SINGULAR MODULI FOR REAL QUADRATIC FIELDS: A RIGID ANALYTIC APPROACH

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Abstract. A *rigid meromorphic cocycle* is a class in the first cohomology of the discrete group $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ with values in the multiplicative group of non-zero rigid meromorphic functions on the p-adic upper half plane $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$. Such a class can be evaluated at the real quadratic irrationalities in $\mathcal{H}_p$, which are referred to as "RM points". Rigid meromorphic cocycles can be envisaged as the real quadratic counterparts of Borcherds' singular theta lifts: their zeroes and poles are contained in a finite union of Γ-orbits of RM points, and their RM values are conjectured to lie in ring class fields of real quadratic fields. These RM values enjoy striking parallels with the CM values of modular functions on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$: in particular they seem to factor just like the differences of classical singular moduli, as described by Gross and Zagier. A fast algorithm for computing rigid meromorphic cocycles to high p-adic accuracy leads to convincing numerical evidence for the algebraicity and factorisation of the resulting singular moduli for real quadratic fields.

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Introduction

Drinfeld's p-adic upper half-plane, a rigid analytic space whose $\mathbb{C}_p$-points are identified with $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$, offers an enticing framework for explicit class field theory for real quadratic fields, since it contains a large supply $\mathcal{H}_p^{\mathrm{RM}}$ of *real multiplication* (RM) points belonging to real quadratic fields in which the prime p is either inert or ramified. Let $\mathcal{M}^\times$ denote the multiplicative group of *rigid meromorphic functions* on $\mathcal{H}_p$, consisting of ratios of non-zero rigid analytic functions. The discrete group $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$ acts on $\mathcal{H}_p$ by Möbius transformations, inducing an action on $\mathcal{M}^\times$ (written either on the right or on the left) by the rule

$$(1) \qquad (f|\gamma)(\tau) := (\gamma^{-1}f)(\tau) := f\left(\frac{a\tau+b}{c\tau+d}\right), \qquad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A naive attempt at explicit class field theory for real quadratic fields could proceed by examining the RM values of Γ-invariant functions in $\mathcal{M}^\times$. However, because the Γ-orbits in $\mathcal{H}_p$

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are dense for the rigid analytic topology, any such function is constant, i.e.,

$$\mathrm{H}^0(\Gamma, \mathcal{M}^\times) = \mathbb{C}_p^\times. \tag{2}$$

It is then natural to consider the first cohomology group $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$ instead. A class in this group is said to be *parabolic* if its restriction to the subgroup $\Gamma_\infty \subset \Gamma$ of upper triangular matrices is trivial, and is said to be *quasi-parabolic* if this restriction lies in $\mathrm{H}^1(\Gamma_\infty, \mathbb{C}_p^\times)$. The groups of such classes are denoted by $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times)$ and $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ respectively.

Definition 1. *A class in* $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ *is called a* rigid meromorphic cocycle *for* Γ.

A rigid meromorphic cocycle is thus a function $J : \Gamma \longrightarrow \mathcal{M}^\times$ satisfying

$$J(\gamma_1\gamma_2) = J(\gamma_1) \times \gamma_1 J(\gamma_2),$$

taken modulo 1-coboundaries, of the form $\xi(\gamma) = \gamma f \div f$, with $f \in \mathcal{M}^\times$, and admitting a quasi-parabolic representative, whose values on Γ_∞ consist of constant functions. This representative is even unique (up to torsion), because $\mathcal{M}$ contains no translation-invariant elements.

This article initiates the study of rigid meromorphic cocycles, with special emphasis on their application to the analytic construction of class fields of real quadratic fields.

The relevance of Definition 1 for explicit class field theory rests on the fact that rigid meromorphic cocycles can be meaningfully *evaluated* at RM points. More precisely, the RM points in $\mathcal{H}_p$ are characterised by the fact that their *associated order*

$$\mathcal{O}_\tau := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}[1/p]) \text{ such that } a\tau + b = c\tau^2 + d\tau \right\}$$

is isomorphic to a $\mathbb{Z}[1/p]$-order in the real quadratic field $K = \mathbb{Q}(\tau)$, via the inclusion

$$\iota : \mathcal{O}_\tau \longrightarrow K, \qquad \iota \begin{pmatrix} a & b \\ c & d \end{pmatrix} = c\tau + d.$$

The stabiliser of τ in Γ is generated up to torsion by a fundamental unit of norm one in $\mathcal{O}_\tau$. It is called the *automorph* of τ, and denoted γ_τ. The *value* of a rigid meromorphic cocycle J at an RM point τ is then defined to be

$$J[\tau] := J(\gamma_\tau)(\tau) \ \in \ \mathbb{C}_p \cup \{\infty\}, \tag{3}$$

a numerical invariant which depends only on the class of J in cohomology (and not on the choice of cocycle representing it) and on the Γ-orbit of τ. The cocycle J thus gives rise to a function

$$J \ : \ \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}} \longrightarrow \mathbb{C}_p \cup \{\infty\}. \tag{4}$$

Conjecture 1 below asserts that it takes algebraic values that lie in (composita of) abelian extensions of real quadratic fields, thus behaving in many key respects like the function $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^{\mathrm{CM}} \longrightarrow \mathbb{C} \cup \{\infty\}$ induced by the classical j-function, or by any other meromorphic modular function defined over $\bar{\mathbb{Q}}$.

Let S be the standard matrix of order 2 in $\Gamma/\langle\pm 1\rangle$ that fixes $i = \sqrt{-1}$.

Definition 2: *A* rigid meromorphic period function *is the value at* S *of the quasi-parabolic representative of a rigid meromorphic cocycle.*

The assignment $J \mapsto j := J(S)$ identifies $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ with the multiplicative group $\mathcal{R}^\times$ of rigid meromorphic period functions, and any function in $\mathcal{R}^\times$ satisfies the functional relations

$$(5) \qquad j(-1/z) = j(z)^{-1}, \qquad j(p^2z) = j(z), \qquad \frac{j(z+1)}{j(z)} = j\left(-\frac{z+1}{z}\right).$$

The main result of the first three chapters is

Theorem 1. *The group $\mathcal{R}^\times$ is of infinite rank. The zeroes and poles of any $j \in \mathcal{R}^\times$ are contained in a finite union of Γ-orbits of RM points in $\mathcal{H}_p$.*

Theorem 1 suggests that rigid meromorphic period functions might be viewed as the real quadratic counterpart of Borcherds' singular theta lifts of modular forms of weight $1/2$, insofar as the latter are meromorphic modular functions with divisor concentrated at CM points.

Let H_τ denote the *ring class field* (in the *narrow sense*) associated to the order $\mathcal{O}_\tau$. It is an abelian extension of $K = \mathbb{Q}(\tau)$ whose Galois group over K is identified via global class field theory with the narrow class group $\mathrm{Pic}^+(\mathcal{O}_\tau)$ of projective oriented $\mathcal{O}_\tau$-modules. If j is any rigid meromorphic period function and J is its associated rigid meromorphic cocycle, Theorem 1 implies that the field

$$H_j = H_J := \mathrm{Compositum}_{j(\tau)=\infty}(H_\tau)$$

is a finite extension of $\mathbb{Q}$; it is called the *field of definition* of j, or of J. The main conjecture of this paper, which is discussed in greater detail in Chapter 4, is

Conjecture 1. *If J is a rigid meromorphic cocycle, and $\tau \in \mathcal{H}_p$ is an RM point, then the value $J[\tau]$ is an algebraic number belonging to the compositum of H_J and H_τ.*

Conjecture 1 gives ample motivation for the systematic study of rigid meromorphic cocycles. This study is carried out in Chapters 1, 2 and 3, where Theorem 1 is proved by giving a complete classification of rigid meromorphic period functions. These functions, and their additive counterparts known as rigid meromorphic period functions of weight two, are reminiscent of the "rational period functions" that are studied in [Kn], [Ash], [CZ], and can be classified along similar lines. The classification obtained in Chapter 3 is constructive and leads to explicit product expansions for rigid meromorphic period functions. To describe these, for any $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$, let

$$(6) \qquad \Sigma_\tau := \{w \in \Gamma\tau \text{ such that } ww' < 0\},$$

where w' is the algebraic conjugate of $w \in K := \mathbb{Q}(\tau)$. The subset Σ_τ of $\Gamma\tau$ contains only finitely many integer translates of any given $w \in \Gamma\tau$, and it can in fact be shown that it is a discrete subset of the full Γ-orbit, relative to the rigid p-adic topology on $\mathcal{H}_p$. After fixing a real embedding of K, let $\delta_\infty(w) \in \{-1, 1\}$ denote the sign of $w \in \Sigma_\tau$.

A prime p is said to be *monstrous* if it divides the cardinality of the Monster sporadic simple group, or equivalently (by a famous observation of Andrew Ogg) if the quotient of the modular curve $X_0(p)$ by its Atkin Lehner involution has genus zero, which occurs precisely when

$$p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

One of the illustrative results of Chapter 3 is

Theorem 2. *Let p be a monstrous prime, and let τ be any RM point in $\mathcal{H}_p$. The infinite product*

$$j_\tau^+(z) := \prod_{w\in\Sigma_\tau} \left(\frac{t_w(z)}{t_{pw}(z)}\right)^{\delta_\infty(w)}, \qquad \text{where } t_w(z) = \begin{cases} z-w & \text{if } |w|\le 1, \\ z/w-1 & \text{if } |w|>1, \end{cases}$$

converges to a rigid meromorphic period function on $\mathcal{H}_p$.

The constructions described in the first three chapters also lead to a complete classification of rigid meromorphic period functions for arbitrary p: the multiplicative group of such functions is spanned by suitable Hecke translates of simple generalisations of the example in Theorem 2.

Chapter 3 concludes by describing an efficient algorithm for computing rigid meromorphic period functions to high p-adic accuracy, in terms of their images in the Tate algebra of a suitable affinoid subset of $\mathcal{H}_p$. This algorithm makes it feasible to compute the RM values of rigid meromorphic cocycles to hundreds of significant digits, and has been used to test Conjecture 1 numerically in a variety of situations.

For example, if p is a monstrous prime that is inert in $\mathbb{Q}(\sqrt{5})$, i.e., if

$$p \in \{2,3,7,13,17,23,47\}, \tag{7}$$

and $\varphi := (1+\sqrt{5})/2$ is the golden ratio, the rigid meromorphic period function j_φ^+ and associated rigid meromorphic cocycle J_φ^+ defined by setting $\tau = \varphi$ in Theorem 2 can be viewed as a convincing analogue of the j-function, whose divisor is concentrated on the CM point $(1+\sqrt{-3})/2$ of smallest negative discriminant. Calculations performed to 100 digits of 3-adic and 13-adic accuracy (the primes $p = 3, 13$ being precisely those in (7) which are also inert in $\mathbb{Q}(\sqrt{2})$) suggest that

$$J_\varphi^+[2\sqrt{2}] \stackrel{?}{=} \begin{cases} (33+56\sqrt{-1})/(5\cdot 13) & \text{in } \mathbb{C}_3; \\ (1+2\sqrt{-2})/3 & \text{in } \mathbb{C}_{13}, \end{cases}$$

consistent with the fact that the order $\mathbb{Z}[2\sqrt{2}]$ has narrow class number two, and that its associated narrow ring class field is equal to $\mathbb{Q}(\sqrt{2},\sqrt{-1})$. Likewise, the real quadratic field $K := \mathbb{Q}(\sqrt{223})$ has narrow class number 6, and the element $\sqrt{223}$ can be viewed as belonging to $\mathcal{H}_7$, $\mathcal{H}_{13}$, and $\mathcal{H}_{47}$. The value of J_φ^+ at $\tau = \sqrt{223}$, computed to 100 digits of p-adic accuracy for $p \in \{7, 13, 47\}$, appears to satisfy the following sextic polynomials:

$$\begin{aligned} p = 7. \quad & 282525425x^6 + 27867770x^5 + 414793887x^4 - 128906260x^3 + 414793887x^2 \\ & \qquad +27867770x + 282525425, \\ p = 13. \quad & 464800x^6 + 1275520x^5 + 1614802x^4 + 1596283x^3 + 1614802x^2 \\ & \qquad +1275520x + 464800, \\ p = 47. \quad & 4x^6 + 4x^5 + x^4 - 2x^3 + x^2 + 4x + 4. \end{aligned}$$

All three of these polynomials have the Hilbert class field of K as their splitting field, confirming Conjecture 1 in a setting where H_τ is non-abelian over $\mathbb{Q}$.

A number of patterns emerge from the above experiments, notably:

• The RM value $J_\varphi^+[\tau]$ appears to belong to $H_\tau^{\sigma_p\sigma_\infty=1}$, where σ_p and σ_∞ denote frobenius elements at the prime p and ∞ respectively. In that sense the situation is even a bit better than predicted by Conjecture 1, which only asserts that $J_\varphi^+[\tau]$ should be defined over the compositum of H_τ with the field of definition $H_\varphi = \mathbb{Q}(\sqrt{5})$ of J_φ^+. The experiments, which have been carried out for numerous other examples as well, suggest that j_φ^+ should really be viewed as being "defined over $\mathbb{Q}$" rather than over $\mathbb{Q}(\sqrt{5})$.

• The primes that occur in the factorisation of $J_{\varphi}^{+}[\tau]$ lie above rational primes that are inert or ramified in both the real quadratic fields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\tau)$, and divide an integer of the form $\frac{5\mathrm{Disc}(\tau)-m^2}{4} \geq 0$.

• The value of $J_{\varphi}^{+}[\tau]$–and even the precise field over which it is defined–depends very much on the monstrous prime p relative to which it is computed. However, if p and q are two monstrous primes for which φ and τ belong to both $\mathcal{H}_p$ and $\mathcal{H}_q$, the primes above q very often (but not always!) occur in the factorisation of the p-adic $J_{\varphi}^{+}[\tau]$ with the same multiplicity as the primes above p in the factorisation of the selfsame q-adic invariant.

In an attempt to better understand this last feature, Chapter 4 focusses on a prime $p \in \{2,3,5,7,13\}$, i.e., a prime for which the modular curve $X_0(p)$ has genus zero. For each pair (τ_1, τ_2) of RM points in $\mathcal{H}_p$ with associated ring class fields $H_1 = H_{\tau_1}$ and $H_2 = H_{\tau_2}$ a *p-adic arithmetic intersection number*

$$J_p(\tau_1, \tau_2) \overset{?}{\in} H_{12} := H_1 H_2$$

is defined. Roughly speaking, it is the value $\hat{J}_{\tau_1}[\tau_2]$, where $\hat{J}_{\tau_1}$ is a simple modification of the cocycle $J_{\tau_1}^{+}$ of Theorem 2, with zeroes and poles concentrated in $\Gamma\tau_1$. The quantity $J_p(\tau_1, \tau_2)$ seems to enjoy many of the same properties as the difference

$$J_\infty(\tau_1, \tau_2) := j(\tau_1) - j(\tau_2), \qquad j(q) = \frac{1}{q} + 744 + 196884q + \cdots$$

of "classical" singular moduli studied in [GZ1], and is conjectured to admit analogous factorisations.

The prediction made in Conjecture 4.26 of Chapter 4 can be loosely paraphrased as follows:

Conjecture 2. *The p-adic intersection number $J_p(\tau_1, \tau_2) \overset{?}{\in} H_{12}$ is divisible only by primes of H_{12} lying above rational primes that are non-split in both of the real quadratic fields $K_1 := \mathbb{Q}(\tau_1)$ and $K_2 := \mathbb{Q}(\tau_2)$, and divide a positive integer of the form $\frac{D_1 D_2 - x^2}{4}$. If q is such a prime, then the valuations of $J_p(\tau_1, \tau_2)$ at the primes above q are determined by certain q-*weighted topological intersection numbers *of modular geodesics attached to τ_1 and τ_2 on the Shimura curve arising from the indefinite quaternion algebra ramified at q and p.*

This prediction resonates closely with the factorisations described in [GZ1], where the valuation at q of $J_\infty(\tau_1, \tau_2)$ is determined by a similar intersection of 0-cycles attached to the CM points τ_1 and τ_2 on the 0-dimensional Shimura variety arising from the definite quaternion algebra ramified at q and ∞.

Chapter 5 recalls the construction of the Gross–Stark units given in [DD] and the Stark–Heegner points of [Da], which are conjecturally defined over ring class fields of real quadratic fields, and explains how these constructions can be recast in the framework of RM values of rigid analytic cocycles "modulo suitable periods".

In closing, it is worth noting that the infinite rank group $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ admits no non-trivial finite rank Hecke stable subspaces. Rigid meromorphic cocycles and their RM values thus bear no direct relationship to Hecke eigenforms and to special values of L-functions with Euler products, unlike the Gross-Stark units and the Stark-Heegner points of Chapter 5, which are expected to satisfy analogues of the Kronecker limit formula and the Gross-Zagier formula. In that sense, the main thesis of this paper — that rigid meromorphic cocycles play the role of meromorphic modular functions in extending the theory of complex multiplication to real quadratic fields — breaks more decisively from the tradition of the Stark conjectures than either [DD] or [Da], where the leading terms of motivic L-functions continue to play a central role.

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1. Meromorphic cocycles of weight two

This chapter introduces additive counterparts of the rigid meromorphic cocycles and their associated rigid meromorphic period functions that were described in the introduction. These are referred to as rigid meromorphic cocycles and period functions *of weight* $k \geq 0$. The main results of this chapter are Theorems 1.22 and 1.23, which together give the full classification of rigid meromorphic period functions of weight two up to rigid analytic period functions of the same weight. The techniques are greatly inspired by the classification of *rational period functions* carried out by Knopp [Kn], Ash [Ash] and Choie-Zagier [CZ].

1.1. The p-adic upper half-plane.

We begin by recalling some facts about the p-adic upper half plane as a rigid analytic space.

Let $\mathcal{T}$ denote the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose vertices are in bijection with the homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$, two vertices being joined by an (unordered) edge if they admit representative lattices contained in each other with index p. Write $\mathcal{T}_0$, $\mathcal{T}_1$, and $\mathcal{T}_1^*$ for the set of vertices, unordered edges, and ordered edges respectively of $\mathcal{T}$. If $e \in \mathcal{T}_1^*$ is an ordered edge, we denote by $s(e)$ and $t(e) \in \mathcal{T}_0$ its *source* and *target* vertices respectively. The group Γ acts on $\mathcal{T}$ through its natural left action on $\mathbb{Q}_p^2$, viewing the latter as column vectors. The "standard vertex" $v_0 \in \mathcal{T}_0$ associated to the lattice $\mathbb{Z}_p^2$ has $\mathrm{SL}_2(\mathbb{Z})$ as its stabiliser for this action. A vertex is said to be *even* if its distance to v_0 is even, and is said to be *odd* otherwise. The set of even and odd vertices are denoted $\mathcal{T}_0^+$ and $\mathcal{T}_0^-$ respectively. Likewise, an ordered edge in $\mathcal{T}_1^*$ is said to be even if its source is even and odd if its source is odd. The subsets of even and odd oriented edges are denoted $\mathcal{T}_1^+$ and $\mathcal{T}_1^-$ respectively, so that we have the decompositions

$$\mathcal{T}_0 = \mathcal{T}_0^+ \sqcup \mathcal{T}_0^-, \qquad \mathcal{T}_1^* = \mathcal{T}_1^+ \sqcup \mathcal{T}_1^-.$$

The p-adic upper half plane $\mathcal{H}_p$ may be thought of as a tubular neighbourhood of the Bruhat–Tits tree $\mathcal{T}$ via the natural "reduction map"

$$\mathrm{red} : \mathcal{H}_p \longrightarrow \mathcal{T},$$

which maps $\mathcal{H}_p(\hat{\mathbb{Q}}_p^{\mathrm{nr}})$ to $\mathcal{T}_0$, where $\hat{\mathbb{Q}}_p^{\mathrm{nr}}$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$. The inverse image of a vertex $v \in \mathcal{T}_0$, denoted $\mathcal{A}_v$, is called a *vertex affinoid* in $\mathcal{H}_p$, and the inverse image of an edge $e \in \mathcal{T}_1$ is an annulus denoted by $\mathcal{W}_e$. The vertex affinoid corresponding to the standard vertex v_0 is called the *standard affinoid*: it is the complement in $\mathbb{P}_1(\mathbb{C}_p)$ of the $p+1$ mod p residue discs around the points in $\mathbb{P}_1(\mathbb{F}_p)$. The edge $e_0 \in \mathcal{T}_1$ with stabiliser $\Gamma_0(p)$ is the image under the reduction map of the *standard annulus*

$$\mathcal{W}_{e_0} = \{z \in \mathbb{C}_p \quad \text{with } 1 < |z| < p\}.$$

If v is a vertex and $e_1, \ldots e_{p+1}$ are the distinct edges in $\mathcal{T}_1$ having v as an endpoint, then the union

$$\mathcal{W}_v := \mathcal{A}_v \cup \bigcup_j \mathcal{W}_{e_j} := \mathrm{red}^{-1}\left(v \cup \bigcup_j e_j\right)$$

is called the *standard wide open subset* attached to v.

Let $\mathcal{T}^{\leq n}$ denote the subgraph of $\mathcal{T}$ spanned by the vertices of distance $\leq n$ from the standard vertex, and write $\mathcal{H}_p^{\leq n}$ for the affinoid subdomain of $\mathcal{H}_p$ consisting of those points reducing to $\mathcal{T}^{\leq n}$. Likewise let $\mathcal{T}^{<n}$ denote the subgraph of $\mathcal{T}$ containing all the vertices of distance $\leq n-1$ as well as all the edges containing at least one of these vertices, and write $\mathcal{H}_p^{<n}$ for the wide open subspace of $\mathcal{H}_p$ consisting of those points reducing to $\mathcal{T}^{<n}$. The subsets $\mathcal{H}_p^{\leq n}$ define an admissible cover

$$\mathcal{H}_p = \bigcup_{n \geq 0} \mathcal{H}_p^{\leq n}$$

of the p-adic upper half-plane by affinoid subsets.

Of course, the actions of Γ on $\mathcal{T}$ and on $\mathcal{H}_p$ by Möbius transformations are compatible under the reduction map. In particular, for all $\gamma \in \Gamma$,

$$\mathcal{A}_{\gamma v} = \gamma \mathcal{A}_v, \qquad \mathcal{W}_{\gamma e} = \gamma \mathcal{W}_e.$$

A $\mathbb{C}_p$-valued function on $\mathcal{H}_p$ is said to be *rigid analytic* if its restriction to any affinoid subset $\mathcal{A}$ of $\mathcal{H}_p$ is a uniform limit, relative to the supremum norm, of rational functions on $\mathbb{P}_1(\mathbb{C}_p)$ having poles outisde of $\mathcal{A}$. The space $\mathcal{O}$ of rigid analytic functions on $\mathcal{H}_p$ is endowed with a natural topology arising from its expression as the inverse limit of the affinoid algebras $\mathcal{O}(\mathcal{H}_p^{\leq n})$, which are Banach spaces for their supremum norms. Let $\mathcal{M}$ denote the fraction field of $\mathcal{O}$. Its elements are called *rigid meromorphic functions* on $\mathcal{H}_p$.

If $\tau \in \mathcal{H}_p$ is an RM point, then there is a primitive integral binary quadratic form $F_\tau(x, y)$ satisfying $F(\tau, 1) = 0$, which is unique up to sign. The *discriminant* of τ is the discriminant of this binary quadratic form. The discriminant is an invariant for the action of $\mathrm{SL}_2(\mathbb{Z})$, but not of Γ, which only preserves the prime-to-p part of $\mathrm{disc}(\tau)$.

Proposition 1.1. *If τ is an RM point of discriminant $D_0 p^n$, where D_0 is prime to p, then τ reduces to a point of $\mathcal{T}$ at distance $n/2$ from v_0. In particular:*

1. *If $n = 2m$ is even, then τ reduces to a vertex of $\mathcal{T}$, and belongs to one of the affinoids in $\mathcal{H}_p^{\leq m} - \mathcal{H}_p^{<m}$.*
2. *If $n = 2m+1$ is odd, then τ reduces to the midpoint of an edge of $\mathcal{T}$, and belongs to one of the annuli in $\mathcal{H}_p^{<m+1} - \mathcal{H}_p^{\leq m}$.*

Proof. Let $Ax^2 + Bxy + Cy^2$ be the primitive integral binary quadratic form of discriminant $D = D_0 p^n$ having τ as a root. Let ϖ be an element of $\mathcal{O}_{\mathbb{C}_p}$ of normalised p-adic valuation $n/2$. The natural image of $\tau = \frac{-B+\sqrt{D}}{2A}$ in $\mathbb{P}_1(\mathcal{O}_{\mathbb{C}_p}/\varpi)$ agrees with the image of $[-B : 2A] \in \mathbb{P}_1(\mathbb{Q}_p)$

under the natural composition

$$\mathbb{P}_1(\mathbb{Q}_p) \hookrightarrow \mathbb{P}_1(\mathbb{C}_p) = \mathbb{P}_1(\mathcal{O}_{\mathbb{C}_p}) \longrightarrow \mathbb{P}_1(\mathcal{O}_{\mathbb{C}_p}/\varpi),$$

while its image in $\mathbb{P}_1(\mathcal{O}_{\mathbb{C}_p}/\varpi p^\epsilon)$ for any $\epsilon > 0$ does not lie in the image of $\mathbb{P}_1(\mathbb{Q}_p)$. The proposition follows. □

1.2. **Modular symbols.** The parabolic cohomology of Γ with values in a $\mathrm{GL}_2(\mathbb{Q}_p)$-module Ω admits a concrete description in terms of Γ-invariant modular symbols, which will also provide a natural bridge between rigid meromorphic cocycles and the rigid meromorphic period functions evoked in the introduction.

The action of Γ on Ω shall be written (both on the right and on the left according to convenience) as

$$(m,\gamma) \mapsto m|\gamma, \quad (\gamma, m) \mapsto \gamma m := m|\gamma^{-1}, \qquad m \in \Omega, \quad \gamma \in \Gamma.$$

Definition 1.2. An Ω-valued modular symbol is a function

$$m : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \Omega,$$

satisfying

$$m\{r,s\} = -m\{s,r\}, \qquad m\{r,s\} + m\{s,t\} = m\{r,t\} \qquad \text{for all } r,s,t \in \mathbb{P}_1(\mathbb{Q}).$$

The space of Ω-valued modular symbols is denoted $\mathrm{MS}(\Omega)$. It is endowed with a natural action of $\mathrm{PGL}_2(\mathbb{Q})$ by the rule

$$(m|\gamma)\{r,s\} := (m\{\gamma r, \gamma s\})\,|\gamma.$$

The space of Γ-invariant modular symbols, denoted

$$\mathrm{MS}^\Gamma(\Omega) := \mathrm{H}^0(\Gamma, \mathrm{MS}(\Omega)),$$

is the set of modular symbols satisfying the Γ-invariance property

$$m\{\gamma r, \gamma s\} = m\{r,s\}|\gamma^{-1} = \gamma m\{r,s\}, \quad \text{for all } \gamma \in \Gamma.$$

It is equipped with the usual action of the Hecke operators T_n (for $(n,p)=1$) defined in terms of the double coset

$$\Gamma \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \sqcup_j \Gamma \gamma_j$$

by setting

$$(m|T_n) = \sum_j (m|\gamma_j).$$

The normaliser of Γ in $\mathrm{PGL}_2(\mathbb{Q})$ is the group $\mathrm{PGL}_2(\mathbb{Z}[1/p])$, and the determinant induces an isomorphism

$$\det : \mathrm{PGL}_2(\mathbb{Z}[1/p])/\Gamma \longrightarrow \mathbb{Z}[1/p]^\times/(\mathbb{Z}[1/p]^\times)^2 = \{1,-1,p,-p\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The elements w_∞ and w_p associated to classes of matrices of determinant -1 and p respectively generate this quotient, and give rise to involutions on $\mathrm{MS}^\Gamma(\Omega)$ by the rules

$$(8) \qquad (m|w_\infty)\{r,s\} := (m\{w_\infty r, w_\infty s\})\,|w_\infty, \qquad (m|w_p)\{r,s\} := (m\{w_p r, w_p s\})\,|w_p.$$

A Γ-invariant modular symbol which is in the $+1$ (resp. -1) eigenspace for the involution w_∞ is said to be even (resp. odd). Likewise, it is said to be p-even (resp. p-odd) if it is in the $+1$ (resp. -1) eigenspace for the involution w_p.

The following lemma relates Γ-invariant modular symbols to the corresponding parabolic cohomology groups.

Lemma 1.3. *There is a natural exact sequence*

$$0 \longrightarrow \Omega^\Gamma \longrightarrow \Omega^{\Gamma_\infty} \longrightarrow \mathrm{MS}^\Gamma(\Omega) \longrightarrow \mathrm{H}^1(\Gamma, \Omega) \longrightarrow \mathrm{H}^1(\Gamma_\infty, \Omega). \tag{9}$$

In particular, there is a canonical Hecke-equivariant surjection $\delta : \mathrm{MS}^\Gamma(\Omega) \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma, \Omega)$ *which is an isomorphism when* $\Omega^\Gamma = \Omega^{\Gamma_\infty}$.

Proof. Let $\mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega)$ be the Γ-module of Ω-valued functions on $\mathbb{P}_1(\mathbb{Q})$, equipped with the natural Γ-action arising from the action of Γ on $\mathbb{P}_1(\mathbb{Q})$ by Möbius transformations. It fits into the exact sequence of $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow \Omega \longrightarrow \mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega) \stackrel{d}{\longrightarrow} \mathrm{MS}(\Omega) \longrightarrow 0, \tag{10}$$

where $df\{r, s\} := f(s) - f(r)$. The lemma follows from taking the long exact Γ-cohomology sequence associated to this short exact sequence and invoking Shapiro's Lemma to identify $\mathrm{H}^i(\Gamma, \mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \Omega))$ with $\mathrm{H}^i(\Gamma_\infty, \Omega)$. $\square$

The cohomology class $\Phi := \delta(\Phi_0)$ associated to $\Phi_0 \in \mathrm{MS}^\Gamma(\Omega)$ is defined by choosing a base point $r \in \mathbb{P}_1(\mathbb{Q})$ and setting

$$\Phi(\gamma) = \Phi_0\{r, \gamma r\}.$$

The parabolic representative of Φ which vanishes on Γ_∞ is obtained by choosing $r = \infty$ in this assignment.

