\tau^v\{r,s\}$ is of degree zero, we find that

$$\mathrm{ord}_p\left[\Delta_\tau^v\{r,s\};\ (r)-(s)\right] = j \cdot \deg(\Delta_\tau^v\{r,s\}) = 0.$$

When $v = v_j$ is on the path $\Upsilon(r,s)$ the quantity (48) is itself a product of factors indexed by the points w in the support of $\Delta_\tau^v\{r,s\}$. The valuation of $[w;(r)-(s)]$ is

$$\begin{array}{cl} j & \text{if } \ e \text{ does not lie on } \Upsilon(r,s), \\ j-1/2 & \text{if } \ e = (v_{j-1}, v_j), \\ j+1/2 & \text{if } \ e = (v_j, v_{j+1}). \end{array}$$

When $j \neq 0$ we again have $\Delta_\tau^v\{r,s\} \cap \Omega^{[n]}(r,s) = \Delta_\tau^v\{r,s\}$, which has degree zero. Taking the weighted sum of all these valuations, it follows that

$$\begin{array}{rcl} \mathrm{ord}_p\left[\Delta_\tau^v\{r,s\};\ (r)-(s)\right] & = & -\frac{1}{2}c_\tau\{r,s\}(v_j,v_{j-1}) + \frac{1}{2}c_\tau\{r,s\}(v_j,v_{j+1}) \\ & = & \frac{1}{2}c_\tau\{r,s\}(v_{j-1},v_j) + \frac{1}{2}c_\tau\{r,s\}(v_j,v_{j+1}). \end{array}$$

When $j = 0$ on the other hand, the valuation simply becomes $\frac{1}{2}c_\tau\{r,s\}(v_0,v_1)$. This proves the desired result for $\mathrm{Obs}(J_\tau^+)[r,s]$. The result for $\mathrm{Obs}(J_\tau^-)[r,s]$ follows in exactly the same way, interchanging even an odd vertices. □

5.4. **A reciprocity law.** The main result of this section is the following reciprocity law.

Theorem 5.12. *Let $\tau \in \mathcal{H}_p$ be an RM point, and (r,s) a pair of elements in $\mathbb{P}_1(\mathbb{Q})$, and let*

$$\begin{array}{rcl} J_\tau & \in & \mathrm{H}^1(\Gamma, \mathcal{M}^\times/C_p^\times), \\ J_{r,s} & \in & \mathrm{H}^1(\Gamma,\ \mathcal{A}^\times/C_p^\times), \end{array}$$

be the associated rigid meromorphic, resp. analytic, theta-cocycles. Then

$$\mathrm{Obs}(J_\tau)[r,s] = J_{r,s}[\tau] \pmod{\text{torsion}}.$$

Proof. Suppose first that τ is unramified. By Proposition 5.8, it is enough to show that

$$J_{r,s}[\tau] = \lim_{n \longrightarrow \infty} [\Pi_\tau^{[n]}\{r,s\}; (r)-(s)].$$

We begin by treating the case where τ is even. We can then assume without loss of generality, after translating τ by a suitable element of Γ, that τ reduces to the vertex $v_\circ$. The automorph γ_τ then belongs to $\mathrm{SL}_2(\mathbb{Z})$, and one has

$$J_{r,s}[\tau] = J_{r,s}^\circ(\gamma_\tau)(\tau),$$

where $J_{r,s}^\circ$ is the lift of $J_{r,s}$ to an element of $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathcal{A}^\times)$. By the formula for $J_{r,s}^\circ$ given in Proposition 3.22,

$$J_{r,s}[\tau] = \prod_{n=0}^{\infty} J_{r,s}^{(n)}(\gamma_\tau)(\tau), \quad \text{where} \quad J_{r,s}^{(n)}(\gamma_\tau)(z) := \prod_{(u,v)\in\Sigma_{r,s}^{(n)}} [z;(u)-(v)]^{[u,v]\cdot[\xi,\gamma_\tau\xi]}. \tag{49}$$

Let $(\Gamma/\Gamma_{r,s})_n$ denote the set of $\alpha \in \Gamma/\Gamma_{r,s}$ satisfying $\mathrm{ord}_p(\det(\alpha r, \alpha s)) \leq n$. We may then rewrite (49) as asserting that $J_{r,s}[\tau] = \lim_{n\to\infty} J_{r,s}^{\leq n}(\gamma_\tau)(\tau)$, where

$$J_{r,s}^{\leq n}(\gamma_\tau)(\tau) = \prod_{\alpha\in(\Gamma/\Gamma_{r,s})_n} [\tau; (\alpha r) - (\alpha s)]^{[\alpha r, \alpha s]\cdot[\xi, \gamma_\tau \xi]}. \tag{50}$$

