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p-adic L-functions and the coniveau filtration on Chow groups

By *Massimo Bertolini* at Essen, *Henri Darmon* at Montreal and *Kartik Prasanna* at Ann Arbor
With an appendix by *Brian Conrad* at Stanford

Abstract. We give evidence for the refined version of the Beilinson–Bloch conjecture involving coniveau filtrations, by studying several infinite families of CM motives (indexed by the integers $r \geq 1$) that are irreducible of Hodge type $(2r+1, 0) + (0, 2r+1)$ and whose L-functions vanish at the center. In each case, we construct a corresponding algebraic cycle that is homologically trivial but nontrivial in the top graded piece for the coniveau filtration. Consequently, we obtain new explicit examples of cycles on varieties over number fields that are nontorsion in the Griffiths group. The proof of the main result relies crucially on p-adic Hodge theoretic methods, and in particular on the relation between images of homologically trivial cycles under the p-adic Abel–Jacobi map and the values of p-adic L-functions.

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Introduction

The aim of this article is to apply the main result of [3] to questions surrounding the Beilinson–Bloch (henceforth, BB) conjecture ([1, 5]) in the theory of algebraic cycles, leading

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to the construction of explicit varieties over number fields with nontorsion Griffiths groups. The cycles that we construct correspond to the vanishing at the central point of the L-functions of certain motives[1)], as predicted by the BB conjectures. The particular motives studied in this article are the simplest interesting ones to which these conjectures apply, namely those attached to Hecke characters of an imaginary quadratic field. In these examples, we construct a homologically trivial cycle that is not just nonzero in the Chow group (with $\mathbf{Q}$-coefficients) but that is also nonzero in its top graded piece for the coniveau filtration, and consequently nontorsion in the Griffiths group.

For X a proper smooth variety over a field k, let $\mathrm{CH}^j(X)$ denote the Chow group of codimension j cycles modulo rational equivalence, and $\mathrm{CH}^j(X)_0$ the subgroup of classes represented by cycles that are homologically trivial. The group $\mathrm{CH}^j(X)_{0,\mathbf{Q}} := \mathrm{CH}^j(X)_0 \otimes \mathbf{Q}$ is equipped with a decreasing *coniveau* filtration ([6,20]), denoted $N^i\mathrm{CH}^j(X)_{\mathbf{Q}}$. The definition of this filtration is recalled in Section 2.1 below, but for the purposes of this introduction we remind the reader that

$$N^0\mathrm{CH}^j(X)_{\mathbf{Q}} = \mathrm{CH}^j(X)_{0,\mathbf{Q}}, \quad N^j\mathrm{CH}^j(X)_{\mathbf{Q}} = 0,$$

and

$$N^{j-1}\mathrm{CH}^j(X)_{\mathbf{Q}} = \mathrm{CH}^j(X)_{\mathrm{alg}} \otimes \mathbf{Q},$$

where $\mathrm{CH}^j(X)_{\mathrm{alg}}$ is the subgroup of cycles that are algebraically equivalent to zero. Finally, the jth Griffiths group of X is

$$\mathrm{Gr}^j(X) := \mathrm{CH}^j(X)_0/\mathrm{CH}^j(X)_{\mathrm{alg}}.$$

To illustrate our main results, let K be an imaginary quadratic field, and suppose that K has class number 1, odd discriminant $-D$ and that the only roots of unity in K are ± 1. There is a finite list of such K, namely those with $D \in \{7, 11, 19, 43, 67, 163\}$. For any positive integer r, let W_r denote the Kuga–Sato variety of dimension $r+1$ over the modular curve $X_1(D)$. Thus W_r is a nonsingular projective variety defined over $\mathbf{Q}$ and fibered over $X_1(D)$, the fibers being isomorphic to the rth power of a varying elliptic curve. Let A be an elliptic curve over $\mathbf{Q}$ with complex multiplication by the maximal order of K and (minimal) conductor D^2. For any positive integer r, let $X_{2r-1,1}$ be the variety over $\mathbf{Q}$ of dimension $2r+1$ given by

$$X_{2r-1,1} := W_{2r-1} \times A.$$

In Section 4.1, we construct (extending the construction in [3]) *generalized Heegner cycles*

$$\tilde{\Delta}^{+}_{2r-1,1}, \tilde{\Delta}^{-}_{2r-1,1} \in \mathrm{CH}^{r+1}(X_{2r-1,1/K})_0 \otimes \mathbf{Q},$$

that are defined over K and that are respectively in the ± 1-eigenspaces for the action of complex conjugation. Roughly, the construction is based on the fact that a copy of A^{2r-1} occurs as a fiber of W_{2r-1} over a CM point on $X_1(D)$; writing this fiber as A'^{2r-1} with $A' \simeq A$, these cycles are obtained by applying certain projectors to the cycle

$$(\text{diagonal}) \times (\text{graph of } \sqrt{-D})^{r-1} \subset (A' \times A) \times (A' \times A')^{r-1} = A'^{2r-1} \times A \subset W_{2r-1} \times A,$$

and taking a suitable trace. The following theorem is a simple but nontrivial consequence of our main result, formulated in Theorem 4.3.1 (cf. Corollary 4.3.5 and its proof).

[1)] We use the word *motives* to mean *compatible systems of realizations.*

Theorem 1. *Suppose that $r \in \mathbf{Z}^{\geq 1}$ is odd if $D = 7$ and is even if $D = 11, 19, 43, 67, 163$. Then $\tilde{\Delta}^{+}_{2r-1,1}$ and $\tilde{\Delta}^{-}_{2r-1,1}$ are both nonzero in the top graded piece of the coniveau filtration, i.e., the classes*

$$[\tilde{\Delta}^{+}_{2r-1,1}], [\tilde{\Delta}^{-}_{2r-1,1}] \in N^0\mathrm{CH}^{r+1}(X_{2r-1,1/K})_{\mathbf{Q}}/N^1\mathrm{CH}^{r+1}(X_{2r-1,1/K})_{\mathbf{Q}}$$

are nonzero. In particular, these cycles are nonzero in $\mathrm{Gr}^{r+1}(X_{2r-1,1/K}) \otimes \mathbf{Q}$.

In other words, the theorem states that there is no proper closed subscheme Y of $X_{2r-1,1}$ such that either $\tilde{\Delta}^{+}_{2r-1,1}$ or $\tilde{\Delta}^{-}_{2r-1,1}$ is rationally equivalent to a cycle that is supported on Y and is homologically trivial on Y. This should be viewed as giving evidence for a refined version of the BB conjecture (see [6] and Section 2 below) relating cycles to L-functions. We now briefly explain this connection.

The field K of class number one has attached to it a so-called *canonical Hecke character* ψ of type $(1,0)$. The character ψ has conductor $\mathfrak{d}_K := (\sqrt{-D})$, namely it satisfies

$$\psi((\alpha)) = \alpha \quad \text{if } \alpha \equiv 1 \bmod \mathfrak{d}_K.$$

In fact, ψ is just the Hecke character associated with the elliptic curve A by the theory of complex multiplication. Thus there is an equality of L-functions:

$$L(A,s) = L(\psi,s) = L(\theta_\psi, s),$$

where $\theta_\psi := \sum_{\mathfrak{a}} \psi(\mathfrak{a}) e^{2\pi i \mathrm{N}\mathfrak{a}\cdot z} \in S_2(\Gamma_0(D^2))$ (the sum being over integral ideals in $\mathcal{O}_K$) is the modular form of weight 2 associated with A. Let us consider positive integral powers ψ^t of the Hecke character ψ and the associated L-functions $L(\psi^t, s)$. We are interested in the case where $L(\psi^t, s)$ is self-dual and such that the center of its functional equation is an integer, so that the BB conjecture applies at this point. This occurs exactly when t is odd, so we write $t = 2r+1$, with r a nonnegative integer. An easy computation using the fact that ψ^2 is an unramified Hecke character of infinity type $(2,0)$ shows that

$$\text{(I-1)} \qquad \operatorname{sign} L(\psi^{2r+1}, s) = \begin{cases} \operatorname{sign} L(\psi,s), & \text{if } r \text{ is even,} \\ -\operatorname{sign} L(\psi,s), & \text{if } r \text{ is odd.} \end{cases}$$

As for the sign of $L(\psi,s)$, it so happens that

$$\text{(I-2)} \qquad \operatorname{sign} L(\psi,s) = \begin{cases} +1, & \text{if } D = 7, \\ -1, & \text{if } D \in \{11, 19, 43, 67, 163\}. \end{cases}$$

Let f be the modular form

$$f = \theta_{\psi^{2r}} = \sum_{\mathfrak{a}} \psi^{2r}(\mathfrak{a}) e^{2\pi i \mathrm{N}\mathfrak{a}\cdot z} \in S_{2r+1}(\Gamma_0(D), \varepsilon_K),$$

where ε_K is the quadratic Dirichlet character of conductor D attached to the extension $K/\mathbf{Q}$. By [35], the form f contributes to the cohomology of the Kuga–Sato variety W_{2r-1}, in fact to the middle cohomology $H^{2r}(W_{2r-1})$. By the Künneth formula, the motive $H^{2r+1}(X_{2r-1,1})$ then contains

$$\text{(I-3)} \qquad M_f \otimes M_{\theta_\psi} = M_{\theta_{\psi^{2r}}} \otimes M_{\theta_\psi} = M_{\theta_{\psi^{2r+1}}} \oplus M_{\theta_{\psi^{2r-1}}}(-1)$$

as a submotive, where we write M_f and $M_{\theta_{\psi^i}}$ for the motives over $\mathbf{Q}$ attached to f and θ_{ψ^i} respectively. The BB conjectures then imply, on account of (I-1) and (I-2), that

$$\operatorname{rank} \mathrm{CH}^{r+1}(X_{2r-1,1})_{0,\mathbf{Q}} \geq 1$$

for *odd* values of r when $D = 7$ and for *even* values of r if $D \neq 7$. Moreover, since $M_{\theta_{\psi^{2r+1}}}$ has Hodge type $(2r+1, 0) + (0, 2r+1)$, the refined version of the BB conjecture (recalled below in Section 2.1) implies that top graded piece of this Chow group for the coniveau filtration is nontrivial. Theorem 1 thus provides evidence for this conjecture by constructing an explicit nonzero element $\tilde{\Delta}^{+}_{2r-1,1}$ in the quotient $\mathrm{CH}^{r+1}(X_{2r-1,1})_{0,\mathbf{Q}}/N^1\mathrm{CH}^{r+1}(X_{2r-1,1})_{\mathbf{Q}}$. In fact, it does somewhat more, namely construct also a similar element $\tilde{\Delta}^{-}_{2r-1,1}$ over K that is in the minus-eigenspace for complex conjugation; this is also predicted by the behavior of the L-function, since the L-function of $M_{\theta_{\psi^{2r+1}}}$ over K is simply the square of its L-function over $\mathbf{Q}$.