Let

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix} \tag{11}$$

denote the standard matrices in Γ satisfying $S^2 = U^3 = -1$. Given $m \in \mathrm{MS}^\Gamma(\Omega)$, the element $\omega := m\{0, \infty\} \in \Omega$ satisfies the so-called *two and three-term relations*

$$\omega + S\omega = 0, \qquad \omega + U\omega + U^2\omega = 0, \tag{12}$$

which follow from the modular symbol relations

$$m\{0, \infty\} + m\{\infty, 0\} = 0, \qquad m\{0, \infty\} + m\{\infty, 1\} + m\{1, 0\} = 0$$

after noting that S interchanges 0 and ∞ while U induces the cyclic permutation $(0\ 1\ \infty)$ on these three elements of $\mathbb{P}_1(\mathbb{Q})$. Let $\Omega^\dagger \subset \Omega$ be the set of elements satisfying (12).

The proposition below is a well known assertion about the cohomology of $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 1.4. *The assigment* $m \mapsto m\{0, \infty\}$ *identifies* $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\Omega)$ *with* $\Omega^\dagger$.

Proof. A pair of elements $(a/b, c/d)$ of $\mathbb{P}_1(\mathbb{Q})$ (expressed in lowest terms, with the convention that $\infty = 1/0$) is said to be *unimodular* if $ad - bc = \pm 1$. The fact that any two elements of $\mathbb{P}_1(\mathbb{Q})$ can be inserted into a *unimodular sequence*, in which all consecutive terms form unimodular pairs, implies that a modular symbol is completely determined by its values on such pairs. The injectivity of the assignment $m \mapsto m\{0, \infty\}$ then follows from the fact that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the set of unimodular pairs. To prove surjectivity, observe that any $\omega \in \Omega^\dagger$ determines a well-defined function m on the set of unimodular pairs by setting

$$m\left\{\frac{a}{b}, \frac{c}{d}\right\} := \begin{pmatrix} c & a \\ d & b \end{pmatrix} \omega,$$

where $(a/b, c/d)$ have been adjusted so that the matrix appearing on the right belongs to $\mathrm{SL}_2(\mathbb{Z})$. If r and s are arbitrary elements of $\mathbb{P}_1(\mathbb{Q})$, the unimodular sequences joining r and s are far from unique, but the theory of Farey sequences implies that any two unimodular sequences admit a common refinement, where a refinement is obtained by making a finite number of replacements of the form

$$\frac{a}{b}, \frac{c}{d}, \frac{-a}{-b} \quad \rightsquigarrow \quad \frac{a}{b}, \qquad\qquad \frac{a}{b}, \frac{c}{d} \quad \rightsquigarrow \quad \frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d}.$$

The two and three-term relations satisfied by ω imply that

$$m\left\{\frac{a}{b}, \frac{c}{d}\right\} + m\left\{\frac{c}{d}, \frac{a}{b}\right\} = 0, \qquad m\left\{\frac{a}{b}, \frac{c}{d}\right\} = m\left\{\frac{a}{b}, \frac{a+c}{b+d}\right\} + m\left\{\frac{a+c}{b+d}, \frac{c}{d}\right\},$$

for all unimodular pairs $(a/b, c/d)$. It follows that m extends *uniquely* to an $\mathrm{SL}_2(\mathbb{Z})$-invariant Ω-valued modular symbol, and hence that the map $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\Omega) \to \Omega^\dagger$ is surjective. Proposition 1.4 follows. □

Let $\Omega^\ddagger \subset \Omega^\dagger$ denote the image of the group $\mathrm{MS}^\Gamma(\Omega)$ under the assignment $m \mapsto m\{0, \infty\}$.

Lemma 1.5. *If ω belongs to $\Omega^\dagger$, then ω satisfies the two and three term relations in* (12) *along with the further relation*

$$D\omega = \omega. \tag{13}$$

Proof. This follows directly from the fact that both 0 and ∞ are fixed by the diagonal matrices. □

Remark 1.6. The equation (13) does not characterise $\Omega^\ddagger$: in general, its elements may need to satisfy further relations, which are less simple to write down explicitly and whose complexity presumably grows as a function of p.

1.3. Basic definitions.

For all $k \geq 0$, the continuous weight k action (cf. [ST, Section 1]) of the group $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ on $\mathcal{O}$ and on $\mathcal{M}$ is given by

$$(f|_k\gamma)(\tau) \ := \ (\gamma^{-1} \,{}_k\, f)(\tau) \ := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right), \qquad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{14}$$

The underlying addditive groups of $\mathcal{O}$ and $\mathcal{M}$ endowed with this weight k action are denoted $\mathcal{O}_k$ and $\mathcal{M}_k$ respectively, with the convention that $\mathcal{O}$ and $\mathcal{M}$ will be used to denote $\mathcal{O}_0$ and $\mathcal{M}_0$ respectively.

The following is the additive counterpart of Definition 1 of the Introduction:

Definition 1.7. A rigid meromorphic (resp. analytic) cocycle of weight $k \geq 0$ is a class in $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}_k)$ (resp. in $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{O}_k)$).

Remark 1.8. The multiplicative group $\mathrm{H}^1(\Gamma, \mathcal{M}^\times)$ should not be confused with the vector space $\mathrm{H}^1(\Gamma, \mathcal{M})$. Although elements of the latter can be evaluated at RM points just as well as their multiplicative counterparts, it will be shown later (cf. Prop. 2.14) that

$$\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}) = 0,$$

and hence no interesting class invariants for real quadratic fields are to be extracted from the additive theory.

Of greatest importance for our study are the rigid meromorphic cocycles of weight two, which are related to rigid meromorphic cocycles via the map

$$\mathrm{dlog} : \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times) \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}_2)$$

arising from the logarithmic derivative

$$\mathrm{dlog} : \mathcal{M}^\times \longrightarrow \mathcal{M}_2, \qquad \mathrm{dlog}(g) = g'/g,$$

which is compatible with the Γ-actions on source and target and therefore induces a map on the associated parabolic cohomology groups. The first step in classifying rigid meromorphic cocycles will be to do the same for their additive, weight two counterparts, with the advantage that the latter are endowed with a natural $\mathbb{C}_p$-linear structure.

Let us first specialise the discussion of the previous section on modular symbols to the setting where $\Omega = \mathcal{M}^\times$ or $\mathcal{M}_k$ for $k \geq 0$.

Lemma 1.9. *The Γ_∞-invariants of $\mathcal{M}^\times$ and $\mathcal{M}_k$ are given by*

$$H^0(\Gamma_\infty, \mathcal{M}^\times) = \mathbb{C}_p^\times, \qquad \mathrm{H}^0(\Gamma_\infty, \mathcal{M}_k) = \begin{cases} \mathbb{C}_p & \text{if } k = 0; \\ 0 & \text{if } k > 0. \end{cases}$$

Proof. This follows from the Weierstrass preparation theorem: any translation invariant rigid analytic function must be constant on the standard affinoid and hence everywhere by analytic continuation. □

Corollary 1.10. *For all $k \geq 0$, the map δ of Lemma 1.3 induces isomorphisms*

$$\mathrm{MS}^\Gamma(\mathcal{M}^\times) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times), \qquad \mathrm{MS}^\Gamma(\mathcal{M}_k) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}_k). \tag{15}$$

Proof. Lemma 1.9 implies that

$$\mathrm{H}^0(\Gamma, \mathcal{M}^\times) = \mathrm{H}^0(\Gamma_\infty, \mathcal{M}^\times), \qquad \mathrm{H}^0(\Gamma, \mathcal{M}_k) = \mathrm{H}^0(\Gamma_\infty, \mathcal{M}_k),$$

and the corollary follows from Lemma 1.3. □

Corollary 1.10 allows us to work with elements of $\mathrm{MS}^\Gamma(\mathcal{M}_k)$ in studying rigid meromorphic cocycles of weight k, with the advantage that many arguments tend to become more transparent when couched in the language of modular symbols.

Recall the multiplicative group $\mathcal{R}^\times \subset \mathcal{M}^\times$ of rigid meromorphic period functions given after Definition 2 of the introduction, which is identified with $\mathrm{MS}^\Gamma(\mathcal{M}^\times)$ via the assignment $J \mapsto J\{0, \infty\}$. The following is the additive counterpart of Definition 2:

Definition 1.11. A *rigid meromorphic period function of weight k* is the value at S of the parabolic representative of a rigid meromorphic cocycle of weight k.

Let $\mathcal{R}_k$ denote the $\mathbb{C}_p$-vector space of rigid meromorphic period functions of weight k. The assignment $\Phi \mapsto \varphi := \Phi\{0, \infty\}$ identifies $\mathrm{MS}^\Gamma(\mathcal{M}_k)$ with $\mathcal{R}_k$. Just as in the multiplicative setting, one has:

Lemma 1.12. *A function $\varphi \in \mathcal{R}_k$ satisfies the two and three term relations*

$$\varphi\left(-\frac{1}{z}\right) = -z^k \varphi(z), \qquad \varphi(z) + z^{-k}\varphi\left(\frac{z-1}{z}\right) + (z-1)^{-k}\varphi\left(\frac{-1}{z-1}\right) = 0,$$

as well as the further linear relation

$$\varphi(p^2 z) = p^{-k}\varphi(z).$$

In conclusion, we have obtained canonical maps

$$\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times) = \mathrm{MS}^\Gamma(\mathcal{M}^\times) \subset \mathcal{R}^\times, \qquad \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}_k) = \mathrm{MS}^\Gamma(\mathcal{M}_k) = \mathcal{R}_k.$$

1.4. **Prelude: rational cocycles and period functions.** The classification of rigid meromorphic cocycles and rigid meromorphic period functions of weight two which will be described in the next section parallels closely – and its proof is strongly inspired by – the classification of so called *rational modular cocycles* and their associated *rational period functions* that were introduced by Marvin Knopp and arise, notably, in the work of Knopp [Kn], Ash [Ash] , Choie–Zagier [CZ], and Duke–Imamoglu–Toth [DIT].

Let C be an algebraically closed field of characteristic 0, and let $\mathcal{M}_k^{\rm rat}$ denote the C-vector space of rational functions on $\mathbb{P}_1(C)$, endowed with the weight k action of $\mathrm{SL}_2(\mathbb{Z})$ as defined in (14) above.

Definition 1.13. A *rational modular cocycle* of weight k is a class in $\mathrm{H}^1_{\rm par}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_k^{\rm rat})$. A *rational period function* of weight k is an element of $(\mathcal{M}_k^{\rm rat})^\dagger$, i.e., a rational function $\phi \in \mathcal{M}_k^{\rm rat}$ satisfying the two and three term relations

$$\phi\left(-\frac{1}{z}\right) = -z^k\phi(z), \quad \phi(z) + z^{-k}\phi\left(\frac{z-1}{z}\right) + (z-1)^{-k}\phi\left(\frac{-1}{z-1}\right) = 0, \qquad \text{for all } z \in C.$$

Proposition 1.4 shows that the assignment $\Phi \mapsto \phi := \Phi\{0,\infty\}$ identifies $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathcal{M}_k^{\rm rat})$ with the space of rational period functions of weight k.

We now focus on the case $k = 2$, where the assignment $r(z) \mapsto r(z)dz$ identifies $\mathcal{M}_2^{\rm rat}$ with the space $\Omega^1_{\rm rat}$ of rational differentials on $\mathbb{P}_{1/C}$. We will henceforth view rational period functions of weight two interchangeably as elements of $\Omega^1_{\rm rat}$ or as functions on $\mathcal{H}_p$ endowed with the weight two Γ-action.

We begin by giving some examples of rational period functions of weight two.

Lemma 1.14. *The function $\phi^\circ_\infty(z) := \frac{1}{z}$ is a rational period function of weight two.*

Proof. Consider the function

$$\Phi^\circ_\infty : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \Omega^1_{\rm rat}$$

given by

$$\Phi^\circ_\infty\{r,s\} := \omega\{r,s\},$$

where $\omega\{r,s\}$ is the unique rational differential with poles concentrated at r and s and residues 1 and -1 respectively at these points. The fact that Φ°_∞ is an $\mathrm{SL}_2(\mathbb{Z})$-invariant modular symbol follows directly from this description. For instance, its $\mathrm{SL}_2(\mathbb{Z})$ invariance can be seen from the calculation

$$\Phi^\circ_\infty\{\gamma r, \gamma s\} = \omega\{\gamma r, \gamma s\} = \gamma\omega\{r,s\},$$

the last equality following from the fact that both differentials have the same poles and residues. The lemma follows after noting that

$$\Phi^\circ_\infty\{0,\infty\} = \frac{dz}{z} = \phi^\circ_\infty(z)dz.$$

□

To construct more interesting examples, let τ be any real quadratic irrationality in $\mathbb{R}$, and let $F_\tau(x,y)$ be the primitive integral binary quadratic form for which $F(\tau,1) = 0$. The $\mathrm{SL}_2(\mathbb{Z})$-orbit of τ is dense in $\mathbb{R}$, but the subset

$$\Sigma^\circ_\tau := \{w \in \mathrm{SL}_2(\mathbb{Z})\cdot\tau \ \text{ such that } \ ww' < 0\} \tag{16}$$

is finite and non-empty. This is because its elements are the roots of $F(z,1)$ were $F(x,y) = Ax^2+Bxy+Cy^2$ is a primitive integral quadratic form in the same class as $F_\tau(x,y)$, and hence of a fixed positive discriminant, satisfying $AC < 0$, and the coefficients of such a quadratic form are bounded in absolute value.

The set Σ_τ° is endowed with a natural sign function $\delta_\infty : \Sigma_\tau^\circ \longrightarrow \pm 1$, which partitions Σ_τ° into its subsets of positive and negative elements respectively. These sets are of equal cardinality since they are interchanged by the involution $z \mapsto -1/z$.

More generally, if r and s are elements of $\mathbb{P}_1(\mathbb{Q})$, let $\gamma(r,s)$ denote the geodesic on $\bar{\mathcal{H}} := \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ joining r to s and oriented in the direction from r to s. The complement of this geodesic in $\bar{\mathcal{H}}$ is partitioned into two disjoint connected subsets

$$\bar{\mathcal{H}} - \gamma(r,s) := \mathcal{H}^+(r,s) \cup \mathcal{H}^-(r,s),$$

labelled with the convention that, as one is travelling along $\gamma(r,s)$ in the direction from r to s, the region $\mathcal{H}^+(r,s)$ is to one's right and the region $\mathcal{H}^-(r,s)$ is to one's left. If $w \in \mathrm{SL}_2(\mathbb{Z}) \cdot \tau$ is any real quadratic irrationality in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of τ, we say that it is *linked* to $\gamma(r,s)$ if it and its algebraic conjugate w' belong to distinct connected components of $\bar{\mathcal{H}} - \gamma(r,s)$, and write $\Sigma_\tau^\circ(r,s) \subset \mathrm{SL}_2(\mathbb{Z}) \cdot \tau$ for the set of w which are linked to $\gamma(r,s)$ in this way. The set $\Sigma_\tau^\circ(r,s)$ is endowed with the sign function $\delta_{r,s}$ defined by

$$\delta_{r,s}(w) = \begin{cases} 1 & \text{if } w \in \mathcal{H}^+(r,s), \\ -1 & \text{if } w \in \mathcal{H}^-(r,s), \end{cases} \tag{17}$$

which partitions $\Sigma_\tau^\circ(r,s)$ into its subsets of positive and negative elements respectively, and

$$\Sigma_\tau^\circ(0,\infty) = \Sigma_\tau^\circ, \qquad \delta_{0,\infty} = \delta_\infty.$$

Let $\mathrm{Div}(\mathbb{P}_1(C))$ denote the free abelian group consisting of finite formal $\mathbb{Z}$-linear combinations of points of $\mathbb{P}_1(C)$, let $\mathrm{Div}^0(\mathbb{P}_1(C))$ denote its subgroup of degree zero divisors, and define

$$\Delta_\tau^\circ\{r,s\} := \sum_{w \in \Sigma_\tau^\circ(r,s)} \delta_{r,s}(w)[w] \quad \in \quad \mathrm{Div}(\mathbb{P}_1(C)). \tag{18}$$

Lemma 1.15. *The function*

$$\Delta_\tau^\circ : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \mathrm{Div}(\mathbb{P}_1(C)),$$

is an element of $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathrm{Div}^0(\mathbb{P}_1(C)))$.

Proof. To check the modular symbol property of Δ_τ°, observe that for all $r,s,t \in \mathbb{P}_1(\mathbb{Q})$, the element $w \in \mathrm{SL}_2(\mathbb{Z})\tau$ is linked to (r,t) if and only if it is linked to either (r,s) or to (s,t), but not to both, and that, in the latter case,

$$\delta_{r,s}(w) + \delta_{s,t}(w) = 0.$$

The $\mathrm{SL}_2(\mathbb{Z})$ equivariance

$$\Delta_\tau^\circ\{\gamma r, \gamma s\} = \gamma \Delta_\tau^\circ\{r,s\}$$

follows directly from the definitions. Finally, since an $\mathrm{SL}_2(\mathbb{Z})$-invariant modular symbol is completely determined by its value on the unimodular pair $(0,\infty)$, all the divisors $\Delta_\tau^\circ\{r,s\}$ inherit from $\Delta_\tau^\circ\{0,\infty\}$ the property of being of degree zero. □

Lemma 1.16. *The function*

$$\phi_\tau^\circ(z) := \sum_{w \in \Sigma_\tau^\circ} \delta_\infty(w) \frac{1}{z-w}$$

is a rational period function of weight two.

Proof. Consider the function

$$\Phi_\tau^\circ : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \Omega^1_{\mathrm{rat}}$$

given by

$$\Phi_\tau^\circ\{r,s\} := \sum_{w \in \Sigma_\tau^\circ(r,s)} \delta_{r,s}(w) \frac{dz}{z-w} =: \omega_\tau^\circ\{r,s\}. \tag{19}$$

The differential $\omega_\tau^\circ\{r,s\}$ is the unique rational differential on $\mathbb{P}_1(C)$ whose residual divisor is equal to $\Delta_\tau^\circ\{r,s\}$. It follows from Lemma 1.15 that Φ_τ° defines an element of $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\Omega^1_{\mathrm{rat}})$. Lemma 1.16 now follows from the fact that

$$\phi_\tau^\circ(z)dz = \Phi_\tau^\circ\{0,\infty\}.$$

□

Lemmas 1.14 and 1.16 have exhibited an explicit collection of distinct rational period functions ϕ_τ° of weight two, as τ ranges over the infinite index set

$$I = \{\infty\} \cup \ (\mathrm{SL}_2(\mathbb{Z})\backslash C^{\mathrm{RM}}),$$

where C^{RM} denotes the collection of real quadratic irrationalities in C. The following classification of rational period functions of weight two, whose statement can be read off by setting $k = 1$ in Theorem 1 of [CZ], asserts that the ϕ_τ° form a basis for the C-vector space of rational period functions of weight two.

Theorem 1.17 (Knopp, Ash, Choie-Zagier)**.** *Any rational period function of weight two is a finite linear combination of the functions ϕ_∞° of Lemma 1.14 and ϕ_τ° of Lemma 1.16.*

Since some of the steps of the proof will be used in our later classification of rigid meromorphic period functions, it is worth briefly recalling them here.

Let ϕ be any rational period function and let $\Sigma_\phi \subset C$ denote its set of poles. The two and three term relations satisfied by ϕ imply that

$$w \in \Sigma_\phi \qquad \Rightarrow \qquad S(w) \in \Sigma_\phi \quad \text{and} \quad U(w) \in \Sigma_\phi \text{ or } U^2(w) \in \Sigma_\phi. \tag{20}$$

Recall the sets Σ_τ° described in (16) for $\tau \in C^{\mathrm{RM}}$, and set $\Sigma_\infty^\circ = \{0,\infty\}$.

Lemma 1.18. *If Σ_ϕ is any finite subset of C satisfying* (20)*, then the set Σ_ϕ is a finite union of the sets of the form Σ_τ° with τ ranging over a finite subset $I_\phi \subset I$.*

Proof. This is just a restatement of Lemma 2 of [CZ], whose proof relies solely on the fact that Σ satisfies (20). Although it is formulated as a statement about rational period functions over $\mathbb{C}$, the argument carries over to the more abstract setting where $\mathbb{C}$ is replaced by any algebraically closed field C of characteristic zero, by fixing an embedding $C \longrightarrow \mathbb{C}$. □

Concerning the behaviour of ϕ at its poles, one has the following:

Lemma 1.19. *The differential $\phi(z)dz$ has only simple poles. Given any $\tau \in I$ for which $\Sigma_\tau^\circ \subset \Sigma_\phi$, there is a $\lambda_\tau \in C$ satisfying*

$$\mathrm{res}_w \phi(z)dz = \begin{cases} -\lambda_\tau & \text{if } w < 0, \\ \lambda_\tau & \text{if } w > 0, \end{cases} \qquad \text{for all } w \in \Sigma_\tau^\circ.$$

Proof. The proof, which is described in Lemmas 4 and 5 of [CZ], exploits the invariance of the principal part of ϕ at w under any non-trivial matrix of $\mathrm{SL}_2(\mathbb{Z})$ which fixes w. More precisely, consider the Laurent expansion $\phi_w(z)$ around $z = w$, and write

$$\phi_w(z) = \mathrm{PP}_w(z) + O(1) = (z-w)^{-m} + O\left((z-w)^{-m+1}\right),$$

where $\mathbb{P}_w(z)$ denotes the *principal part* of ϕ at w, a polynomial of some degree $m \geq 1$ in $(z-w)^{-1}$ with no constant term. Let γ be a generator of the stabiliser of w in $\mathrm{SL}_2(\mathbb{Z})$. It is shown in Lemmas 4 and 5 of [CZ] that $\mathrm{PP}_w|_2\gamma = \mathrm{PP}_w$, while

$$(\phi|_2\gamma)(z) = (rw+s)^{2-2m}(z-w)^{-m} + O\left((z-w)^{-m+1}\right), \qquad \text{where } \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

The quantity $(rw+s)$ is a fundamental unit in an appropriate quadratic order of $\mathbb{Q}(w)$, and is hence non-torsion in $C^\times$. It follows that $2-2m = 0$, i.e., that $m = 1$ and therefore that ϕ has at most simple poles. The two and three term relations satisfied by ϕ immediately imply (in light of Lemma 1.20 below) that all of its residues are equal up to sign on any given Σ_τ°, and the two term relation shows that the sign of the residue depends only on the sign of $w \in \Sigma_\tau^\circ$. □

Proof of Theorem 1.17. Let ϕ be a rational period function. Write $\Sigma_\phi = \cup_{\tau \in I_\phi} \Sigma_\tau^\circ$, where I_ϕ is the finite subset of I given in Lemma 1.18. Let $(\lambda_\tau)_{\tau \in I_\phi}$ be the vector of scalars indexed by $\tau \in I_\phi$ determined by Lemma 1.19. The difference $\phi - \sum_{\tau \in I_\phi} \lambda_\tau \phi_\tau^\circ$ is a rational function without singularities, and hence is constant. It follows that

$$\phi = \sum_{\tau \in I_\phi} \lambda_\tau \phi_\tau^\circ,$$

since there are no constant rational period functions of weight two. The theorem follows.

We record the following closure property of the sets Σ_τ° refining (20):

Lemma 1.20. *For all positive $w \in \Sigma_\tau^\circ$,*

1. *the negative element $S(w) = -1/w$ also belongs to Σ_τ°;*
2. *the set $\{U(w), U^2(w)\}$ contains exactly one element $w^\flat \in \Sigma_\tau^\circ$, which is negative, and given by*

$$w^\flat = \begin{cases} U^2(w) = \frac{w-1}{w} & \text{if } 0 < w < 1; \\ U(w) = \frac{1}{1-w} & \text{if } w > 1. \end{cases}$$

Proof. The first statement is clear. For the second, observe that U cyclically permutes the elements 0, 1, and $\infty \in \mathbb{P}_1(\mathbb{R})$, and hence does the same to the open intervals $(0,1)$, $(1,\infty)$, and $(-\infty, 0)$. It follows that, if (w', w) belongs to $(-\infty,0)\times(0,1)$, the translate $U(w)$ has positive norm while $U^2(w)$ is a negative element of Σ_τ°. Likewise, if (w',w) belongs to $(-\infty,0)\times(1,\infty)$, the translate $U^2(w)$ has positive norm while $U(w)$ is a negative element of Σ_τ°. □

1.5. **Classification of rigid meromorphic cocycles of weight two.** We will now adapt the ideas of the previous section to classify elements of $\mathrm{MS}^\Gamma(\mathcal{M}_2)$. Recall that the rigid meromorphic period function $\varphi := \varPhi\{0,\infty\}$ attached to a rigid meromorphic cocycle $\varPhi$ satisfies the properties

$$\varphi|(1+S) = 0, \qquad \varphi|(1+U+U^2) = 0, \qquad \varphi|D = \varphi, \tag{21}$$

where the matrices S, U and $D \in \Gamma$ are defined in (11). The matrix $P \in \mathrm{GL}_2(\mathbb{Z}[1/p])$ defined by

$$P := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies $P^2 = D$ in $\mathrm{PGL}_2(\mathbb{Z}[1/p])$ and induces an involution on the space of rigid meromorphic period functions of weight two, defined by

$$\varpi_p(\varphi)(z) = -\varphi|P(z) = -p\varphi(pz).$$

(Note the presence of the minus sign in this definition.) A rigid meromorphic period function is said to be p-even (resp. p-odd) if is satisfies

$$\varpi_p(\varphi) = \varphi, \qquad (\text{resp. } \varpi_p(\varphi) = -\varphi). \tag{22}$$

As in the previous section, a rigid meromorphic period function of weight two shall be viewed as an element of the space Ω^1 of rigid meromorphic differentials on $\mathcal{H}_p$, and a rigid meromorphic cocycle, as an element of $\mathrm{MS}^\Gamma(\Omega^1)$.

We begin by constructing some basic examples of rigid meromorphic period functions of weight two, modelled on the construction of the rational period functions ϕ_τ° of Lemma 1.16. Let τ be any RM point in $\mathcal{H}_p$, and fix an embedding of the real quadratic field $\mathbb{Q}(\tau)$ into $\mathbb{R}$. Recall that the image of τ in $\mathcal{T}$ under the reduction map belongs either to $\mathcal{T}_0$ or is the midpoint of an edge in $\mathcal{T}_1$. The Γ-orbit of τ is dense in $\mathcal{H}_p$ for the rigid analytic topology. As in (16), consider the subset

$$\Sigma_\tau := \{w \in \Gamma \cdot \tau \ \text{ such that } \ ww' < 0\}. \tag{23}$$

It is endowed with the sign function δ_∞ defined as in Section 1.4. Other notations and terminologies similar to those in Section 1.4 are also adopted. Notably, for each $r, s \in \mathbb{P}_1(\mathbb{Q})$, let $\Sigma_\tau(r,s) \subset \Gamma \cdot \tau$ denote the set of $w \in \Gamma\tau$ which are linked to $\gamma(r,s)$ in the sense of that section, and let $\delta_{r,s}$ denote the same sign function as in (17). Finally, imitating (18), we let

$$\Delta_\tau\{r,s\} := \sum_{w \in \Sigma_\tau(r,s)} \delta_{r,s}(w)[w] \quad \in \quad \mathrm{Div}(\mathcal{H}_p). \tag{24}$$

The sum in this equation is to be viewed as an *infinite* formal sum of points in $\mathcal{H}_p$, and $\mathrm{Div}(\mathcal{H}_p)$ simply denotes the Γ-module of such formal sums. For the same reason as in Lemma 1.15, the function Δ_τ defines a Γ-invariant, $\mathrm{Div}(\mathcal{H}_p)$-valued modular symbol.

A subset of $\mathcal{H}_p$ is said to be *discrete* if its intersection with each affinoid subset of $\mathcal{H}_p$ is finite. The module of divisors on $\mathcal{H}_p$ with discrete support is denoted $\mathrm{Div}^\dagger(\mathcal{H}_p)$.