Since γ_τ belongs to the group $\mathrm{SL}_2(\mathbb{Z})$ which preserves the determinant of (r, s), it acts on the set $(\Gamma/\Gamma_{r,s})_n$ by left multiplication, and therefore

$$J_{r,s}^{\leq n}(\gamma_\tau)(\tau) = \prod_{\alpha\in\gamma_\tau^{\mathbb{Z}}\backslash(\Gamma/\Gamma_{r,s})_n} \prod_{j=-\infty}^{\infty} [\tau; (\gamma_\tau^j \alpha r) - (\gamma_\tau^j \alpha s)]^{[\gamma_\tau^j \alpha r, \gamma_\tau^j \alpha s]\cdot[\xi, \gamma_\tau \xi]} \tag{51}$$

$$= \prod_{\alpha\in\gamma_\tau^{\mathbb{Z}}\backslash(\Gamma/\Gamma_{r,s})_n} \prod_{j=-\infty}^{\infty} [\tau; (\alpha r) - (\alpha s)]^{[\alpha r, \alpha s]\cdot[\gamma_\tau^{-j} \xi, \gamma_\tau^{-j+1} \xi]} \tag{52}$$

$$= \prod_{\alpha\in\gamma_\tau^{\mathbb{Z}}\backslash(\Gamma/\Gamma_{r,s})_m} [\tau; (\alpha r) - (\alpha s)]^{[\alpha r, \alpha s]\cdot[\tau', \tau]} \tag{53}$$

$$= \prod_{\alpha\in\gamma_\tau^{\mathbb{Z}}\backslash(\Gamma/\Gamma_{r,s})_m} [(\alpha^{-1}\tau); (r) - (s)]^{[r,s]\cdot[\alpha^{-1}\tau', \alpha^{-1}\tau]} \pmod{p^{\mathbb{Z}}}. \tag{54}$$

In this series of equalities, (52) and (54) follow from the $\mathrm{SL}_2(\mathbb{Z})$ and Γ-equivariance properties of the expression $[z; (r) - (s)]$, respectively, and (53) follows from the identity

$$\begin{aligned} \sum_{j=-\infty}^{\infty} [\alpha r, \alpha s] \cdot [\gamma_\tau^{-j} \xi, \gamma_\tau^{-j+1} \xi] &= \lim_{M \longrightarrow \infty} [\alpha r, \alpha s] \cdot [\gamma_\tau^{-M} \xi, \gamma_\tau^{M+1} \xi] \\ &= [\alpha r, \alpha s] \cdot [\tau', \tau], \end{aligned}$$

since τ is the attractive fixed point in $\mathcal{H}$ for γ_τ and τ' is its repulsive fixed point.

Since the subgraph $\mathcal{T}_{[v_0, v_{2e}]}$ is a fundamental region in $\mathcal{T}$ for the action of the matrix γ_{rs}, the set of matrices $\alpha \in \Gamma$ for which $\alpha^{-1} v_\circ \in \mathcal{T}_{[v_0, v_{2e}]}$ gives system of representatives for the quotient $\Gamma/\Gamma_{r,s}$. After identifying the quotient with this particular system of representatives, the natural map $\alpha \mapsto \alpha^{-1}\tau$ yields a bijection

$$\Gamma_\tau\backslash(\Gamma/\Gamma_{r,s})_m \longrightarrow (\Gamma\tau) \cap \Omega^{(m)}(r, s),$$

where we recall that $\Omega^{(m)}$ is the truncated fundamental region for the action of $\Gamma_{r,s}$. We may then rewrite (54) as

$$J_{r,s}^{\leq m}(\gamma_\tau)(\tau) = \prod_{w\in(\Gamma\tau)\cap\Omega^{(m)}(r,s)} [w; (r) - (s)]^{\delta_{r,s}(w)} = [\Pi_\tau^{\leq m}\{r, s\}; (r) - (s)] \pmod{p^{\mathbb{Z}}}.$$

We have therefore proved that

$$J_{r,s}[\tau] = \mathrm{Obs}(J_\tau)[r, s],$$

up to powers of p. On the other hand,

(1) The left hand side belongs to $\mathcal{O}_{C_p}^\times$, by Cor. 3.23.