We now say a few words about the proof of Theorem 1. The content of this theorem is that for each imaginary quadratic field K above, there exists an explicit infinite family of varieties $\mathcal{F}_K$ of increasing dimension (or more precisely, an infinite family of irreducible motives) for which a part of the refined BB conjecture can be verified. The main ingredient in the proof is the relation between p-adic Abel–Jacobi images of cycles and p-adic L-functions that is proved in [3]. This relation can be used to reduce the validity of Theorem 1 for any given variety X in $\mathcal{F}_K$ to the nonvanishing of the Katz p-adic L-function associated with K at a specific point outside the range of interpolation that is determined by the dimension of X. If the value in question turns out to be a p-adic unit, then by the yoga of p-adic L-functions, this nonvanishing mod p, and therefore the validity of the theorem, propagates to all the varieties in $\mathcal{F}_K$ whose dimension lies in the same congruence class as $\dim(X)$ mod $2(p-1)$. Happily, it turns out that we can find for each K in our list above a suitable prime p such that this nonvanishing mod p holds for all the relevant congruence classes mod $2(p-1)$, thereby reducing the theorem, which applies to an infinite set of varieties of unbounded dimension, to a finite (and simple!) computation.

Remark 2. In the main text, we prove a somewhat more general result, namely for any i such that $0 \leq i \leq r-1$ and $i \equiv r-1 \bmod 2$, we construct cycles $\tilde{\Delta}^{\pm}_{r+i,r-i}$ on the variety

$$X_{r+i,r-i} := W_{r+i} \times A^{r-i},$$

and show that these are nontrivial in N^0/N^1. This is the content of Corollary 4.3.5 below.

Remark 3. Theorem 1 suggests that for the Hecke characters ψ^{2r+1} considered above, whenever the sign of the functional equation is -1, the L-function vanishes to order exactly one, i.e.,

$$\operatorname{sign} L(\psi^{2r+1}, s) = -1 \implies L'(\psi^{2r+1}, r+1) \neq 0. \tag{I-4}$$

We do not know how to prove this at this point. In fact, (I-4) would follow if one could

(i) prove a precise relation between the (Arakelov) height of the cycle $\tilde{\Delta}^{\pm}_{2r-1,1}$ and the derivative $L'(\psi^{2r+1}, r+1)$, and

(ii) if one knew the non-degeneracy of the height pairing, at least on the lines spanned by $\tilde{\Delta}^{\pm}_{2r-1,1}$.

Of these, (i) is certainly within reach by the methods of [38], but (ii) would seem to require a genuinely new idea. It would be very interesting therefore to prove (I-4) directly, say by purely analytic means.

The reader might wonder what happens in the complementary cases, namely r even when $D = 7$ and r odd otherwise. In these cases,

$$\operatorname{sign} L(\psi^{2r+1}, s) = +1, \quad \text{while} \quad \operatorname{sign} L(\psi^{2r-1}, s) = -1.$$

Since $M_{\theta_{\psi^{2r-1}}}$ has Hodge type $(2r-1, 0) + (0, 2r-1)$, one expects that the projection of the cycle $\tilde{\Delta}^{\pm}_{2r-1,1}$ on the (f, θ_ψ)-component lands in $N^1\mathrm{CH}^{r+1}(X_{2r-1,1})_{\mathbf{Q}}$ but is nonzero in the quotient N^1/N^2. A weaker statement is that one can apply some projector to $\tilde{\Delta}^{\pm}_{2r-1,1}$ to yield a cycle with this property. However, it seems very hard to construct such a projector since this is essentially equivalent to verifying an open case of the Tate conjecture. The same difficulty persists in the more general situation of Remark 2. Consider for example the extreme case $i = 0$ in Remark 2, so that r is odd and $D \neq 7$ and the relevant L-function is

$$L(M_{\theta_{\psi^{r+1}}} \otimes M_{\theta_{\psi^r}}, s) = L(\psi^{2r+1}, s) L(\psi, s - r).$$

Since

$$\operatorname{sign} L(\psi^{2r+1}, s) = +1, \quad \text{while} \quad \operatorname{sign} L(\psi, s) = -1,$$

and M_{θ_ψ} has Hodge type $(1, 0) + (0, 1)$, one should expect that the $(\theta_{\psi^{r+1}}, \theta_{\psi^r})$-component of $\tilde{\Delta}^{\pm}_{r,r}$ lands in $N^r\mathrm{CH}^{r+1}(X_{r,r})$, that is, comes by correspondence from a divisor of degree zero on a curve. Since $L(\psi, s)$ is the L-function of the elliptic curve A, such a curve should naturally contain A as a factor in its Jacobian and one therefore gets from $\tilde{\Delta}^{\pm}_{r,r}$ a (conjectural) construction of rational points on A. This idea is explored in detail in [4], which formulates such a conjectural *motivic* construction precisely and shows (unconditionally) that its realization in p-adic étale cohomology yields classes that come from rational points. On the other hand, its realization in Betti cohomology yields points on $A(\mathbf{C})$ that can be experimentally verified to be rational in many cases, but which cannot be proved to be rational except in a handful of low-dimensional examples where the relevant case of the Tate conjecture can be verified by hand.

Remark 4. It may be worth pointing out here the analogy with the Gross–Zagier theorem, in which in certain instances the Rankin–Selberg motive being considered splits as a sum of rank-2 motives and the associated L-function factors as a product of two degree-2 L-functions. The sign of the degree-4 L-function is -1 and the signs of the two individual factors determine the action of complex conjugation on the cycle. For example, the simplest case of such a factorization is

$$L(E_{/K}, s) = L(E, s) \cdot L(E, \varepsilon_K, s),$$

where E is an elliptic curve over $\mathbf{Q}$, and the signs of the individual factors determine whether the corresponding Heegner point is defined over K or $\mathbf{Q}$. In our case, again we have such a factorization but the individual signs seem to determine a somewhat more sophisticated invariant of the cycle, namely its exact position in the coniveau filtration.

The paper ends with two miscellaneous but related results. The first (in Section 5.1) treats similar phenomena for some non-CM motives. The results here use an explicit calculation of the relevant non-CM form (and of a derivative of this form) at a CM point, and in particular that the algebraic parts of these computed values are p-units for a suitably chosen prime p. This requires identifying the computed values as algebraic numbers, and here we make essential use of Appendix B by Brian Conrad that provides integrality results for modular forms in terms of the q-expansion at the cusp at infinity. The second miscellaneous result is a new proof by the methods of this paper of Bloch's theorem that the Chow group of codimension-2 cycles on the Jacobian of the Fermat quartic over $\mathbf{Q}$ contains a cycle (the Ceresa cycle) that is homologically trivial but nontorsion in the Griffiths group. This result is of historical importance since it provided the first concrete evidence for the generalization of the Birch–Swinnerton-Dyer conjecture to the BB conjecture. The following section describes some of this history and the relation between Bloch's method and the one of this article.

1. Cycles and the Griffiths group

Let X be a smooth projective variety over $\mathbf{C}$ of dimension n. For $1 \le j \le n$, let

$$\mathrm{cl} : \mathrm{CH}^j(X) \to H^{2j}(X, \mathbf{C})$$

denote the cycle class map, $\mathrm{CH}^j(X)_0$ its kernel (the group of homologically trivial cycles) and

$$\mathrm{AJ} : \mathrm{CH}^j(X)_0 \to J^j(X)$$

the Abel–Jacobi map into the jth Griffiths intermediate Jacobian of X. Recall that $J^j(X)$ is the intermediate Jacobian of the Hodge structure $H^{2j-1}(X)$, i.e.,

$$J^j(X) := (\mathrm{Fil}^j H^{2j-1}(X, \mathbf{C}))^\vee / H_{2j-1}(X, \mathbf{Z}) = \Bigl(\bigoplus_{i \ge j} H^{i,2j-1-i}(X)^\vee\Bigr) / H_{2j-1}(X, \mathbf{Z}).$$

In the extreme cases $j = 1$ and $j = n$, $J^j(X)$ is an abelian variety, namely the identity component of the Picard variety of X and, respectively, the Albanese of X. In general, $J^j(X)$ is only a complex torus. The *maximal abelian subvariety* $J^j(X)_{\mathrm{alg}}$ of $J^j(X)$ is defined to be the complex subtorus of $J^j(X)$ which is the intermediate Jacobian of the largest sub-Hodge structure of $H^{j,j-1}(X) \oplus H^{j-1,j}(X)$. This terminology is motivated by the fact that the image of $\mathrm{CH}^j(X)_{\mathrm{alg}}$ (the subgroup of $\mathrm{CH}^j(X)_0$ consisting of cycles algebraically equivalent to zero) under AJ is a complex subtorus of $J^j(X)$ that is contained in $J^j(X)_{\mathrm{alg}}$ and has the structure of an abelian variety, and further, the Hodge conjecture implies that this image is equal to $J^j(X)_{\mathrm{alg}}$. The jth Griffiths group $\mathrm{Gr}^j(X)$ is the quotient

$$\mathrm{Gr}^j(X) := \mathrm{CH}^j(X)_0 / \mathrm{CH}^j(X)_{\mathrm{alg}}.$$

This quotient is at most countable, as can be shown using the theory of Hilbert schemes; in particular, this implies that AJ has no chance of being surjective if

$$\mathrm{Fil}^j H^{2j-1}(X) > H^{j,j-1}(X).$$

In fact, for $j = n$ and $j = 1$, the group $\mathrm{Gr}^j(X)$ is always zero since algebraic and homological equivalence coincide for 0-cycles (trivially) and for divisors by the Lefschetz $(1,1)$-theorem.

However, in contrast to these cases, for other values of j, it was only in the late 1960s that Griffiths showed in an important series of articles [17] that $\mathrm{Gr}^j(X)$ can be nonzero and indeed even nontorsion. The particular example that Griffiths studied was Gr^2 of the *very general* hypersurface of degree 5 in $\mathbb{P}^4$. Griffiths' work was later generalized by Clemens [10], who showed for the same example that $\mathrm{Gr}^2(X)$ is not even finitely generated. Yet another example is due to Ceresa [8], who proved that for C the very general curve of genus $g \geq 3$, the *Ceresa cycle* $C - [-1]^*C$ in $X = \mathrm{Jac}(C)$ is homologically trivial and nontorsion in $\mathrm{Gr}^{g-1}(X)$. However, the arguments in all these cases made essential use of transcendental elements in the base field and assumed the variety is very general, that is, lies outside of a countable union of proper locally closed subsets of the parameter space. This left open the question of giving an explicit example of such a phenomenon and also whether the Griffiths group could be nontorsion for a variety defined over a number field. These questions were settled by work of Bruno Harris and Spencer Bloch, and led to the first nontrivial evidence for the BB conjecture relating the behavior of L-functions at the central point to algebraic cycles. We will quickly recall their work since the motives we deal with later in this article are generalizations of exactly the examples considered by Harris and Bloch.

Let C be the Fermat quartic

$$T_0^4 + T_1^4 = T_2^4.$$

Then C is a curve of genus 3 defined over $\mathbf{Q}$. Let X be the Jacobian of C. Choose an embedding of C in X and let

$$\Xi = [C] - [(-1)^*C] \in \mathrm{Gr}^2(X).$$

The class of Ξ in $\mathrm{Gr}^2(X)$ is independent of the choice of embedding of C in X. Harris [19] proved that in fact Ξ is *nonzero* in $\mathrm{Gr}^2(X)$ by showing that its image under AJ is not contained in $J^2(X)_{\mathrm{alg}}$. Since X is a threefold, this is equivalent to showing the following: writing $\Xi = \partial\Gamma$ for a singular 3-chain Γ, the functional

$$H^{3,0}(X) \to \mathbf{C}, \quad \omega \mapsto \int_\Gamma \omega$$

is not in the image of $H_3(X, \mathbf{Z})$ in $H^{3,0}(X)^\vee$. In this special case, it turns out that the image of $H_3(X, \mathbf{Z})$ is a lattice (of periods) in $H^{3,0}(X)^\vee$ (since X is isogenous to a product of three elliptic curves with complex multiplication by the ring of integers of the same imaginary quadratic field) and the problem can therefore be reduced to showing that the ratio of a certain iterated integral to a period (see [19, equation (1)]) is not an integer. This in turn can be verified by a numerical computation; in fact, if one could show that this ratio (which experimentally is $1.24178\ldots$) is not rational then it would follow that Ξ is even *nontorsion* in $\mathrm{Gr}^2(X)$. Such an approach to showing Ξ is nontorsion however seems out of reach even at this point of time.