Lemma 1.21. *For all $r, s \in \mathbb{P}_1(\mathbb{Q})$, the sets $\Sigma_\tau(r,s)$ are discrete. Furthermore, the finite intersection $\Sigma_\tau(r,s) \cap \mathcal{A}_v$ contains equal numbers of positive and negative elements, and likewise for $\Sigma_\tau(r,s) \cap \mathcal{W}_v$.*

Proof. Proposition 1.1 implies that the intersection $\Sigma_\tau \cap \mathcal{H}_p^{\leq n}$ is finite for all $n \geq 0$, since it consists of *RM* points that are roots of reduced binary quadratic forms of bounded discriminant. Letting

$$\Delta_\tau^v\{r,s\} := \sum_{w \in \Sigma_\tau\{r,s\} \cap \mathcal{A}_v} \delta_{r,s}(w)[w], \qquad \text{for } v \in \mathcal{T}_0, \quad r, s \in \mathbb{P}_1(\mathbb{Q}),$$

it follows that $\Delta_\tau^v\{0,\infty\}$ has finite support, for all $v \in \mathcal{T}_0$. The Γ-equivariance property

$$\gamma \Delta_\tau^v\{0,\infty\} = \Delta_\tau^{\gamma v}\{\gamma 0, \gamma\infty\}$$

then implies that $\Delta_\tau^v\{r,s\}$ has finite support for all $v \in \mathcal{T}_0$ and all unimodular pairs (r,s) of elements of $\mathbb{P}_1(\mathbb{Q})$, since the group Γ acts transitively on the latter. But then the same conclusion must hold for all pairs (r,s), by the additivity properties of modular symbols. The discreteness of $\Sigma_\tau(r,s)$ follows. To verify the second assertion in the lemma, consider the functions

$$\deg_{\mathcal{A}_v},\ \deg_{\mathcal{W}_v} : \mathrm{Div}^\dagger(\mathcal{H}_p) \longrightarrow \mathbb{Z}, \qquad \deg_{\mathcal{A}_v}(\Delta) := \deg(\Delta|_{\mathcal{A}_v}), \quad \deg_{\mathcal{W}_v}(\Delta) := \deg(\Delta|_{\mathcal{W}_v}).$$

These functions are $\Gamma_v := \mathrm{Stab}_\Gamma(v)$-equivariant and hence induce maps

$$\deg_{\mathcal{A}_v},\ \deg_{\mathcal{W}_v} : \mathrm{MS}^\Gamma(\mathrm{Div}^\dagger(\mathcal{H}_p)) \longrightarrow \mathrm{MS}^{\Gamma_v}(\mathbb{Z}).$$

Since $\Gamma_v \simeq \mathrm{SL}_2(\mathbb{Z})$, there are no non-trivial Γ_v-invariant modular symbols, and hence, for all $v \in \mathcal{T}_0$,

$$\deg_{\mathcal{A}_v}(\Delta_\tau\{r,s\}) = \deg(\Delta_\tau^v\{r,s\}) = 0.$$

The lemma follows since the degree of $\Delta_\tau^v\{r,s\}$ is precisely the difference between the number of positive and negative elements in $\Sigma_\tau(r,s)\cap\mathcal{A}_v$, and likewise when $\mathcal{A}_v$ is replaced by $\mathcal{W}_v$. □

The following lemma is the natural extension of Lemma 1.16 to the setting of rigid meromorphic period functions:

Theorem 1.22. *For any $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$, the infinite sum*

$$\varphi_\tau(z) := \sum_{w\in\Sigma_\tau} \delta_\infty(w)\frac{1}{z-w}$$

converges to a rigid meromorphic period function of weight two.

Proof. The infinite sum in the statement of the theorem is the limit as $h \longrightarrow \infty$ of the rational functions

$$(25)\qquad \sum_{w\in\Sigma_\tau^{\leq h}} \delta_\infty(w)\frac{1}{z-w}, \qquad \text{where } \Sigma_\tau^{\leq h} := \Sigma_\tau \cap \mathcal{H}_p^{\leq h}.$$

Assume first for simplicity that τ, and hence all $w \in \Gamma\tau$, reduce to vertices of $\mathcal{T}$. By Lemma 1.21, the functions in (25) are sums of terms of the form

$$\frac{1}{z-w_1} - \frac{1}{z-w_2},$$

where w_1 and w_2 belong to the same vertex affinoid $\mathcal{A}_v$, and $v \in \mathcal{T}_0$ is of distance $N \leq h$ from v_0. This term is regular on $\mathcal{H}_p^{\leq n}$ if $N > n$, and for $z \in \mathcal{H}_p^{\leq n}$ it satisfies the inequalities

$$\left|\frac{1}{z-w_1}\right| \leq p^n, \qquad \left|\frac{1}{z-w_2}\right| \leq p^n, \qquad \left|\frac{1}{z-w_1} - \frac{1}{z-w_2}\right| \leq p^{2n-N}.$$

For all $n \geq 0$, the restrictions of the rational functions in (25) to $\mathcal{H}_p^{\leq n}$ therefore form a Cauchy sequence relative to the sup norm on this affinoid, and hence converge to a rigid meromorphic function on $\mathcal{H}_p$. By the same reasoning, the infinite sums

$$(26)\qquad \Phi_\tau\{r,s\} := \sum_{w\in\Sigma_\tau(r,s)} \delta_{r,s}(w)\frac{dz}{z-w}$$

converge to rigid meromorphic differentials on $\mathcal{H}_p$, with residual divisor equal to $\Delta_\tau\{r,s\}$. In particular, the function

$$\Phi_\tau \ : \ \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \Omega^1$$

is an Ω^1-valued modular symbols, just as in the proof of Lemma 1.16. Theorem 1.22 now follows from the fact that

$$\varphi_\tau(z) = \Phi_\tau\{0,\infty\}.$$

To handle the case when τ (and hence all w in its Γ-orbit) reduce to midpoints of edges of $\mathcal{T}$ rather than to vertices, which happens precisely when τ is defined over a real quadratic field in which p is ramified, it suffices to replace the system of affinoids $\{\mathcal{A}_v\}_{v\in\mathcal{T}_0}$ by the system of wide open subsets $\{\mathcal{W}_v\}_{v\in\mathcal{T}_0^+}$ in the above argument. □

Theorem 1.22 provides an explicit collection of rigid meromorphic period functions φ_τ of weight two, as τ ranges over the infinite index set

$$I^{(p)} = \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}.$$

These functions are linearly independent, since their residual divisors have disjoint support.

The following result extends Theorem 1.17 to the rigid meromorphic setting:

Theorem 1.23. *Any rigid meromorphic period function of weight two is a finite linear combination of the functions φ_τ of Theorem 1.22 and of a rigid analytic period function of weight two.*

Proof. Let φ be a rigid meromorphic period function of weight two. Any such φ can be written as the average of $\varphi^+ := \varphi + \varpi_p(\varphi)$ and $\varphi^- := \varphi - \varpi_p(\varphi)$, which are p-even and p-odd respectively. Hence we may assume without loss of generality that φ satisfies (22), and even, for the sake of definiteness, that it is p-even, since the case where it is p-odd will be disposed of by the same argument. Let $\Sigma_\varphi \subset \mathcal{H}_p$ be the set of poles of φ. While the invariance of φ under the matrix D shows that Σ_φ is either empty or infinite, the intersection

$$\Sigma_\varphi^{<1} := \Sigma_\varphi \cap \mathcal{H}_p^{<1}$$

is finite, since a rigid differential on $\mathcal{H}_p$ has finitely many poles when restricted to any affinoid. Since $\mathcal{H}_p^{<1}$ is preserved by the action of $\mathrm{SL}_2(\mathbb{Z})$, the two and three term relations satisfied by φ imply that the set $\Sigma_\varphi^{<1}$ satisfies the closure properties of (20). It follows from Lemma 1.18 that

$$\Sigma_\varphi^{<1} = \cup_{\tau\in I_\varphi} \Sigma_\tau^\circ,$$

where

$$I_\varphi \subset \mathrm{SL}_2(\mathbb{Z})\backslash(\mathcal{H}_p^{\mathrm{RM}} \cap \mathcal{H}_p^{<1})$$

is a finite set, and Σ_τ° is defined as in (16), but is now being viewed as a subset of $\mathcal{H}_p$. Lemma 1.19, whose proof applies just as well, mutatis mutandis, to the setting where φ is a rigid meromorphic period function, shows that φ has only simple poles on $\mathcal{H}_p^{<1}$, and that for each $\tau \in I_\varphi$, there is a $\lambda_\tau \in \mathbb{C}_p$ satisfying

$$\mathrm{res}_w \varphi(z)dz = \begin{cases} \lambda_\tau & \text{if } w > 0 \\ -\lambda_\tau & \text{if } w < 0, \end{cases} \qquad \text{for all } w \in \Sigma_\tau.$$

The difference

$$\varphi - \sum_{\tau\in I_\varphi} \lambda_\tau \varphi_\tau^+$$

is a p-even rigid meromorphic period function having no singularities in $\mathcal{H}_p^{<1}$. Theorem 1.23 now follows from Proposition 1.24 below. □

Proposition 1.24. *Let φ be any rigid meromorphic period function of weight two. Assume that it satisfies (22), i.e., that it is either p-odd or p-even. If φ is regular on $\mathcal{H}_p^{<1}$, then it is regular everywhere.*

Proof. Suppose that φ has a pole at $\tau \in \mathcal{H}_p$, and hence at all $w \in \Sigma_\tau^\circ$. Since τ does not belong to $\mathcal{H}_p^{<1}$, Proposition 1.1 implies that it is an RM point of discriminant $D_0 p^n$ with $n \geq 2$ and $p \nmid D_0$. Let

$$D = \begin{cases} D_0 & \text{if } n \text{ is even,} \\ D_0 p & \text{if } n \text{ is odd .} \end{cases} \qquad m = [n/2] \geq 1.$$

The set Σ_τ° then contains an element of the form $p^m w_0$, where w_0 is an RM point in $\mathcal{H}_p$ of discriminant D. The invariance property (22) of φ shows that φ is singular at w_0 as well.

But w_0 belongs to $\mathcal{H}_p^{<1}$ by Proposition 1.1, contradicting the regularity assumption that was made on φ. □

Theorem 1.23 classifies the rigid meromorphic period functions only up to rigid analytic period functions of weight two. The latter will be classified in turn in Chapter 2.

The following describes the action of the Hecke operators on the cocycles Φ_τ:

Lemma 1.25. *If ℓ is a prime not dividing p or the discriminant D of $\tau \in \mathcal{H}_p^{\mathrm{RM}}$, then $T_\ell(\Phi_\tau)$ is a linear combination of cocycles of the form $\Phi_{\tau'}$ where τ' is of discriminant D or $D\ell^2$, and involves at least one $\Phi_{\tau'}$ in which τ' is of discriminant $D\ell^2$.*

The proof of this lemma is by a direct calculation, and proceeds along the same lines as in [Ge], where the argument is explained in the setting of rational period functions. As in loc. cit., Lemma 1.25 has the following important corollary when combined with Theorem 1.23:

Corollary 1.26. *Any finite-dimensional subspace of $\mathrm{MS}^\Gamma(\mathcal{M}_2)$ which is stable under the Hecke operators is contained in $\mathrm{MS}^\Gamma(\mathcal{O}_2)$. In particular, if θ is a non-zero Hecke operator and $\Phi \in MS^\Gamma(\mathcal{M}_2)$ belongs to the kernel of θ, then Φ belongs to $\mathrm{MS}^\Gamma(\mathcal{O}_2)$.*

For future reference, it is also worth recording the following corollary of the fact that rigid meromorphic period functions of weight two have at worst simple poles:

Corollary 1.27. *Any rigid meromorphic modular cocycle of weight zero is analytic, i.e., the natural inclusion $\mathrm{MS}^\Gamma(\mathcal{O}) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{M})$ is an isomorphism.*

Proof. The image of the derivative $d : \mathcal{M} \longrightarrow \mathcal{M}_2$ consists of rigid meromorphic functions with vanishing residues, and the image of the induced map

$$d : \mathrm{MS}^\Gamma(\mathcal{M}) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{M}_2)$$

on Γ-invariant modular symbols is therefore contained in $\mathrm{MS}^\Gamma(\mathcal{O}_2)$. It follows that any $\mathcal{M}$-valued Γ-invariant modular symbol is necessarily $\mathcal{O}$-valued, as claimed. □

2. Analytic cocycles of weight two

This chapter completes Theorem 1.23 of Chapter 1 by describing the space of rigid analytic cocycles of weight two. The main result is Theorem 2.12 of Section 2.5, which shows that $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{O}_2)$ is finite dimensional and closely related to the space of classical modular forms of weight two for the Hecke congruence group $\Gamma_0(p)$. The techniques used to prove this theorem differ markedly from those of the previous chapter, relying heavily on ideas of Stevens and Schneider-Teitelbaum. The reader with a single minded interest in the applications of rigid meromorphic cocycles to explicit class field theory may elect to skip this chapter on a first reading, since none of the material it contains is required to understand the subsequent chapters.

2.1. **Rigid analytic functions and boundary distributions.** The classification of elements of $\mathrm{MS}^\Gamma(\mathcal{O}_k) = \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{O}_k)$ rests on the fact that $\mathcal{O}_k$ is isomorphic to a space of locally analytic distributions on the boundary $\mathbb{P}_1(\mathbb{Q}_p)$ of $\mathcal{H}_p$. Assume henceforth that $k = 2$ for simplicity, although the results described below can certainly be extended to more general positive even weights.

The dual of the space of locally analytic functions on $\mathbb{P}_1(\mathbb{Q}_p)$, equipped with the strong topology of uniform convergence on compact open subsets, is called the space of *locally analytic distributions* on $\mathbb{P}_1(\mathbb{Q}_p)$, and is denoted $\mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$. Given $\mu \in \mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$, the notation

$$\mu(h) =: \int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t) d\mu(t)$$

shall be adopted. More generally, if U is a compact open subset of $\mathbb{P}_1(\mathbb{Q}_p)$ and 1_U is its characteristic function, we define

$$\int_U h(t) d\mu(t) := \int_{\mathbb{P}_1(\mathbb{Q}_p)} 1_U(t) h(t) d\mu(t).$$

A distribution $\mu \in \mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ satisfying $\mu(1) = 0$, where 1 denotes the constant function 1 on $\mathbb{P}_1(\mathbb{Q}_p)$, is said to be *of total volume zero*, and the space of such locally analytic distributions is denoted $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$.

The group $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts naturally on $\mathcal{D}(\mathbb{P}_1(\mathbb{Q}_p))$ and on $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$ via the weight zero action on locally analytic functions on $\mathbb{P}_1(\mathbb{Q}_p)$, defined as in (14). More precisely,

$$\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t) d(\mu|\gamma)(t) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} h(\gamma t) d\mu(t), \qquad \text{where } \gamma t := \frac{at+b}{ct+d}.$$

To any rigid analytic function $f \in \mathcal{O}_2$, we attach a locally analytic distribution μ_f on $\mathbb{P}^1(\mathbb{Q}_p)$ by setting, for all analytic functions $h(t)$ on a compact open $U_e \subset \mathbb{P}_1(\mathbb{Q}_p)$,

$$\int_{U_e} h(t) d\mu_f(t) := \mathrm{res}_e(f(z)h(z)dz).$$

Here, res_e denotes the p-adic annular residue along the oriented annulus $\mathcal{W}_e$. The distribution μ_f is called the *boundary distribution* attached to f. It is a direct consequence of the residue theorem that μ_f belongs to $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$.

Proposition 2.1. *The map $f \mapsto \mu_f$ induces a topological isomorphism*

$$\mathrm{BD} : \mathcal{O}_2 \xrightarrow{\sim} \mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)),$$

which is compatible with the $\mathrm{PGL}_2(\mathbb{Q}_p)$*-actions on both sides.*

Proof. Setting $k = 2$ in the statement of Theorem 2.2.1 of [DT], the dual of the map denoted I_2 in loc.cit. induces an isomorphism

$$\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)) \longrightarrow \mathcal{O}_2,$$

in light of the fact that $\mathcal{O}_2$ is a reflexive Frechet space and hence is identified with its double dual. The "boundary distribution map" BD is just the inverse of this isomorphism. □

The map BD induces an isomorphism on the parabolic cohomology groups, denoted by the same symbol by a slight abuse of notation:

$$\mathrm{BD} : \mathrm{MS}^{\Gamma}(\mathcal{O}_2) \xrightarrow{\sim} \mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))). \tag{27}$$

This reduces the problem of understanding $\mathrm{MS}^{\Gamma}(\mathcal{O}_2)$ to that of classifying the Γ-invariant modular symbols with values in $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$. An element μ of the latter is simply a collection of distributions $\mu\{r, s\}$ on $\mathbb{P}_1(\mathbb{Q}_p)$, indexed by elements $r, s \in \mathbb{P}_1(\mathbb{Q})$, which satisfy the usual modular symbol relations

$$\mu\{r, s\} = -\mu\{s, r\}, \qquad \mu\{r, s\} + \mu\{s, t\} = \mu\{r, t\},$$

together with the equivariance property

$$\int_{\gamma B} h(t)d\mu\{r,s\}(t) = \int_B h(\gamma t)d\mu\{\gamma^{-1}r,\gamma^{-1}s\}(t), \qquad \text{for all } \gamma\in\Gamma. \tag{28}$$

If μ is an eigensymbol for the involutions w_∞ and w_p defined in (8), then the invariance property of (28) even holds for all $\gamma \in \mathrm{PGL}_2(\mathbb{Z}[1/p])$, up to a sign which depends on the determinant of γ and on the parity and the p-parity of μ:

$$\int_{\gamma B} h(t)d\mu\{r,s\}(t) = \pm\int_B h(\gamma t)d\mu\{\gamma^{-1}r,\gamma^{-1}s\}(t), \qquad \text{for all } \gamma\in\mathrm{PGL}_2(\mathbb{Z}[1/p]). \tag{29}$$

2.2. **Restriction to $\mathbb{Z}_p$.** The compact open subset $\mathbb{Z}_p \subset \mathbb{P}_1(\mathbb{Q}_p)$ is a ball whose stabiliser in Γ is the usual congruence group $\Gamma_0(p)$. The restriction map $\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)) \longrightarrow \mathcal{D}(\mathbb{Z}_p)$ to the space of distributions on $\mathbb{Z}_p$ therefore induces a map on modular symbols:

$$\mathrm{res}_{\mathbb{Z}_p} : \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)).$$

The target of this map is called the space of *overconvergent modular symbols* of weight two and level p.

Lemma 2.2. *The map* $\mathrm{res}_{\mathbb{Z}_p}$ *is injective.*

Proof. The matrix $\iota_p := \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ interchanges $\mathbb{Z}_p$ and its complement $\mathbb{Z}_p' := \mathbb{P}_1(\mathbb{Q}_p) - \mathbb{Z}_p$, and normalises $\Gamma_0(p)$. It therefore induces mutually inverse isomorphisms

$$\mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) \stackrel{\iota_p}{\longleftrightarrow} \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p'))$$

for which the diagram

$$\begin{array}{ccc} \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) & \xrightarrow{w_p} & \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) \\ \downarrow{\scriptstyle \mathrm{res}_{\mathbb{Z}_p}} & & \downarrow{\scriptstyle \mathrm{res}_{\mathbb{Z}_p'}} \\ \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) & \xrightarrow{\iota_p} & \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p')) \end{array}$$

commutes. In particular, the involution w_p interchanges the kernels of $\mathrm{res}_{\mathbb{Z}_p}$ and of $\mathrm{res}_{\mathbb{Z}_p'}$, and it suffices to show that $\mathrm{res}_{\mathbb{Z}_p'}$ is injective. If μ is in the kernel of $\mathrm{res}_{\mathbb{Z}_p'}$, then

$$\mu\{r,s\}|_{\mathbb{Z}_p'} = 0, \qquad \text{for all } r,s\in\mathbb{P}_1(\mathbb{Q}). \tag{30}$$

The domain $\mathbb{P}_1(\mathbb{Q}_p)$ admits a decomposition as a disjoint union of $p+1$ open balls,

$$\mathbb{P}_1(\mathbb{Q}_p) = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_{p-1} \sqcup \mathbb{Z}_p', \tag{31}$$

where $B_j \subset \mathbb{Z}_p$ is the mod p residue disc of $-j$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the collection $\{B_0, B_1, \ldots, \mathbb{Z}_p'\}$. Let $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix satisfying $\mathbb{Z}_p' = \gamma_j B_j$. Then for all $j = 0,\ldots,p-1$, and for all $r,s\in\mathbb{P}_1(\mathbb{Q})$,

$$\mu\{r,s\}|_{B_j} = \mu\{r,s\}|_{\gamma_j^{-1}\mathbb{Z}_p'} = (\mu\{\gamma_j r,\gamma_j s\}|_{\mathbb{Z}_p'})|\gamma_j = 0,$$

where the last equality follows from (30). It now follows from (31) that $\mu\{r,s\} = 0$ as a distribution on $\mathbb{P}_1(\mathbb{Q}_p)$, for all $r,s\in\mathbb{P}_1(\mathbb{Q})$. The lemma follows. □

The space of overconvergent modular symbols is equipped with a Hecke operator U_p, defined explicitly by

$$\int_{\mathbb{Z}_p} h(t)d(U_p\mu)\{r,s\}(t) := \sum_{j=0}^{p-1}\int_{\mathbb{Z}_p} h(\alpha_j^{-1}t)d\mu\{\alpha_j r,\alpha_j s\}(t), \qquad \text{where } \alpha_j = \begin{pmatrix} 1 & j \\ 0 & p\end{pmatrix}. \tag{32}$$

The space $\mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))$ decomposes as a direct sum

$$\mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p)) = \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))^+ \oplus \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))^-,$$

where $\mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))^\epsilon$ denotes, for $\epsilon \in \{+,-\}$, the ϵ-eigenspace for the action of the involution w_p. Let $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p=\epsilon}$ denote the space of overconvergent modular symbols on which U_p acts as multiplication by ϵ.

Proposition 2.3. *The map* $\mathrm{res}_{\mathbb{Z}_p}$ *induces Hecke-equivariant inclusions*

$$\mathrm{res}_{\mathbb{Z}_p} : \mathrm{MS}^\Gamma(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))^\epsilon \hookrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p=\epsilon}.$$

Proof. For $j = 0, 1, \ldots, p-1$, let $B_j \subset \mathbb{Z}_p$ denote, as in the proof of Lemma 2.2, the residue class of $-j$ modulo p, so that

$$\mathbb{Z}_p = B_0 \sqcup B_1 \sqcup \cdots \sqcup B_{p-1}, \qquad \alpha_j B_j = \mathbb{Z}_p,$$

with α_j as in (32). By the additivity of the distribution $\mu\{r,s\} \in \mathcal{D}(\mathbb{Z}_p)$, we have, for any locally analytic function h on $\mathbb{Z}_p$:

$$\begin{aligned} \int_{\mathbb{Z}_p} h(t) d\mu\{r,s\}(t) &= \sum_{j=0}^{p-1} \int_{B_j} h(t) d\mu\{r,s\}(t) = \sum_{j=0}^{p-1} \int_{\alpha_j^{-1}\mathbb{Z}_p} h(t) d\mu\{r,s\}(t) \\ &= \epsilon \sum_{j=0}^{p-1} \int_{\mathbb{Z}_p} h(\alpha_j^{-1} t) d\mu\{\alpha_j r, \alpha_j s\}(t), \end{aligned}$$

where the last equality follow from (29) in light of the fact that the matrices $\alpha_j \in \mathrm{PGL}_2(\mathbb{Z}[1/p])$ have determinant p. The proposition now follows from the definition of the U_p operator given in (32). □

Proposition 2.3 shows that U_p preserves the image of $\mathrm{res}_{\mathbb{Z}_p}$ and that the minimal polynomial of its restriction to this space divides $x^2 - 1$. Composing the map BD of (27) with the restriction map $\mathrm{res}_{\mathbb{Z}_p}$ thus gives an injection

$$\mathrm{MS}^\Gamma(\mathcal{O}_2) \hookrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p^2=1}. \tag{33}$$

2.3. **Stevens' control theorem.** The "total measure" map $\mathcal{D}(\mathbb{Z}_p) \longrightarrow \mathbb{C}_p$ which sends μ to $\mu(1)$ induces a "weight two specialisation map"

$$\rho : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p), \tag{34}$$

which is compatible with the actions of the Hecke operators on both sides.

Theorem 2.4 (Stevens)**.** *The weight two specialisation map* ρ *induces an isomorphism*

$$\rho : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{U_p^2=1} \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p). \tag{35}$$

Proof. The *ordinary subspace* of a Hecke module M is the direct summand of it on which the U_p operator acts with slope zero, and is denoted M^{ord}. The *control theorem for overconvergent modular symbols* (cf. the case $k = 0$ of Theorem 1.1 of [PS]) asserts that ρ induces an isomorphism

$$\rho^{\mathrm{ord}} : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p))^{\mathrm{ord}} \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p)^{\mathrm{ord}}.$$

But $\mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p)$ is isomorphic as a Hecke module to the direct sum of an "Eisenstein line" with two copies of the space of modular forms of weight two on $\Gamma_0(p)$. Since all such modular forms are new at p, it follows that U_p^2 acts as the identity on this space, and that

$$\mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p)^{\mathrm{ord}} = \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p)^{U_p^2=1} = \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p).$$

The theorem follows. □

Corollary 2.5. *The map*

$$\eta := \rho \circ \mathrm{res}_{\mathbb{Z}_p} \circ \mathrm{BD} : \mathrm{MS}^{\Gamma}(\mathcal{O}_2) \hookrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p)$$

is injective.

Proof. This follows from Propositions 2.1 and 2.3 combined with Theorem 2.4. □

Our goal in the remainder of this chapter is to show that the map η is surjective as well.

2.4. **The residue map.** Let Ω be a Γ-module.

Definition 2.6. A function $c : \mathcal{T}_1^* \longrightarrow \Omega$ is said to be *harmonic* if it satisfies

$$c(\bar{e}) = -c(e), \text{ for all } e \in \mathcal{T}_1^*, \quad \text{and} \quad \sum_{s(e)=v} c(e) = 0, \text{ for all } v \in \mathcal{T}_0.$$

The space of harmonic functions on $\mathcal{T}_1^*$ with values in Ω is denoted $C_{\mathrm{har}}(\Omega)$. The action of Γ on $\mathcal{T}$ induces a natural right action of Γ on the space $C_{\mathrm{har}}(\Omega)$. In what follows we will be primarily interested in the case where $\Omega = \mathbb{Z}$ or a bounded subgroup of $\mathbb{C}_p$, equipped with the trivial action of Γ.

Remark 2.7. Elsewhere in the literature (e.g., in [Te]) it is cusomary to refer to harmonic functions as *harmonic cocycles.* Since the noun "cocycle" already being used in its more standard form in this article, a more transparent terminology was chosen for Definition 2.6.

One can associate to any $f \in \mathcal{O}_2$ a harmonic function $c_f \in C_{\mathrm{har}}(\mathbb{C}_p)$ by the rule

$$c_f(e) = \partial_e f(z)dz, \qquad \text{for } e \in \mathcal{T}_1^*,$$

where

$$\partial_e : \mathcal{O}(\mathcal{W}_e) \longrightarrow \mathbb{C}_p$$

is the p-adic annular residue on the space of rigid differentials on the oriented annulus $\mathcal{W}_e$. The $\mathrm{PGL}_2(\mathbb{Q}_p)$-equivariant map

$$\partial : \mathcal{O}_2 \longrightarrow C_{\mathrm{har}}(\mathbb{C}_p)$$

sending f to c_f is called the *residue map.* The same notation and terminology is used to describe the induced map

$$\partial : \mathrm{MS}^{\Gamma}(\mathcal{O}_2) \longrightarrow \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathbb{C}_p)) \tag{36}$$

on modular symbols. Let e_0 denote the standard edge of $\mathcal{T}^*$, whose stabiliser is $\Gamma_0(p)$ and whose associated open ball in $\mathbb{P}_1(\mathbb{Q}_p)$ is $\mathbb{Z}_p$. The evaluation at e_0

$$\mathrm{ev}_{e_0} : C_{\mathrm{har}}(\mathbb{C}_p) \longrightarrow \mathbb{C}_p$$

is $\Gamma_0(p)$-equivariant, and hence induces a map

$$\mathrm{ev}_{e_0} : \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathbb{C}_p)) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p). \tag{37}$$

Our strategy to show the surjectivity of η will be to prove the surjectivity of the maps ev_{e_0} and ∂ that fit into the following commutative diagram:

$$\begin{array}{ccc} \mathrm{MS}^{\Gamma}(\mathcal{O}_2) & \xrightarrow{\partial} & \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathbb{C}_p)) \\ \downarrow{\scriptstyle \mathrm{BD}} & \searrow^{\eta} & \downarrow{\scriptstyle \mathrm{ev}_{e_0}} \\ \mathrm{MS}^{\Gamma}(\mathcal{D}_0(\mathbb{P}_1(\mathbb{Q}_p))) & & \\ \downarrow{\scriptstyle \mathrm{res}_{\mathbb{Z}_p}} & & \\ \mathrm{MS}^{\Gamma_0(p)}(\mathcal{D}(\mathbb{Z}_p)) & \xrightarrow{\rho} & \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p). \end{array} \tag{38}$$

The surjectivity of ev_{e_0} is elementary:

Lemma 2.8. *The map* $ev_{e_0} : \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathbb{C}_p)) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p)$ *is an isomorphism.*

Proof. The injectivity of ev_{e_0} follows from much the same argument as in the proof of the injectivity of the map $\mathrm{res}_{\mathbb{Z}_p}$ given in Lemma 2.2. Namely, an element c of its kernel satisfies

$$c\{r,s\}(e_0) = 0 \qquad \text{for all } r,s \in \mathbb{P}_1(\mathbb{Q}).$$

Since Γ acts transitively on $\mathcal{T}_1^+$, it then follows from the Γ-equivariance of c that $c\{r,s\}(e) = 0$ for all $e \in \mathcal{T}_1^+$, and hence, for all $e \in \mathcal{T}_1^*$ by the harmonicity of $c\{r,s\}$. To check surjectivity, given $c_0 \in \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p)$, define $c \in \mathrm{MS}^{\Gamma}(\mathbb{C}_p)$ by setting, for all $e = \gamma^{-1}e_0 \in \mathcal{T}_1^+$,

$$c\{r,s\}(e) := c_0\{\gamma r, \gamma s\} \qquad \text{for all } r,s \in \mathbb{P}_1(\mathbb{Q}).$$

Although γ is only well-defined up to left multiplication by elements of $\Gamma_0(p)$, the $\Gamma_0(p)$-invariance of c_0 ensures that the value of $c\{r,s\}(e)$ does not depend on the choice of γ, and one checks directly that $\mathrm{ev}_{e_0}(c) = c_0$. □

2.5. **The Schneider–Teitelbaum transform.** We now show that the residue map ∂ in (36) is surjective. The main ingredient for achieving this is the integral "p-adic Poisson transform" of Schneider and Teitelbaum which allows one to recover certain elements of $\mathcal{O}_2$ from their associated boundary distributions.