(2) The right hand side belongs to $\mathcal{O}_{C_p}^\times$ by Corollary 5.9.

The theorem follows. The arguments for τ ramified are similar. □

Remark 5.13. It is in general not expected that the toric values of the lifting obstruction $\mathrm{Obs}(J_\tau)[r,s]$ are algebraic numbers. However, the reciprocity law of Theorem 5.12 implies that they are very closely related to the *Gross–Stark units* and *Stark–Heegner points* discussed in § 3. To see why, note that the reciprocity law equates the toric value to the RM value of the toric cocycle $J_{r,s}$, which is an analytic theta cocycle, and therefore a combination of the cocycles defined in § 3.7. Through the Schneider–Teitelbaum lift, the precise linear combination may be computed easily in homology, which was done in [DPV1]. For instance, for $(r,s)=(0,\infty)$ one finds that

$$J_{0,\infty} = \left(\frac{12}{p-1}\right)\cdot J_{\mathrm{DR}} + \sum_f L_{\mathrm{alg}}(f,1)\cdot J_f^-, \tag{55}$$

where the sum runs over a basis of normalised eigenforms for $S_2(\Gamma_0(p))$, and additive notation is used to denote the group operation in $\mathrm{H}^1(\Gamma,\mathcal{A}^\times/C_p^\times)\otimes\overline{\mathbb{Q}}_p$. The quantity $L_{\mathrm{alg}}(f,1)$ is the algebraic part of the L-value attached to f, and may easily be computed in practice.

The explicit description provided by Proposition 5.8 makes the toric values of the lifting obstructions eminently suited to explicit computation, using the modular symbol algorithms that were described in [DV21, § 2.5]. We illustrate this here with a few examples.

Example 5.14. The smallest discriminant of a rigid meromorphic theta-symbol whose parabolic lifting obstruction has non-trivial toric values is $D=12$, which has class number one but narrow class number two. The primes $p=5,7$ are both inert for this discriminant, so that $\tau:=(1+\sqrt{3})/2$ is contained in the corresponding p-adic upper half plane $\mathcal{H}_p$. For these primes all terms of (55) corresponding to cusp forms must vanish, since $S_2(\Gamma_0(p))=0$. This means that the p-adic cocycle $J_{0,\infty}$ is a multiple of the Dedekind–Rademacher cocycle J_{DR} for both of these primes, and as such it should have algebraic RM values. Numerical calculations, carried out to 100 digits of p-adic precision, indicate that

$$\mathrm{Obs}(J_\tau)[0,\infty] \stackrel{?}{=} -1+2i \in \mathbb{Q}_5, \qquad \mathrm{Obs}(J_\tau)[0,\infty]^3 \stackrel{?}{=} (-13+3\sqrt{-3})/2 \in \mathbb{Q}_7\,. \tag{56}$$

When $p=7$ and $D=321$, the smallest positive discriminant of narrrow class number 6 in which p is inert, there are three distinct rigid meromorphic cocycles J_i (for $i=1,2,3$) of discriminant 321, whose zeroes and poles are concentrated on the Γ-orbits of $\pm\tau_i$ where

$$\tau_1=\frac{-17+\sqrt{321}}{2}, \qquad \tau_2=\frac{-17+\sqrt{321}}{4}, \qquad \tau_3=\frac{-17+\sqrt{321}}{8}.$$

Their cuspidal values, calculated to 20 digits of 7-adic accuracy, are

$$\begin{aligned}
\mathrm{Obs}(J_1)[0,\infty] &= 11055762063642167 && (\mathrm{mod}\ 7^{20}),\\
\mathrm{Obs}(J_2)[0,\infty] &= 27863515261720344 - 24701001956851703\sqrt{321} && (\mathrm{mod}\ 7^{20}),\\
\mathrm{Obs}(J_3)[0,\infty] &= 35228448313023684 - 11567417813120589\sqrt{321} && (\mathrm{mod}\ 7^{20}).
\end{aligned} \tag{57}$$

The first quantity belongs to $\mathbb{Q}_7$ while the second and third are conjugate to each other (up to inversion) over the unramified quadratic extension of $\mathbb{Q}_7$. All three quantities are 7-adic units, but to 200 digits of 7-adic accuracy it appears that, up suitable powers of 7, all three of them satisfy the same sextic polynomial with rational coefficients

$$7^4x^6 - 20976x^5 - 270624x^4 + 526859689x^3 - 649768224x^2 - 120922465776x + 7^{16}, \tag{58}$$

whose splitting field is the narrow Hilbert class field of $\mathbb{Q}(\sqrt{321})$.