The reasons to expect Ξ to be nontorsion in $\mathrm{Gr}^2(X)$ come from a refined version of the Beilinson–Bloch conjecture. We will now recall what the relevant L-functions are. The abelian variety X is isogenous over $\mathbf{Q}$ to the product $E \times E \times E'$, where

$$E : y^2 = x^3 - x, \quad E' : y^2 = x^3 + x.$$

The curves E and E' have conductors 32 and 64 respectively, and have CM by the ring of integers of the imaginary quadratic field $K = \mathbf{Q}(i)$. Let ψ and ψ' be the associated Grössen-characters of K of type $(1, 0)$. These have the property that their central characters $\varepsilon_\psi, \varepsilon_{\psi'}$

(that is, their restrictions to the ideles of $\mathbf{Q}$) equal ε_K, the quadratic Dirichlet character associated with the extension $K/\mathbf{Q}$. Further, $\psi' = \psi\mu$, for a certain quadratic character μ of K, namely that attached to the quadratic extension $\mathbf{Q}(\zeta_8)/K$.

For any Hecke character χ of K, we write M_{θ_χ} for the motive over $\mathbf{Q}$ of rank 2 whose associated Galois representation is $\mathrm{Ind}_{\mathbf{Q}}^K(\chi)$. We also write $V_\ell(M_{\theta_\chi})$ for the ℓ-adic realization of M_{θ_χ}. On account of the extra complex multiplications, the motive $H^3(X)$ splits as a sum $I \oplus M$, where $h^{3,0}(I) = 0$ and M is of type $(3,0)+(0,3)$. Indeed, I is just a sum of three copies of $M_{\theta_{\psi'}}(-1)$, while $M = M_{\theta_\chi}$, where $\chi := \psi^2\psi' = \psi^3\mu$. Thus

$$L(H^3(X), s) = L(\psi', s-1)^3 \cdot L(\chi, s).$$

It turns out in this case that the central point is $s = 2$, and $L(\psi', 1) \neq 0$, while $L(\chi, s)$ vanishes to order 1 at $s = 2$. The BB conjecture then predicts that the rank of $\mathrm{CH}^2(X)_0$ is ≥ 1. In addition, since M_{θ_χ} is purely of type $(3,0)+(0,3)$, one might expect moreover that the rank of the quotient $\mathrm{Gr}^2(X)$ is also ≥ 1. This expectation is made precise in the refined version of the BB conjecture involving coniveau filtrations, recalled below. That Bloch [5] was able to verify this expectation in this case was for him the first nontrivial evidence for the BB conjecture[2)] in higher dimensions.

Bloch's method is to study the map

$$\mathrm{CH}^2(X)_0 \xrightarrow{\mathrm{AJ}_{\text{ét}}} H^1(\overline{\mathbf{Q}}/K, H^3(X_{\overline{\mathbf{Q}}}, \mathbf{Z}_\ell(2))) \xrightarrow{P_*} H^1(\overline{\mathbf{Q}}/K, H^3(X_{\overline{\mathbf{Q}}}, \mathbf{Z}_\ell(2))),$$

where $\mathrm{AJ}_{\text{ét}}$ is the étale Abel–Jacobi map, and P_* is induced by a projector P that cuts out the motive M_{θ_χ}. (Such a projector exists on account of the extra complex multiplications.) Let $\gamma_\ell := \mathrm{AJ}_{\text{ét}}(\Xi)$ and $\beta_\ell := P_*(\gamma_\ell)$. Also, let $p \neq \ell$ be a prime at which X has good reduction. By restricting β_ℓ to a decomposition group at a prime above p and using a clever geometric construction in characteristic p, Bloch is able to show that $\beta_\ell \neq 0$, from which it easily follows that Ξ is nontorsion in $\mathrm{Gr}^2(X)$. This idea was extended by Schoen [33] and combined with a calculation of the complex Abel–Jacobi map to show that if W is the Kuga–Sato threefold over the modular curve $X(N)$, $N \geq 3$, then the Griffiths group of X over $\overline{\mathbf{Q}}$ is not finitely generated. There is no connection with L-functions made in loc. cit., however, another article of Schoen [34] studies the BB conjecture for a specific Kuga–Sato threefold over an infinite collection of number fields and provides some evidence in support of the conjecture.

Our approach to constructing examples of varieties over number fields with nontorsion Griffiths groups is based on Bloch's idea of using the étale Abel–Jacobi map. A key difference is that we study the restriction of β_ℓ to a decomposition group D_p at a prime above p in the case $\ell = p$ rather than $\ell \neq p$. Via the comparison theorems of p-adic Hodge theory, the restriction of β_p (resp. γ_p) to D_p can be viewed as a linear form on $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_{\theta_\chi}(2))$ (resp. on $\mathrm{Fil}^0 D_{\mathrm{dR}}(H^3(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(2))) = \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X_{\mathbf{Q}_p})$), and to show that Ξ is nontorsion in $\mathrm{Gr}^2(X)$, it suffices to show that β_p is not zero (resp. that γ_p is not zero on $H^{3,0}(X) \subset \mathrm{Fil}^2 H^3_{\mathrm{dR}}(X, \mathbf{Q}_p)$). In this way, the method begins to resemble that of Harris, except with the usual Abel–Jacobi map replaced by the p-adic Abel–Jacobi map. However, the p-adic Abel–Jacobi map has the distinct advantage that its target is torsion-free, since there are no periods (in a naive sense) to deal with! Further, it turns out that in the p-adic case, there are precise relations between

[2)] In fact, Bloch does not call it a conjecture at all but rather a *recurrent fantasy*, ranking it above *idle speculation* on account of this one example!

the coordinates of the Abel–Jacobi images in suitable bases and the values of certain p-adic L-functions at points outside the range of interpolation. The main theorem of [3] provides an instance of this relation for certain Rankin–Selberg motives, which can be specialized to yield motives of the sort considered above. In fact, there are already tenuous hints of such a relation for the motives attached to CM Hecke characters in the second article of Bloch [6], where he proves a Coates–Wiles type theorem showing that the nonvanishing of the restriction of the p-adic AJ-class of a homologically trivial cycle to a decomposition group at p implies the vanishing of a central L-value. Our results are in the opposite direction, namely, we show that the vanishing of an L-value (for sign reasons) corresponds (via the nonvanishing of a p-adic L-value) to the nonvanishing of the p-adic AJ class of an explicit homologically trivial cycle.

2. Coniveau and the refined Beilinson–Bloch conjecture

In this section alone, we let k denote a number field, while K will denote the completion of k at a finite place above the rational prime p. This will allow us to import notations from p-adic Hodge theory more easily.

2.1. The refined conjecture. We recall, following [6, Section 1], the refined version of the BB conjecture involving coniveau filtrations. Let $G_k := \mathrm{Gal}(\bar{k}/k)$ and X a smooth projective variety over k of dimension n. Let $\mathrm{CH}^j(X)$ (resp. $\mathrm{CH}^j(X)_0$) denote the Chow group of cycles (resp. homologically trivial cycles) of codimension j on X defined over k modulo rational equivalence. The BB conjecture predicts that for any $1 \leq j \leq n$, $\mathrm{CH}^j(X)_0$ is a finitely generated abelian group and

$$\dim_{\mathbf{Q}} \mathrm{CH}^j(X)_{0,\mathbf{Q}} = \mathrm{ord}_{s=j}\, L(H^{2j-1}(X_{\bar{k}}), s).$$

For the refined version, we need to recall the definitions of the *coniveau* filtrations. On the space $H^*(X_{\bar{k}}, \mathbf{Q}_\ell)$, this is the Galois stable filtration defined by

$$N^i H^*(X_{\bar{k}}, \mathbf{Q}_\ell) := \lim_{\substack{Y \subset X_{\bar{k}} \\ \mathrm{codim}\, Y \geq i}} \ker(H^*(X_{\bar{k}}, \mathbf{Q}_\ell) \to H^*(X_{\bar{k}} - Y, \mathbf{Q}_\ell)),$$

the limit being taken over closed subschemes $Y \subset X_{\bar{k}}$ of codimension $\geq i$. Define $N^i\mathrm{CH}^j(X)_{\mathbf{Q}}$ to consist of those classes $\gamma \in \mathrm{CH}^j(X)_{\mathbf{Q}}$ satisfying the following property: there exists a cycle c representing γ and a closed subscheme $Y \subset X_{\bar{k}}$ of codimension $\geq i$ such that $\mathrm{supp}(c) \subset Y$ and c is homologous to zero on Y. Thus $N^0\mathrm{CH}^j(X)_{\mathbf{Q}} = \mathrm{CH}^j(X)_{0,\mathbf{Q}}$ and $N^j\mathrm{CH}^j(X)_{\mathbf{Q}} = 0$. Further, $N^{j-1}\mathrm{CH}^j(X)_{\mathbf{Q}} = \mathrm{CH}^j(X)_{\mathrm{alg},\mathbf{Q}}$ (see [6, Lemma 1.1]), hence

$$\mathrm{Gr}^j(X)_{\mathbf{Q}} = N^0\mathrm{CH}^j(X)_{\mathbf{Q}}/N^{j-1}\mathrm{CH}^j(X)_{\mathbf{Q}}.$$

Conjecture 2.1.1 ([6, p. 381]). *Let*

$$\mathrm{gr}^i_N H^* := N^i H^*(X_{\bar{k}}, \mathbf{Q}_\ell)/N^{i+1}.$$

Then $L(\mathrm{gr}^i_N H^*, s)$ is independent of ℓ for any value of $*$, and

$$\dim_{\mathbf{Q}} \mathrm{gr}^i_N \mathrm{CH}^j(X)_{\mathbf{Q}} = \mathrm{ord}_{s=j}\, L(\mathrm{gr}^i_N H^{2j-1}, s).$$

2.2. Abel–Jacobi. Conjecture 2.1.1 is motivated by studying the ℓ-adic Abel–Jacobi map

$$\mathrm{AJ}_\ell : \mathrm{CH}^j(X)_{0,\mathbf{Q}} \to H^1(G_k, H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_\ell(j))),$$

defined as in [29, Section 1] for example, and its relation to the coniveau filtration. The following suggestive proposition in that direction is due to Bloch ([6, Proposition 1.5]).