Let $C^b_{\mathrm{har}}(\mathbb{C}_p) \subset C_{\mathrm{har}}(\mathbb{C}_p)$ denote the subspace of *bounded* harmonic functions, i.e., those whose values lie in a bounded subset of $\mathbb{C}_p$. A element $c \in C^b_{\mathrm{har}}(\mathbb{C}_p)$ can be parlayed into a bounded linear functional μ_c on the space of locally constant functions on $\mathbb{P}^1(\mathbb{Q}_p)$, by setting

$$\int_{U_e} 1 d\mu_c := c(e).$$

The boundedness of μ_c implies that it extends uniquely to a *measure* on $\mathbb{P}_1(\mathbb{Q}_p)$, i.e., a continuous functional on the space of continuous functions on $\mathbb{P}_1(\mathbb{Q}_p)$ endowed with the sup norm. This extension exploits the fact that every continuous function $h(t)$ on $\mathbb{P}_1(\mathbb{Q}_p)$ is a uniform limit of locally constant functions to express $\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t)d\mu_c(t)$ as a limit of (finite) Riemann sums.

Proposition 2.9 (Schneider, Teitelbaum)**.** *There is a unique* Γ*-equivariant splitting of the residue map* ∂ *on* $C^b_{\mathrm{har}}(\mathbb{C}_p)$*, i.e., a map* $\mathrm{ST} : C^b_{\mathrm{har}}(\mathbb{C}_p) \longrightarrow \mathcal{O}_2$ *for which the diagram*

(39)
$$\begin{array}{ccccc} \mathcal{O}_2 & \xrightarrow{\quad\partial\quad} & C_{\mathrm{har}}(\mathbb{C}_p) & \longrightarrow & 0 \\ & \nwarrow & \cup & & \\ & \mathrm{ST} & C^b_{\mathrm{har}}(\mathbb{C}_p) & & \end{array}$$

commutes.

Proof. The map ST is constructed by integrating a "Poisson kernel" against this measure, as in [Te], namely, one sets

$$\mathrm{ST}(c)(z) = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_c(t), \qquad \text{for all } z \in \mathcal{H}_p. \tag{40}$$

See [Te] for more details. □

Lemma 2.10. *The natural inclusion* $\mathrm{MS}^{\Gamma}(C^b_{\mathrm{har}}(\mathbb{C}_p)) \hookrightarrow \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathbb{C}_p))$ *is an isomorphism.*

Proof. Given $c \in \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathbb{C}_p))$, consider its image in $\mathrm{H}^1(\Gamma_0(p), \mathbb{C}_p)$ under the map ev_{e_0}. Since $\Gamma_0(p)$ is finitely generated, there is a bounded subset $\Omega \subset \mathbb{C}_p$ for which $\mathrm{ev}_{e_0}(c) \in \mathrm{H}^1(\Gamma_0(p), \Omega)$. But then the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\Omega)) & \xrightarrow{\mathrm{ev}_{e_0}} & \mathrm{MS}^{\Gamma_0(p)}(\Omega) \\ \downarrow & & \downarrow \\ \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathbb{C}_p)) & \xrightarrow{\mathrm{ev}_{e_0}} & \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p), \end{array}$$

in which the horizontal arrows are isomorphisms by Lemma 2.8, implies that c belongs to $\mathrm{MS}^\Gamma(C_{\mathrm{har}}(\Omega)) \subset \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathbb{C}_p))$. □

Corollary 2.11. *The residue map*

$$\partial : \mathrm{MS}^\Gamma(\mathcal{O}_2) \longrightarrow \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathbb{C}_p))$$

of (36) *is an isomorphism.*

Proof. The injectivity of ∂ is apparent from the fact that the injective map η in the commutative diagram (38) factors through it. Given $c \in \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathbb{C}_p))$, the harmonic functions $c\{r, s\}$ belong to $C^b_{\mathrm{har}}(\mathbb{C}_p)$ for all $r, s \in \mathbb{P}_1(\mathbb{Q})$, by Lemma 2.10. We may therefore set

$$f\{r, s\} := \mathrm{ST}(c\{r, s\}) \in \mathcal{O}_2.$$

The assignment $(r, s) \mapsto f\{r, s\}$ is an element of $\mathrm{MS}^\Gamma(\mathcal{O}_2)$ satisfying $\partial(f) = c$, and the result follows. □

Theorem 2.12. *The map* η *of* (38) *gives a Hecke-equivariant isomorphism*

$$\eta : \mathrm{MS}^\Gamma(\mathcal{O}_2) \xrightarrow{\sim} \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p).$$

Proof. This follows immediately from Lemma 2.8 and Corollary 2.11. □

Definition 2.13. The inverse of the isomorphism η, denoted

$$L^{\mathrm{ST}} : \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p) \xrightarrow{\sim} \mathrm{MS}^\Gamma(\mathcal{O}_2)$$

is called the *Schneider–Teitelbaum lift.*

We close this section by recording the following consequence of the injectivity of the residue map on $\mathrm{MS}^\Gamma(\mathcal{O}_2)$:

Proposition 2.14. *The spaces* $\mathrm{MS}^\Gamma(\mathcal{O})$ *and* $\mathrm{MS}^\Gamma(\mathcal{M})$ *of rigid analytic and meromorphic cocycles of weight zero are trivial.*

Proof. The image of the map $d : \mathrm{MS}^\Gamma(\mathcal{O}) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{O}_2)$ consists of modular symbols with values in the exact rigid differentials, which have trivial residues, and hence this image is contained in the kernel of the residue map ∂. Since Corollary 2.11 asserts that ∂ is injective, it follows that, for all $f \in \mathrm{MS}^\Gamma(\mathcal{O})$ and for all $r, s \in \mathbb{P}_1(\mathbb{Q})$, the function $f\{r, s\}$ is a constant, and hence that f belongs to $\mathrm{MS}^\Gamma(\mathbb{C}_p)$, which is trivial. The triviality of $\mathrm{MS}^\Gamma(\mathcal{O})$ follows, and that of $\mathrm{MS}^\Gamma(\mathcal{M})$ is then a consequence of Corollary 1.27. □

2.6. **The universal cocycle.** There are precisely two conjugacy classes of parabolic subgroups of $\Gamma_0(p)$, the group P_∞ consisting of the upper triangular matrices, which stabilise the cusp ∞, and the group P_0 consisting of lower triangular matrices, which stabilise the cusp 0. The $\Gamma_0(p)$-module $\mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \mathbb{C}_p)$ therefore decomposes as a direct sum of the two induced modules

$$\mathcal{F}(\mathbb{P}_1(\mathbb{Q}), \mathbb{C}_p) \quad = \quad \mathrm{Ind}_{P_\infty}^{\Gamma_0(p)} \mathbb{C}_p \ \oplus \ \mathrm{Ind}_{P_0}^{\Gamma_0(p)} \mathbb{C}_p,$$

and the $\Gamma_0(p)$-cohomology of the exact sequence (10) with $\Omega = \mathbb{C}_p$ leads to a short exact sequence

$$0 \longrightarrow \mathbb{C}_p \longrightarrow \mathrm{MS}^{\Gamma_0(p)} \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma_0(p), \mathbb{C}_p) \longrightarrow 0. \tag{41}$$

The one-dimensional kernel of the penultimate arrow in this sequence is spanned by the modular symbol m_{univ} defined by

$$m_{\mathrm{univ}}\{r, s\} = \begin{cases} 1 & \text{if } r \sim 0 \text{ and } s \sim \infty, \\ -1 & \text{if } r \sim \infty \text{ and } s \sim 0, \\ 0 & \text{otherwise}, \end{cases} \tag{42}$$

where $r \sim s$ means that $r, s \in \mathbb{P}_1(\mathbb{Q})$ are $\Gamma_0(p)$-equivalent. We now proceed to describe the the Schneider-Teitelbaum lift of m_{univ}.

Given $r, s \in \mathbb{Q}$, consider the function

$$f_{\mathrm{univ}}\{r, s\}(z) \ := \ \mathrm{dlog}\left(\frac{z - s}{z - r}\right) \ = \ \frac{dz}{z - s} - \frac{dz}{z - r} \ \in \ \mathcal{O}_2, \tag{43}$$

extending its definition to $r, s \in \mathbb{P}_1(\mathbb{Q})$ by adopting the conventions $z - \infty := 1$ and $\frac{dz}{z-\infty} := 0$.

Lemma 2.15. *The rule f_{univ} sending (r, s) to $f_{\mathrm{univ}}\{r, s\}$ is an element of $\mathrm{MS}^\Gamma(\mathcal{O}_2)$.*

Proof. The function

$$(r, s) \mapsto f^\times_{\mathrm{univ}}\{r, s\} := \left(\frac{z - s}{z - r}\right), \tag{44}$$

where the right hand side is viewed as an element of $\mathcal{O}^\times/\mathbb{C}_p^\times$, clearly satisfies the modular symbol properties, since $f^\times_{\mathrm{univ}}\{r, s\}$ is characterised as the unique rational function (up to scaling) whose divisor is $(s) - (r)$. Furthermore, for all $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$, the rational functions $f^\times_{\mathrm{univ}}\{\gamma r, \gamma s\}(\gamma z)$ and $f^\times_{\mathrm{univ}}\{r, s\}(z)$ have the same divisor, and hence are equal (up to a multiplicative constant). It follows that $f^\times_{\mathrm{univ}}$ can be viewed as an element of $\mathrm{MS}^\Gamma(\mathcal{O}^\times/\mathbb{C}_p^\times)$, and therefore that its logarithmic derivative f_{univ} is an element of $\mathrm{MS}^\Gamma(\mathcal{O}_2)$. ☐

The rigid analytic modular symbol f_{univ} is called the *universal modular cocycle* because it arises from the restriction to Γ of a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant modular symbol.

Lemma 2.16. *The class f_{univ} is the Schneider-Teitelbaum lift of m_{univ}, i.e,*

$$\eta(f_{\mathrm{univ}}) = m_{\mathrm{univ}}.$$

Proof. The standard annulus

$$\mathcal{W}_{e_0} = \{z \in \mathbb{C}_p \text{ such that } 1 < |z| < p\}$$

attached to the edge e_0 divides $\mathbb{P}_1(\mathbb{C}_p)$ into two components, whose intersections with $\mathbb{P}_1(\mathbb{Q}_p)$ are $\mathbb{Z}_p$ and $\mathbb{Z}_p'$ respectively. Observe that r belongs to $\mathbb{Z}_p'$ if and only if $r \sim \infty$, and that r belongs to $\mathbb{Z}_p$ if and only if $r \sim 0$. It follows that

$$\partial_{\mathcal{W}_{e_0}}(f_{\mathrm{univ}}\{r, s\}) = \sum_{t \in \mathbb{Z}_p' \cap \{r, s\}} \partial_t(f_{\mathrm{univ}}\{r, s\}) = \begin{cases} 1 & \text{if } r \sim 0 \text{ and } s \sim \infty, \\ -1 & \text{if } r \sim \infty \text{ and } s \sim 0, \\ 0 & \text{otherwise}, \end{cases}$$

The lemma follows from (42). □

Corollary 2.17. *The Schneider-Teitelbaum lift induces an isomorphism*

$$L^{\rm ST}_{\rm par} : {\rm H}^1_{\rm par}(\Gamma_0(p), \mathbb{C}_p) \longrightarrow {\rm H}^1_{\rm par}(\Gamma, \mathcal{O}_2)/\mathbb{C}_p f_{\rm univ}.$$

3. Multiplicative cocycles

This chapter focusses on the multiplicative theory, parlaying the understanding of the space $H^1_{\rm par}(\Gamma, \mathcal{M}_2)$ gained in the previous chapters into a description of $H^1_f(\Gamma, \mathcal{M}^\times)$, which maps to the former by the logarithmic derivative map

$${\rm dlog} : H^1_f(\Gamma, \mathcal{M}^\times) \longrightarrow H^1_{\rm par}(\Gamma, \mathcal{M}_2).$$

3.1. **Prelude: rational multiplicative cocycles.** The rational cocycle Φ°_τ of weight two and its associated rational period function ϕ°_τ described in equation (19) of Section 1.4 have natural multiplicative counterparts defined by

$$(45) \qquad \bar{J}^\circ_\tau\{r,s\} := \prod_{w\in\Sigma^\circ_\tau(r,s)} (z-w)^{\delta_{r,s}(w)}, \qquad \bar{\jmath}^\circ_\tau := \bar{J}^\circ_\tau\{0,\infty\} = \prod_{w\in\Sigma^\circ_\tau} (z-w)^{\delta_\infty(w)}.$$

These functions satisfy

$${\rm dlog}\, \bar{J}^\circ_\tau\{r,s\} = \Phi^\circ_\tau\{r,s\}, \qquad {\rm dlog}\, \bar{\jmath}^\circ_\tau = \phi^\circ_\tau,$$

which characterise them up to multiplicative scalars. It follows that $\bar{J}^\circ_\tau$ can be viewed as an ${\rm SL}_2(\mathbb{Z})$-invariant modular symbol with values in the quotient $\mathcal{M}^\times_{\rm rat}/C^\times$, where $\mathcal{M}^\times_{\rm rat}$ is the multiplicative group of non-zero rational functions on $\mathbb{P}_1(C)$. The obstruction to lifting $\bar{J}^\circ_\tau \in {\rm H}^1_{\rm par}({\rm SL}_2(\mathbb{Z}), \mathcal{M}^\times_{\rm rat}/C^\times)$ to an element of ${\rm H}^1_f({\rm SL}_2(\mathbb{Z}), \mathcal{M}^\times_{\rm rat})$— i.e, a genuine multiplicative, but not necessarily parabolic, cocycle—is trivial, since ${\rm H}^2({\rm SL}_2(\mathbb{Z}), C^\times) = 1$, while the ambiguity in making this lift lies in ${\rm H}^1({\rm SL}_2(\mathbb{Z}), C^\times) = {\rm hom}({\rm SL}_2(\mathbb{Z}), \mu_{12})$. Therefore one obtains a rational multiplicative cocycle $J^\circ_\tau \in {\rm H}^1_f({\rm SL}_2(\mathbb{Z}), \mathcal{M}^\times_{\rm rat})$ attached to any ${\rm SL}_2(\mathbb{Z})$-orbit of real quadratic irrationalities, which is well-defined up to 12-torsion and satisfies

$$(46) \qquad J^\circ_\tau(\gamma) = \bar{J}^\circ_\tau(\gamma) \pmod{C^\times}.$$

Note that J°_τ is not just a cohomology class but a specific cocycle, characterised by the fact that its values on the parabolic subgroup P_∞ of upper triangular matrices in ${\rm SL}_2(\mathbb{Z})$ are constant functions.

It will be useful to have explicit formulae for J°_τ in terms of the rational period function $\bar{\jmath}^\circ_\tau$. The following lemma examines the extent to which the latter fails to satisfy the two and three term relations:

Lemma 3.1. *The function $\bar{\jmath}^\circ_\tau$ satisfies*

$$\begin{array}{lcl} \bar{\jmath}^\circ_\tau|(1+S) & = & \pm\xi_\tau^2, \\ \bar{\jmath}^\circ_\tau|(1+U+U^2) & = & \pm\xi_\tau^3\times\varepsilon_\tau^3, \end{array} \qquad \text{with} \quad \xi_\tau := \prod_{w\in\Sigma^\circ_\tau,\ w>0} w,$$

where ε_τ is the unique fundamental unit of $\mathcal{O}_\tau$ of norm 1 in the interval $(0,1)$.

Proof. We invoke Lemma 1.20 to write

$$\bar{\jmath}^\circ_\tau(z) = \prod_{w\in\Sigma^\circ_\tau} (z-w)^{\delta_\infty(w)} = \prod_{w\in\Sigma^\circ_\tau,\ w>0} t^{(2)}_w(z) = \prod_{w\in\Sigma^\circ_\tau,\ w>0} t^{(3)}_w(z),$$

where the rational functions $t_w^{(2)}$ and $t_w^{(3)}$ are obtained by grouping together the factors that are in the same orbits for the groups $\{1,S\}$ and $\{1,U,U^2\}$ respectively, i.e.,

$$t_w^{(2)}(z) = \frac{z-w}{z+1/w}, \qquad t_w^{(3)}(z) = \begin{cases} \frac{z-w}{z-(w-1)/w} & \text{if } 0<w<1, \\ \frac{z-w}{z-1/(1-w)} & \text{if } w>1. \end{cases}$$

The functions $t_w^{(2)}$ and $t_w^{(3)}$ satisfy the two and three term relations, respectively, up to scalars. More precisely, a direct if slightly tedious computation reveals that

$$t_w^{(2)}|(1+S) = -w^2, \qquad t_w^{(3)}|(1+U+U^2) = \begin{cases} -w^3 & \text{if } 0<w<1, \\ (w-1)^3 & \text{if } w>1. \end{cases}$$

It follows that

$$\jmath_\tau^\circ|(1+S) = \pm\,\xi_\tau^2, \qquad \jmath_\tau^\circ|(1+U+U^2) = \pm\,\xi_\tau^3 \;\times \prod_{0<w<1} w^3 \;\times \prod_{-1<w'<0} w^{-3},$$

where the two products above, and those in the next equation, are taken over the relevant subsets of $w \in \Sigma_\tau^\circ$. The theory of cycles of reduced quadratic irrationalities implies, as explained in [Za1, Section 6], that

$$\prod_{0<w<1} w \;\times \prod_{-1<w'<0} w^{-1} \;=\; \varepsilon_\tau,$$

where ε_τ is the unique norm 1 fundamental unit of $\mathcal{O}_\tau$ in the interval $(0,1)$. □

Remark: The undetermined sign in the above statement could have been made explicit, but since it only reflects a 2-torsion ambiguity in the multiplicative group, it plays an accessory role in what follows, and there is little to be gained from being more precise.

Lemma 3.2. *The cocycle* $J_\tau^\circ \in \mathrm{H}^1_f(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_{\mathrm{rat}}^\times/\pm 1)$ *is determined by the relations*

$$J_\tau^\circ(S) = \xi_\tau^{-1}\cdot \jmath_\tau^\circ, \qquad J_\tau^\circ(U) = \xi_\tau^{-1}\varepsilon_\tau^{-1}\cdot \jmath_\tau^\circ.$$

Proof. The function $J_\tau^\circ(S)$ is the unique scalar multiple (up to ± 1) of $\jmath_\tau^\circ$ satisfying the two-term relation, while $J_\tau^\circ(U)$ is the unique scalar multiple (up to cube roots of unity) of that same function satisfying the three term relation. The result follows directly from Lemma 3.1. □

The lift J_τ° need not be (and, in fact, never is, as we shall see) parabolic, but one does have:

Lemma 3.3. *The parabolic class* $\bar{J}_\tau^\circ \in \mathrm{H}^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_{\mathrm{rat}}^\times/C^\times)$ *lifts to a class*

$$\hat{J}_\tau^\circ \in \mathrm{H}^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_{\mathrm{rat}}^\times/\varepsilon_\tau^{\mathbb{Z}}),$$

where ε_τ *is a fundamental unit of norm one in the order* $\mathcal{O}_\tau$.

Proof. Let $\Lambda_\tau \subset K := \mathbb{Q}(\tau)$ be a rank two projective $\mathcal{O}_\tau$-module attached to the class of τ (for instance, the $\mathbb{Z}$-module $[\tau,1]$ spanned by τ and 1) and let $\mathcal{B}_\tau$ denote the set of positive $\mathbb{Z}$-bases of Λ_τ, where a basis $[\omega_1,\omega_2]$ is said to be *positive* if $\omega_1\omega_2' - \omega_1'\omega_2 > 0$. The assignment $[\omega_1,\omega_2] \mapsto \omega_1/\omega_2$ defines a surjective map

$$\pi : \mathcal{B}_\tau \longrightarrow \mathrm{SL}_2(\mathbb{Z})\tau$$

which is compatible with the natural Γ-action on both sets, and whose fibers are principal homogeneous spaces for the group $\varepsilon_\tau^{\mathbb{Z}}$. Given $w \in \Gamma_\tau$, set

$$t_w(z) = w_2 z - w_1,$$

where $[w_1, w_2]$ is any element of $\mathcal{B}_\tau$ satisfying $\pi([w_1, w_2]) = w$. The function t_w has divisor $(w) - (\infty)$, and is well defined up to multiplication by elements of $\varepsilon_\tau^{\mathbb{Z}}$. It also satisfies the pleasant transformation formula

$$t_{\gamma w}(\gamma z) = (cz+d)^{-1} t_w(z) \pmod{\varepsilon_\tau^{\mathbb{Z}}}, \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{47}$$

One can use these distinguished functions with prescribed divisor to refine the function $\bar{J}_\tau^\circ\{r, s\}$ of (45) by setting

$$\hat{J}_\tau^\circ\{r, s\} = \prod_{w \in \Sigma_\tau^\circ(r,s)} t_w(z)^{\delta_{r,s}(w)} \pmod{\varepsilon_\tau^{\mathbb{Z}}}. \tag{48}$$

It is immediate from (47) and the fact that that the divisors $\Delta^\circ\{r, s\}$ are of degree zero that $\hat{J}_\tau^\circ$ satisfies the rule

$$\hat{J}_\tau^\circ\{\gamma r, \gamma s\}(\gamma z) = \hat{J}_\tau^\circ\{r, s\}(z) \pmod{\varepsilon_\tau^{\mathbb{Z}}}, \qquad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}),$$

and thus defines an element of $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathcal{M}_{\mathrm{rat}}^\times/\varepsilon_\tau^{\mathbb{Z}}) = \mathrm{H}^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_{\mathrm{rat}}^\times/\varepsilon_\tau^{\mathbb{Z}})$ lifting the class $\bar{J}_\tau^\circ$. The lemma follows. □

Lemma 3.3 implies that the cocycle J_τ° is parabolic modulo $\varepsilon_\tau^{\mathbb{Z}}$. The following lemma makes this conclusion more precise.

Lemma 3.4. *The class $J_\tau^\circ \in \mathrm{H}^1_f(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_{\mathrm{rat}}^\times)$ satisfies*

$$J_\tau^\circ(T) = \varepsilon_\tau \pmod{\mu_{12}},$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the standard generator of P_∞.

Proof. Since $T^{-1} = SU$, Lemmas 3.2 and 3.1 imply that

$$J_\tau^\circ(T)^{-1} = J_\tau^\circ(S) \times S J_\tau^\circ(U) = \xi_\tau^{-2} \varepsilon_\tau^{-1} (1+S) \bar{J}_\tau^\circ = \varepsilon_\tau^{-1}.$$

The lemma follows. □

It is natural to study the quantity

$$J^\circ(\tau_1, \tau_2) := \hat{J}_{\tau_1}^\circ[\tau_2] \pmod{\varepsilon_1^{\mathbb{Z}}, \varepsilon_2^{\mathbb{Z}}}$$

associated to real quadratic elements τ_1 and τ_2. The following proposition, which shows that it is antisymmetric, can be viewed as a "Weil reciprocity formula" for the rational parabolic cocycles $\hat{J}_\tau^\circ$.

Proposition 3.5. *For all real irrationalities τ_1 and τ_2 with associated orders $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, we have*

$$\hat{J}_{\tau_1}^\circ[\tau_2] = \hat{J}_{\tau_2}^\circ[\tau_1]^{-1} \pmod{\mathcal{O}_1^\times \mathcal{O}_2^\times},$$

i.e., $J(\tau_1, \tau_2) = J(\tau_2, \tau_1)^{-1}$ modulo the group of units in $\mathcal{O}_1^\times$ and $\mathcal{O}_2^\times$.

Proof. Following the notations that were used in the proof of Lemma 3.3, let $[w_1^{(1)}, w_2^{(1)}]$ and $[w_1^{(2)}, w_2^{(2)}]$ be elements of $\mathcal{B}_{\tau_1}$ and $\mathcal{B}_{\tau_2}$ respectively, satisfying

$$\pi([w_1^{(1)}, w_2^{(1)}]) = \tau_1, \qquad \pi([w_1^{(2)}, w_2^{(2)}]) = \tau_2,$$

and define

$$\det(\tau_1, \tau_2) := \det \begin{pmatrix} w_1^{(1)} & w_1^{(2)} \\ w_2^{(1)} & w_2^{(2)} \end{pmatrix},$$

which is well-defined modulo the group generated by the units $\pm\varepsilon_1$ and $\pm\varepsilon_2$. Since

$$t_{\tau_1}(\tau_2) = -(w_2^{(2)})^{-1} \cdot \det(\tau_1, \tau_2),$$

we can invoke (48) to write:

$$J^\circ(\tau_1, \tau_2) = \hat{J}^\circ_{\tau_1}[\tau_2] = \hat{J}^\circ_{\tau_1}\{r, \gamma_2 r\}(\tau_2) = \prod_{\gamma \in \mathrm{SL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} \det(\gamma\tau_1, \tau_2)^{\delta_{r,\gamma_2 r}(\gamma\tau_1)},$$

where the replacement of $t_{\gamma\tau_1}(\tau_2)$ by $\det(\gamma\tau_1, \tau_2)$ is justified by that fact that the intersection number $\delta_{r,\gamma_2 r}(w) \in \{-1, 0, 1\}$ satisfies

$$\sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} \delta_{r,\gamma_2 r}(\gamma\tau_1) = 0.$$

Let $\delta(w_1, w_2) \in \{-1, 0, 1\}$ be the signed intersection between the geodesic on $\mathcal{H}$ going from w_1 to w_1', and the geodesic from w_2 to w_2'. Then

$$\sum_{j=-\infty}^{\infty} \delta_{\gamma_2^j r, \gamma_2^{j+1} r}(\gamma\tau_1) = \delta(\gamma\tau_1, \tau_2).$$

Therefore we obtain

$$J^\circ(\tau_1, \tau_2) \quad = \prod_{\gamma \,\in\, \gamma_2^{\mathbb{Z}}\backslash \mathrm{SL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} \det(\gamma\tau_1, \tau_2)^{\delta(\gamma\tau_1, \tau_2)}. \tag{49}$$

Proposition 3.5 can be deduced from (49) in light of the fact that, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\det(\gamma\tau_1, \tau_2) = \det(\tau_1, \gamma^{-1}\tau_2) = -\det(\gamma^{-1}\tau_2, \tau_1),$$

and that

$$\delta(\gamma\tau_1, \tau_2) = \delta(\tau_1, \gamma^{-1}\tau_2) = -\delta(\gamma^{-1}\tau_2, \tau_1).$$

□

It will be useful later to have a formula for the valuation of $J^\circ(\tau_1, \tau_2)$ at certain rational primes p. Recall that two RM points τ_1 and τ_2 of discriminants D_1 and D_2 respectively correspond to embeddings φ_1 and φ_2 of the associated orders $\mathcal{O}_1$ and $\mathcal{O}_2$ into the matrix ring $M_2(\mathbb{Z})$. The *intersection multiplicity at* p of φ_1 and φ_2 is defined by setting

$$[\varphi_1 \cdot \varphi_2]_p := \max t \geq 0 \text{ s.t. } \varphi_1(\mathcal{O}_1),\ \varphi_2(\mathcal{O}_2) \text{ have the same image in } M_2(\mathbb{Z}/p^t\mathbb{Z}). \tag{50}$$

To motivate the following definition, we remark that the finite sum

$$\sum_{\gamma \in \gamma_2^{\mathbb{Z}}\backslash \mathrm{SL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} \delta(\gamma\tau_1, \tau_2)$$

can be interpreted as the topological intersection of the closed geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ attached to τ_1 and τ_2, and hence is equal to 0 since $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ is a Zariski open subset of a curve of genus zero.

Definition 3.6. The p-weighted intersection number of τ_1 and τ_2 is the sum (involving only finitely many non-zero terms)

$$(\varphi_1 \cdot \varphi_2)_{p\infty} := \sum_{\gamma \in \gamma_2^{\mathbb{Z}}\backslash \mathrm{SL}_2(\mathbb{Z})/\gamma_1^{\mathbb{Z}}} [\gamma\varphi_1\gamma^{-1} \cdot \varphi_2]_p \cdot \delta(\gamma\tau_1, \tau_2). \tag{51}$$

Proposition 3.7. *Let $p \nmid D_1 D_2$ be a rational prime which is inert in K_1 and in K_2. Then*

$$\mathrm{ord}_p\, J^\circ(\tau_1, \tau_2) = (\varphi_1, \varphi_2)_{p\infty}.$$

Proof. This follows directly from the formula (49) for $J^\circ(\tau_1,\tau_2)$ in light of the fact that $\mathrm{ord}_p \det(\gamma\tau_1,\tau_2) = [\gamma\varphi_1\gamma^{-1}\cdot\varphi_2]_p$. □

Remark 3.8. In their suggestive work [DIT] on linking numbers and modular cocycles, Duke, Imamoglu and Toth remark that the quantities $J^\circ(\tau_1,\tau_2)$ are closely related to the linking number between the modular geodesics attached to τ_1 and τ_2 on the threefold $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$. As noted in the acknowledgements, the use of multiplicative rational cocycles in [DIT] was an important source of inspiration for this paper.

3.2. **A review of p-adic theta-functions.** This section briefly recalls the theory of rigid analytic theta functions following the treatment in [GVdP].

Given $w\in\mathbb{P}^1(\mathbb{C}_p)$, let $t_w(z)$ denote the linear polynomial on $\mathbb{P}^1(\mathbb{C}_p)$ defined by

$$t_w(z) := \begin{cases} z-w & \text{if} \quad |w|\le 1, \\ z/w-1 & \text{if} \quad |w|>1, \\ 1 & \text{if} \quad w=\infty. \end{cases} \tag{52}$$

Meromorphic functions on $\mathcal{H}_p$ with prescribed divisors can be constructed in a systematic way using the following adaptation of a result of Gerritzen–van der Put [GVdP, Lemma 2.2].