Example 5.15. Let $\tau = 2\sqrt{2}$, which has discriminant 32, and is contained in the 11-adic upper half plane. Let $\bar{J}_\tau$ be the 11-adic theta cocycle attached to τ. Then the reciprocity law proved in Theorem 5.12 implies that

$$\mathrm{Obs}(J_\tau)[0,\infty] = \frac{6}{5} J_{\mathrm{DR}}[\tau] + \frac{1}{5} J_E^-[\tau]$$

where E is the modular curve of level 11 given explicitly in Weierstraß form by

$$E : y^2 + y = x^3 - x^2 - 10x - 20$$

This means that whereas the quantity $\mathrm{Obs}(J_\tau)[0,\infty]$ itself need not be algebraic, we may apply suitable Hecke operators to obtain algebraic quantities, according to the results discussed in § 3.7. We observed to a high 11-adic precision that

$$\begin{cases} \mathrm{Obs}((T_2+2)J_\tau)[0,\infty] &=& J_{\mathrm{DR}}[\tau]^6 &=& \left(\frac{\sqrt{-2}-3}{11}\right)^6 & \in \mathbb{C}_{11}^\times \\ \mathrm{Obs}((T_2-3)J_\tau)[0,\infty] &=& J_E^-[\tau]^{-1} &=& (2\sqrt{-2}, 4\sqrt{-2}-5) & \in \mathbb{C}_{11}^\times / q_E^{\mathbb{Z}} \end{cases}$$

6. Generating series

6.1. **A Gross–Kohnen–Zagier formula for Stark–Heegner points.** Let $-d_1$ be a fixed negative fundamental discriminant prime to p. If $-d_2$ is a second negative discriminant such that $(-d_2/p) \neq (-d_1/p)$, we can associate to the pair $(-d_1, -d_2)$ the RM divisors $\mathbb{D}^\pm_{-d_1,-d_2}$ as in (2). If J is any analytic theta-cocycle, the values $J[\mathbb{D}^\pm_{-d_1,-d_2}]$ are defined in the obvious way, by extending the definition of the RM values $J[\tau_Q]$ by multiplicativity.

Let $\mathbb{M}(p) \subset M_{3/2}(4p) \otimes \mathbb{Q}_p$ denote Kohnen's subspace tensored with $\mathbb{Q}_p$, consisting of modular forms whose fourier coefficients are supported on integers $d_2 \equiv 0, 3 \pmod 4$. It can be decomposed as

$$\mathbb{M}(p) = \mathbb{M}(p)^+ \oplus \mathbb{M}(p)^-,$$

where $\mathbb{M}(p)^+$ (resp. $\mathbb{M}(p)^-$) consists of modular forms whose fourier coefficients are supported on integers d_2 for which $(-d_2/p) \neq 1$ (resp. $(-d_2/p) \neq -1$). [1] Then $\mathbb{M}(p)^+$ is the $+1$-eigenspace for the U_{p^2} operator (Cf. [Gr87, §12]), i.e., a form $g = \sum c(D) q^D$ in this space satisfies

$$c(Dp^{2m}) = c(D). \tag{59}$$

The following result can be viewed as an analogue of the Gross–Kohnen–Zagier theorem for Stark–Heegner points:

Theorem 6.1. *Let J be any analytic theta-cocycle.*

(1) If $\left(\frac{-d_1}{p}\right) = 1$, then the generating series

$$\Theta_{-d_1,J}(q) := \sum_{d_2 \geq 0} \log_p J[\mathbb{D}^+_{-d_1,-d_2}] q^{d_2}$$

belongs to $\mathbb{M}(p)^+$.

[1] Note that over $\mathbb{C}$, the space obtained by the conditions defining $\mathbb{M}(p)^+$ is denoted instead $M_{\mathbb{C}}^*$ in [Gr87] and its cuspidal subspace is denoted $S_{3/2}(p)^-$ in [Ko82].