Proposition 2.2.1. *The map* AJ_ℓ *is compatible with the coniveau filtrations and induces natural maps*

$$\mathrm{gr}^i_N \mathrm{CH}^j(X)_{\mathbf{Q}} \to H^1(G_k, \mathrm{gr}^i_N H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_\ell(j))).$$

This proposition implies there are natural maps δ_i for $0 \le i \le j$ such that the diagram below commutes (note that the vertical arrows on the right are inclusions since the quotient $N^i H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_\ell(j))/N^{i+1}$ is pure of weight -1, hence has no nonzero G_k-invariants):

(2.2.1)
$$\begin{array}{ccc}
\mathrm{CH}^j(X)_{0,\mathbf{Q}} & \xrightarrow{\delta_0 = \mathrm{AJ}_\ell} & H^1(G_k, H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_\ell(j))) \\
\cup \uparrow & & \cup \uparrow \\
N^i \mathrm{CH}^j(X)_{0,\mathbf{Q}} & \xrightarrow{\delta_i} & H^1(G_k, N^i H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_\ell(j))) \\
\cup \uparrow & & \cup \uparrow \\
N^{i+1} \mathrm{CH}^j(X)_{0,\mathbf{Q}} & \xrightarrow{\delta_{i+1}} & H^1(G_k, N^{i+1} H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_\ell(j))).
\end{array}$$

Suppose that v is a prime of k at which X has good reduction, lying over the rational prime p. Fix a place $\bar{v}$ of $\bar{k}$ above v and let G_v denote the decomposition group of $\bar{v}$ over v. Take $\ell = p$, and consider the composite map

(2.2.2)
$$\begin{aligned}
\delta_{0,v} : \mathrm{CH}^j(X)_{0,\mathbf{Q}} &\xrightarrow{\delta_0} H^1(G_k, H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_p(j))) \\
&\xrightarrow{\mathrm{res}_v} H^1(G_v, H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_p(j))).
\end{aligned}$$

We will need to study the image of the coniveau filtration under the map $\delta_{0,v}$, for which we need some notations and results from p-adic Hodge theory, recalled in the following subsection.

2.3. The Bloch–Kato exponential and logarithm. We summarize some results from p-adic Hodge theory, referring the reader to [7,28] for more details. Let K be a finite extension of $\mathbf{Q}_p$ and $G_K := \mathrm{Gal}(\overline{K}/K)$. We denote by K_0 the maximal unramified extension of $\mathbf{Q}_p$ in K and by σ the absolute Frobenius acting on K_0 so that σ induces the pth power map on the residue field of K_0. Let V be a p-adic representation of G_K, i.e., a finite-dimensional $\mathbf{Q}_p$-vector space with a continuous action of G_K. Associated with V are certain vector spaces (over K_0 or K) with additional structures:

(i) $D_{\mathrm{cris}}(V) := (V \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})^{G_K}$, a K_0-vector space equipped with a σ-semilinear bijective map $\varphi : D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V)$.

(ii) $D_{\mathrm{st}}(V) := (V \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})^{G_K}$, a K_0-vector space equipped with a σ-semilinear bijective map $\varphi : D_{\mathrm{st}}(V) \to D_{\mathrm{st}}(V)$ and a K_0-linear map $N : D_{\mathrm{st}}(V) \to D_{\mathrm{st}}(V)$ (called the monodromy operator) satisfying $N\varphi = p\varphi N$. We let φ_K denote the K_0-linear map $\varphi^{[K_0:\mathbf{Q}_p]}$.

(iii) $D_{\mathrm{dR}}(V) := (V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{G_K}$, a K-vector space equipped with a decreasing filtration

$$(\mathrm{Fil}^r D_{\mathrm{dR}}(V))_{r\in\mathbf{Z}},$$

satisfying $\bigcup \mathrm{Fil}^r D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V)$, and $\bigcap \mathrm{Fil}^r D_{\mathrm{dR}}(V) = 0$.

These are related by

- $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)^{N=0}$,
- the injective canonical maps

$$D_{\mathrm{cris}}(V) \otimes_{K_0} K \to D_{\mathrm{dR}}(V) \tag{2.3.1}$$

and

$$D_{\mathrm{st}}(V) \otimes_{K_0} K \to D_{\mathrm{dR}}(V), \tag{2.3.2}$$

- $\dim_{K_0} D_{\mathrm{cris}}(V) \le \dim_{K_0} D_{\mathrm{st}}(V) \le \dim_K D_{\mathrm{dR}}(V) \le \dim_{\mathbf{Q}_p}(V)$.

The representation V is said to be crystalline (resp. semistable, resp. de Rham) if $\dim_{K_0} D_{\mathrm{cris}}(V)$ (resp. $\dim_{K_0} D_{\mathrm{st}}(V)$, resp. $\dim_K D_{\mathrm{dR}}(V)$) is equal to $\dim_{\mathbf{Q}_p}(V)$. Thus

$$V \text{ crystalline} \implies V \text{ semistable} \implies V \text{ de Rham}.$$

Further, if V is semistable, then the map (2.3.2) is an isomorphism, while if V is crystalline, then $N = 0$ on $D_{\mathrm{st}}(V)$, $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)$ and both (2.3.1) and (2.3.2) are isomorphisms.

Finally, we need to recall certain subspaces of $H^1(G_K, V)$. The subspace $H^1_f(G_K, V)$ is defined by

$$H^1_f(G_K, V) = \ker(H^1(G_K, V) \to H^1(G_K, V \otimes B_{\mathrm{cris}})).$$

Likewise, the subspaces $H^1_{\mathrm{st}}(G_K, V)$ and $H^1_g(G_K, V)$ are defined by replacing B_{cris} by B_{st} and B_{dR} respectively. Concretely, if V is a crystalline (resp. semistable, resp. de Rham) representation, then an element ξ of $H^1_f(G_K, V)$ (resp. $H^1_{\mathrm{st}}(G_K, V)$, resp. $H^1_g(G_K, V)$) is represented by an extension of p-adic representations:

$$0 \to V \to V' \to \mathbf{Q}_p \to 0, \tag{2.3.3}$$

where V' is also crystalline (resp. semistable, resp. de Rham). It is known that if V is semistable, then

$$H^1_{\mathrm{st}}(G_K, V) = H^1_g(G_K, V)$$

([28, Proposition 1.24]), though we will not need this fact below.

Suppose that V is a de Rham representation satisfying $D_{\mathrm{cris}}(V)^{\varphi_K=1} = 0$. Then the Bloch–Kato exponential ([7, Definition 3.10 and Theorem 4.1 (ii)]) gives an isomorphism

$$\exp : D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V) \simeq H^1_f(G_K, V).$$

The inverse isomorphism will be denoted

$$\log : H^1_f(G_K, V) \simeq D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V).$$

If further V is crystalline, the log map has the following alternate description which is very useful. Namely, let $\xi \in H^1_f(G_K, V)$ and suppose ξ is represented by an extension as in (2.3.3) above, so that V' is also crystalline. Applying the functor D_{dR} yields exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & D_{\mathrm{dR}}(V) \arrow[r] & D_{\mathrm{dR}}(V') \arrow[r] & K \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Fil}^0 D_{\mathrm{dR}}(V) \arrow[r] \arrow[u] & \mathrm{Fil}^0 D_{\mathrm{dR}}(V') \arrow[r] \arrow[u] & K \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

Pick elements $x \in \mathrm{Fil}^0 D_{\mathrm{dR}}(V')$ and $y \in D_{\mathrm{cris}}(V')^{\varphi_K=1} \subset D_{\mathrm{dR}}(V')$ such that both map to the element $1 \in K$. Then $x - y \in D_{\mathrm{dR}}(V)$ and its class in $D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)$ is well defined since $D_{\mathrm{cris}}(V)^{\varphi_K=1} = 0$. The alternate description of $\log(\xi)$ is then

$$\log(\xi) = [x - y] \in D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V). \tag{2.3.4}$$

2.4. Coniveau and the Hodge filtration. We now apply the tools from the previous subsection to study the map $\delta_{0,v}$ defined in (2.2.2) above. Let $V = H^{2j-1}(X_{\bar{k}}, \mathbf{Q}_p(j))$ and $K = k_v$, and identify G_v with G_K. By [29, Theorem 3.1 (i)], the image of the map $\delta_{0,v}$ is contained in $H^1_f(G_K, V)$. Since V is a de Rham representation (in fact even crystalline) and $D_{\mathrm{cris}}(V)^{\varphi_K=1} = 0$ (by [25]), the log map gives an isomorphism

$$\log : H^1_f(G_K, V) \simeq D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V).$$

As a filtered vector space, $D_{\mathrm{dR}}(V)$ is isomorphic to $H^{2j-1}_{\mathrm{dR}}(X_K)$ equipped with the usual Hodge filtration shifted by j. Thus

$$\mathrm{Fil}^0 D_{\mathrm{dR}}(V) = \mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K).$$

By Poincaré duality, there is a canonical isomorphism

$$\mathrm{PD} : D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V) = H^{2j-1}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K) \simeq (\mathrm{Fil}^{j^*} H^{2j^*-1}_{\mathrm{dR}}(X_K))^\vee,$$

where for any $j \le \dim(X)$, the integer j^* is defined by $j + j^* = \dim(X) + 1$. Finally, let us define

$$\alpha_v := \log \circ \delta_{0,v} : \mathrm{CH}^j(X)_{0,\mathbf{Q}} \to H^{2j-1}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K)$$

and

$$\beta_v := \mathrm{PD} \circ \log \circ \delta_{0,v} : \mathrm{CH}^j(X)_{0,\mathbf{Q}} \to (\mathrm{Fil}^{j^*} H^{2j^*-1}_{\mathrm{dR}}(X_K))^\vee.$$

The following key proposition relates the coniveau filtration on the Chow group to the Hodge filtration on de Rham cohomology.

Proposition 2.4.1. *For any integer i in the range $0 \le i \le j$,*

$$\alpha_v(N^i \mathrm{CH}^j(X)_{0,\mathbf{Q}}) \subseteq \mathrm{Fil}^i H^{2j-1}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K).$$

The following corollary is immediate:

Corollary 2.4.2. *For any integer i in the range $0 \le i \le j$, $\beta_v(N^i \mathrm{CH}^j(X)_{0,\mathbf{Q}})$ annihilates $\mathrm{Fil}^{i^*} H^{2j^*-1}_{\mathrm{dR}}(X_K)$.*

Remark 2.4.3. In all our applications below, we will always have $j = j^*$, so that $\dim(X) = 2j - 1$ and the cycles are in the middle (arithmetic) dimension. We state this case separately below.

Corollary 2.4.4. *Suppose that* $j = j^*$, *so that* $\dim(X) = 2j-1$. *Then:*

- $\beta_v(N^i\mathrm{CH}^j(X)_{0,\mathbf{Q}})$ *annihilates* $\mathrm{Fil}^{2j-i} H^{2j-1}_{\mathrm{dR}}(X_K)$,
- $\beta_v(N^1\mathrm{CH}^j(X)_{0,\mathbf{Q}})$ *annihilates* $\mathrm{Fil}^{2j-1} H^{2j-1}_{\mathrm{dR}}(X_K) = H^0(X_K, \Omega^{2j-1})$.

Proof of Proposition 2.4.1. Let $\gamma \in N^i\mathrm{CH}^j(X)_{0,\mathbf{Q}}$ and let (Y,c) be a pair consisting of a closed subscheme $f : Y \subset X_{\bar{k}}$ of codimension i and a cycle c on $X_{\bar{k}}$ representing γ such that $\mathrm{supp}(c)$ is contained in Y and c is homologous to zero on Y. We need to show that $\alpha_v(\gamma)$ lies in $\mathrm{Fil}^i H^{2j-1}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K)$. Replacing γ by an integer multiple, we may assume (as in the proof of [6, Proposition 1.5]) that Y and c are defined over k.