Lemma 3.9. *Let (w_i^+) and (w_i^-) be sequences of points in $\mathcal{H}_p$ satisfying the following:*

(i) *For any $\varepsilon>0$, and for all i large enough relative to ε,*

$$\begin{array}{ll} |w_i^+-w_i^-|<\varepsilon & \text{if } |w_i|\le 1, \\ |1/w_i^+-1/w_i^-|<\varepsilon & \text{if } |w|>1. \end{array}$$

(ii) *The sets of $w_i^\pm$ are discrete, i.e., for all $n\ge 0$, the affinoid $\mathcal{H}_p^{\le n}$ contains finitely many of the w_i^+ and w_i^-.*

Then the infinite product

$$J(z)=\prod_{i=1}^{\infty}\left(\frac{t_{w_i^+}(z)}{t_{w_i^-}(z)}\right) \tag{53}$$

converges to a rigid meromorphic function on $\mathcal{H}_p$ with zeroes only at the w_i^+ and poles only at the w_i^-, whose logarithmic derivative is

$$\mathrm{dlog}\, J(z)=\sum_{i=1}^{\infty}\left(\frac{dz}{z-w_i^+}-\frac{dz}{z-w_i^-}\right).$$

Proof. The infinite product in (53) converges to a rigid meromorphic function on $\mathcal{H}_p$ because its general factor converges uniformly to 1 on any affinoid $\mathcal{H}_p^{\le n}$. More precisely, we have

$$\left|\frac{t_{w_i^+}(z)}{t_{w_i^-}(z)}-1\right|\le p^{n-N}, \qquad \text{for all } w_i^+,w_i^-\in\mathcal{H}_p^{\ge N}, z\in\mathcal{H}_p^{\le n}. \tag{54}$$

All of the other properties of $J(z)$ are a direct consequence of the definitions. □

For this section (and this section only!), let Γ be a subgroup of $\mathrm{PSL}_2(\mathbb{Q}_p)$ acting discretely and without fixed points on $\mathcal{H}_p$ by Möbius transformations. This excludes finite index subgroups of $\mathrm{SL}_2(\mathbb{Z}[1/p])$, whose non-trivial fixed points consist of the RM points in $\mathcal{H}_p$. Examples of such discrete groups arise for instance from suitable finite index subgroups of p-arithmetic groups $R_1^\times$ consisting of the elements of norm 1 in a maximal $\mathbb{Z}[1/p]$-order R in a definite quaternion algebra B over $\mathbb{Q}$ which is split at p so that $B\otimes\mathbb{Q}_p$ can be identified with $M_2(\mathbb{Q}_p)$. After fixing such an identification, the group

$$\Gamma := R_1^\times/\langle\pm 1\rangle\subset\mathrm{PSL}_2(\mathbb{Q}_p)$$

acts on $\mathcal{H}_p$ with discrete orbits. The quotient $\Gamma\backslash\mathcal{H}_p$ can be identified with the $\mathbb{C}_p$-points of a complete rigid analytic curve X over $\mathbb{Q}_p$: a Shimura curve, which has a model over $\mathbb{Q}$ and enjoys many of the same rich arithmetic properties as classical modular curves.

Let Δ be a divisor of degree 0 on $\mathcal{H}_p$ (say, $\Delta = w^+ - w^-$). After enumerating the elements of $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_i, \ldots\}$, one can show that the sequences

$$w_i^+ := \gamma_i(w^+), \qquad w_i^- = \gamma_i(w^-)$$

satisfy the conditions in Lemma 3.9, and hence that the function

$$\bar{J}_{w^+-w^-}(z) := \prod_{\gamma\in\Gamma}\left(\frac{t_{\gamma w^+}(z)}{t_{\gamma w^-}(z)}\right) \tag{55}$$

converges to a meromorphic function on $\mathcal{H}_p$ which is rigid analytic on $\mathcal{H}_p - \Gamma w^+ - \Gamma w^-$, and has zeroes and poles on Γw^+ and Γw^- respectively.

The definition of $\bar{J}_{w^+-w^-}$ can be extended by multiplicativity to allow the replacement of $w^+ - w^-$ by any degree zero divisor Δ on $\Gamma\backslash\mathcal{H}_p$. The function $\bar{J}_\Delta$ is Γ-invariant *up to multiplicative scalars*:

$$\bar{J}_\Delta \in \mathrm{H}^0(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times),$$

but need not be Γ-invariant itself. An arbitrary lift J_Δ of $\bar{J}_\Delta$ to $\mathcal{M}^\times$ satisfies the transformation formula

$$J_\Delta(\gamma z) = \kappa_\Delta(\gamma) J_\Delta(z),$$

where $\kappa_\Delta \in \mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)$ is the *period function* attached to J_Δ. This class represents the obstruction to lifting the image of $\bar{J}_\Delta$ to an element of $\mathrm{H}^0(\Gamma, \mathcal{M}^\times)$, and encodes the image of Δ in the Jacobian of X over $\mathbb{C}_p$. More precisely, taking the Γ-cohomology of the exact sequence

$$0 \longrightarrow \mathbb{C}_p^\times \longrightarrow \mathcal{M}^\times \longrightarrow \mathcal{M}^\times/\mathbb{C}_p^\times \longrightarrow 0$$

yields

$$(\mathcal{M}^\times)^\Gamma \longrightarrow (\mathcal{M}^\times/\mathbb{C}_p^\times)^\Gamma \stackrel{\kappa}{\longrightarrow} \mathrm{H}^1(\Gamma, \mathbb{C}_p^\times)/Q \longrightarrow 0, \tag{56}$$

where Q is the period lattice of $X := \Gamma\backslash\mathcal{H}_p$ spanned by the elements of the form κ_Δ as Δ ranges over the (lifts to $\mathcal{H}_p$ of) principal divisors on $X(\mathbb{C}_p)$. In particular, J_Δ is a Γ-invariant function if and only if $\kappa_\Delta \in Q$, i.e., the image of Δ in $\mathrm{Div}^0(X)$ is a principal divisor.

3.3. **Rigid meromorphic cocycles.** We return to the original setting of rigid meromorphic cocycles, where $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ acts on $\mathcal{H}_p$ by Möbius transformations and on $\mathcal{M}^\times$ with the weight zero action.

Recall the definition given in (26) of the rigid meromorphic cocycle Φ_τ of weight two attached to $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$. For each $r, s \in \mathbb{P}_1(\mathbb{Q})$, let w_i^+ and w_i^- be a complete list of the positive and negative elements of $\Sigma_\tau(r,s)$, paired together so that w_i^- and w_i^+ belong to the same wide open subset of the form $\mathcal{W}_v$, with $v \in \mathcal{T}_0$, for all $i \geq 0$. This collection of elements satisfies the conditions in Lemma 3.9, and hence, letting $t_w(z)$ be the rational functions given in (52), the infinite products

$$\bar{\jmath}_\tau := \prod_{w\in\Sigma_\tau} t_w(z)^{\delta_\infty(w)}, \qquad \bar{J}_\tau\{r,s\} := \prod_{w\in\Sigma_\tau(r,s)} t_w(z)^{\delta_{r,s}(w)}, \tag{57}$$

converge to rigid meromorphic functions satisfying

$$\mathrm{dlog}\, \bar{\jmath}_\tau = \varphi_\tau, \qquad \mathrm{dlog}\, \bar{J}_\tau\{r,s\} = \Phi_\tau\{r,s\}. \tag{58}$$

The function $\bar{J}_\tau\{r,s\}$ is completely determined by (58) up to multiplication by a non-zero scalar in $K_p^\times$, where K_p is the completion of $K = \mathbb{Q}(\tau)$ at the unique prime of K above p. Hence the system of $\bar{J}_\tau\{r,s\}$ determines an element

$$\bar{J}_\tau \ \in \ \mathrm{MS}^\Gamma(\mathcal{M}^\times/K_p^\times) = \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/K_p^\times).$$

The cocycle $\bar{J}_\tau$ is called the *projective rigid meromorphic cocycle* attached to τ.

Because $\varphi_\tau|_2(D-1) = 0$, it follows that

$$\bar{\jmath}_\tau|(D-1) = \bar{\jmath}_\tau(p^2z)/\bar{\jmath}_\tau(z) \in K_p^\times \tag{59}$$

is a scalar that does not depend on z, and is in fact an intrinsic invariant of φ_τ which does not depend on the choice of an anti-logarithmic derivative of φ_τ. As before, recall that $v_p(w) = \mathrm{ord}_p(w)$ denotes the p-adic valuation of $w \in K_p^\times$. The following lemma computes the multiplicative period (59) attached to φ_τ.

Lemma 3.10. *For all $z \in \mathcal{H}_p$,*

$$\bar{\jmath}_\tau(p^2z)/\bar{\jmath}_\tau(z) = \prod_{v_p(w)\in[0,2)} w^{\delta_\infty(w)},$$

where the product on the right is taken over the $w \in \Sigma_\tau$ for which $0 \le v_p(w) < 2$.

Proof. We may write

$$\bar{\jmath}_\tau(z) = \prod_{\Sigma_\tau, w^+>0} \frac{t_{w^+}(z)}{t_{w^-}(z)},$$

where w^+, w^- have the same valuation. An elementary calculation shows that

$$\frac{t_{w^+}(p^2z)}{t_{w^-}(p^2z)} \times \frac{t_{w^-/p^2}(z)}{t_{w^+/p^2}(z)} = \begin{cases} w^+/w^- & \text{if} \quad v_p(w^+) \in [0,2) \\ 1 & \text{if} \quad v_p(w^+) \notin [0,2) \end{cases}$$

from which the lemma follows. □

Recall, following Definition 1 of the introduction, that $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ consists of classes represented by rigid meromorphic multiplicative cocycles whose restriction to Γ_∞ take values in the group of constant functions. It fits into the long exact cohomology sequence

$$0 \longrightarrow \mathrm{H}^1(\Gamma, K_p^\times) \longrightarrow \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times) \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/K_p^\times) \xrightarrow{\delta} \mathrm{H}^2(\Gamma, K_p^\times). \tag{60}$$

Let $\mathbb{T} = \mathbb{Z}[T_2, T_3, T_5, \ldots]$ be the algebra of Hecke operators whose elements are polynomials in the Hecke operators T_ℓ with $\ell \neq p$ a prime, and let

$$\mathcal{I}_p := \mathrm{Ann}_{\mathbb{T}}(M_2(\Gamma_0(p), \mathbb{Z})),$$

where $M_2(\Gamma_0(p), \mathbb{Z})$ is the space of weight two modular forms on $\Gamma_0(p)$ with integral fourier coefficients.

Lemmas 3.11 and 3.12 below analyze the structure of the $\mathbb{T}$-modules arising in (60).

Lemma 3.11. (1) *The group $\mathrm{H}^1(\Gamma, K_p^\times)$ is finite of exponent* 12.

(2) *There is a natural map*

$$\eta : \mathrm{H}^1(\Gamma_0(p), K_p^\times) \longrightarrow \mathrm{H}^2(\Gamma, K_p^\times)$$

whose kernel and cokernel are of exponent 12.

(3) *The module $\mathrm{H}^2(\Gamma, K_p^\times)$ is a torsion $\mathbb{T}$-module which is annihilated by $12 \cdot \mathcal{I}_p$.*

Proof. Let $\mathcal{F}(\mathcal{T}_0, K_p^\times)$ and $\mathcal{F}(\mathcal{T}_1, K_p^\times)$ denote the Γ-modules of $K_p^\times$-valued functions on the sets of vertices and edges of the Bruhat-Tits tree $\mathcal{T}$. Recall that every edge $e \in \mathcal{T}_1$ contains a unique positive vertex $v_+ \in \mathcal{T}_0^+$ and a unique negative vertex $v_- \in \mathcal{T}_0^-$. For all $f \in \mathcal{F}(\mathcal{T}_0, K_p^\times)$, one can define a function $df \in \mathcal{F}(\mathcal{T}_1, K_p^\times)$ by setting $df(e) = f(v_-) - f(v_+)$. The map d fits into the short exact sequence

$$1 \longrightarrow K_p^\times \longrightarrow \mathcal{F}(\mathcal{T}_0, K_p^\times) \stackrel{d}{\longrightarrow} \mathcal{F}(\mathcal{T}_1, K_p^\times) \longrightarrow 1, \tag{61}$$

which provides a resolution of $K_p^\times$ by the induced Γ-modules

$$\begin{aligned} \mathcal{F}(\mathcal{T}_0, K_p^\times) &= \mathcal{F}(\mathcal{T}_0^+, K_p^\times) \oplus \mathcal{F}(\mathcal{T}_0^-, K_p^\times) = \mathrm{Ind}_{\mathrm{SL}_2(\mathbb{Z})}^\Gamma K_p^\times \oplus \mathrm{Ind}_{\mathrm{SL}_2'(\mathbb{Z})}^\Gamma K_p^\times, \\ \mathcal{F}(\mathcal{T}_1, K_p^\times) &= \mathrm{Ind}_{\Gamma_0(p)}^\Gamma (K_p^\times), \end{aligned}$$

where

$$\mathrm{SL}_2'(\mathbb{Z}) = P^{-1}\mathrm{SL}_2(\mathbb{Z})P = \left\{ \begin{pmatrix} a & b/p \\ pc & d \end{pmatrix} \text{ with } a,b,c,d \in \mathbb{Z} \right\}, \qquad \Gamma_0(p) = \mathrm{SL}_2(\mathbb{Z}) \cap \mathrm{SL}_2'(\mathbb{Z}).$$

Taking the Γ-cohomology of (61) and invoking Shapiro's lemma yields the long exact sequence

$$\begin{aligned} 1 \;\longrightarrow\;& \mathrm{H}^1(\Gamma, K_p^\times) \longrightarrow \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), K_p^\times) \oplus \mathrm{H}^1(\mathrm{SL}_2'(\mathbb{Z}), K_p^\times) \\ \longrightarrow\;& \mathrm{H}^1(\Gamma_0(p), K_p^\times) \stackrel{\eta}{\longrightarrow} \mathrm{H}^2(\Gamma, K_p^\times) \longrightarrow \mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), K_p^\times) \oplus \mathrm{H}^2(\mathrm{SL}_2'(\mathbb{Z}), K_p^\times) \longrightarrow \cdots \end{aligned}$$

The first two statements in the lemma follow after noting that the first and second cohomology of $\mathrm{SL}_2(\mathbb{Z})$ with values in $K_p^\times$ has exponent 12. Eichler-Shimura theory, which asserts that

$$\mathcal{I}_p := \mathrm{Ann}_{\mathbb{T}}(\mathrm{H}^1(\Gamma_0(p), K_p^\times)) = \mathrm{Ann}_{\mathbb{T}}(M_2(\Gamma_0(p), \mathbb{Z})),$$

implies the third statement. □

A point $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$ is said to be *fundamental* if its associated order is the *maximal* $\mathbb{Z}[1/p]$-order of the real quadratic field $\mathbb{Q}(\tau)$, i.e., if it is the root of a binary quadratic form whose discriminant, up to powers of p, is equal to a fundamental discriminant.

Lemma 3.12. *The quotient*

$$\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/K_p^\times)/\,\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{O}^\times/K_p^\times)$$

is torsion-free over $\mathbb{T}$. The classes $\bar{J}_\tau$, as τ ranges over the fundamental elements of $\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$, are multiplicatively independent over $\mathbb{T}$.

Proof. The logarithmic derivative identifies $\mathcal{M}^\times/K_p^\times$ with the group $\mathcal{M}_2^{\mathbb{Z}}$ of rigid meromorphic differentials on $\mathcal{H}_p$ having at worst simple poles and integer residues. (In the classical terminology, these are referred to as *differentials of the third kind.*) Given any $\bar{J} \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/K_p^\times)$, let $\Phi := \mathrm{dlog}\,\bar{J} \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}_2)$ be its logarithmic derivative. If θ is a non-zero element of $\mathbb{T}$, then $\bar{J}|\theta$ can only be regular if the same is true of $\Phi|_2\theta$. But then Φ must be regular, by Corollary 1.26, which implies that $\bar{J}$ has to be regular as well. The first assertion in the proposition follows. The second is an immediate consequence of Lemma 1.25, which implies that the rigid meromorphic differentials of the form $\varphi_\tau|_2\theta$, as τ ranges over the primitive elements of $\Gamma\backslash\mathcal{H}_p^{\mathrm{RM}}$, have non-trivial, mutually disjoint residual divisors. □

We are now ready to prove the main theorem of this section, from which one recovers Theorem 1 of the introduction.

Theorem 3.13. *For all primes p, the group $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ is of infinite rank over $\mathbb{Z}$. The zeroes and poles of a rigid meromorphic period function are contained in a finite collection of Γ-orbits of RM points of $\mathcal{H}_p$.*

Proof. The first assertion follows immediately from Lemma 3.11, which asserts that the left and rightmost modules in (60) are torsion $\mathbb{T}$-modules (with a specific annihilator $\mathcal{I}_p$), combined with Lemma 3.12, which asserts that the term $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/K_p^\times)$ has a non-finitely generated, $\mathbb{T}$-torsion free quotient. As to the second assertion, it follows immediately from Theorem 1.23 applied to the logarithmic derivative of an element of $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$. □

As in section 3.1, the class $\delta(\bar{J}_\tau) \in \mathrm{H}^2(\Gamma, K_p^\times)$ represents the obstruction to lifting $\bar{J}_\tau \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/K_p^\times)$ to a genuine multiplicative cocycle in $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$. By the second statement in Lemma 3.11, we may write

$$\delta(\bar{J}_\tau^{12}) = \eta(\kappa_\tau),$$

where $\kappa_\tau \in \mathrm{H}^1(\Gamma_0(p), K_p^\times)$ is well defined up to the 12-torsion group $\ker \eta$. The class κ_τ measures the obstruction to lifting the class $\bar{J}_\tau \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/K_p^\times)$ (or rather, its 12-th power) to a genuine rigid meromorphic cocycle.

Definition 3.14. The class κ_τ is called the *lifting obstruction* attached to the class $\bar{J}_\tau$.

It will be useful to have an explicit description of the lifting obstruction κ_τ. Recall the standard vertex $v_0 \in \mathcal{T}_0$ whose stabiliser in Γ is $\mathrm{SL}_2(\mathbb{Z})$, and the standard edge $e = (v_0, v_0')$ whose stabiliser in Γ is $\Gamma_0(p)$. The restriction of $\bar{J}_\tau^{12} \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times/K_p^\times)$ to the groups

$$\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Stab}_\Gamma(v_0), \qquad \mathrm{SL}_2'(\mathbb{Z}) = \mathrm{Stab}_\Gamma(v_0')$$

lift (uniquely, up to 12 torsion) to classes

$$J_\tau^{v_0} \in \mathrm{H}^1_f(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}^\times), \qquad J_\tau^{v_0'} \in \mathrm{H}^1_f(\mathrm{SL}_2'(\mathbb{Z}), \mathcal{M}^\times),$$

which are related by the rule

$$J_\tau^{v_0'}(\gamma) = J_\tau^{v_0}(P\gamma P^{-1}).$$

The restriction to $\Gamma_0(p) = \mathrm{SL}_2(\mathbb{Z}) \cap \mathrm{SL}_2'(\mathbb{Z})$ of the ratio $J_\tau^{v_0}/J_\tau^{v_0'}$ lies in the kernel of the natural map

$$\mathrm{H}^1(\Gamma_0(p), \mathcal{M}^\times) \longrightarrow \mathrm{H}^1(\Gamma_0(p), \mathcal{M}^\times/K_p^\times),$$

which is equal to $\mathrm{H}^1(\Gamma_0(p), K_p^\times)$, and, for all $\gamma \in \Gamma_0(p)$,

$$\kappa_\tau(\gamma) = J_\tau^{v_0}(\gamma)/J_\tau^{v_0'}(\gamma) = J_\tau^{v_0}(\gamma)/J_\tau^{v_0}(P\gamma P^{-1}). \tag{62}$$

Assume below that p does not divide the discriminant of the field $K = \mathbb{Q}(\tau)$, i.e., that the elements of $\Gamma\tau \subset \mathcal{H}_p$ reduce to vertices of $\mathcal{T}$. Recall that the element τ is then said to be *even* if these images are even vertices, and is said to be *odd* otherwise.

Proposition 3.15. *For all $\tau \in \mathcal{H}_p^{\mathrm{RM}}$, the class $J_\tau^{v_0} \in \mathrm{H}^1_f(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}^\times)$ satisfies*

$$J_\tau^{v_0}(T) = \varepsilon_\tau^{(p)},$$

where $\varepsilon_\tau^{(p)}$ is the unique element of $K_p^\times$ of norm 1 satisfying

$$\varepsilon_\tau^{(p)} \equiv \begin{cases} \varepsilon_\tau \pmod{p} & \text{if } \tau \text{ is even,} \\ 1 \pmod{p} & \text{if } \tau \text{ is odd,} \end{cases} \qquad (\varepsilon_\tau^{(p)})^{1-p^2} = \varepsilon_\tau^{1+p}.$$

Proof. Any $w \in \Gamma\tau$ is the root of a unique (up to sign) primitive integral binary quadratic form, whose discriminant is of the form Dp^{2m} with $p \nmid D$. The exponent m is called the *level* of w. It is an even integer if τ is even, and an odd integer if τ is odd, which is constant on $\mathrm{SL}_2(\mathbb{Z})$-orbits but not, of course, on the full Γ-orbit of τ. Upon setting

$$\Sigma_\tau^{(m)}(r,s) = \{w \in \Sigma_\tau(r,s) \text{ with } \mathrm{level}(w) = m\},$$

the sets $\Sigma_\tau(r,s)$ decompose as a disjoint union of the form

$$\Sigma_\tau(r,s) = \begin{cases} \Sigma_\tau^{(0)}(r,s) \sqcup \Sigma_\tau^{(2)}(r,s) \sqcup \Sigma_\tau^{(4)}(r,s) \sqcup \cdots & \text{if } \tau \text{ is even;} \\ \Sigma_\tau^{(1)}(r,s) \sqcup \Sigma_\tau^{(3)}(r,s) \sqcup \Sigma_\tau^{(5)}(r,s) \sqcup \cdots & \text{if } \tau \text{ is odd.} \end{cases}$$

It follows that
(63)

$$\bar{J}_\tau\{r,s\}(z) = \prod_{m=0}^{\infty} \bar{J}_\tau^{(m)}\{r,s\}(z), \qquad \text{where} \qquad \bar{J}_\tau^{(m)}\{r,s\}(z) := \prod_{w \in \Sigma_\tau^{(m)}(r,s)} t_w(z)^{\delta_{r,s}(w)},$$

adopting the convention that $\bar{J}_\tau^{(m)}\{r,s\} = 1$ if τ and m are of different parity.

For any fixed $m \geq 0$, the finite set $\Sigma_\tau^{(m)}(r,s)$ decomposes as a finite union of $\mathrm{SL}_2(\mathbb{Z})$-orbits, of the form

$$\Sigma_\tau^{(m)}(r,s) = \Sigma_{\tau_1}^\circ(r,s) \sqcup \cdots \sqcup \Sigma_{\tau_d}^\circ(r,s),$$

where $\tau_1, \ldots, \tau_d$ are the distinct representatives of the $\mathrm{SL}_2(\mathbb{Z})$ orbits of RM points of discriminant Dp^{2m} which are Γ-equivalent to τ. Recall the classes $J_\tau^\circ \in \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathcal{M}_{\mathrm{rat}}^\times)$ associated to $\tau \in \mathrm{SL}_2(\mathbb{Z})\backslash C^{\mathrm{RM}}$ in equation (46) of Section 3.1, and set

$$J_\tau^{(m)} := J_{\tau_1}^\circ \times \cdots \times J_{\tau_d}^\circ. \tag{64}$$

By Lemma 3.4, we have

$$J_{\tau_j}^\circ(T) = \varepsilon_m = \varepsilon_\tau^{e_m}, \tag{65}$$

where ε_m is the fundamental unit of the real quadratic order of discriminant Dp^{2m}. One has $e_0 = 1$ and, for $m \geq 1$, the exponent e_m is given by the *class number formula*

$$h^+(Dp^{2m})e_m = p^{m-1}\,(p+1)\,h^+(D),$$

which implies that

$$e_m = \begin{cases} 1 & \text{if } m = 0; \\ (p+1)p^{m-1}d^{-1} & \text{if } m \geq 1. \end{cases} \tag{66}$$

By combining (64), (65), and (66), one obtains

$$J_\tau^{(m)}(T) := J_{\tau_1}^\circ(T) \times \cdots \times J_{\tau_d}^\circ(T) = \varepsilon_m^d = \begin{cases} \varepsilon_\tau & \text{if } m = 0; \\ \varepsilon_\tau^{(p+1)p^{m-1}} & \text{if } m \geq 1. \end{cases}$$

The uniqueness of $J_\tau^{v_0}$ implies that

$$J_\tau^{v_0} = J_\tau^{(0)} \times J_\tau^{(1)} \times J_\tau^{(2)} \times \cdots \tag{67}$$

It follows that

$$J_\tau^{v_0}(T) = \begin{cases} \varepsilon_\tau^{1+(p+1)p+(p+1)p^3+(p+1)p^5+\cdots} & \text{if } \tau \text{ is even;} \\ \varepsilon_\tau^{(p+1)+(p+1)p^2+(p+1)p^4+(p+1)p^6+\cdots} & \text{if } \tau \text{ is odd.} \end{cases}$$

The infinite series expressing the exponents in the equation above converge in the group $\mathbb{Z}/(p+1)\mathbb{Z} \times \mathbb{Z}_p$ to the elements $(1, 1/(1-p))$ when τ is even, and to $(0, 1/(1-p))$ when τ is odd. The proposition follows. □

Theorem 3.16. *For all $\tau \in \mathcal{H}_p^{\mathrm{RM}}$, the class κ_τ satisfies*

$$\kappa_\tau(T) = \varepsilon_\tau.$$

Proof. This follows directly from (62) and from Proposition 3.15, in light of the identity

$$PTP^{-1} = T^p.$$

□

3.4. **Explicit examples.** Although Theorem 3.13 guarantees a large supply of rigid meromorphic cocycles for any prime p, it is useful for numerical experiments to single out some notably simple instances of these objects, in which complicated Hecke translates of the basic projective cocycles $\bar{J}_\tau$ need not be invoked. Such constructions are available for the small primes given in Definitions 3.17 and 3.19 below.

Definition 3.17. A prime p is said to be *genus zero prime* if the modular curve $X_0(p)$ has genus zero, i.e., if $p = 2$, 3, 5, 7, or 13.

Theorem 3.16 implies that the cocycles $\bar{J}_\tau$ themselves never lift to an element of $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$. However, one has:

Theorem 3.18. *If p is a genus zero prime, then the cocycle $\bar{J}_\tau$ lifts to a cocycle $\hat{J}_\tau \in \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times/\varepsilon_\tau^{\mathbb{Z}})$, where ε_τ is the fundamental unit of the real quadratic order attached to τ. This lift is well-defined up to a* 12 *torsion class, and*

$$\hat{J}_\tau(T) \;=\; \varepsilon_\tau^{(p)} \pmod{\mu_{12}}, \tag{68}$$

$$\hat{J}_\tau(S) \;=\; \pm(\xi_\tau^{(p)})^{-1} \times \bar{\jmath}_\tau \pmod{\mu_{12}}, \qquad \textit{where } \xi_\tau^{(p)} := \prod_{v_p(w)\in[0,2), w>0} w. \tag{69}$$

Proof. When p is a genus zero prime, the abelianisation of $\Gamma_0(p)$ is generated by the image of the parabolic matrix T, and hence the existence of the lift $\hat{J}_\tau$ follows from Theorem 3.16. It follows from Theorem 3.15 that $\hat{J}_\tau(T) = \varepsilon_\tau^{(p)}$. To calculate $\hat{J}_\tau(S)$, note that we may write

$$\bar{\jmath}_\tau(z) = \prod_{w\in\Sigma_\tau, w>0} \frac{t_w(z)}{t_{Sw}(z)}.$$

A direct calculation shows that

$$\frac{t_w(z)}{t_{Sw}(z)} \times \frac{t_w(Sz)}{t_{Sw}(Sz)} = \begin{cases} -w^2 & \text{if} \quad w \in \mathcal{O}_{\mathbb{C}_p}^\times \\ -1 & \text{if} \quad w \notin \mathcal{O}_{\mathbb{C}_p}^\times \end{cases}$$

from which it follows that $(\xi_\tau^{(p)})^{-1} \times \bar{\jmath}_\tau$ is a lift of $\bar{\jmath}_\tau$ to $\mathcal{M}^\times$ which satisfies the 2-term relation. Since $\hat{J}_\tau(S)$ is the unique such lift, up to sign, the lemma follows. □

Definition 3.19. A prime p is said to be *monstrous* if it satisfies one of the following equivalent conditions:

(1) p divides the cardinality of the Monster sporadic simple group;
(2) the modular curve $X_0(p)/w_p$ has genus zero;
(3) p is equal to $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59$, or 71.

(The equivalence of the first and second conditions, first observed by Andrew Ogg, is part of the empirical panoply of "monstrous moonshine".)

Theorem 3.20. *If p is a monstrous prime and τ is any RM point in $\mathcal{H}_p$ of discriminant prime to p, then the p-even projective cocycle*

$$(1 + \varpi_p)\bar{J}_\tau = \bar{J}_\tau/\bar{J}_{p\tau} \tag{70}$$

lifts to a uniquely determined rigid meromorphic cocycle $J_\tau^+ \in \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times/\mu_{12})$ whose associated rigid meromorphic period function j_τ^+ is given in Theorem 2 of the Introduction.