(2) If $\left(\frac{-d_1}{p}\right) = -1$*, then the generating series*

$$\Theta_{-d_1,J}(q) := \sum_{d_2 \geq 0} \log_p J[\mathbb{D}^-_{-d_1,-d_2}] q^{d_2}$$

belongs to $\mathbb{M}(p)^-$*.*

Sketch of proof. Assume first that $\left(\frac{-d_1}{p}\right) = 1$. Since the Hecke algebra of weight two and level p acts semisimply on the space of rigid analytic theta cocycles, we may assume, after tensoring this space with $\bar{\mathbb{Q}}$, that J is a Hecke eigenclass. If J is the class J_f^+ or J_f^- attached to a cusp form f of weight 2 on $\Gamma_0(p)$, then $J_f^+[\mathbb{D}^+_{-d_1,-d_2}] = 1$, while the modularity assertion for J_f^- follows from [DT08, Theorem 5.1 (a)]. More precisely, after replacing the fundamental discriminant $-D_2$ of loc.cit. by $-d_1$ and the varying discriminants $-D_1$ by $-d_2$, this theorem asserts that there is a modular form $\theta := \sum b(d_2) q^{d_2} \in \mathbb{M}(p)^+$ for which

$$\log_p J_f^-[\mathbb{D}^+_{-d_1,-d_2}] = b(d_2),$$

for all d_2 satisfying

$$\text{ord}_p(d_2) \leq 1, \quad d_2 = 0, 3 \pmod 4, \quad \left(\frac{-d_2}{p}\right) \neq 1. \tag{60}$$

In applying [DT08, Theorem 5.1 (a)], it must be noted that somewhat stronger assumptions are made in its statement than needed. Namely:

• It is assumed that $-d_1$ and $-d_2$ are both fundamental and relatively prime to each other. This stronger hypothesis on $-d_2$ is only used in the proof of [DT08, Theorem 5.1 (b)], where it is necessary that the character ω_{-d_1} be a *genus character* in order to invoke [BD09, Thm. 1]. The proof of [DT08, Theorem 5.1 (a)], where only Kohnen's formula is used, applies to the more general $-d_2$ of Theorem 6.1.

• It is assumed that f has rational fourier coefficients and hence corresponds to an elliptic curve, but the proof applies just as well to arbitrary cuspidal eigenforms after extending scalars to the field generated by the fourier coefficients of f.

• Finally, Theorem 5.1 of [DT08] applies only to the cuspidal eigenclasses, leaving out the boundary theta-symbol $J_\sharp$ of § 3.7.1 and the Dedekind–Rademacher cocycle J_{DR} of § 3.7.3. The case where $J = J_\sharp$ is trivial since $J_\sharp[\mathbb{D}^+_{-d_1,-d_2}] = 1$, while the case where $J = J_{\text{DR}}$ is handled in [Pa10, Theorem 5.2.] and the discussion that follows it, which shows that the associated generating series $\Theta_{-d_1,J_{\text{DR}}}(q)$ is an Eisenstein series of weight $3/2$ on $\Gamma_0(4p)$.

The modularity of the coefficients $\log_p J_f^-[\mathbb{D}^+_{-d_1,-d_2}]$ is thus established, for all rigid analytic theta-cocycles J and for all d_2 satisfying (60). The result then follows for all d_2 from the rule (3) satisfied by the quantities $\mathbb{D}^+_{-d_1,-d_2}$, combined with the fact that the fourier coefficients of forms in $\mathbb{M}(p)^+$ satisfy the same relation, by (59).

This handles the first assertion. The argument for the case $\left(\frac{-d_1}{p}\right) = -1$ proceeds along the same lines. □

Remark 6.2. While the argument above falls somewhat short of being a complete proof because of its reliance on the slightly weaker results in [BD09] and [DT08], a simpler and more direct route to Theorem 6.1 is available, which will be treated in forthcoming work. It builds on the approach

of [BDG], and produces the generating series $\Theta_{-d_1,J}(q)$ as the first derivative of a p-adic family of Kudla-Millson theta series in weight $3/2$.