Let us suppose for the moment that Y is smooth over k and has good reduction at v. Writing $W := H^{2(j-i)-1}(Y_{\bar{k}}, \mathbf{Q}_p(j-i))$, there is a commutative diagram:

$$\begin{tikzcd}
\mathrm{CH}^{j-i}(Y)_{0,\mathbf{Q}} \arrow[r, "\mathrm{res}_v \circ \mathrm{AJ}_Y"] \arrow[d, "f_*"] & H^1_f(G_K, W) \arrow[r, "\log", "\simeq"'] \arrow[d, "f_*"] & D_{\mathrm{dR}}(W)/\mathrm{Fil}^0 = H^{2(j-i)-1}_{\mathrm{dR}}(Y_K)/\mathrm{Fil}^{j-i} \arrow[d, "f_*"] \\
\mathrm{CH}^j(X)_{0,\mathbf{Q}} \arrow[r, "\delta_{0,v} = \mathrm{res}_v \circ \mathrm{AJ}_X"] & H^1_f(G_K, V) \arrow[r, "\log", "\simeq"'] & D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 = H^{2j-1}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^j.
\end{tikzcd}$$

Since the pushforward (Gysin) map f_* on the far right takes $\mathrm{Fil}^{\cdot}$ to $\mathrm{Fil}^{\cdot+i}$ (as can be seen by embedding K in $\mathbf{C}$ and using Hodge theory), it follows that the image of $\mathrm{CH}^{j-i}(Y)_{0,\mathbf{Q}}$ under $\log \circ \delta_{0,v} \circ f_*$ is contained in $\mathrm{Fil}^i H^{2j-1}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K)$, hence $\alpha_v(\gamma)$ is also contained in the same subspace.

In general, Y may be singular and in any case may have bad reduction, so we argue somewhat differently. Let $Z := \mathrm{supp}(c)$ and write $\bar{X}, \bar{Y}, \bar{Z}$ for $X_{\bar{k}}, Y_{\bar{k}}, Z_{\bar{k}}$ respectively. Also let $\xi_\gamma := \delta_{0,v}(\gamma)$. Then (as in [6, Proposition 1.5], again) the Abel–Jacobi map (restricted to the place v) associates with c an element

$$\xi_c \in H^1(G_K, H^{2j-1}_{\bar{Y}}(\bar{X})(j))$$

which maps to ξ_γ under the natural map

$$H^1(G_K, H^{2j-1}_{\bar{Y}}(\bar{X})(j)) \to H^1(G_K, H^{2j-1}(\bar{X})(j)).$$

Both ξ_γ and ξ_c may be viewed as extensions coming from a long exact sequence of cohomology. Indeed, there is a commutative diagram:

$$\begin{tikzcd}
0 = H^{2j-1}_{\bar{Z}}(\bar{X}, \mathbf{Q}_p(j)) \arrow[r] & H^{2j-1}(\bar{X}, \mathbf{Q}_p(j)) \arrow[r] & H^{2j-1}(\bar{X}-\bar{Z}, \mathbf{Q}_p(j)) \arrow[r] & H^{2j}_{\bar{Z}}(\bar{X}, \mathbf{Q}_p(j)) \\
0 = H^{2j-1}_{\bar{Z}}(\bar{X}, \mathbf{Q}_p(j)) \arrow[r] \arrow[u, equal] & H^{2j-1}_{\bar{Y}}(\bar{X}, \mathbf{Q}_p(j)) \arrow[r] \arrow[u] & H^{2j-1}_{\bar{Y}-\bar{Z}}(\bar{X}-\bar{Z}, \mathbf{Q}_p(j)) \arrow[r] \arrow[u] & H^{2j}_{\bar{Z}}(\bar{X}, \mathbf{Q}_p(j)) \arrow[u, equal]
\end{tikzcd}$$

and the extensions ξ_γ and ξ_c are obtained by pulling back the first and second lines by the map

$$\mathbf{Q}_p \to H^{2j}_{\bar{Z}}(\bar{X}, \mathbf{Q}_p(j)), \quad 1 \mapsto \mathrm{cl}(c).$$

(Note that $\mathrm{cl}(c)$ maps to 0 in the next term of each row, since c is homologically trivial on Y.)

Thus, writing $W := H^{2j-1}_{\bar{Y}}(\bar{X}, \mathbf{Q}_p(j))$, we have a commutative diagram of G_K-representations

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & V' \arrow[r] & \mathbf{Q}_p \arrow[r] & 0 \\ 0 \arrow[r] & W \arrow[r] \arrow[u] & W' \arrow[r] \arrow[u] & \mathbf{Q}_p \arrow[r] \arrow[u, equal] & 0, \end{tikzcd}$$

where the top row represents ξ_γ and the bottom row represents ξ_c.

Since V is crystalline and $\xi_\gamma \in H^1_f(G_K, V)$, the representation V' is crystalline. In contrast with V and V', the representations W and W' need not be crystalline. Nevertheless, by a theorem of Kisin ([26, Theorem 3.2]), W and W' are at least potentially semistable, so after making a finite extension L/K, we may assume ξ_c is semistable, that is,

$$\mathrm{res}_{K/L}\, \xi_c \in H^1_{\mathrm{st}}(G_L, W).$$

Applying the functors D_{st} and D_{dR} (over L), we see that there are commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & D_{\mathrm{st}}(V) \arrow[r] & D_{\mathrm{st}}(V') \arrow[r] & L_0 \arrow[r] & 0 \\ 0 \arrow[r] & D_{\mathrm{st}}(W) \arrow[r] \arrow[u, "\iota"] & D_{\mathrm{st}}(W') \arrow[r] \arrow[u, "\iota"] & L_0 \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & D_{\mathrm{dR}}(V) \arrow[r] & D_{\mathrm{dR}}(V') \arrow[r] & L \arrow[r] & 0 \\ 0 \arrow[r] & D_{\mathrm{dR}}(W) \arrow[r] \arrow[u, "\iota"] & D_{\mathrm{dR}}(W') \arrow[r] \arrow[u, "\iota"] & L \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

with the rows being exact. Pick elements $x' \in \mathrm{Fil}^0 D_{\mathrm{dR}}(W')$ and $y' \in D_{\mathrm{st}}(W')^{\varphi_L=1}$ such that both x' and y' map to the element $1 \in L$. Let

$$x = \iota(x'), \quad y = \iota(y'),$$

so that

$$x \in \mathrm{Fil}^0 D_{\mathrm{dR}}(V'), \quad y \in D_{\mathrm{st}}(V')^{\varphi_L=1} = D_{\mathrm{cris}}(V')^{\varphi_L=1},$$

and x, y both map to the element $1 \in L$. By the alternate description of the log map, we have

$$\begin{aligned} \log(\mathrm{res}_{K/L}(\xi_\gamma)) &= [x - y] \\ &= [\iota(x') - \iota(y')] \\ &= [\iota(x' - y')] \in D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 = H^{2j-1}_{\mathrm{dR}}(X_L)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_L). \end{aligned}$$

But $x' - y' \in D_{\mathrm{dR}}(W)$ and the map $\iota : D_{\mathrm{dR}}(W) \to D_{\mathrm{dR}}(V)$ is identified (via the p-adic comparison theorem, see [26, Theorem 3.3]) with the canonical map

$$\iota : H^{2j-1}_{Y_L,\mathrm{dR}}(X_L) \to H^{2j-1}_{\mathrm{dR}}(X_L).$$

Since Y has codimension $\geq i$ in X, the image of this map lands in $\mathrm{Fil}^i H^{2j-1}_{\mathrm{dR}}(X_L)$, as can be seen for example by embedding L in $\mathbf{C}$ and using usual (mixed) Hodge theory (see [31, Section 7.1 and Corollary 7.4]). We have thus shown

$$\log(\mathrm{res}_{K/L}(\xi_\gamma)) \in \mathrm{Fil}^i H^{2j-1}_{\mathrm{dR}}(X_L)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_L).$$

But then the commutativity of the diagram

$$\begin{array}{ccc} H^1_f(G_K, V) & \xrightarrow{\log} & H^{2j-1}_{\mathrm{dR}}(X_K)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K) \\ \Big\downarrow{\scriptstyle \mathrm{res}_{K/L}} & & \Big\downarrow{\scriptstyle \mathrm{can}} \\ H^1_f(G_L, V) & \xrightarrow{\log} & H^{2j-1}_{\mathrm{dR}}(X_L)/\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_L) \end{array}$$

implies that

$$\begin{aligned} \alpha_v(\gamma) = \log(\xi_\gamma) \in (H^{2j-1}_{\mathrm{dR}}(X_K) \cap \mathrm{Fil}^i H^{2j-1}_{\mathrm{dR}}(X_L))/&\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K) \\ &\| \\ \mathrm{Fil}^i H^{2j-1}_{\mathrm{dR}}(X_K)/&\mathrm{Fil}^j H^{2j-1}_{\mathrm{dR}}(X_K). \qquad \square \end{aligned}$$

3. *p*-adic *L*-functions

In this section, we summarize the main results about p-adic L-functions that will be used in the following sections. The reader may refer to [2, Sections 3A–3D] for a more detailed discussion. Henceforth, we view all number fields as being contained in a fixed algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. A complex embedding $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and an embedding $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}_\ell$ for each rational prime ℓ are chosen once and for all. Let K be an imaginary quadratic field of discriminant $-D$, and let p be a fixed rational prime which is *split* in K. Write $\mathfrak{p}$ for the prime above p corresponding to the distinguished embedding $K \hookrightarrow \bar{\mathbf{Q}}_p$. Also write $\mathfrak{d}_K$ for the ideal $(\sqrt{-D})$ in $\mathcal{O}_K$.

3.1. The Katz *p*-adic *L*-function. Let $\mathfrak{c}$ be an integral ideal of K which is prime to p, and let $\Sigma(\mathfrak{c})$ denote the set of all Hecke characters of K of conductor dividing $\mathfrak{c}$. For $\nu \in \Sigma(\mathfrak{c})$, write $L_{\mathfrak{c}}(\nu, s)$ for the Hecke L-function attached to ν with the Euler factors at the primes dividing $\mathfrak{c}$ removed. Denote by $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ the set of critical characters, i.e., those characters ν in $\Sigma(\mathfrak{c})$ for which $L(\nu^{-1}, 0)$ is a critical value in the sense of Deligne. This set is the disjoint union

$$\Sigma_{\mathrm{crit}}(\mathfrak{c}) = \Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c}) \sqcup \Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c}),$$

where

$$\begin{aligned} \Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c}) &:= \{\nu \in \Sigma(\mathfrak{c}) \text{ of infinity type } (\ell_1, \ell_2) \text{ with } \ell_1 \le 0,\ \ell_2 \ge 1\}, \\ \Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c}) &:= \{\nu \in \Sigma(\mathfrak{c}) \text{ of infinity type } (\ell_1, \ell_2) \text{ with } \ell_1 \ge 1,\ \ell_2 \le 0\}. \end{aligned}$$

Viewing the elements of $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ as p-adic characters, this set is endowed with the natural topology of uniform convergence with respect to the p-adic norm. Write $\hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c})$ for the completion of $\Sigma_{\mathrm{crit}}(\mathfrak{c})$ relative to this topology. Note that $\Sigma^{(1)}_{\mathrm{crit}}(\mathfrak{c})$ and $\Sigma^{(2)}_{\mathrm{crit}}(\mathfrak{c})$ are both dense subsets of $\hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c})$.