Proof. The proof of Lemma 1.4 of [Da] explains that the "lifting obstruction" map

$$\eta^{-1} \circ 12 \circ \delta : \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}^\times / K_p^\times) \longrightarrow \mathrm{H}^1(\Gamma_0(p), K_p^\times)$$

where δ is the map of (60) and η is given in the second part of Lemma 3.11, intertwines the involution ϖ_p on the domain with the Atkin Lehner involution at p on the target. When p is a monstrous prime, the subspace $\mathrm{H}^1(\Gamma_0(p), K_p^\times)^{\varpi_p = 1}$ is trivial. It follows that the projective cocycle in (70) lifts to a genuine rigid meromorphic cocycle in $\mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$. □

3.5. **The efficient calculation of rigid meromorphic cocycles.** To simplify the presentation, we assume henceforth that p is inert in $K = \mathbb{Q}(\tau)$. Our ultimate aim is to be able to compute the values of a rigid meromorphic cocycle J at any z in $\mathcal{H}_p^{\mathrm{RM}}$. We go about this via a series of simplifications. The first crucial remark is that every element of $\mathcal{H}_p$ is $\tilde{\Gamma}$-equivalent to an element of $\mathcal{H}_p^{\leq 1}$. Hence, by the Γ-equivariance property mentionned right after (3) in the Introduction, and proved in Lemma 4.3 below, it is enough to be able to evaluate the RM values of J at such elements of $\mathcal{H}_p$. We may assume without loss of generality that $z > 1$ and $-1 < z' < 0$, in which case the continued fraction

$$z = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}} \qquad (a_i \geq 1)$$

is purely periodic, of some length $n \geq 1$. If γ_z is the unique generator of Γ_z for which z is a stable fixed point, then we have

$$\gamma_z = \gamma_n \cdot \gamma_{n-1} \ \cdots \ \gamma_1, \qquad \text{where} \quad \gamma_i = ST^{(-1)^i a_i},$$

Using the cocycle relations, we compute that

$$J(\gamma_z) \ = \ J(T)^{-a_1 + a_2 - a_3 + \ldots (-1)^n a_n} \ \times \ \prod_{i=0}^{n} J(S)^{(\gamma_n \cdots \gamma_i)^{-1}}. \tag{71}$$

In the special case where $J = \hat{J}_\tau$, it was shown above that $\hat{J}_\tau(T) = \varepsilon_\tau^{(p)}$, so that it is enough to efficiently compute the rigid meromorphic function $\hat{J}_\tau(S)$.

To compute $\hat{\jmath}_\tau := \hat{J}_\tau(S)$, it is enough by (69) to compute $\bar{\jmath}_\tau$. The infinite product expansion (57) defining $\bar{\jmath}_\tau$ gives a theoretically effective way to evaluate it at arbitrary $\tau \in \mathcal{H}_p$, but this method is hardly efficient. Indeed, the estimate (54) shows that the evaluation of $\bar{\jmath}_\tau(z)$ for $z \in \mathcal{H}_p^0$ to M significant digits of p-adic accuracy requires the infinite product defining it to be taken over all

$$w \in \Sigma_\tau(r,s)^{\leq M} = \Sigma_\tau(r,s) \cap \mathcal{H}_p^{\leq M}.$$

The latter set has size roughly p^M, and it is impractical to take a product over such an index set for even moderate values of M, whereas many of the experiments that will be reported on later required p-adic precision on the order of hundreds of digits. This section describes how rigid meromorphic period functions can be calculated and stored efficiently on a computer, in a way that enables the calculation of their RM values to large p-adic accuracy.

We first describe a polynomial time recursive algorithm for computing $\bar{\jmath}_\tau$, which is somewhat in the spirit of the algorithm based on overconvergent modular symbols for computing the rigid analytic cocycles described in [DP] and in Chapter 2. Recall from (63) the decomposition

$$\bar{J}_\tau\{r,s\} = \bar{J}_\tau^{(0)}\{r,s\} \times \bar{J}_\tau^{(1)}\{r,s\} \times \bar{J}_\tau^{(2)}\{r,s\} \times \cdots$$

By the estimate (54) we have that

$$\bar{J}_\tau^{(m)}\{0,\infty\}(\mathcal{A}) \subseteq 1 + p^{2m}\mathcal{O}_{\mathbb{C}_p}.$$

This implies that in order to evaluate $\bar{\jmath}_\tau$ at a point in $\mathcal{A}$ to a p-adic accuracy of p^m, it suffices to evaluate the finite collection of rational functions $\bar{J}_\tau^{(t)}\{0,\infty\}$ for $t \le m-1$. To compute $\bar{J}_\tau^{(m)}\{0,\infty\}$, the key idea is to represent it as a multiplicative Mittag–Leffler expansion on the standard wide open rather than as a rational function. More precisely, for all $m \ge 1$:

$$\bar{J}_\tau^{(m)}\{r,s\} = \prod_{a=0}^{p-1} F_a^{(m)}\{r,s\} \times F_\infty^{(m)}\{r,s\}, \qquad \text{where } F_a^{(m)}\{r,s\}(z) = \prod_{w\in\Sigma_\tau^{(m)}\cap(a+p\mathcal{O}_{\mathbb{C}_p})} t_w(z)^{\delta_{r,s}}$$

The explicit knowledge of the multiplicative Mittag–Leffler expansion suffices for the explicit evaluation of $\bar{J}_\tau^{(m)}\{0,\infty\}$, and therefore $\bar{\jmath}_\tau$, at points of the standard affinoid $\mathcal{A}$. This is particularly convenient, since the functions $F_a^{(m)}\{r,s\}$ satisfy the following recursion formulae, for $a = 0,1,\dots,p-1$ given by

$$(72) \qquad F_a^{(m+1)}\{0,\infty\}(z) = \prod_{\ell=0}^{p-1} F_\ell^{(m)}\left\{-\frac{a}{p},\infty\right\}\left(\frac{z-a}{p}\right) \pmod{K_p^\times}$$

whereas for $a=\infty$ we have

$$(73) \qquad F_\infty^{(m+1)}\{0,\infty\}(z) = \prod_{\ell=1}^{p-1} F_\ell^{(m)}\{0,\infty\}(pz) \times F_\infty^{(m)}\{0,\infty\}(pz) \pmod{K_p^\times}$$

These recursion formulae follow from the observation that both sides define rational functions with the same divisor, and must therefore be equal up to a constant. Observe that

(1) For each fixed m and a, the function $F_a^{(m)}\{r,s\}$ is a modular symbol in $\mathrm{MS}(\mathcal{M}^\times)$.
(2) For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$F_{\gamma a}^{(m)}\{\gamma r,\gamma s\}(\gamma z) = F_a^{(m)}\{r,s\}(z) \pmod{K_p^\times}.$$

Note that the function $F_a^{(m)}\{-a/p,\infty\}$ is easily expressed, up to a multiplicative constant, as a combination of the functions $F_a^{(m)}\{0,\infty\}$ by finding a unimodular path from $-a/p$ to ∞, and using the two observations above. Using the recursions (72) and (73), this allows us to compute the functions $F_a^{(m+1)}\{0,\infty\}$ from the functions $F_a^{(m)}\{0,\infty\}$, up to a multiplicative constant. To determine this constant, set

$$t_a = \frac{1}{z-a} \quad \text{for} \quad a = 0,\dots,p-1 \qquad \text{and} \qquad t_\infty = z.$$

It is straightforward to check that

$$F_a^{(m)}\{0,\infty\} \in 1 + p\mathcal{O}_{\mathbb{C}_p}\langle t_a\rangle, \qquad a \in \{0,\dots,p-1,\infty\}$$

so that the implicit constant is easily found in practice, by normalising the right hand side to have constant term 1.

We summarise this discussion in the following steps, which describe how to compute the values $\hat{J}_\tau[z]$ at an RM point z in the standard affinoid $\mathcal{A}$, up to precision p^M:

- **Step 1.** Compute the rational function $\hat{J}_\tau^\circ\{0,\infty\}$, as well as the $p+1$ power series

$$F_a^{(1)}\{0,\infty\}(t_a) \in 1 + p\mathcal{O}_{\mathbb{C}_p}\langle t_a\rangle, \quad a = 0,1,\dots,p-1,\infty$$

- **Step 2.** Use (72) and (73), as well as the modular symbol relations for the functions $F_a(r,s)$ to compute for any $2 \le m \le n-1$ the power series

$$F_a^{(m)}\{0,\infty\}(t_a) \in 1 + p\mathcal{O}_{\mathbb{C}_p}\langle t_a\rangle \quad a = 0,1,\dots,p-1,\infty$$

up to precision p^M. Store the data of $\bar{\jmath}_\tau := \bar{J}_\tau\{0,\infty\}$ to that accuracy, expressing it as a product of $p+1$ power series in the variables t_a, up to precision t_a^n.

- **Step 3.** Compute the quantity $\hat{J}_\tau(S) = (\xi_\tau^{(p)})^{-1} \times \hat{\jmath}_\tau$ via the identity

$$\left(\xi_\tau^{(p)}\right)^2 = \prod_{a=1}^{p-1} F_a\{0,\infty\}(0).$$

- **Step 4.** Compute $\hat{J}_\tau[z] = \hat{J}_\tau(\gamma_z)(z)$ via (71) and the identity $\hat{J}_\tau(T) = \varepsilon_\tau^{(p)}$.

This algorithm has been implemented in `magma`, and the resulting code is available on the authors' webpages. It will be exploited in the next chapter to give numerical examples in support of the proposed conjectures on RM values of rigid meromorphic cocycles.

4. Real quadratic singular moduli

This chapter is devoted to the most important notion explored in this paper: the *values* of rigid meromorphic cocycles at RM points, which are conjecturally defined over composita of ring class fields of real quadratic fields, and otherwise exhibit many striking parallels with singular moduli arising in the classical theory of complex multiplication.

4.1. **RM values of rigid meromorphic cocycles.** Let $J \in \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ be a rigid meromorphic cocycle. By Theorem 1.23, its logarithmic derivative is of the form

$$\mathrm{dlog}\, J = \Phi_0 + \sum_{\tau \in \Sigma_J} \lambda_\tau \Phi_\tau^\pm,$$

where $\Phi_0 \in \mathrm{H}^1_f(\Gamma, \mathcal{O}_2)$ is a rigid analytic cocycle of weight two and Σ_J is a finite subset of the orbit space $\tilde{\Gamma}\backslash\mathcal{H}_p^{\mathrm{RM}}$.

Given any $\tau \in \mathcal{H}_p^{\mathrm{RM}}$, the *discriminant* of τ, denoted D_τ, is the prime-to-p part of the discriminant of any primitive integral binary quadratic form having τ as a root. This discriminant is well-defined on $\tilde{\Gamma}$-orbits, i.e.,

$$D_{\gamma\tau} = D_\tau, \qquad \text{for all } \gamma \in \tilde{\Gamma}, \tau \in \mathcal{H}_p^{\mathrm{RM}}.$$

Let H_τ denote the *narrow ring class field* attached to the order of discriminant D_τ. It is an abelian extension of $K := \mathbb{Q}(\tau)$ whose Galois group over K is canonically identified with the class group in the narrow sense of the order of discriminant D_τ.

Definition 4.1. The *field of definition* of J, denoted H_J, is the compositum of the narrow ring class fields H_τ, as τ ranges over the set Σ_J.

As explained in the introduction, one of the principal interests of rigid meromorphic cocycles is that they can be meaningfully evaluated at RM points. Recall the *automorph* $\gamma_\tau \in \mathcal{O}_\tau^\times$ of $\tau \in \mathcal{H}_p^{\mathrm{RM}}$ that was defined in the introduction.

Definition 4.2. The *value* of J at an RM point τ is the element

$$J[\tau] := J(\gamma_\tau)(\tau).$$

The following lemma shows how the values of J vary over $\tilde{\Gamma}$-orbits.

Lemma 4.3. *For all $\gamma \in \Gamma$ and all $\tau \in \mathcal{H}_p^{\mathrm{RM}}$,*

$$J[\gamma\tau] = J[\tau].$$

If γ belongs to $\tilde{\Gamma} - \Gamma$, then

$$J[\gamma\tau] = \begin{cases} J[\tau] & \text{if } J \text{ is } p\text{-even;} \\ J[\tau]^{-1} & \text{if } J \text{ is } p\text{-odd.} \end{cases}$$

Proof. If γ belongs to Γ, then the first part follows from the fact that the automorph of $\gamma\tau$ is $\gamma\gamma_\tau\gamma^{-1}$, and hence that

$$J[\gamma\tau] \;=\; J(\gamma\gamma_\tau\gamma^{-1})(\gamma\tau) \;=\; J(\gamma)(\gamma\tau)\times J(\gamma_\tau)(\tau)\times J(\gamma^{-1})(\tau) \;=\; J[\tau].$$

Since the group $\widetilde{\Gamma}/\Gamma$ is generated by w_p and w_∞, the second part follows by an direct calculation from the classification in Theorem 1.23.

□

The main conjecture of this section concerns the algebraicity of the RM values of rigid meromorphic cocycles, and was already stated as Conjecture 1 in the Introduction:

Conjecture 4.4. *Let $J \in \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ be a rigid meromorphic cocycle, and $\tau \in \mathcal{H}_p^{\mathrm{RM}}$. Then $J[\tau]$ is contained in the compositum of H_J and H_τ.*

The following examples describe the calculations of RM values for various small discriminants, using the computational techniques from Section 3.5.

Example 4.5. The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$, which is a root of the binary quadratic form $x^2 + xy - y^2$ of discriminant 5, is the simplest real quadratic irrationality and it is therefore natural to examine the RM values of the rigid meromorphic cocycle J_φ^+ attached to it, which, (for p a monstrous prime) is the only interesting rigid meromorphic cocycle whose zeroes and poles are concentrated in the $\widetilde{\Gamma}$-orbit of the golden ratio.

Some of the values of J_φ^+, at the RM points of discriminants 8 and 892, were already described in the introduction. The algorithms of Section 3.5 were also used to compute the value of the 2-adic cocycle J_φ^+ at the RM points of discriminant 21 to 1000 significant digits, yielding

$$J_\varphi^+\left[\frac{-3+\sqrt{21}}{2}\right] = \frac{37 \pm 48\sqrt{-3}}{7\cdot 13} \qquad (\mathrm{mod}\ 2^{1000}).$$

This experimental finding is consistent with Conjecture 4.4, since the ring class field of discriminant 21 is $\mathbb{Q}(\sqrt{-3}, \sqrt{-7})$.

We also computed J_φ^+ at $p = 7$ and $p = 17$ to 400 and 100 significant digits respectively, as well as the RM values of these cocycles at the four classes of RM points of discriminant 96, whose associated ring class field is $\mathbb{Q}(\sqrt{2}, \sqrt{-3}, \sqrt{-1})$. In this way we found

$$\begin{aligned} J_\varphi^+[2\sqrt{6}] &= \tfrac{3\pm 8\sqrt{2}\pm 12\sqrt{-1}\pm 2\sqrt{-2}}{17} \qquad (\mathrm{mod}\ 7^{400}),\\ J_\varphi^+[2\sqrt{6}] &= \tfrac{2\pm 1\sqrt{-3}\pm 3\sqrt{2}\pm 2\sqrt{-6}}{7} \qquad (\mathrm{mod}\ 17^{100}). \end{aligned}$$

Notice that the 7-adic valuation of the 17-adic invariant is equal to the 17-adic valuation of the corresponding 7-adic invariant. This phenomenon will be addressed in Section 4.4. Just as in the introduction, the values $J_\varphi^+[\tau]$ seem to be defined over H_τ rather than $H_\tau(\sqrt{5})$, an observation that will be explained by the Shimura reciprocity law formulated in Section 4.2 below.

Finally, Table 4.6 below lists the values of the cocycle J_φ^+ at a few arguments in $\mathbb{Q}(\varphi)$, for the primes $p = 2, 3, 7, 13, 17$, and 23. This is the full list of the monstrous primes that are inert in $\mathbb{Q}(\varphi)$, with the exception of the largest prime $p = 47$, which was omitted for lack of space and because the column attached to this prime is the least varied: all its entries are equal to 1 with the exception of

$$J_\varphi^+[11\varphi] \stackrel{?}{=} \frac{3+\sqrt{-55}}{2^3} \qquad \text{in } \mathbb{C}_{47}.$$

τ	$p=2$	$p=3$	$p=7$	$p=13$	$p=17$	$p=23$
3φ	$\frac{-313+713\sqrt{-3}}{2\cdot 7^2\cdot 13}$	$-$	$\frac{1+\sqrt{-15}}{4}$	$\frac{-1+\sqrt{-15}}{4}$	1	1
4φ	$-$	$\frac{174+832\sqrt{-1}}{2\cdot 5^2\cdot 17}$	$\frac{-10+24\sqrt{-1}}{2\cdot 13}$	$\frac{-4+6\sqrt{-5}}{2\cdot 7}$	$\frac{-2-\sqrt{-5}}{3}$	1
6φ	$-$	$-$	$\frac{-34+8\sqrt{-15}}{2\cdot 23}$	$\frac{67+3\sqrt{-15}}{2^2\cdot 17}$	$\frac{-1+15\sqrt{-3}}{2\cdot 13}$	$\frac{2+8\sqrt{-3}}{2\cdot 7}$
7φ	$\frac{8693+1675\sqrt{-35}}{2\cdot 3\cdot 13^3}$	$\frac{1129+357\sqrt{-7}}{2^6\cdot 23}$	$-$	$\frac{-3+\sqrt{-7}}{2^2}$	$\frac{-1-3\sqrt{-7}}{2^3}$	$\frac{-1+\sqrt{-35}}{2\cdot 3}$
9φ	$\frac{18012458+56391392\sqrt{-3}}{2\cdot 7^4\cdot 13\cdot 37\cdot 43}$	$-$	$\frac{-14+8\sqrt{-15}}{2\cdot 17}$	$\frac{-61+5\sqrt{-15}}{2^6}$	$\frac{1+4\sqrt{-3}}{7}$	1
11φ	$\frac{1394644289+132949133\sqrt{-11}}{2\cdot 3\cdot 5\cdot 23\cdot 37\cdot 47\cdot 53}$	$\frac{-3826843+133719\sqrt{-55}}{2^5\cdot 13^2\cdot 17\cdot 43}$	$\frac{-106+32\sqrt{-11}}{2\cdot 3\cdot 5^2}$	1	$\frac{5-\sqrt{-11}}{2\cdot 3}$	$\frac{-3+\sqrt{-55}}{2^3}$

TABLE 4.6. The values of the p-adic cocycle $J_{\varphi}^{+}[n\varphi]$.

Example 4.7. The next positive discriminant after 5 is 8, corresponding to the field $K := \mathbb{Q}(\sqrt{2})$. Its narrow class number is 1, so that once again the essentially unique rigid meromorphic cocycle with zeroes and poles in the $\tilde{\Gamma}$-orbit of $\sqrt{2}$ is $J_{\sqrt{2}}^{+}$.

There are four distinct classes of RM points τ_{105} of discriminant 105, and the monstrous primes that are inert for both 8 and 105 are precisely $p = 11, 19, 29$. We compute that

$$
\begin{aligned}
J_{\sqrt{2}}^{+}[\tau_{105}] &\stackrel{?}{=} \frac{2\pm 10\sqrt{-3}\pm 15\sqrt{5}\pm\sqrt{-15}}{2\cdot 19} && \pmod{11^{100}},\\
J_{\sqrt{2}}^{+}[\tau_{105}] &\stackrel{?}{=} \frac{6\pm 3\sqrt{-7}\pm 7\sqrt{5}\pm 2\sqrt{-35}}{2\cdot 11} && \pmod{19^{100}},\\
J_{\sqrt{2}}^{+}[\tau_{105}] &\stackrel{?}{=} 1 && \pmod{29^{100}}.
\end{aligned}
$$

When $p = 11$, these are the four roots of $19x^4 - 4x^3 - 21x^2 - 4x + 19$, whereas for $p = 19$ these are the roots of $11x^4 - 12x^3 + 3x^2 - 12x + 11$. Both sets generate distinct fields of degree 4 over $\mathbb{Q}$, and the compositum of either field with $\mathbb{Q}(\sqrt{105})$ is the ring class field of discriminant 105. As in the previous example, notice the linear independence with the field of definition K, and the reciprocity occurring in the denominators, both of which will be discussed in Section 4.2 and 4.4. To conclude the discussion of discriminant 8 cocycles, Table 4.8 below lists the values of $J_{\sqrt{2}}^{+}$ at small integer multiples of $\sqrt{2}$, for all the monstrous primes which are inert in $\mathbb{Q}(\sqrt{2})$.

τ	$p=3$	$p=5$	$p=11$	$p=13$	$p=19$	$p=29$	$p=59$
$2\sqrt{2}$	$\frac{204+253\sqrt{-1}}{5^2\cdot 13}$	$\frac{7-6\sqrt{-2}}{11}$	$\frac{3-4\sqrt{-1}}{5}$	$\frac{1-2\sqrt{-2}}{3}$	1	1	1
$3\sqrt{2}$	$-$	$\frac{11++21\sqrt{-3}}{2\cdot 19}$	$\frac{-1+15\sqrt{-3}}{2\cdot 13}$	$\frac{5+4\sqrt{-6}}{11}$	$\frac{-1+2\sqrt{-6}}{5}$	1	1
$4\sqrt{2}$	$\frac{6063-7216\sqrt{-1}}{5^2\cdot 13\cdot 29}$	$\frac{-31+8\sqrt{-2}}{3\cdot 11}$	$\frac{3+4\sqrt{-1}}{5}$	$\frac{41-28\sqrt{-2}}{3\cdot 19}$	$\frac{5+12\sqrt{-1}}{13}$	$\frac{1+2\sqrt{-2}}{3}$	1
$5\sqrt{2}$	1	$-$	1	1	1	1	1

TABLE 4.8. Some values of the p-adic cocycle $J_{\sqrt{2}}^{+}[n\sqrt{2}]$.

Example 4.9. The real irrationality $\sqrt{3}$ has discriminant 12, its associated narrow ring class field is the biquadratic field $H_{\sqrt{3}} = \mathbb{Q}(\sqrt{3}, \sqrt{-1})$, and it defines an RM point in the standard affinoid of $\mathcal{H}_p$, for any prime $p \equiv 5, 7 \mod 12$. For each such p, one may consider the rigid meromorphic cocycle $J_{\sqrt{3}}^{+}$. This cocycle was computed to a 5-adic accuracy of 5^{200}. Table 4.10 below lists the minimal polynomials of its values at a few τ of small discriminant, as well as the number field defined by these polynomials.

τ	Minimal polynomial of $J^+_{\sqrt{3}}[\tau]$	Field
$\sqrt{2}$	$9x^4 - 36x^3 + 40x^2 + 12x + 9$	$\mathbb{Q}(\zeta_8)$
$\frac{1+\sqrt{13}}{2}$	$2401x^8 + 19404x^7 + 72589x^6 + 166716x^5 + 121944x^4$ $-166716x^3 + 72589x^2 - 19404x + 2401$	$\mathbb{Q}(\sqrt{-1}, \sqrt{3}, \sqrt{13})$
$\frac{1+\sqrt{17}}{2}$	$194481x^8 - 1100736x^7 + 20364174x^6 - 71994624x^5 + 840839779x^4$ $+71994624x^3 + 20364174x^2 + 1100736x + 194481$	$\mathbb{Q}(\sqrt{-1}, \sqrt{3}, \sqrt{17})$

TABLE 4.10. Some RM values $J^+_{\sqrt{3}}[\tau]$, for $p = 5$.

Example 4.11. Now let $\omega_{13} := \frac{1+\sqrt{13}}{2}$ be the RM point of discriminant 13 in $\mathcal{H}_p^{\mathrm{RM}}$ for $p = 5$, 11, 19, and 59, which are monstrous primes which are inert in both $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{2})$. Table 4.12 collects a few values of the cocycles $J^+_{\omega_{12}}$ for those primes, at RM points of the form $\tau = n\sqrt{2}$ for those small values of n for which the order $\mathcal{O}_\tau$ has wide class number 1.

τ	$p = 5$	$p = 11$	$p = 19$	$p = 59$
$\sqrt{2}$	1	1	1	1
$2\sqrt{2}$	$\frac{47+144\sqrt{-2}}{11\cdot 19}$	$\frac{3-4\sqrt{-1}}{5}$	$\frac{3-4\sqrt{-1}}{5}$	1
$3\sqrt{2}$	$\frac{121-551\sqrt{-3}}{2\cdot 13\cdot 37}$	$\frac{11+21\sqrt{-3}}{2\cdot 19}$	$\frac{5-4\sqrt{-6}}{11}$	1
$4\sqrt{2}$	$\frac{2806273-1604736\sqrt{-2}}{11\cdot 59\cdot 67\cdot 83}$	$\frac{57-176\sqrt{-1}}{5\cdot 37}$	$\frac{5-12\sqrt{-1}}{13}$	$\frac{3+4\sqrt{-1}}{5}$
$7\sqrt{2}$	$\frac{13349623871+1962731160\sqrt{-7}}{11^2\cdot 37\cdot 109\cdot 149\cdot 197}$	$\frac{118393-8328\sqrt{-14}}{5^2\cdot 59\cdot 83}$	$\frac{93+95\sqrt{-7}}{2^2\cdot 67}$	$\frac{37+9\sqrt{-7}}{2^2\cdot 11}$
$8\sqrt{2}$	$\frac{1920792095831+651036999168\sqrt{-2}}{11^3\cdot 19^2\cdot 59\cdot 227\cdot 331}$	$\frac{1312-1425\sqrt{-1}}{13\cdot 149}$	$\frac{43+924\sqrt{-1}}{5^2\cdot 37}$	$\frac{3+4\sqrt{-1}}{5}$
$9\sqrt{2}$	$\frac{1012867083636287+3520320389376383\sqrt{-3}}{2\cdot 13^2\cdot 37^2\cdot 229\cdot 349\cdot 397\cdot 421}$	$\frac{11387+12320\sqrt{-3}}{19^2\cdot 67}$	$\frac{43+4100\sqrt{-6}}{11^2\cdot 83}$	1
$11\sqrt{2}$	$\frac{189808743946255480996 9+25021359226682861760\sqrt{-22}}{13\cdot 19^2\cdot 109\cdot 149\cdot 293\cdot 461\cdot 541\cdot 557\cdot 613}$	–	$\frac{209711-130467\sqrt{-11}}{2\cdot 5^2\cdot 59\cdot 163}$	$\frac{3+4\sqrt{-22}}{19}$

TABLE 4.12. The values of the p-adic cocycle $J^+_{\omega_{13}}[n\sqrt{2}]$ for $1 \le n \le 11$.

4.2. **The Shimura reciprocity law.** We begin by briefly recalling the classical Shimura reciprocity law in the setting of the theory of complex multiplication. Let $D < 0$ be a negative discriminant and let $H/\mathbb{Q}$ be the associated ring class field of $K = \mathbb{Q}(\sqrt{D})$, whose Galois group canonically splits as the semi-direct product:

$$\mathrm{Gal}(H/\mathbb{Q}) \simeq \mathrm{Gal}\,(H/K) \rtimes \langle \mathrm{Fr}_\infty \rangle = \mathrm{Cl}(D) \rtimes \langle \mathrm{Fr}_\infty \rangle, \tag{74}$$

where $\mathrm{Cl}(D)$ is the class group of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of positive definite binary quadratic forms of discriminant D, equipped with the usual Gaussian composition, and the identifications

$$\mathrm{rec} : \mathrm{Cl}(D) \longrightarrow \mathrm{Gal}\,(H/K), \qquad \mathrm{rec} : \mathrm{Cl}(D) \rtimes \langle \mathrm{Fr}_\infty \rangle \longrightarrow \mathrm{Gal}\,(H/K) \rtimes \langle \mathrm{Fr}_\infty \rangle$$

arises from global class field theory. There is a canonical bijection between $\mathrm{Cl}(D) \rtimes \langle \mathrm{Fr}_\infty \rangle$ and the set of $\mathrm{SL}_2(\mathbb{Z})$-orbits of CM points of discriminant D on the union of the upper and lower half planes $\mathcal{H}^\pm$, defined by

$$g := [a, b, c] \cdot \mathrm{Fr}_\infty^\delta \longmapsto \tau_g := (-1)^\delta \left(\frac{-b + \sqrt{D}}{2a} \right), \tag{75}$$

where $[a, b, c]$ denotes the class of the binary quadratic form $ax^2 + bxy + cy^2$.

Let J be a meromorphic modular function on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ with fourier expansion coefficients in a field H_J, extended to a meromorphic function on the union $\mathcal{H}^\pm$ of upper and lower complex upper half-planes by requiring $J(-z) = J(z)$. If τ is any CM point for which H_τ is linearly disjoint from H_J, then restriction of automorphisms induces isomorphisms

$$G_D := \mathrm{Gal}\,(H_J H_\tau/H_J) = \mathrm{Gal}\,(H_\tau/\mathbb{Q}) \stackrel{\mathrm{rec}}{\longleftarrow} \mathrm{Cl}(D) \rtimes \langle \mathrm{Fr}_\infty \rangle. \tag{76}$$

The Shimura reciprocity law can then be stated as

$$J(\tau_{gh}) = J(\tau_h)^{\mathrm{rec}(g)^{-1}}, \qquad \text{for all } g, h \in \mathrm{Cl}(D) \rtimes \langle \mathrm{Fr}_\infty \rangle. \tag{77}$$

Turning to the RM setting, let $D > 0$ be a discriminant for which p is inert, and let $H/\mathbb{Q}$ be the ring class field associated to D, whose Galois group can be described as a semi-direct product via the formula, which is almost identical to (74):

$$\mathrm{Gal}\,(H/\mathbb{Q}) \simeq \mathrm{Gal}\,(H/K) \rtimes \langle \mathrm{Fr}_p \rangle = \mathrm{Cl}(D) \rtimes \langle \mathrm{Fr}_p \rangle. \tag{78}$$

the latter identification arising, as before, from the isomorphism rec of global class field theory. As in (75), there is a canonical bijection between $\mathrm{cl}(D) \rtimes \langle \mathrm{Fr}_p \rangle$ and the set of Γ-orbits of RM points of discriminant D on $\mathcal{H}_p$, defined by

$$g := [a, b, c] \cdot \mathrm{Fr}_p^\delta \longmapsto \tau_g := p^\delta \left(\frac{-b + \sqrt{D}}{2a} \right). \tag{79}$$

Recall that by Conjecture 4.4 the RM values $J[\tau]$ of a rigid meromorphic cocycle J should be algebraic, contained in the compositum of the field of definition H_J of J and the ring class field H_τ of τ. If these two fields are linearly disjoint, then one has the same identifications as in (76):

$$\mathrm{Gal}\,(H_J H_\tau/H_J) = \mathrm{Gal}\,(H_\tau/\mathbb{Q}) = \mathrm{Cl}(D) \rtimes \langle \mathrm{Fr}_p \rangle.$$

The conjectural Shimura reciprocity law is the statement:

Conjecture 4.13. *For all $g \in \mathrm{Cl}(D) \rtimes \langle \mathrm{Fr}_p \rangle$ as as above,*

$$J[\tau_{gh}] = J[\tau_h]^{\mathrm{rec}(g)^{-1}}.$$

We now present a number of examples that lend credence to this conjecture.