Remark 6.3. When $\left(\frac{-d_1}{p}\right) = 1$, the generating series

$$\Theta^{\flat}_{-d_1,J}(q) := \sum_{d_2 \geq 0} \log_p J[\mathbb{D}^{-}_{-d_1,-d_2}]q^{d_2}$$

is not a classical modular form of weight $3/2$, but is best envisaged as a *p-adic mock modular form* whose "shadow" is a suitable kernel of the Shimura–Shintani correspondence between $\mathbb{M}(p)^-$ and $S_2(p)^-$, where $S_2(p)^-$ denotes the (-1)-eigenspace for the U_p operator on forms of weight two and level p. The full generating series

$$\sum_{d_2 \geq 0} \log_p J[\mathbb{D}_{-d_1,-d_2}]q^{d_2}$$

exhibits a somewhat subtle behaviour: it is a classical modular form of weight $3/2$ along $\mathbb{M}(p)^+$, and a "p-adic mock modular form" along $\mathbb{M}(p)^-$. For details, see the discussion in [DT08].

Corollary 6.4. *Let $\iota := \left(\frac{-d_1}{p}\right)$. For all $\phi = \sum_{d \gg 0} c(d)q^d \in M^{!!}_{1/2}(4p)$, the divisor*

$$\mathbb{D}_{-d_1,\phi} := \sum_{d_2 \geq 0} c(-d_2)\mathbb{D}^{\iota}_{-d_1,-d_2}$$

satisfies

$$\log_p J[\mathbb{D}_{-d_1,\phi}] = 0, \qquad \text{for all } J \in \mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times).$$

Proof. This is a direct consequence of Serre duality in light of Theorem 6.1. Namely, the value

$$\log_p J[\mathbb{D}_{-d_1,\phi}] = \sum_{d_2 \geq 0} c(-d_2) \cdot \log_p J[\mathbb{D}^{\iota}_{-d_1,-d_2}]$$

is the zero-th fourier coefficient of $\phi(q)\Theta_{-d_1,J}(q)$, a weakly holomorphic modular form of weight 2 and level p with poles only at the cusp ∞. This coefficient is the residue of the associated regular differential

$$\phi(q)\Theta_{-d_1,J}(q)\frac{dq}{q} \in \Omega^1(X_0(p) - \{\infty\}),$$

and is therefore zero by the residue theorem. □

6.2. **A principality criterion.** Let $\mathbb{D}$ be an arbitrary RM divisor. The following criterion for $\mathbb{D}$ to be principal plays a crucial role in our argument.

Proposition 6.5. *If $\log_p J[\mathbb{D}] = 0$ for all analytic theta-symbols $J \in \mathrm{MS}^{\Gamma}(\mathcal{A}^\times/C_p^\times)$, then $\mathbb{D}$ is a principal divisor, i.e., there is a meromorphic modular symbol $J_{\mathbb{D}} \in \mathrm{H}^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$ satisfying*

$$\mathrm{Divisor}(J_{\mathbb{D}}) = \mathbb{D}.$$

Proof. The hypothesis implies in particular that

$$\log_p J_{r,s}[\mathbb{D}] = 0, \qquad \text{for all } (r,s) \in \mathbb{P}_1(\mathbb{Q})^2.$$

By the reciprocity law of Theorem 5.12, it follows that

$$\log_p \mathrm{Obs}(J_{\mathbb{D}})[r,s] = 0, \qquad \text{for all } (r,s) \in \mathbb{P}_1(\mathbb{Q})^2,$$

where $J_{\mathbb{D}}$ is the rigid meromorphic theta-symbol having $\mathbb{D}$ as a divisor. Proposition 5.7 then implies that $\log_p \mathrm{Obs}(J_{\mathbb{D}}) = 0$, and therefore that $J_{\mathbb{D}}$ lifts to a meromorphic modular symbol in $\mathrm{H}^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$. The proposition follows. □

6.3. **A real quadratic Borcherds lift.** The results of the previous sections can now be combined to prove the following result. Recall that we have fixed a negative fundamental discriminant $-d_1$ and set $\iota := \left(\frac{-d_1}{p}\right)$.

Theorem 6.6. *For all $\phi = \sum_{d \gg -\infty} c_\phi(d) q^d \in M^{!!}_{1/2}(4p)$ with integer Fourier coefficients, there is a rigid meromorphic cocycle $J_{-d_1,\phi} \in \mathrm{H}^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$ satisfying*

$$\mathrm{Divisor}(J_{-d_1,\phi}) = \sum_{d_2 \geq 0} c_\phi(-d_2) \mathbb{D}^\iota_{-d_1,-d_2},$$

where the sum is taken over the negative discriminants $-d_2$ satisfying $\left(\frac{-d_2}{p}\right) \neq \iota$.