Let A be an elliptic curve defined over the Hilbert class field H of K such that

$$A_{\mathbf{C}} \simeq \mathbf{C}/\mathcal{O}_K,$$

and let $\omega_A \in \Omega^1(A_{/H})$ be a nowhere vanishing differential. We further assume that A has been chosen to have good reduction at the prime above p, and that ω_A extends to a nowhere vanishing differential on the Néron model $\mathcal{A}/\mathcal{O}_{H,\mathfrak{p}}$. (In the applications in this article, we shall only use the case when K has class number one, the elliptic curve A is defined over $\mathbf{Q}$ and ω_A

is a global Néron differential on A.) We fix once and for all an isomorphism $\hat{\mathcal{A}} \simeq \hat{\mathbb{G}}_m$, where $\hat{\mathcal{A}}$ denotes the formal completion of $\mathcal{A}$ along the identity section. The choice of ω_A along with the isomorphisms $A \simeq \mathbf{C}/\mathcal{O}_K$ and $\hat{\mathcal{A}} \simeq \hat{\mathbb{G}}_m$ determines a complex period $\Omega \in \mathbf{C}^\times$ and a p-adic period $\Omega_p \in \mathbf{C}_p^\times$ ([13, Chapter II, Section 4.4]), the latter being a p-adic unit.

Theorem 3.1.1 ([22, Section 9], [13, Chapter II, Section 4.14]). *There exists a unique p-adically continuous $\mathbf{C}_p$-valued function $\mathcal{L}_{p,\mathfrak{c}}(\nu)$ on $\hat{\Sigma}_{\mathrm{crit}}(\mathfrak{c})$ satisfying the interpolation property*

$$\frac{\mathcal{L}_{p,\mathfrak{c}}(\nu)}{\Omega_p^{\ell_1-\ell_2}} = \left(\frac{\sqrt{D}}{2\pi}\right)^{\ell_2}(\ell_1-1)!\left(1-\frac{\nu(\mathfrak{p})}{p}\right)(1-\nu^{-1}(\bar{\mathfrak{p}}))\frac{L_{\mathfrak{c}}(\nu^{-1},0)}{\Omega^{\ell_1-\ell_2}} \tag{3.1.1}$$

for all $\nu \in \Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c})$ of infinity type (ℓ_1, ℓ_2).

3.2. p-adic Rankin L-series. Here we summarize the properties of the p-adic Rankin L-series of [3, Section 5]. Let N be a positive integer, let ε be a Dirichlet character mod N and let $f \in S_k(\Gamma_0(N), \varepsilon)$ be a normalized newform of weight $k \geq 2$ and character ε. Assume that the pair (f, K) satisfies the *Heegner hypothesis*, i.e., $\mathcal{O}_K$ contains an ideal $\mathfrak{N}$ such that $\mathcal{O}_K/\mathfrak{N} = \mathbf{Z}/N\mathbf{Z}$. Fix once and for all such a cyclic ideal $\mathfrak{N}$ of norm N.

Let χ be a Hecke character of K of infinity type (ℓ_1, ℓ_2), and let ε_χ denote its central character. Recall that ε_χ is the finite order character of $\mathbf{A}_{\mathbf{Q}}^\times$ determined by the equation

$$\chi|_{\mathbf{A}_{\mathbf{Q}}^\times} = \varepsilon_\chi \cdot \mathbf{N}^{\ell_1+\ell_2},$$

where $\mathbf{N}$ is the norm character. We say that χ as above is *central critical* for f if

$$\ell_1 + \ell_2 = k \quad \text{and} \quad \varepsilon_\chi = \varepsilon.$$

In this case, $s = 0$ is the central critical point for the complex Rankin L-series $L(f, \chi^{-1}, s)$.

Fix a rational integer c prime to pN and write $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ for the set of central critical characters satisfying the conditions of [2, Definition 3.10]. This set is the disjoint union

$$\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon) = \Sigma_{\mathrm{cc}}^{(1)}(c, \mathfrak{N}, \varepsilon) \sqcup \Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon),$$

where $\Sigma_{\mathrm{cc}}^{(1)}(c, \mathfrak{N}, \varepsilon)$ (resp. $\Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon)$) is defined to consist of those characters in $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ of infinity type $(k+j, -j)$ with $-(k-1) \leq j \leq -1$ (resp. $j \geq 0$.) The domain of classical interpolation for the p-adic Rankin L-series attached to f and K is the set $\Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon)$. For $\chi \in \Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon)$, we define the *algebraic part* of $L(f, \chi^{-1}, 0)$ to be

$$L^{\mathrm{alg}}(f, \chi^{-1}, 0) := w(f, \chi)^{-1} C(f, \chi, c) \cdot \frac{L(f, \chi^{-1}, 0)}{\Omega^{2(k+2j)}}, \tag{3.2.1}$$

where $w(f, \chi)^{-1}$ and $C(f, \chi, c)$ are the constants defined in [3, equation (5.1.11)] and [3, Theorem 4.6], respectively. The set $\Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon)$ is dense in the p-adic completion $\hat{\Sigma}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ of $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$.

Proposition 3.2.1 ([3, Proposition 5.10]). *Suppose that D and c are odd integers. There is a unique $\mathbf{C}_p$-valued, p-adically continuous function $L_p(f, \chi)$ on $\hat{\Sigma}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ satisfying the interpolation property*

$$L_p(f, \chi) := \Omega_p^{2(k+2j)}(1 - \chi^{-1}(\bar{\mathfrak{p}})a_p(f) + \chi^{-2}(\bar{\mathfrak{p}})\varepsilon(p)p^{k-1})^2 L^{\mathrm{alg}}(f, \chi^{-1}, 0) \tag{3.2.2}$$

for all χ in $\Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon)$ of infinity type $(k+j, -j)$.

Remark 3.2.2. The hypotheses that D and c are odd are made in [3, Proposition 5.10] to simplify some of the computations, and are certainly not crucial. However, they will suffice for our applications below.

3.3. Factorization of the p-adic Rankin L-series. We specialize the p-adic L-function of the previous subsection to the case that f is itself a theta series associated with a Hecke character of K; then this p-adic L-function factors as a product of two Katz p-adic L-functions.

Let ψ be a Hecke character of K of infinity type $(k-1, 0)$, with $k \geq 2$, and consider the associated theta series

$$\theta_\psi := \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}},$$

the sum being taken over the ideals of $\mathcal{O}_K$ prime to the conductor $\mathfrak{f}_\psi$ of ψ. Assume that $\mathfrak{f}_\psi$ is a cyclic ideal $\mathfrak{m}$ of norm M prime to D. Taking $N := MD$, the form $f := \theta_\psi$ belongs to the space of cusp forms $S_k(\Gamma_0(N), \varepsilon)$, with $\varepsilon = \varepsilon_K \varepsilon_\psi$. Moreover, the pair (f, K) satisfies the Heegner hypothesis, the ideal $\mathfrak{N} := \mathfrak{d}_K \mathfrak{m}$ being cyclic of norm N.

Proposition 3.3.1 ([2, Theorem 3.17]). *For all $\chi \in \Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$, we have a factorization*

$$L_p(\theta_\psi, \chi) = \frac{w(\theta_\psi, \chi)^{-1} w_K}{2c^{k+2j-1}} \prod_{q \mid c} \frac{q - \varepsilon_K(q)}{q-1} \times \mathcal{L}_{p, c\mathfrak{d}_K}(\psi^{-1}\chi) \times \mathcal{L}_{p, c\mathfrak{d}_K \mathfrak{m}}(\psi^{*-1}\chi), \tag{3.3.1}$$

where ψ^* is the Hecke character defined by $\psi^*(x) = \psi(\bar{x})$.

If χ belongs to the range $\Sigma^{(2)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ of classical interpolation for $L_p(\theta_\psi, \cdot)$, then $\psi^{-1}\chi$ belongs to $\Sigma^{(2)}_{\mathrm{crit}}(c\mathfrak{d}_K)$ and $\psi^{*-1}\chi$ belongs to $\Sigma^{(2)}_{\mathrm{crit}}(c\mathfrak{d}_K\mathfrak{m})$ (see [2, Lemma 3.16]), so that both these characters are in the range of classical interpolation for the Katz p-adic L-function. As a consequence, the proof of the above proposition reduces to obtaining a similar factorization formula for complex special values, which is a simple consequence of the Artin formalism for these L-series. The usefulness of the proposition lies in applying the equality above to characters χ in $\Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$, which are outside the range of interpolation. In this case, the character $\psi^{*-1}\chi$ still lies in $\Sigma^{(2)}_{\mathrm{crit}}(c\mathfrak{d}_K\mathfrak{m})$ but $\psi^{-1}\chi$ lies in $\Sigma^{(1)}_{\mathrm{crit}}(c\mathfrak{d}_K)$ and is outside the range of interpolation for the Katz p-adic L-function $\mathcal{L}_{p,c\mathfrak{d}_K}$.

Remark 3.3.2. We observe that the critical characters $\psi^{-1}\chi$ and $\psi^{*-1}\chi$ arising as above when χ belongs to $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ are *self-dual*. (See [2, Definition 3.4 and Lemma 3.16].)

Corollary 3.3.3. *Suppose that χ belongs to $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$. Then*

$$L_p(\theta_\psi, \chi) \neq 0 \iff \mathcal{L}_{p,\mathfrak{c}}(\psi^{-1}\chi) \neq 0 \text{ and } \mathcal{L}_{p,\mathfrak{c}}(\psi^{*-1}\chi) \neq 0,$$

where $\mathfrak{c}$ is any ideal in $\mathcal{O}_K$, prime to p and divisible by $c\mathfrak{d}_K\mathfrak{m}$.

4. Generalized Heegner cycles and coniveau

4.1. Generalized Heegner cycles. In this subsection, we will give a slight variant of the construction of generalized Heegner cycles given in [3]. At first we work in the more general setting of Section 3.2. Later, in Section 4.3, we specialize the discussion to the case of CM forms as in Section 3.3.

The setup will thus be as in Section 3.2, except we take $c = 1$ since this suffices for our applications. Let N be a positive integer, K an imaginary quadratic field (of discriminant $-D$) that satisfies the Heegner condition with respect to N, i.e., all the primes dividing N are either split or ramified in K and if $q^2 \mid N$, then q is split in K. Thus there exists a cyclic ideal $\mathfrak{N}$ of norm N in $\mathcal{O}_K$, i.e., $\mathcal{O}_K/\mathfrak{N} = \mathbf{Z}/N\mathbf{Z}$. We fix such an ideal $\mathfrak{N}$ in what follows. We also fix once and for all an elliptic curve $A \simeq \mathbf{C}/\mathcal{O}_K$ defined over the Hilbert class field H of K, and with CM by $\mathcal{O}_K$, and a generator t of the cyclic group $A[\mathfrak{N}]$. The pair (A, t) defines a point P on the modular curve $X_1(N)$, that is defined over an abelian extension of K.

For $\mathfrak{a}$ an ideal in $\mathcal{O}_K$, let us write $A_{\mathfrak{a}}$ for the elliptic curve $\mathbf{C}/\mathfrak{a}^{-1}$ and let $\varphi_{\mathfrak{a}}$ denote the canonical isogeny of degree $\mathrm{N}\mathfrak{a}$,

$$\varphi_{\mathfrak{a}} : A = \mathbf{C}/\mathcal{O}_K \to \mathbf{C}/\mathfrak{a}^{-1} = A_{\mathfrak{a}}.$$

Define a variety X_{r_1,r_2} of dimension $r_1 + r_2 + 1$ by

$$X_{r_1,r_2} = W_{r_1} \times A^{r_2},$$

where W_{r_1} is the Kuga–Sato variety of dimension $r_1 + 1$ over $X_1(N)$. (We refer the reader to [3, Section 2 and Appendix] for a more explicit description of this variety and its cohomology.) The only cases that will interest us are when $r_1 \geq r_2$ and $r_1 \equiv r_2 \bmod 2$. Write $r_1 + r_2 = 2s$ and $r_1 - r_2 = 2u$. For each ideal $\mathfrak{a}$ in $\mathcal{O}_K$ that is prime to $\mathfrak{N}$, we construct a cycle

$$\Delta_{r_1,r_2,\mathfrak{a}} \in \mathrm{CH}^{s+1}(X_{r_1,r_2})_{0,\mathbf{Q}}$$

as follows.