Example 4.14. Let φ be the golden ratio and let $\tau_1, \ldots, \tau_6$ be the roots of the narrow equivalence classes of binary quadratic forms of discriminant 321, which has narrow class number 6. The monstrous prime 23 is inert in both the real quadratic fields $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{321})$. Let $J_\varphi^+ \in \mathrm{H}^1_f(\Gamma, \mathcal{M}^\times)$ be the rigid meromorphic cocycle attached to φ as in Theorem 3.20. Since J_φ^+ is p-even, its values on the RM points τ and $p\tau$ coincide. The Shimura reciprocity conjecture therefore predicts that the 6 values $J[\tau_j]$ lie in the Hilbert class field of $\mathbb{Q}(\sqrt{321})$ and are permuted by $\mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$. Using the algorithms described in Section 3.5, we have verified that the values $J_\varphi^+[\tau_j]$ for $j = 1, \ldots, 6$ agree with the distinct roots of the polynomial

$$63x^6 - 6x^5 + x^4 + 76x^3 + x^2 - 6x + 63 = 0,$$

to within fifty 23-adic digits. The roots of this polynomial generate the Hilbert class field of $\mathbb{Q}(\sqrt{321})$.

Example 4.15. The discriminants $D_1 = 13$ and $D_2 = 621 = 3^2 \cdot 69$ have narrow class numbers 1 and 6 respectively. The prime $p = 71$ is inert in both real quadratic fields, and it is the largest prime factor of the order of the Monster group. The values of the cocycle $J_{\omega_{13}}^+$, where

ω_{13} is an RM point of discriminant 13, were computed on the six RM points of discriminant 621, and ostensibly (namely, modulo 71^{30}) give the distinct roots of the polynomial

$$7x^6 + 6x^5 + 6x^4 + 10x^3 + 6x^2 + 6x + 7 = 0,$$

whose splitting field is the ring class field of conductor 3 of $\mathbb{Q}(\sqrt{69})$.

4.3. **p-adic intersection numbers.** Let $p \in \{2, 3, 5, 7, 13\}$ be a genus zero prime and let τ_1 and τ_2 be two RM points of $\mathcal{H}_p$ with discriminants D_1 and D_2 respectively.

Definition 4.16. The *p-adic intersection number* of τ_1 and τ_2 is the quantity

$$J_p(\tau_1, \tau_2) := \hat{J}_{\tau_1}[\tau_2] \in \mathbb{C}_p^\times / \langle \varepsilon_{\tau_1}^{\mathbb{Z}} \rangle,$$

where $\hat{J}_{\tau_1} \in H^1_f(\Gamma, \mathcal{M}^\times / \varepsilon_1^{\mathbb{Z}})$ is the rigid meromorphic cocycle of Theorem 3.18.

The following proposition summarises a few of the basic properties of the p-adic intersection number.

Proposition 4.17. *The invariants $J_p(\tau_1, \tau_2)$ satisfy:*
(1) $J_p(-\tau_1, -\tau_2) = J_p(\tau_1, \tau_2)^{-1} \pmod{\varepsilon_1^{\mathbb{Z}}}$;
(2) $J_p(p\tau_1, p\tau_2) = J_p(\tau_1, \tau_2) \pmod{\varepsilon_1^{\mathbb{Z}}}$.

Proof. To show the first part, let D be the diagonal matrix with entries 1 and -1. This matrix normalises Γ and hence the cocycle $\hat{J}'_{\tau_1}$ determined by

$$\hat{J}'_{\tau_1}(\gamma)(z) := \hat{J}_{-\tau_1}(D\gamma D^{-1})(-z)$$

belongs to $H^1_f(\Gamma, \mathcal{M}^\times / \varepsilon_1^{\mathbb{Z}})$. A direct calculation shows that

$$\operatorname{dlog} \hat{J}'_{\tau_1} = -\operatorname{dlog} \hat{J}_{\tau_1}.$$

It follows from the uniqueness of the rigid meromorphic cocycle $\hat{J}_{\tau_1}$ that

$$\hat{J}'_{\tau_1} = \hat{J}_{\tau_1}^{-1} \pmod{\varepsilon_1^{\mathbb{Z}}},$$

and hence, evaluating at τ_2, that

$$\hat{J}_{-\tau_1}[-\tau_2] = \hat{J}_{\tau_1}[\tau_2]^{-1} \pmod{\varepsilon_1^{\mathbb{Z}}}.$$

The first part of the proposition follows. The second assertion is proved by a similar reasoning and is left to the reader. $\square$

Remark 4.18. Proposition 3.5 suggests that the invariant J_p satisfies the antisymmetry

$$J_p(\tau_1, \tau_2) = J_p(\tau_2, \tau_1)^{-1} \pmod{\langle \varepsilon_1^{\mathbb{Z}}, \varepsilon_2^{\mathbb{Z}} \rangle},$$

which indeed is verified on numerous examples.

Since $\hat{J}_{\tau_1}$ is not quite a rigid meromorphic cocycle but only a cocycle "modulo $\mathcal{O}_{K_1}^\times$", it falls slightly outside the purview of the conjectures formulated in the previous two sections. Nonetheless, we conjecture that it satisfies a natural extension of the Shimura reciprocity law, which we only state, for simplicity, in the case where the discriminants D_1 and D_2 are relatively prime, so that the associated ring class fields H_1 and H_2 are linearly disjoint over $\mathbb{Q}$. As before, let rec denote the reciprocity map of global class field theory:

$$G_{D_1,D_2} := (\mathrm{Cl}(D_1) \rtimes \langle \mathrm{Fr}_\infty \rangle) \times (\mathrm{Cl}(D_2) \rtimes \langle \mathrm{Fr}_\infty \rangle) \xrightarrow{\mathrm{rec}} \mathrm{Gal}\,(H_1/\mathbb{Q}) \times \mathrm{Gal}\,(H_2/\mathbb{Q}) = \mathrm{Gal}\,(H_{12}/\mathbb{Q}).$$

Conjecture 4.19 (Shimura reciprocity)**.** *Let $h = (h_1, h_2)$ be any element of G_{D_1,D_2}. Then $J_p(\tau_{h_1}, \tau_{h_2})$ belongs to H_{12} and, for all $g = (g_1, g_2) \in G_{D_1,D_2}$,*

$$J_p(\tau_{g_1 h_1}, \tau_{g_2 h_2}) = J_p(\tau_{h_1}, \tau_{h_2})^{\mathrm{rec}(g)^{-1}} \pmod{\varepsilon_1^{\mathbb{Z}}}.$$

Example 4.20. Let $(D_1, D_2) = (5, 32)$ which have class numbers 1 and 2 respectively. As previously, let φ denote the golden ratio. We computed the quantities $J_3(\varphi, 2\sqrt{2})$ and $J_3(\varphi, -2\sqrt{2})$ to 800 3-adic digits, obtaining

$$J_3(\varphi, 2\sqrt{2}) \stackrel{?}{=} (-70 + 35\sqrt{5} - 40\sqrt{2} + 40\sqrt{-1} + 16\sqrt{10} - 20\sqrt{-5} - 70\sqrt{-2} + 28\sqrt{-10})/65$$
$$J_3(\varphi, -2\sqrt{2}) \stackrel{?}{=} (-70 + 35\sqrt{5} - 40\sqrt{2} - 40\sqrt{-1} + 16\sqrt{10} + 20\sqrt{-5} + 70\sqrt{-2} - 28\sqrt{-10})/65$$

modulo $\varepsilon_1^{\mathbb{Z}}$. Both these values lie in H_{12}, and the Shimura reciprocity law even predicts that they are complex conjugates of each other in this case, which is clearly seen to be satisfied.

Remark 4.21. The Shimura reciprocity law combined with the properties of $J_p(\tau_1, \tau_2)$ stated in Proposition 4.17 imply certain restrictions on the Galois-theoretic behaviour of these intersection numbers. For instance, the Shimura reciprocity law implies that $J_p(\tau_1, \tau_2)$ and $J_p(-\tau_1, -\tau_2)$ are complex conjugates of each other relative to any complex embedding of H_{12}. (Indeed, the complex conjugation is independent of the choice of complex embedding since Fr_∞ is a well defined central involution in $\mathrm{Gal}\,(H_{12}/\mathbb{Q})$). It then follows from Part 1 of Proposition 4.17 that the $J_p(\tau_1, \tau_2)$ are complex numbers of norm 1 relative to any complex embedding of H_{12}, i.e., that

$$J_p(\tau_1, \tau_2)^{\mathrm{Fr}_\infty} = J_p(\tau_1, \tau_2)^{-1}.$$

In particular, the p-adic intersection number is forced to be trivial whenever H_1 and H_2 are both totally real, which occurs when the class numbers in the wide and narrow sense agree for both D_1 and D_2.

4.4. **Gross-Zagier style factorisations.** The goal of this section is to propose a conjectural recipe for the prime factorisations of the p-adic intersection numbers $J_p(\tau_1, \tau_2)$, modelled on the analogous recipe in [GZ1] for the factorisation of $J_\infty(\tau_1, \tau_2)$ when τ_1 and τ_2 are CM points on the complex upper half plane.

We begin by recalling the latter, in a form that best lends itself to an extension to the real quadratic setting. If τ_1 and τ_2 are CM points of $\mathcal{H}$ with associated ring class fields H_1 and H_2, the quantity $J_\infty(\tau_1, \tau_2)$ belongs to the compositum $H_{12} = H_1H_2$. It will be assumed that complex and q-adic embeddings of H_{12} for all primes q have been fixed at the outset, so that one can speak of the normalised valuation at q of $J_\infty(\tau_1, \tau_2)$ for any rational prime q.

Let q be such a prime and let B be the definite quaternion algebra ramified at q and ∞. A *q-oriented maximal order* in B is a maximal order $R \subset B$ equipped with a surjective homomorphism $\iota : R \longrightarrow \mathbb{F}_{q^2}$ called the "orientation at q". Likewise, a q-oriented quadratic order is a quadratic order $\mathcal{O}$ equipped with a similar structure. An orientation at q in this sense exists if and only if q does not divide the conductor of $\mathcal{O}$ and is inert in its fraction field. The q-oriented orders form a category in which the morphisms from (R_1, ι_1) to (R_2, ι_2) are ring homomorphisms $\varphi : R_1 \longrightarrow R_2$ satisfying $\iota_2\varphi = \iota_1$.

Definition 4.22. A *q-oriented optimal embedding* of a q-oriented quadratic order $\mathcal{O} \subset K$ into B is a pair (φ, R), where $\varphi : K \longrightarrow B$ is an algebra homomorphism and R is a maximal q-oriented order in B, satisfying $\varphi(K) \cap R = \varphi(\mathcal{O})$, and for which φ is compatible with the q-orientations on $\mathcal{O}$ and on R.

Write $\mathrm{Emb}(\mathcal{O}, B)$ for the set of oriented optimal embeddings of $\mathcal{O}$ into B. The multiplicative group $B^\times$ acts on this set by the rule

$$b \star (\varphi, R) := (b\varphi b^{-1}, bRb^{-1}),$$

and the set of $B^\times$-orbits for this action is denoted $\Sigma(\mathcal{O}, B)$. Letting D be the discriminant of $\mathcal{O}$, the class group $\mathrm{Cl}(D)$ of that discriminant acts naturally on $\Sigma(\mathcal{O}, B)$ by setting, for any projective $\mathcal{O}$-module $\mathfrak{a} \subset K$:

(80) $\quad \mathfrak{a} \star (\varphi, R) = (\varphi, R'), \qquad$ where $R' := \{b \in B \text{ s.t. } \varphi(a^{-1}) b \varphi(a) \in R \text{ for all } a \in \mathfrak{a}\}$.

Recall the set $\mathcal{H}^D$ of CM points on the complex upper half plane of discriminant D, and let H be the associated ring class field. The quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^D$ is also equipped with a simply transitive action of $\mathrm{Cl}(D)$, which is compatible with the action of $\mathrm{Gal}\,(H/K)$ on the singular moduli $j(\tau)$ via the reciprocity map of global class field theory.

Lemma 4.23. *The choice of complex and q-adic embeddings of H determines a canonical bijection*

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^D \longrightarrow \Sigma(\mathcal{O}, B)$$

which is compatible with the simply transitive actions of $\mathrm{Cl}(D)$ *on both sides.*

Proof. Of crucial importance in constructing this canonical bijection is the fact that for all $\tau \in \mathcal{H}^D$, the complex number $j(\tau)$ (which can be viewed as an element of H via the chosen embedding of H into $\mathbb{C}$) is the j-invariant of an elliptic curve E/H with complex multiplication, admitting a smooth integral model over $\mathcal{O}_H[1/D]$ and equipped with a canonical identification $\mathcal{O} = \mathrm{End}_H(E)$, in which $\lambda \in \mathcal{O}$ is sent to the unique endomorphism of E acting as multiplication by λ on its cotangent space. Since q is inert in K, the unique prime of K that lies above q splits completely in H/K. Hence $j(\tau)$ can be viewed (via our chosen q-adic embedding of H) as an element of the unramified quadratic extension C_q of $\mathbb{Q}_q$, with residue field $\mathbb{F}_{q^2}$. Let $\bar{E}$ denote the special fiber of E over the residue field $\mathbb{F}_{q^2}$. It is a *supersingular* elliptic curve, whose endomorphism ring is isomorphic to a maximal order R in the quaternion algebra B ramified at q and ∞, equipped with a q-orientation describing the action of endomorphisms on the cotangent space of $\bar{E}$. The quadratic order $\mathcal{O} \subset \mathrm{End}(\bar{E})$ is equipped with a q-orientation for the same reason. To any $\tau \in \mathcal{H}^D$ one can thus associate an optimal embedding $\varphi_\tau : \mathcal{O} \longrightarrow R$ of q-oriented orders by taking the composition

$$\varphi_\tau : \mathcal{O} = \mathrm{End}(E) \hookrightarrow \mathrm{End}(\bar{E}) \simeq R.$$

The order R is well defined up to conjugation in $B^\times$, and hence the image of the pair (φ_τ, R) in $\Sigma(\mathcal{O}, B)$ is well-defined. The lemma follows. ☐

The *intersection multiplicity at q* of two elements $(\varphi_1, R_1) \in \mathrm{Emb}(\mathcal{O}_1, B)$ and $(\varphi_2, R_2) \in \mathrm{Emb}(\mathcal{O}_2, B)$ is defined by setting $[\varphi_1 \cdot \varphi_2]_q := 0$ if $R_1 \neq R_2$ (as q-oriented orders) and, if $R_1 = R_2 =: R$, setting

(81) $\quad [\varphi_1 \cdot \varphi_2]_q := \ \max t \geq 1 \text{ s.t. } \varphi_1(\mathcal{O}),\ \varphi_2(\mathcal{O}_2) \text{ have the same image in } R/q^{t-1}R.$

This definition can be extended to the classes in $\Sigma(\mathcal{O}_1, B)$ and $\Sigma(\mathcal{O}_2, B)$ represented by (φ_1, R_1) and (φ_2, R_2) respectively, by setting

(82)
$$(\varphi_1 \cdot \varphi_2)_q := \sum_{b \in B_1^\times} [\varphi_1 \cdot b\varphi_2 b^{-1}]_q.$$

Observe that all but finitely many of the terms in the above sum are 0, because the normaliser of a given maximal oriented order R in $B_1^\times$ is equal to $R^\times$, which (since B is a definite quaternion algebra) is a discrete subgroup of a compact Lie group and hence is finite.

Theorem 4.24 (Gross-Zagier)**.** *Let $\tau_1 \in \mathcal{H}^{D_1}$ and $\tau_2 \in \mathcal{H}^{D_2}$ be CM points, and let $q \nmid D_1 D_2$ be a rational prime. If q is split in either K_1 or K_2, then* $\mathrm{ord}_q\, J_\infty(\tau_1, \tau_2) = 0$. *Otherwise, let $\varphi_1 \in \Sigma(\mathcal{O}_1, B)$ and $\varphi_2 \in \Sigma(\mathcal{O}_2, B)$ be the classes of q-oriented optimal embeddings associated to τ_1 and τ_2 respectively via Lemma 4.23. Then*

$$\mathrm{ord}_q\, J_\infty(\tau_1, \tau_2) = (\varphi_1 \cdot \varphi_2)_q.$$

Let us now turn to the factorisation of $J_p(\tau_1, \tau_2)$ where τ_1 and τ_2 are RM points of $\mathcal{H}_p$. Assume for simplicity that p is *inert* in the real quadratic fields $K_1 = \mathbb{Q}(\tau_1)$ and $K_2 = \mathbb{Q}(\tau_2)$.

In contrast with the study of $\mathrm{ord}_q J_p(\tau_1, \tau_2)$ for $q \neq p$, which is at least as deep as the assertion that $J_p(\tau_1, \tau_2)$ is algebraic, the calculation of $\mathrm{ord}_p J_p(\tau_1, \tau_2)$ is entirely elementary and turns out to be instructive. We will therefore start with a formula for this valuation, which can be phrased in terms of embeddings of the real quadratic orders attached to τ_1 and τ_2 in the maximal order $R = M_2(\mathbb{Z})$ of the global split quaternion algebra $B = M_2(\mathbb{Q})$. The RM points τ_1 and τ_2 of discriminants D_1 and D_2 (which are prime to p by definition) have associated $\mathbb{Z}[1/p]$-orders of the form $\mathcal{O}_1[1/p]$ and $\mathcal{O}_2[1/p]$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are the orders of discriminant D_1 and D_2 respectively. These points thus give rise to optimal embeddings of $\mathbb{Z}[1/p]$-orders

$$\varphi_1 : \mathcal{O}_1[1/p] \longrightarrow R[1/p], \qquad \varphi_2 : \mathcal{O}_2[1/p] \longrightarrow R[1/p],$$

where $R := M_2(\mathbb{Z})$ is the standard maximal order of $M_2(\mathbb{Q})$, which is conjugate to any other maximal order. If τ_1 and τ_2 reduce to distinct vertices of $\mathcal{T}$, then we set

$$[\varphi_1, \varphi_2]_p = 0.$$

Otherwise, we will assume without loss of generality that τ_1 and τ_2 both reduce to the standard vertex of $\mathcal{H}_p$, so that they induce a pair of optimal embeddings

$$\varphi_1 : \mathcal{O}_1 \longrightarrow R, \qquad \varphi_2 : \mathcal{O}_2 \longrightarrow R.$$

Consider now the classes in $\Sigma(\mathcal{O}_1, R)$ and $\Sigma(\mathcal{O}_2, R)$ represented by these oriented optimal embeddings, and recall the p-weighted intersection multiplicity $(\varphi_1 \cdot \varphi_2)_{p\infty}$ of (51) in Definition (3.6). The valuation at p of $J_p(\tau_1, \tau_2)$ is intimately connected to this quantity:

Theorem 4.25. *Let τ_1 and τ_2 be RM points on $\mathcal{H}_p$ with associated quadratic orders $\mathcal{O}_1$ and $\mathcal{O}_2$, attached to classes of optimal embeddings $\varphi_1 \in \Sigma(\mathcal{O}_1, R)$ and $\varphi_2 \in \Sigma(\mathcal{O}_2, R)$. Then*

$$\mathrm{ord}_p J_p(\tau_1, \tau_2) = (\varphi_1 \cdot \varphi_2)_{p\infty}.$$

Proof. By definition,

$$J_p(\tau_1, \tau_2) = \hat{J}_{\tau_1}[\tau_2] = \hat{J}_{\tau_1}\{r, \gamma_2 r\}(\tau_2).$$

Furthermore,

$$\hat{J}_{\tau_1}\{r, \gamma_2 r\}(\tau_2) = \prod_{w_1 \in \Sigma_{\tau_1}(r, \gamma_2 r)} t_{w_1}(\tau_2) \pmod{\mathcal{O}_{\mathbb{C}_p}^\times}.$$

However, one can observe that

$$\mathrm{ord}_p t_{w_1}(\tau_2) = 0 \quad \text{if } \mathrm{level}(w_1) \neq 0.$$

It follows that

$$\mathrm{ord}_p \hat{J}_{\tau_1}\{r, \gamma_2 r\}(\tau_2) = \mathrm{ord}_p \prod_{w_1 \in \Sigma^\circ_{\tau_1}(r, \gamma_2 r)} t_{w_1}(\tau_2) = \mathrm{ord}_p J^\circ_{\tau_1}[\tau_2] = \mathrm{ord}_p J^\circ(\tau_1, \tau_2).$$

The theorem now follows from Proposition 3.7. □

We now turn to the (conjectural!) arithmetic intersection number of $J_p(\tau_1, \tau_2)$ at a rational prime $q \neq p$. For simplicity, it will be assumed also that $q \nmid D_1 D_2$. The recipe for the q-adic valuation of this p-adic invariant involves the quaternion algebra B ramified at q and p. Because B is an indefinite quaternion algebra, all the maximal orders in B are conjugate to each other. Let R be a fixed choice of maximal order, and fix an identification of $B \otimes \mathbb{R}$ with $M_2(\mathbb{R})$. Via this identification, the multiplicative group $R_1^\times$ acts discretely and co-compactly on $\mathcal{H}$, and the compact Riemann surface $R_1^\times \backslash \mathcal{H}$ is identified with the set of complex points of the *Shimura curve of level pq*.

Just as before, the sets $\Sigma(\mathcal{O}_1, R)$ and $\Sigma(\mathcal{O}_2, R)$ of $R_1^\times$-conjugacy classes of oriented optimal embeddings are equipped with natural fixed-point-free actions of the class groups $\mathrm{Cl}(D_1)$ and $\mathrm{Cl}(D_2)$ respectively, and have the same cardinality as $\mathcal{H}_p^{D_1}$ and $\mathcal{H}_p^{D_2}$ respectively. Hence, one can fix bijections

$$\Gamma\backslash\mathcal{H}_p^{D_1} \xrightarrow{\sim} \Sigma(\mathcal{O}_1, R), \qquad \Gamma\backslash\mathcal{H}_p^{D_2} \xrightarrow{\sim} \Sigma(\mathcal{O}_2, R) \tag{83}$$

which are compatible with the actions of $\mathrm{Cl}(D_1)$ and $\mathrm{Cl}(D_2)$ on both sides. Given $\tau_1 \in \mathcal{H}_p^{D_1}$ and $\tau_2 \in \mathcal{H}_p^{D_2}$, let φ_1 and φ_2 be the optimal embeddings associated to τ_1 and τ_2 under these bijections, and let γ_1 and $\gamma_2 \in R_1^\times$ be the images of the fundamental units of $\mathcal{O}_1^\times$ and $\mathcal{O}_2^\times$ under φ_1 and φ_2. The q-weighted intersection number of φ_1 and φ_2 is defined to be

$$(\varphi_1 \cdot \varphi_2)_{q\infty} := \sum_{\gamma \in \gamma_2^{\mathbb{Z}} \backslash R_1^\times / \gamma_1^{\mathbb{Z}}} [\gamma\varphi_1\gamma^{-1} \cdot \varphi_2]_q \cdot \delta(\gamma\tau_1, \tau_2),$$

where the symbol $[\varphi_1 \cdot \varphi_2]_q$ is defined exactly as in (81), and the remaining terms in the expression are otherwise exactly as in Definition 3.6, with $M_2(\mathbb{Z})$ replaced by R.

The following conjecture proposes a formula for $\mathrm{ord}_q J_p(\tau_1, \tau_2)$ involving a synthesis of Theorems 4.24 and 4.25:

Conjecture 4.26. *If q is split in either K_1 or K_2, then* $\mathrm{ord}_q J_p(\tau_1, \tau_2) = 0$. *Otherwise, there is an embedding of H_{12} into $\bar{\mathbb{Q}}_q$ for which*

$$\mathrm{ord}_q J_p(\tau_1, \tau_2) = (\varphi_1 \cdot \varphi_2)_{q\infty},$$

for all $\tau_1 \in \mathcal{H}_p^{D_1}$ and all $\tau_2 \in \mathcal{H}_p^{D_2}$.

Remark 4.27. In the proof of Lemma 4.23 before the statement of Theorem 4.24, we were able to give a precise recipe for the assignment $\tau_1 \mapsto \varphi_1$ and $\tau_2 \mapsto \varphi_2$, which depended on a choice of complex and q-adic embeddings of H_{12}, by relying on the theory of CM elliptic curves and their supersingular reductions at the primes above q. These arithmetic ingredients are (at least for the time being) conspicuously absent in the RM setting, and one must therefore be content with a slightly vaguer formulation, in which the bijections (83) of transitive $\mathrm{Cl}(D_1)$ and $\mathrm{Cl}(D_2)$-sets, and their dependence on choices of p-adic and q-adic embeddings of H_{12}, are not spelled out.

We now give a sampling of the experimental evidence that has been gathered in support of Conjecture 4.26. James Rickards has devised efficient algorithms for calculating the q-weighted topological intersection numbers $(\varphi_1 \cdot \varphi_2)_{q\infty}$ on the Shimura curve of discriminant pq, and has implemented them on the computer. Rickards' programs have generated a wealth of data on q-weighted intersection numbers, running to over 600 pages of tables, which have been invaluable in verifying Conjecture 4.26. The examples below are but a small sample of the experiments that were carried out in support of Conjecture 4.26.

Given pairwise coprime positive discriminants D_1 and D_2 which are non-squares modulo p, let $G_{12} := \mathrm{Cl}(D_1) \times \mathrm{Cl}(D_2)$. For each prime q that is non-split in both K_1 and K_2, Rickards defines elements of the integral group ring $\mathbb{Z}[G_{12}]$ by choosing base points $\varphi_1 \in \mathrm{Emb}(\mathcal{O}_1, R)$ and $\varphi_2 \in \mathrm{Emb}(\mathcal{O}_2, R)$, letting φ_1' be the embedding obtained from φ_1 by conjugating it by an element of norm p, and considering the following sums over $g = (g_1, g_2) \in G_{12}$, viewed as elements of the integral group ring of G_{12}:

$$I_{p,q}(D_1, D_2) = \sum_{g \in G_{12}} (\varphi_1^{g_1} \cdot \varphi_2^{g_2})_{q\infty} \cdot g,$$

$$I'_{p,q}(D_1, D_2) = \sum_{g \in G_{12}} (\varphi_1'^{g_1} \cdot \varphi_2^{g_2})_{q\infty} \cdot g.$$

Instead of directly identifying the quantity $J_p(\tau_1,\tau_2)$ as algebraic numbers, it has turned out to be easier to work with the related quantities

$$\begin{aligned} J_p^+(\tau_1,\tau_2) &= J_p(\tau_1,\tau_2) \div J_p(p\tau_1,\tau_2) = J_{\tau_1}^+[\tau_2], \\ J_p^-(\tau_1,\tau_2) &= J_p(\tau_1,\tau_2) \times J_p(p\tau_1,\tau_2), \end{aligned}$$

which are predicted to lie in slightly smaller field extension of $\mathbb{Q}$. A refinement of Conjecture 4.26 (combined with the Shimura reciprocity conjecture) predicts that, after fixing a prime $\mathfrak{q}$ of H_{12} above q, and setting

$$I_{p,q}^+(D_1,D_2) = I_{p,q}(D_1,D_2) + I'_{p,q}(D_1,D_2), \qquad I_{p,q}^-(D_1,D_2) = I_{p,q}(D_1,D_2) - I'_{p,q}(D_1,D_2),$$

one must have

$$\sum_{g\in G_{12}} \mathrm{ord}_{\mathfrak{q}^g}(J_p^{\pm}(\tau_1,\tau_2))\cdot g = I_{p,q}^{\pm}(D_1,D_2) \pmod{G_{12}}, \tag{84}$$

where the equality in (84) is to be interpreted in the group ring $\mathbb{Z}[G_{12}]$ modulo the multiplication by group-like elements in G_{12}. The coefficients appearing in the group ring element on the left of (84) can be computed from the slopes of the Newton polygon at q of the polynomials in $\mathbb{Z}[x]$ satisfied by $J_p^+(\tau_1,\tau_2)$ and $J_p^-(\tau_1,\tau_2)$ respectively. Our experiments have largely consisted in comparing these Newton slopes with the coefficients that appear in Rickard's group ring elements $I_{p,q}^+(D_1,D_2)$ and $I_{p,q}^-(D_1,D_2)$. The fact that we have consistently obtained a perfect match in hundreds of experiments can be viewed as convincing empirical evidence for Conjecture 4.26.