Proof. Corollary 6.4 implies that $\log_p J[\mathbb{D}_{-d_1,\phi}] = 0$ for all rigid analytic theta-symbols J, and the principality criterion of Proposition 6.5 then shows that the divisor $\mathbb{D}_{-d_1,\phi}$ is principal, i.e., arises as the divisor of some $J_{-d_1,\phi}$ in the space $\mathrm{H}^1(\Gamma, \mathcal{M}^\times) \otimes \mathbb{Q}$. □

Appendix A. The parallel with Mumford's p-adic theta functions

The purpose of this motivational appendix is to draw the parallel between the p-adic theta functions that arise in the theory of p-adic uniformisation of Mumford curves by p-adic Schottky groups acting discretely on $\mathcal{H}_p$, and rigid meromorphic theta-cocycles. Up to a few significant differences between the two settings, one passes from one notion to the other by "shifting the degree of cohomology by one".

Firstly, let $\Gamma \subset \mathrm{SL}_2(\mathbb{Q}_p)$ be a p-adic Schottky group acting freely and discretely on $\mathcal{H}_p$ and on the Bruhat–Tits tree, and let X_Γ be the associated Mumford curve over $\mathbb{Q}_p$, whose $\mathbb{C}_p$-points are identified with the quotient $\Gamma \backslash \mathcal{H}_p$. As before, let $\mathcal{A}^\times$ and $\mathcal{M}^\times$ denote the multiplicative groups of rigid analytic and rigid meromorphic functions on $\mathcal{H}_p$, respectively. The theory of p-adic theta-functions associates to any degree zero divisor Δ of $\mathcal{H}_p$ a rigid meromorphic function

$$\theta_\Delta(z) := \prod_{\gamma \in \Gamma} [(z) - (\eta)\,;\, \gamma\Delta].$$

The divisor of such a function is Γ-invariant. In fact, θ_Δ is invariant modulo scalars, i.e., it belongs to $\mathrm{H}^0(\Gamma, \mathcal{M}^\times / C_p^\times)$. It lifts to an element of $\mathrm{H}^0(\Gamma, \mathcal{M}^\times)$, i.e., to a rational function on X_Γ, if any only if the image of Δ in $X_\Gamma(\mathbb{C}_p)$ is a principal divisor. The obstruction to Δ being principal is measured by the automorphy factor $\kappa_\Delta : \Gamma \longrightarrow \mathbb{C}_p^\times$ of θ_Δ, defined by

$$\theta_\Delta(\gamma z) = \kappa_\Delta(\gamma) \theta_\Delta(z), \qquad \text{for } \gamma \in \Gamma.$$

The group $\mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$ is isomorphic to $(\mathbb{C}_p^\times)^g$ where g is the genus of X_Γ, and the group Π_Γ of automorphy factors arising from elements of $\mathrm{H}^0(\Gamma, \mathcal{A}^\times / \mathbb{C}_p^\times)$ forms a sublattice in $\mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$ which is commensurable with the Tate period lattice of the Jacobian of X_Γ. One obtains a p-adic uniformisation of this Jacobian as the quotient of $\mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$ by Π_Γ, and the lifting obstruction κ_Δ attached to $\Delta \in \mathrm{Div}^0(\mathcal{H}_p)$ encodes the image of Δ in $\mathrm{Jac}(X_\Gamma)$. This discussion is summarised in the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & \mathrm{H}^0(\Gamma, \mathcal{A}^\times / \mathbb{C}_p^\times) & \xrightarrow{\delta} & \Pi_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}_p^\times & \longrightarrow & \mathrm{H}^0(\Gamma, \mathcal{M}^\times) & \longrightarrow & \mathrm{H}^0(\Gamma, \mathcal{M}^\times / \mathbb{C}_p^\times) & \xrightarrow{\delta} & \mathrm{H}^1(\Gamma, \mathbb{C}_p^\times) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Div}} & & \downarrow{\scriptstyle \mathrm{Div}} & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathrm{Prin}(X_\Gamma) & \longrightarrow & \mathrm{Div}^0(X_\Gamma) & \longrightarrow & \mathrm{Jac}(X_\Gamma) & \longrightarrow & 0, \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
\tag{61}
$$

where δ is the connecting homomorphism arising from in the long exact Γ-cohomology exact sequence, and

$$\Pi_\Gamma := \delta(\mathrm{H}^0(\Gamma, \mathcal{A}^\times / C_p^\times)) \subset \mathrm{H}^1(\Gamma, C_p^\times).$$