Let $t_{\mathfrak{a}}$ denote the image of t under the map $\varphi_{\mathfrak{a}}$. Then the pair $(A_{\mathfrak{a}}, t_{\mathfrak{a}})$ defines a point on the modular curve $X_1(N)$, which we denote $P_{\mathfrak{a}}$. The fiber of W_{r_1} over $P_{\mathfrak{a}}$ is canonically isomorphic to $A_{\mathfrak{a}}^{r_1}$. Thus viewing X_{r_1,r_2} as fibered over $X_1(N)$, the fiber over $P_{\mathfrak{a}}$ is

$$(A_{\mathfrak{a}}^{r_1} \times A^{r_2}) = (A_{\mathfrak{a}} \times A)^{r_2} \times (A_{\mathfrak{a}} \times A_{\mathfrak{a}})^u.$$

Now set (with tr meaning transpose),

$$\begin{aligned}
\Gamma_{\mathfrak{a}} &= (\text{graph of } \sqrt{-D})^{\mathrm{tr}} \in Z^1(A_{\mathfrak{a}} \times A_{\mathfrak{a}}), \\
\Gamma_{\varphi,\mathfrak{a}} &= (\text{graph of } \varphi_{\mathfrak{a}})^{\mathrm{tr}} \in Z^1(A_{\mathfrak{a}} \times A), \\
\Gamma_{r_1,r_2,\mathfrak{a}} &= \Gamma_{\varphi,\mathfrak{a}}^{r_2} \times \Gamma_{\mathfrak{a}}^{u},
\end{aligned}$$

and

$$\Delta_{r_1,r_2,\mathfrak{a}} := \epsilon_{X_{r_1,r_2}}(\Gamma_{r_1,r_2,\mathfrak{a}}) \in \mathrm{CH}^{s+1}(X_{r_1,r_2})_{\mathbf{Q}}, \tag{4.1.1}$$

where $\epsilon_{X_{r_1,r_2}} := \epsilon_{W_{r_1}} \epsilon_{A^{r_2}}$ with $\epsilon_{W_{r_1}}$ and $\epsilon_{A^{r_2}}$ being the projectors on W_{r_1} and A^{r_2} described in [3, Section 2.1 and Section 1.4] respectively. (There they are called just ϵ_W and ϵ_A respectively, but here we will need to keep track of the indices.) When $r_1 = r_2 = r$ say, we just write X_r, $\Gamma_{r,\mathfrak{a}}$ and $\Delta_{r,\mathfrak{a}}$ for $X_{r,r}$, $\Gamma_{r,r,\mathfrak{a}}$ and $\Delta_{r,r,\mathfrak{a}}$ respectively.

The same argument as in [3, Section 2.2 and Section 2.3] shows that $\Delta_{r_1,r_2,\mathfrak{a}}$ is homologically trivial on X_{r_1,r_2}. This also follows from the following proposition which shows that the cycles constructed here are not very much more general than the *generalized Heegner cycles* of loc. cit., since they can be obtained by correspondence from the previously defined cycles on a higher-dimensional variety.

Proposition 4.1.1. *There exists an algebraic correspondence (in the ring of correspondences tensor* $\mathbf{Q}$*)*

$$P : X_{r_1} = W_{r_1} \times A^{r_1} \dashrightarrow W_{r_1} \times A^{r_2} = X_{r_1,r_2}$$

such that for all ideals $\mathfrak{a}$ *(prime to* $\mathfrak{N}$*), we have*

$$P_*(\Gamma_{r_1,\mathfrak{a}}) = (\mathrm{N}\mathfrak{a})^u \Gamma_{r_1,r_2,\mathfrak{a}}, \tag{4.1.2}$$

$$P_*(\Delta_{r_1,\mathfrak{a}}) = (\mathrm{N}\mathfrak{a})^u \Delta_{r_1,r_2,\mathfrak{a}}, \tag{4.1.3}$$

as classes in $\mathrm{CH}^{s+1}(X_{r_1,r_2})$*. In particular, since* $\Delta_{r_1,\mathfrak{a}}$ *is homologically trivial on* X_{r_1}*, so is* $\Delta_{r_1,r_2,\mathfrak{a}}$ *on* X_{r_1,r_2}*.*

Proof. Consider the variety

$$Y = W_{r_1} \times A^s = (W_{r_1} \times A^{r_2}) \times A^u = X_{r_1,r_2} \times A^u.$$

We embed Y in the product

$$X_{r_1} \times X_{r_1,r_2} = (X_{r_1,r_2} \times (A \times A)^u) \times X_{r_1,r_2}$$

via the map

$$((\mathrm{id}_{X_{r_1,r_2}}, (\sqrt{-D}, \mathrm{id}_A)^u), \mathrm{id}_{X_{r_1,r_2}}).$$

Then $Y \in Z^{r_1+s+1}(X_{r_1} \times X_{r_1,r_2})$ and it induces a correspondence $P : X_{r_1} \dashrightarrow X_{r_1,r_2}$. Since $p_1^*(\Delta_{r_1})$ intersects transversally with Y in $X_{r_1} \times X_{r_1,r_2}$, it is straightforward to check that the relation (4.1.2) holds using the fact that the degree of $\varphi_{\mathfrak{a}}$ is $N\mathfrak{a}$. To show (4.1.3), we need to study how P interacts with the automorphisms of X_{r_1} that arise in the sum defining $\epsilon_{X_{r_1}}$. For $\sigma \in \Xi_{2u} = \mu_2^{2u} \rtimes S_{2u}$, let δ_σ (resp. $\tilde{\delta}_\sigma$) be the automorphism of $X_{r_1} = W_{r_1} \times A^{r_2} \times A^{2u}$ that acts by σ on the last $2u$ components of the fiber over any point of W_{r_1} (resp. by σ on the last $2u$ copies of A.) Thus we can view $\Xi_{2u} \times \Xi_{2u}$ as a subgroup of $\mathrm{Aut}(X_{r_1})$ via $(\sigma, \tau) \mapsto (\delta_\sigma, \tilde{\delta}_\tau)$. For any $\sigma, \tau \in \Xi_{2u}$, we have

$$P_*(\delta_{\sigma\tau}\tilde{\delta}_\tau \cdot \Gamma_{r_1,\mathfrak{a}}) = P_*(\delta_\sigma \cdot \Gamma_{r_1,\mathfrak{a}}) = \delta_\sigma \cdot P_*(\Gamma_{r_1,\mathfrak{a}}), \tag{4.1.4}$$

where we also write δ_σ for the automorphism of X_{r_1,r_2} defined similarly, i.e., by the action of σ on the last $2u$ components of the fiber over W_{r_1}. Let G_{r_1} (resp. G_{r_1,r_2}) be the group of automorphisms of X_{r_1} (resp. of X_{r_1,r_2}) that occur in the sum defining $\epsilon_{X_{r_1}}$ (resp. $\epsilon_{X_{r_1,r_2}}$), and let μ denote the natural homomorphisms $G_{r_1} \to \mu_2$, $G_{r_1,r_2} \to \mu_2$. Observe that G_{r_1,r_2} is naturally identified as a subgroup of G_{r_1}. Then, viewing Ξ_{2u} as a subgroup of $\mathrm{Aut}(X_{r_1,r_2})$ via $\sigma \mapsto \delta_\sigma$, we have

$$\begin{aligned}
P_*\epsilon_{X_{r_1}}\Gamma_{r_1,\mathfrak{a}} &= P_*\left(\frac{1}{|G_{r_1}|}\sum_{g \in G_{r_1}} \mu(g) g \cdot \Gamma_{r_1,\mathfrak{a}}\right) \\
&= P_*\Bigg(\frac{1}{|G_{r_1}/(\Xi_{2u} \times \Xi_{2u})|}\sum_{g \in G_{r_1}/(\Xi_{2u} \times \Xi_{2u})} \mu(g) g \\
&\qquad \cdot \frac{1}{|\Xi_{2u}|^2}\sum_{(\sigma,\tau) \in \Xi_{2u}} \mu(\delta_\sigma\tilde{\delta}_\tau)\delta_\sigma\tilde{\delta}_\tau \cdot \Gamma_{r_1,\mathfrak{a}}\Bigg) \\
&= \frac{1}{|G_{r_1,r_2}/\Xi_{2u}|}\sum_{g \in G_{r_1,r_2}/\Xi_{2u}} \mu(g) g \\
&\qquad \cdot P_*\left(\frac{1}{|\Xi_{2u}|^2}\sum_{(\sigma,\tau) \in \Xi_{2u}} \mu(\delta_{\sigma\tau}\tilde{\delta}_\tau)\delta_{\sigma\tau}\tilde{\delta}_\tau \cdot \Gamma_{r_1,\mathfrak{a}}\right)
\end{aligned}$$

(by (4.1.4))

$$= \frac{1}{|G_{r_1,r_2}/\Xi_{2u}|} \sum_{g\in G_{r_1,r_2}/\Xi_{2u}} \mu(g)g \cdot \left(\frac{1}{|\Xi_{2u}|} \sum_{\sigma\in\Xi_{2u}} \mu(\delta_\sigma)\delta_\sigma \cdot P_*(\Gamma_{r_1,\mathfrak{a}}) \right)$$

$$= \frac{1}{|G_{r_1,r_2}|} \sum_{g\in G_{r_1,r_2}} \mu(g)g \cdot P_*(\Gamma_{r_1,\mathfrak{a}})$$

(by (4.1.2))

$$= (\mathrm{N}\mathfrak{a})^u \epsilon_{X_{r_1,r_2}} \Gamma_{r_1,r_2,\mathfrak{a}}$$

which proves (4.1.3). □

We now assume that A has good reduction at the prime above p and pick a generator ω_A for $\Omega^1(A_{/H})$ as described in Section 3.1. This determines a period pair $(\Omega, \Omega_p) \in \mathbf{C}^\times \times \mathcal{O}_{\mathbf{C}_p}^\times$ as well as a unique element $\eta_A \in H^1_{\mathrm{dR}}(A_{/H})$ such that

$$[\alpha]^*\eta_A = \alpha^\rho \eta_A \quad \text{and} \quad \langle \omega_A, \eta_A \rangle = 1,$$

where ρ denotes the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$. For any power A^r and for any j such that $0 \le j \le r$, we define $\omega_A^j \eta_A^{r-j}$ by

$$\begin{aligned}\omega_A^j \eta_A^{r-j} &:= \epsilon_{A^r}^*(p_1^*\omega_A \wedge \cdots \wedge p_j^*\omega_A \wedge p_{j+1}^*\eta_A \wedge \cdots \wedge p_r^*\eta_A) \\ &= \binom{r}{j}^{-1} \sum_{I\subset\{1,\ldots,r\}} p_1^*\varpi_{1,I} \wedge \cdots \wedge p_r^*\varpi_{r,I},\end{aligned}$$

where $\varpi_{i,I} := \omega_A$ or η_A according as $i \in I$ or $i \notin I$.