Example 4.28. Let $(D_1,D_2)=(5,473)$ and $p=13$. The RM values with discriminant 473 of J_φ^+ coincide up to 100 digits of 13-adic precision with the roots of the polynomial

$$4995x^6 - 4141x^5 - 1570x^4 + 1443x^3 - 1570x^2 - 4141x + 4995,$$

whereas those of $J_\varphi \times J_{13\varphi}$ satisfy, up to the same precision, the polynomial

$$999x^6 - 2933x^5 + 3361x^4 - 2829x^3 + 3361x^2 - 2933x + 999.$$

We have that $\mathrm{Gal}\,(H_{473}/\mathbb{Q}) \simeq \langle g\rangle \rtimes \langle \mathrm{Fr}_2\rangle$, where g is of order 6. The following table lists the non-trivial intersection numbers computed by James Rickards, as encoded in the group ring elements $I_{q,13}(5,473)$ and $I'_{q,13}(5,473)$, alongside the non-trivial Newton slopes of the ostensibly algebraic numbers $J_{13}^{\pm}(\tau_1,\tau_2(j))$ for $1\le j\le 6$.

q	$I_{q,13}(5,473)$	$I'_{q,13}(5,473)$	$\mathrm{ord}_q\, J_{13}^+(\tau_1,\tau_2^{(j)})$	$\mathrm{ord}_q\, J_{13}^-(\tau_1,\tau_2^{(j)})$
3	$3(1-g^3)$	0	$\mathbf{3}_1, \mathbf{0}_4, -\mathbf{3}_1$	$\mathbf{3}_1, \mathbf{0}_4, -\mathbf{3}_1$
5	$(1-g^3)$	$(1-g^3)$	$\mathbf{2}_1, \mathbf{0}_4, -\mathbf{2}_1$	$\mathbf{0}_6$
37	$(1-g^3)$	0	$\mathbf{1}_1, \mathbf{0}_4, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_4, -\mathbf{1}_1$

As predicted by Conjecture 4.26, the last two columns are the multisets of coefficients appearing in the sum and difference of the group ring elements in the first two columns.

Example 4.29. Let $(D_1,D_2)=(13,621)$ and $p=7$. We have that $\mathrm{Gal}\,(H_{621}/\mathbb{Q}) \simeq \langle g\rangle \rtimes \langle \mathrm{Fr}_7\rangle$, where g is of order 6. The element g^3 corresponde to complex conjugation in $\mathrm{Gal}\,(H_{621}/\mathbb{Q})$. There is a unique $\tau_1 \in \Gamma\backslash\mathcal{H}_7^{13}$, and there are six RM points $\tau_2^{(1)},\ldots\tau_2^{(6)} \in \mathcal{H}_7^{621}$. The resulting invariants $J_7^+(\tau_1,\tau_2^{(j)})$ coincide up to 200 digits of 7-adic precision with the roots of the polynomial

$$4378144x^6 - 5762700x^5 + 9490680x^4 - 11616641x^3 + 9490680x^2 - 5762700x + 4378144.$$

We compute furthermore that the invariants $J_7^-(\tau_1, \tau_2^{(j)})$ satisfy, up to the same precision, the polynomial

$$17932877824x^6 + 69949203456x^5 + 143523182304x^4 + 177833888503x^3 \\ +143523182304x^2 + 69949203456x + 17932877824.$$

The following table shows all the the non-trivial intersection numbers computed by James Rickards, followed by the non-trivial Newton slopes for these two polynomials.

q	$I_{q,7}(13, 621)$	$I'_{q,7}(13, 621)$	$\mathrm{ord}_q\, J_7^+(\tau_1, \tau_2^{(j)})$	$\mathrm{ord}_q\, J_7^-(\tau_1, \tau_2^{(j)})$
2	$(1-g^3)(2+5g+2g^2)$	$(1-g^3)(-3-2g-3g^2)$	$\mathbf{3}_1, \mathbf{1}_2, -\mathbf{1}_2, -\mathbf{3}_1$	$\mathbf{7}_1, \mathbf{5}_2, -\mathbf{5}_2, -\mathbf{7}_1$
41	$(1-g^3)$	0	$\mathbf{1}_1, \mathbf{0}_4, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_4, -\mathbf{1}_1$
47	$(1-g^3)$	0	$\mathbf{1}_1, \mathbf{0}_4, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_4, -\mathbf{1}_1$
71	$(1-g^3)$	0	$\mathbf{1}_1, \mathbf{0}_4, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_4, -\mathbf{1}_1$

Example 4.30. Consider $(D_1, D_2) = (13, 285)$, and set $p = 2$. The narrow class group of discriminant $285 = 3 \cdot 5 \cdot 19$ is isomorphic to the Klein 4-group V_4, generated by involutions s_1, s_2. There is, up to translation by $\tilde{\Gamma}$, a unique $\tau_1 \in \Gamma\backslash\mathcal{H}_2^{13}$, and there are four RM points $\tau_2^{(1)}, \ldots \tau_2^{(4)}$ in any Cl(285)-orbit in $\mathcal{H}_2^{285}$.

We have checked that the 2-adic intersection numbers $J_2^+(\tau_1, \tau_2^{(j)})$ for $j = 1, \ldots, 4$ are distinct, and coincide with 800 digits of 2-adic precision with the roots of the polynomial

$$\begin{aligned} &77360972841758936947502973998239x^4 + 140181070438890831721314135099803x^3 \\ &+209895619549791255199413489899292x^2 + 140181070438890831721314135099803x \\ &+77360972841758936947502973998239, \end{aligned} \tag{85}$$

which generate the extension $\mathbb{Q}(\sqrt{-3}, \sqrt{-19})$ over $\mathbb{Q}$. Likewise, the 2-adic intersection numbers $J_2^-(\tau_1, \tau_2^{(j)})$ are also distinct, and coincide with 800 digits of 2-adic precision with the roots of the polynomial

$$\begin{aligned} &1821488696558254611662551x^4 + 203729098486198913585801x^3 - 3016614164551653876723804x^2 \\ &+203729098486198913585801x + 1821488696558254611662551, \end{aligned} \tag{86}$$

which generate the extension $\mathbb{Q}(\sqrt{57}, \sqrt{-195})$. The constant terms of these polynomials factor as

$$\begin{aligned} 77360972841758936947502973998239 &= 7^7 \cdot 19^2 \cdot 31^2 \cdot 73 \cdot 109^2 \cdot 151^2 \cdot 163 \cdot 397 \cdot 457 \cdot 463, \\ 1821488696558254611662551 &= 7 \cdot 31^2 \cdot 73 \cdot 109^2 \cdot 151^2 \cdot 163 \cdot 397 \cdot 457 \cdot 463. \end{aligned}$$

The first two columns of the table below list the arithmetic intersection numbers computed by James Rickards, and the last two give the Newton slopes for the polynomials (85) and (86) at the primes that arose in these factorisations:

q	$I_{q,2}(13, 285)$	$I'_{q,2}(13, 285)$	$\mathrm{ord}_q\, J_2^+(\tau_1, \tau_2^{(j)})$	$\mathrm{ord}_q\, J_2^-(\tau_1, \tau_2^{(j)})$
7	$(1-s_1)(1+2s_2)$	$(1-s_1)(1+3s_2)$	$\mathbf{5}_1, \mathbf{2}_1, -\mathbf{2}_1, -\mathbf{5}_1$	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$
19	$(1-s_1)(1-s_2)$	$(1-s_1)(1-s_2)$	$\mathbf{2}_2, -\mathbf{2}_2$	$\mathbf{0}_4$
31	$(1-s_1)(1-s_2)$	0	$\mathbf{1}_2, -\mathbf{1}_2$	$\mathbf{1}_2, -\mathbf{1}_2$
73	$(1-s_1)$	0	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$
109	$(1-s_1)(1-s_2)$	0	$\mathbf{1}_2, -\mathbf{1}_2$	$\mathbf{1}_2, -\mathbf{1}_2$
151	$(1-s_1)(1-s_2)$	0	$\mathbf{1}_2, -\mathbf{1}_2$	$\mathbf{1}_2, -\mathbf{1}_2$
163	$(1-s_1)$	0	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$
397	$(1-s_1)$	0	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$
457	$(1-s_1)$	0	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$
463	$(1-s_1)$	0	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$	$\mathbf{1}_1, \mathbf{0}_2, -\mathbf{1}_1$

Once again, the two last columns are precisely the coefficients of the sum and difference, respectively, of the group ring elements $I_{q,2}(13, 285)$ and $I'_{q,2}(13, 285)$ given in the first two columns of the table.

Now, let $p = 7$. We computed that the invariants $J_7^+(\tau_1, \tau_2^{(j)})$ for $j = 1, \ldots, 4$ coincide to at least 200 digits of 7-adic precision with the solutions of

$$1936x^4 + 308x^3 - 1887x^2 + 308x + 1936 = 0.$$

Likewise, the invariants $J_7^-(\tau_1, \tau_2^{(j)})$ satisfied, to the same precision, the polynomial

$$12390400x^4 - 41050240x^3 + 57394209x^2 - 41050240x + 12390400 = 0.$$

The corresponding table in this situation is:

q	$I_{q,7}(13, 285)$	$I'_{q,7}(13, 285)$	$\mathrm{ord}_q\, J_7^+(\tau_1, \tau_2^{(j)})$	$\mathrm{ord}_q\, J_7^-(\tau_1, \tau_2^{(j)})$
2	$2(1 - s_1)(2 + s_2)$	$2(1 - s_1)(-1 - 2s_2)$	$\mathbf{2}_2, \mathbf{-2}_2$	$\mathbf{6}_2, \mathbf{-6}_2$
5	$(1 - s_1)(1 - s_2)$	$(1 - s_1)(-1 + s_2)$	$\mathbf{0}_4$	$\mathbf{2}_2, \mathbf{-2}_2$
11	$(1 - s_1)(1 - s_2)$	0	$\mathbf{1}_2, \mathbf{-1}_2$	$\mathbf{1}_2, \mathbf{-1}_2$

We similarly verified Conjecture 4.26 for all other prime pairs (p, q) in this example.

5. Gross-Stark units and Stark-Heegner points

The goal of this brief concluding chapter is to make the bridge between the constructions of this paper and those of [DD] and [Da], which are based on the RM values of partial lifts of certain rigid *analytic* cocycles of weight two under the logarithmic derivative map

$$\mathrm{dlog} : \mathrm{H}^1_f(\Gamma, \mathcal{O}^\times) \longrightarrow \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{O}_2).$$

These lifts give rise to arithmetically significant rigid analytic cocycles of weight zero *modulo certain p-adic periods*, whose values at RM points lead to analogues of elliptic units and Heegner points in the setting of ring class fields of real quadratic fields.

5.1. Multiplicative cocycles and the multiplicative Schneider-Teitelbaum lift.

The logarithmic derivative gives a natural injection

$$\mathrm{dlog} : \mathcal{O}^\times/\mathbb{C}_p^\times \longrightarrow \mathcal{O}_2$$

sending the local section f to f'/f, where f' denotes the derivative with respect to τ. It induces a similar map

$$\mathrm{dlog} : \mathrm{MS}^\Gamma(\mathcal{O}^\times/\mathbb{C}_p^\times) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{O}_2) \tag{87}$$

on the space of Γ-invariant modular symbols. The space $\mathrm{dlog}(\mathcal{O}^\times) \subset \mathcal{O}_2$ is called the space of *rigid differentials of the third kind* on $\mathcal{H}_p$, and consists of differentials whose image under ∂ are $\mathbb{Z}$-valued harmonic functions on $\mathcal{T}_1^*$. The image of (87) is likewise called the space of rigid analytic modular symbols of the third kind.

Proposition 5.1. *There is a Hecke equivariant map*

$$L^\times_{\mathrm{ST}} : \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z}) \longrightarrow \mathrm{MS}^\Gamma(\mathcal{O}^\times/\mathbb{C}_p^\times)$$

for which the diagram

$$\begin{array}{ccc} \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z}) & \xrightarrow{L^{\times}_{\mathrm{ST}}} & \mathrm{MS}^{\Gamma}(\mathcal{O}^{\times}/\mathbb{C}_p^{\times}) \\ \big\downarrow & & \big\downarrow \mathrm{dlog} \\ \mathrm{MS}^{\Gamma_0(p)}(\mathbb{C}_p) & \xrightarrow{L_{\mathrm{ST}}} & \mathrm{MS}^{\Gamma}(\mathcal{O}_2) \end{array}$$

commutes.

The map $L^{\times}_{\mathrm{ST}}$ is called the *multiplicative Schneider-Teitelbaum lift.* It is constructed as a multiplicative refinement of (40), by setting, for each $m \in \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$,

$$L^{\times}_{\mathrm{ST}}(m)\{r,s\}(z) := \times\!\!\!\!\!\!\int_{\mathbb{P}_1(\mathbb{Q}_p)} (z-t)d\mu_m\{r,s\}(t) := \lim_{\{U_\alpha\}} \prod_\alpha (z-t_\alpha)^{m\{r,s\}(U_\alpha)}, \tag{88}$$

where the limit of "Riemann products" on the right-hand side is taken over finer and finer coverings $\{U_\alpha\}$ of $\mathbb{P}_1(\mathbb{Q}_p)$ by open balls, the point t_α is any sample point in U_α, and

$$m\{r,s\}(U_\alpha) := m\{\gamma r, \gamma s\}, \text{ with } \gamma \in \Gamma, \quad \gamma U_\alpha = \mathbb{Z}_p.$$

Given

$$\bar{J} \in \mathrm{MS}^{\Gamma}(\mathcal{O}^{\times}/\mathbb{C}_p^{\times}) = \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{O}^{\times}/\mathbb{C}_p^{\times}) \subset \mathrm{H}^1(\Gamma, \mathcal{O}^{\times}/\mathbb{C}_p^{\times}),$$

it is natural to consider its lifts to "genuine" multiplicative classes in $\mathrm{H}^1(\Gamma, \mathcal{O}^{\times})$. The obstruction to lifting $\bar{J}$ to such a class lies in $H^2(\Gamma, \mathbb{C}_p^{\times})$ and is the image of $\bar{J}$ under the connecting homomorphism δ in the following long exact cohomology sequence:

$$\cdots \longrightarrow \mathrm{H}^1(\Gamma, \mathbb{C}_p^{\times}) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{O}^{\times}) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{O}^{\times}/\mathbb{C}_p^{\times}) \xrightarrow{\delta} \mathrm{H}^2(\Gamma, \mathbb{C}_p^{\times}) \longrightarrow \cdots$$

Definition 5.2. The class $\kappa := \delta(\bar{J}) \in \mathrm{H}^2(\Gamma, \mathbb{C}_p^{\times})$ is called the *lifting obstruction* attached to $\bar{J}$. A subgroup $Q \subset \mathbb{C}_p^{\times}$ is said to *trivialise* this lifting obstruction if the natural image of κ in $H^2(\Gamma, \mathbb{C}_p^{\times}/Q)$ is trivial.

If Q trivialises the lifting obstruction for $\bar{J}$, then this class lifts to an element of $\mathrm{H}^1(\Gamma, \mathcal{O}^{\times}/Q)$. This lift is unique up to elements of order 12, since the abelianisation of Γ is a quotient of $(\mathbb{Z}/12\mathbb{Z})$ and therefore

$$\mathrm{H}^1(\Gamma, \mathbb{C}_p^{\times}/Q) \subset (\mathbb{C}_p^{\times}/Q)[12].$$

In conclusion, after replacing Q by a slightly larger group (containing Q with finite index) one can thus associate to any modular symbol $m \in \mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$ a canonical rigid analytic cocycle

$$J \in \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{O}^{\times}/Q) \tag{89}$$

of weight zero "modulo Q". The trivialising subgroup Q is a subtle invariant of m and a careful analysis is required to identify it in each case.

The guiding theme of this chapter is that the RM values of rigid analytic modular cocycles obtained in (89) lead to algebraic invariants in ring class fields of the associated real quadratic field.

5.2. **The universal cocycle.** We illustrate this principle with the simple "toy example" of the multiplicative universal modular cocycle

$$\bar{J}_{\mathrm{univ}} = L^{\times}_{\mathrm{ST}}(m_{\mathrm{univ}}),$$

which was already introduced in (44) and is given by

$$\bar{J}_{\mathrm{univ}}\{r,s\}(z) = \left(\frac{z-s}{z-r}\right) \pmod{\mathbb{C}_p^{\times}}.$$

Proposition 5.3. *The lattice Q_{univ} of $\mathbb{C}_p^{\times}$ generated by -1 and p trivialises the lifting obstruction for $\bar{J}_{\mathrm{univ}}$. More precisely, the class $\bar{J}_{\mathrm{univ}}$ admits a canonical lift to a class in $\mathrm{MS}^{\Gamma}(\mathcal{O}^{\times}/Q_{\mathrm{univ}})$.*

Proof. Given r and s in $\mathbb{P}_1(\mathbb{Q})$, let $r = a/b$ and $s = c/d$ be their expressions as fractions in lowest terms, adopting the usual convention that $\infty = 1/0$. Then the expression

$$J_{\mathrm{univ}}\{r,s\}(z) = \pm\frac{dz-c}{bz-a} \tag{90}$$

defines an $\mathrm{SL}_2(\mathbb{Z})$-invariant modular symbol modulo ± 1, since $\mathrm{SL}_2(\mathbb{Z})$ preserves the set of column vectors (a,b) with $\gcd(a,b) = 1$. The image of the column vector (a,b) under a matrix $\gamma \in \Gamma$ is a vector (a',b') for which $\gcd(a,b) \in Q_{\mathrm{univ}}$, and hence (90) defines a Γ-invariant modular symbol with values in $\mathcal{O}^{\times}/Q_{\mathrm{univ}}$. □

We now consider the RM values of the lifted cocycle J_{univ}. If $F(x,s) = Ax^2+Bxy+Cy^2$ is a binary quadratic form of discriminant $D = B^2-4AC$, then its root is $\tau_F = (-B+\sqrt{D})/2A$, while its stabiliser is generated by

$$\gamma_F = \begin{pmatrix} u-Bv & -2Cv \\ 2Av & u+Bv \end{pmatrix}, \qquad u^2 - Dv^2 = 1,$$

where $u+v\sqrt{D}$ is a fundamental solution to Pell's equation. A straightforward calculation shows that

$$J_{\mathrm{univ}}[\tau_F] = J_{\mathrm{univ}}\{r, \gamma_\tau r\}(\tau_F) = u \pm v\sqrt{D} \pmod{\mathbb{Z}}[1/p]^{\times},$$

for any $r \in \mathbb{P}_1(\mathbb{Q})$.

It follows that the cocycle J_{univ} takes algebraic values at RM points, albeit somewhat uninteresting ones, since they always belong to the field of "real multiplication" and are just a power of the fundamental unit in this field. More precisely, $J_{\mathrm{univ}}[\tau]$ is a fundamental unit in the order associated to τ.

To obtain more interesting class invariants it is necessary to consider the RM values of analytic cocycles arising from the multiplicative Schneider-Teitelbaum lifts of more general elements of $\mathrm{MS}^{\Gamma_0(p)}(\mathbb{Z})$.

5.3. **The Dedekind-Rademacher cocycle and Gross-Stark units.** Let

$$\varphi_{\mathrm{DR}} \in \mathrm{H}^1(\Gamma_0(p), \mathbb{Z})$$

be the *Dedekind-Rademacher homomorphism* defined by the rule

$$\varphi_{\mathrm{DR}}\begin{pmatrix} a & b \\ pc & d \end{pmatrix} = \begin{cases} (p-1)b/d & \text{if } c = 0; \\ \frac{(p-1)(a+d)}{cp} + 12\cdot\mathrm{sign}(c)\cdot D^p\left(\frac{a}{p|c|}\right) & \text{if } c \neq 0, \end{cases} \tag{91}$$

where $D^p(x) := D(x) - D(px)$ is the p-stabilisation of the usual Dedekind sum

$$D(a/m) := \sum_{x=1}^{m-1} B_1(x/m)B_1(ax/m), \qquad B_1(x) := x - [x] - 1/2.$$

The vector space $\mathrm{H}^1(\Gamma_0(p), \mathbb{Q})$ is endowed with the usual action of the Hecke operators T_n (with $n \geq 1$), with T_p denoting the Atkin operator which is customarily denoted U_p in the literature. The Dedekind-Rademacher homomorphism belongs to the one-dimensional Hecke eigenspace of vectors $\varphi \in \mathrm{H}^1(\Gamma_0(p), \mathbb{Q})$ satisfying

$$T_n(\varphi) = \sigma_1'(n)\varphi, \qquad \text{with } \sigma_1'(n) = \sum_{d|n, p\nmid d} d. \tag{92}$$

(Cf. for instance [Ma][§II.2].) The class φ_{DR} does not lie in the parabolic cohomology but the multiplicative Schneider Teitelbaum lift admits a simple extension

$$L_{\mathrm{ST}}^{\times} : \mathrm{H}^1(\Gamma_0(p), \mathbb{Z}) \longrightarrow \mathrm{H}^1(\Gamma, \mathcal{O}^{\times}/\mathbb{Q}_p^{\times}).$$

Let

$$\bar{J}_{\mathrm{DR}} := L_{\mathrm{ST}}^{\times}(\varphi_{\mathrm{DR}}) \in \mathrm{H}^1(\Gamma, \mathcal{O}^{\times}/\mathbb{Q}_p^{\times})$$

be the multiplicative Schneider-Teitelbaum transform of φ_{DR}, and let

$$\kappa_{\mathrm{DR}} \in \mathrm{H}^2(\Gamma, \mathbb{Q}_p^{\times})$$

denote the associated lifting obstruction.

Conjecture 5.4. *There is a lattice Q_{DR} containing $p^{\mathbb{Z}}$ with finite index which trivialises κ_{DR}, and for which $\bar{J}_{\mathrm{DR}}$ lifts uniquely to a class $J_{\mathrm{DR}} \in \mathrm{H}^1(\Gamma, \mathcal{O}^{\times}/Q_{\mathrm{DR}})$.*

If $\tau \in \mathcal{H}_p \cap K$ is an RM point belonging to the real quadratic field K, the set of matrices

$$R_\tau := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}[1/p]) \quad \text{such that } c\tau^2 + (d-a)\tau - b = 0 \right\}$$

is a commutative subring of $M_2(\mathbb{Z}[1/p])$ which is isomorphic to a $\mathbb{Z}[1/p]$-order in K (and hence in particular is free of rank two as a $\mathbb{Z}[1/p]$-module). Class field theory identifies the Picard group (in the narrow sense) of $R = R_\tau$ with the Galois group of an abelian extension of K: the narrow ring class field of K attached to R, denoted $H = H_\tau$, whose degree over K is equal to the narrow class number of R. The field H contains the *ring class field* attached to R, a totally real abelian extension H^+ of K whose degree is equal to the class number of R. In particular,

$$[H : H^+] = \begin{cases} 1 & \text{if } R \text{ contains a unit of negative norm;} \\ 2 & \text{otherwise.} \end{cases}$$

The following is a natural refinement of Conjecture 2.14 of [DD]:

Conjecture 5.5. *For any RM point $\tau \in \mathcal{H}_p \cap K$, the value $J_{\mathrm{DR}}[\tau]$ belongs to $\mathcal{O}_H[1/p]^{\times}$, where H is the narrow ring class field of K attached to the order R_τ.*

Remark 5.6. Conjectures 5.4 and 5.5 are suggested by the findings of [DD], which considers a "modified Dedekind-Rademacher symbol"

$$\varphi_{\mathrm{DR}}' \in H^1_{\mathrm{par}}(\Gamma_0(pN), \mathbb{Z}),$$

where N is an auxiliary integer that is prime to p. This class has the same Hecke eigenvalues as the class φ_{DR} at all the good Hecke operators T_n indexed by n that are relatively prime to N, as described in (92). More concretely, it is a linear combination of conjugates of φ_{DR} by integral matrices of determinant dividing N which are upper-triangular modulo p. The multiplicative Schneider-Teitelbaum transform

$$\mathrm{ST}^{\times}(\varphi_{\mathrm{DR}}') \in H^1_{\mathrm{par}}(\Gamma', \mathcal{O}^{\times}/\mathbb{Q}_p^{\times})$$

is invariant under the p-arithmetic congruence subgroup

$$\Gamma' := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \text{with } N|c. \right\}.$$

Prop. 2.8 of [DD] shows that this class lifts uniquely to a class $J'_{\rm DR} \in {\rm MS}^{\Gamma'}(\mathcal{O}^\times)$ up to torsion. Conjecture 2.14 of loc.cit. predicts that the values of $J'_{\rm DR}$ at RM points are powers of Gross-Stark p-units in the idoneous ring class field. This conjecture, which has been extensively tested numerically in [DD], provides a basic instance in which rigid analytic multiplicative cocycles can be used to construct non-trivial invariants in ring class fields of real quadratic fields.

Remark 5.7. Conjecture 5.4 above has the virtue of not relying on any auxiliary level structure, and in that sense is somewhat more natural than the conjectures of [DD]. It would be interesting to refine the techniques of loc.cit. to prove it, make a careful study of the minimal lattice $Q \subset \mathbb{Q}_p^\times$ that trivialises the lifting obstruction for $\bar{J}_{\rm DR}$, and provide numerical evidence for Conjecture 5.5. We have not attempted to carry this out, since Conjectures 5.4 and 5.5 are stated solely for motivation and are tangential to our primary goal of building a theory of singular moduli for real quadratic fields.

5.4. **Elliptic modular cocycles and Stark-Heegner points.** Let E be an elliptic curve of prime conductor p, and let

$$L(E,s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

denote its associated Hasse-Weil L-series. The Shimura Taniyama conjecture proved by Wiles asserts that the generating series

$$f_E(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is the fourier expansion of a modular form of weight two on the Hecke congruence group $\Gamma_0(p)$. There are two distinguished elements

$$\varphi_E^+, \ \varphi_E^- \ \in \ {\rm MS}^{\Gamma_0(p)}(\mathbb{Z}),$$

satisfying

$$T_n\varphi_E^+ = a_n \cdot \varphi_E^+, \qquad T_n\varphi_E^- = a_n \cdot \varphi_E^-, \qquad \text{for all } n \geq 1,$$

and

$$\int_r^s (2\pi i) f_E(z) dz = \varphi_E^+\{r,s\} \cdot \Omega_E^+ + \varphi_E^-\{r,s\} \cdot \Omega_E^-,$$

where Ω_E^+ and Ω_E^- are real and imaginary periods attached to E.

Let

$$\bar{J}_E^+, \ \bar{J}_E^- \quad \in {\rm MS}^{\Gamma}(\mathcal{O}^\times/\mathbb{Q}_p^\times)$$

denote the multiplicative Schneider-Teitelbaum lifts of φ_E^+ and φ_E^- respectively, and let κ_E^+ and $\kappa_E^- \in {\rm H}^2(\Gamma, \mathbb{Q}_p^\times)$ denote the associated lifting obstructions. Recall the Tate p-adic period $q_E \in \mathbb{Q}_p^\times$ attached to E, and let

$$\Psi_{E,p} : \mathbb{C}_p^\times / q_E^{\mathbb{Z}} \longrightarrow E(\mathbb{C}_p)$$

denote the Tate uniformisation of E.

Theorem 1 of [Da] can be stated as follows:

Theorem 5.8. *There are lattices Q_E^+ and $Q_E^- \subset \mathbb{Q}_p^\times$ which are commensurable with the Tate lattice $q_E^{\mathbb{Z}}$ and trivialise κ_E^+ and κ_E^- respectively.*

After slightly enlarging the lattices $Q_E^{\pm}$, the classes $\bar{J}_E^{\pm}$ lift uniquely to classes $J_E^{\pm} \in \mathrm{H}^1(\Gamma, \mathcal{O}^\times/Q_E^{\pm})$. Let t be an integer for which $(Q_E^{\pm})^t \subset q_E^{\mathbb{Z}}$. After replacing the multiplicative cocycles $J_E^{\pm}$ by their t-th powers power and reducing modulo $q_E^{\mathbb{Z}}$, we may view J_E^+ and J_E^- as elements of $\mathrm{H}^1(\Gamma, \mathcal{O}^\times/q_E^{\mathbb{Z}})$, whose values at RM points $\tau \in \mathcal{H}_p$ can then be viewed as elements of $E(\mathbb{C}_p)$ by applying the Tate uniformisation $\Psi_{E,p}$. One thus obtains two p-adic variants

$$J_E^+, \quad J_E^- \quad : \quad \Gamma\backslash\mathcal{H}_p^{\mathrm{RM}} \longrightarrow E(\mathbb{C}_p)$$

of the classical modular parametrisation attached to E.

Conjecture 5.9. *Let E be an elliptic curve of conductor p. For all RM points $\tau \in \mathcal{H}_p$,*

(1) *the point $J_E^+[\tau] \in E(\mathbb{C}_p)$ is defined over the ring class field attached to τ;*
(2) *the point $J_E^-[\tau] \in E(\mathbb{C}_p)$ is defined over the narrow ring class field attached to τ, and is in the (-1)-eigenspace for the action of complex conjugation.*

This conjecture suggests that the multiplicative modular cocycles $J_E^{\pm}$ attached to E carry arithmetic information about E that is just as rich and useful as the classical modular parametrisation, allowing the construction of global points on E that cannot be obtained (as far as we know) from the more classical parametrisations of elliptic curves by modular or Shimura curves. Extensive numerical evidence for this conjecture has been gathered in [Da], [DG], [DP], and (in much more general settings) in [GM1] and [GM2].

Conjecture 5.9 gives further motivation for wanting a better understanding of the RM values of rigid meromorphic cocycles. A genuine understanding of the phenomenon underlying their algebraicity would lead to new perspective on the construction of rational points on elliptic curves, a question which is a major stumbling block in understanding the Birch and Swinnerton-Dyer conjecture, and where so far Heegner points arising from the theory of complex multiplication have provided the only unconditional approach.

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