The analogue of (61) in the setting of rigid meromorphic cocycles is obtained by replacing the p-adic Schottky group Γ by the multiplicative group of a $\mathbb{Z}[1/p]$-order in an *indefinite* quaternion algebra over $\mathbb{Q}$. The Ihara group $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ is the simplest, prototypical example of such a group. It is too large to act discretely on $\mathcal{H}_p$, or on the Bruhat–Tits tree without fixed points. Indeed, the vertex and edge stablisers in Γ are conjugate to $\mathrm{SL}_2(\mathbb{Z})$ and to the Hecke congruence group $\Gamma_0(p)$, respectively. Because of this, the groups $\mathrm{H}^0(\Gamma, \mathcal{A}^\times)$ and $\mathrm{H}^0(\Gamma, \mathcal{M}^\times)$ contain only the constant functions, and it becomes natural to replace the group $\mathrm{H}^0(\Gamma, \mathcal{M}^\times)$ of rigid meromorphic functions on X_Γ when Γ is a p-adic Schottky group, with the group $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$ of rigid meromorphic cocycles.

The theory of theta-cocycles described in § 4 associates to any divisor Δ of $\mathcal{H}_p$ *consisting of RM points* a rigid meromorphic cocycle

$$J_\Delta(z) \in \mathrm{H}^1(\Gamma, \mathcal{M}^\times / C_p^\times)$$

modulo multiplicative scalars, and shows that all such cocycles are obtained in this way. The divisor Δ is said to be *principal* if the class J_Δ lifts to genuine rigid meromorphic cocycle in $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$. The obstruction to Δ being principal is measured by the lifting obstruction $\kappa_\Delta \in \mathrm{H}^1(\Gamma, \mathrm{MS}(C_p^\times))$ and its non parabolic counterpart in $H^2(\Gamma, C_p^\times)$. The group $\mathrm{H}^2(\Gamma, C_p^\times)$ maps with finite kernel to $\mathrm{H}^1(\Gamma_0(p), C_p^\times)$, suggesting that it could serve as the domain for a p-adic uniformisation of $J_0(p)$ (or even, of the generalised Jacobian of the open curve $Y_0(p)$). Indeed, it turns out that the group Π_Γ generated by the lifting obstructions of *analytic* cocycles in $\mathrm{H}^1(\Gamma, \mathcal{A}^\times / C_p^\times)$

forms a sublattice in $\mathrm{H}^2(\Gamma, C_p^\times)$ which is commensurable with two copies of the Tate period lattice of $J_0(p)$, augmented by the discrete group generated by $p^{\mathbb{Z}}$. As explained in [Dar01, §2] and in [Das05], this is essentially a reformulation of the "exceptional zero conjecture" of Mazur, Tate and Teitelbaum [MTT86] which was proved by Greenberg and Stevens [GS93]. One obtains a kind of p-adic uniformisation of two copies of this Jacobian (along with a multiplicative factor of $C_p^\times/p^{\mathbb{Z}}$) as a rigid analytic quotient of $\mathrm{H}^2(\Gamma, C_p^\times)$ by the lattice Π_Γ. The lifting obstruction κ_τ attached to any $\tau \in \mathcal{H}_p^{\mathrm{RM}}$ encodes the images of the Gross–Stark units and the Stark–Heegner points attached to τ, in the generalised Jacobian of $Y_0(p)$.

This discussion can be summarised in the following commutative diagram in the category of abelian groups up to isogeny, where morphisms are decreed to be isomorphisms if they have finite kernels and cokernels:

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1(\Gamma, \mathcal{A}^\times) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathcal{A}^\times/C_p^\times) & \xrightarrow{\delta} & \Pi_\Gamma \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^1(\Gamma, \mathcal{M}^\times) & \longrightarrow & \mathrm{H}^1(\Gamma, \mathcal{M}^\times/C_p^\times) & \xrightarrow{\delta} & \mathrm{H}^2(\Gamma, C_p^\times) \\
\downarrow \mathrm{Div} & & \downarrow \mathrm{Div} & & \downarrow \\
\mathrm{Prin}(\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}) & \longrightarrow & \mathrm{Div}^0(\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}) & \longrightarrow & \mathrm{Jac}(X_0(p))(C_p)^2 \oplus C_p^\times/p^{\mathbb{Z}} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\tag{62}
$$

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