Proposition 4.1.2. *Let F be a number field over which all the structures above can be defined and let v be a place of F at which X_{r_1} and X_{r_1,r_2} have good reduction. Then for any $\omega \in H^{r_1+1}_{\mathrm{dR}}(W_{r_1,F})$, we have*

$$(4.1.5)\quad (\mathrm{N}\mathfrak{a})^u \cdot \beta_v(\Delta_{r_1,r_2,\mathfrak{a}})[\omega\wedge\omega_A^j\eta_A^{r_2-j}] = (2\sqrt{-D})^u \cdot \beta_v(\Delta_{r_1,\mathfrak{a}})[\omega\wedge\omega_A^{j+u}\eta_A^{r_2-j+u}],$$

where β_v is the map defined in Section 2.4.

Proof. We first remark that since $\epsilon_{X_{r_1,r_2}}$ (resp. ϵ_{X_r}) is the product of the commuting idempotents $\epsilon_{W_{r_1}}$ and $\epsilon_{A^{r_2}}$ (resp. $\epsilon_{A^{r_1}}$), we have

$$(4.1.6)\qquad \epsilon_{A^{r_2}} \cdot \Delta_{r_1,r_2,\mathfrak{a}} = \Delta_{r_1,r_2,\mathfrak{a}} \quad \text{and} \quad \epsilon_{A^{r_1}} \cdot \Delta_{r_1,\mathfrak{a}} = \Delta_{r_1,\mathfrak{a}}.$$

Write ϖ for the class

$$p_1^*\omega_A \wedge \cdots \wedge p_j^*\omega_A \wedge p_{j+1}^*\eta_A \wedge \cdots \wedge p_{r_2}^*\eta_A \in H^{r_2}(A^{r_2}).$$

Then using (4.1.6) twice as well as Proposition 4.1.1, we have

$$(4.1.7)\qquad \begin{aligned}(\mathrm{N}\mathfrak{a})^u \cdot \beta_v(\Delta_{r_1,r_2,\mathfrak{a}})[\omega \wedge \omega_A^j\eta_A^{r_2-j}] &= (\mathrm{N}\mathfrak{a})^u \cdot \beta_v(\Delta_{r_1,r_2,\mathfrak{a}})[\omega \wedge \epsilon_{A^{r_2}}^*(\varpi)] \\ &= (\mathrm{N}\mathfrak{a})^u \cdot \beta_v(\Delta_{r_1,r_2,\mathfrak{a}})[\omega \wedge \varpi] \\ &= \beta_v(P_*(\Delta_{r_1,\mathfrak{a}}))[\omega \wedge \varpi] \\ &= \beta_v(\Delta_{r_1,\mathfrak{a}})[P^*(\omega \wedge \varpi)] \\ &= \beta_v(\Delta_{r_1,\mathfrak{a}})[\epsilon_{A^{r_1}}^* P^*(\omega \wedge \varpi)].\end{aligned}$$

Let ϑ denote the cohomology class of the transpose of the graph of $\sqrt{-D}$ on $A \times A$. Then

$$P^*(\omega \wedge \varpi) = \omega \wedge \varpi \wedge (p_{r_2+1}, p_{r_2+2})^* \vartheta \wedge \cdots \wedge (p_{r_1-1}, p_{r_1})^* \vartheta.$$

But the class ϑ on $A \times A$ is of the form

$$\vartheta = \vartheta_1 + \vartheta_2,$$

where

$$\vartheta_1 = (\sqrt{-D}) \cdot \big(p_1^*(\omega_A) \wedge p_2^*(\eta_A) + p_1^*(\eta_A) \wedge p_2^*(\omega_A)\big),$$

and ϑ_2 is in the span of $p_1^*(\omega_A) \wedge p_1^*(\eta_A)$ and $p_2^*(\omega_A) \wedge p_2^*(\eta_A)$. Then

$$\begin{aligned}
\epsilon_{A^{r_1}}^* P^*(\omega \wedge \varpi) &= \epsilon_{A^{r_1}}^* [\omega \wedge \varpi \wedge (p_{r_2+1}, p_{r_2+2})^* \vartheta \wedge \cdots \wedge (p_{r_1-1}, p_{r_1})^* \vartheta] \\
&= \epsilon_{A^{r_1}}^* [\omega \wedge \varpi \wedge (p_{r_2+1}, p_{r_2+2})^* (\vartheta_1 + \vartheta_2) \wedge \cdots \wedge (p_{r_1-1}, p_{r_1})^* (\vartheta_1 + \vartheta_2)] \\
&= \epsilon_{A^{r_1}}^* [\omega \wedge \varpi \wedge (p_{r_2+1}, p_{r_2+2})^* \vartheta_1 \wedge \cdots \wedge (p_{r_1-1}, p_{r_1})^* \vartheta_1] \\
&= (2\sqrt{-D})^u \epsilon_{A^{r_1}}^* [\omega \wedge \varpi \wedge p_{r_2+1}^* \omega_A \wedge p_{r_2+2}^* \eta_A \wedge \cdots \wedge p_{r_1-1}^* \omega_A \wedge p_{r_1}^* \eta_A] \\
&= (2\sqrt{-D})^u \cdot \omega \wedge \omega^{j+u} \eta^{r_2-j+u}.
\end{aligned}$$

The proposition follows by combining this last equality with (4.1.7). □

Theorem 4.1.3. *Let $f \in S_{r_1+2}(\Gamma_0(N), \varepsilon_f)$ be a newform of level N, and let χ be a Hecke character of K of infinity type $(r_2 + 1 - j, 1 + j)$ for some j with $0 \le j \le r_2$ and of (finite) type $(1, \mathfrak{N}, \varepsilon_f)$ (*[3, Definition 4.4]*). Let $p = \mathfrak{p}\bar{\mathfrak{p}}$ be a prime split in K and not dividing N and let v be a place of F lying above $\mathfrak{p}$. Then*

$$\left(\sum_{\mathfrak{a}} \chi^{-1}(\mathfrak{a}) \mathrm{N}\mathfrak{a} \cdot \beta_v(\Delta_{r_1,r_2,\mathfrak{a}})[\omega_f \wedge \omega_A^j \eta_A^{r_2-j}]\right)^2 = (-4D)^u \,\mathcal{E}_p(f, \chi \mathbf{N}_K^u)^{-1} \frac{L_p(f, \chi \mathbf{N}_K^u)}{\Omega_p^{r_2-2j}},$$

where the sum is over a set of representatives $\{\mathfrak{a}\}$ for the ideal classes in $\mathcal{O}_K$ chosen to be prime to $\mathfrak{N}$, and the Euler-type factor $\mathcal{E}_p(f, \chi \mathbf{N}_K^u)$ is given by

$$\mathcal{E}_p(f, \chi \mathbf{N}_K^u) = (1 - \chi^{-1}(\bar{\mathfrak{p}}) p^{-u} a_p(f) + \chi^{-2}(\bar{\mathfrak{p}}) \varepsilon_f(p) p^{r_2+1})^2.$$

Proof. Let χ' denote the character $\chi \mathbf{N}_K^u$. Then χ' is also of finite type $(1, \mathfrak{N}, \varepsilon_f)$ but has infinity type

$$(r_2 + 1 - j + u, 1 + j + u) = (r_1 + 1 - (j + u), 1 + (j + u)),$$

so that χ' lies in $\Sigma_{\mathrm{cc}}^{(1)}(1, \mathfrak{N}, \varepsilon_f)$ in the notation of Section 3.2. By the previous proposition, we have

$$\begin{aligned}
&(\chi^{-1}(\mathfrak{a}) \mathrm{N}\mathfrak{a}) \cdot \beta_v(\Delta_{r_1,r_2,\mathfrak{a}})[\omega_f \wedge \omega_A^j \eta_A^{r_2-j}] \\
&\qquad = (2\sqrt{-D})^u \cdot (\chi')^{-1}(\mathfrak{a}) \mathrm{N}\mathfrak{a} \cdot \beta_v(\Delta_{r_1,\mathfrak{a}})[\omega_f \wedge \omega_A^{j+u} \eta_A^{r_1-(j+u)}].
\end{aligned}$$

The result now follows by the main theorem of [3, Theorem 5.13] applied to the pair (f, χ'). □

Remark 4.1.4. When $u = 0$, so that $r_1 = r_2 = r$ say, one recovers exactly the statement of [3, Theorem 5.13] for the case of conductor $c = 1$.

4.2. Fields of definition. We now systematically deal with questions of rationality for the structures appearing in the previous section:

(i) The variety X_{r_1,r_2} is defined over $\mathbf{Q}$.

(ii) The elliptic curve A (along with its complex multiplication) can be defined over the Hilbert class field H of K.

(iii) The point t is defined over an abelian extension of H that may not be abelian over K. However, the pair (A, t) (up to isomorphism, equivalently the point P_A on $X_1(N)$) is defined over the abelian extension F/K corresponding to the subgroup

$$K^\times W \subseteq \mathbf{A}_K^\times,$$

where

$$W := \{x \in \mathbf{A}_K^\times : x\mathcal{O}_K = \mathcal{O}_K,\ xt = t\}.$$

Let W_0 be the subgroup

$$W_0 := \{x \in \mathbf{A}_K^\times : x\mathcal{O}_K = \mathcal{O}_K\},$$

so that H is the class field corresponding to $K^\times W_0$. Then we have canonical isomorphisms

$$\begin{aligned} \mathrm{Gal}(F/H) &\simeq K^\times W_0/K^\times W \simeq W_0/(W_0 \cap K^\times W) \\ &\simeq (\mathcal{O}_K/\mathfrak{N}\mathcal{O}_K)^\times/\{\pm 1\} \simeq (\mathbf{Z}/N\mathbf{Z})^\times/\{\pm 1\}. \end{aligned} \tag{4.2.1}$$

(iv) The elliptic curve $A_\mathfrak{a}$ can also be defined over H. The pair $(A_\mathfrak{a}, t_\mathfrak{a})$ is defined over F, since the subgroup

$$\{x \in \mathbf{A}_K^\times : x\mathfrak{a} = \mathfrak{a},\ xt_\mathfrak{a} = t_\mathfrak{a}\}$$

is equal to W for any $\mathfrak{a}$ prime to $\mathfrak{N}$.

(v) The isogeny $\varphi_\mathfrak{a}$ is defined over H, while the correspondences $\epsilon_{X_{r_1}}$ and $\epsilon_{A^{r_2}}$ are defined over $\mathbf{Q}$. Thus the cycle $\Delta_{r_1,r_2,\mathfrak{a}}$ is defined over F.

(vi) The character ε_f is a character of $(\mathbf{Z}/N\mathbf{Z})^\times$. Let $\ker(\varepsilon_f)$ denote its kernel and let H_f be the intermediate extension $H \subseteq H_f \subseteq F$ that corresponds to the subgroup

$$(\ker \varepsilon_f)\{\pm 1\}/\{\pm 1\} \subseteq (\mathbf{Z}/N\mathbf{Z})^\times/\{\pm 1\},$$

via the isomorphism (4.2.1) above. Define

$$\tilde{\Delta}_{r_1,r_2,\mathfrak{a}} := \frac{1}{[F : H_f]} \mathrm{Tr}_{F/H_f}(\Delta_{r_1,r_2,\mathfrak{a}}). \tag{4.2.2}$$

Thus $\tilde{\Delta}_{r_1,r_2,\mathfrak{a}}$ is defined over the field H_f.

(vii) Let $\mathbf{Q}(f)$ denote the extension of $\mathbf{Q}$ generated by the Hecke eigenvalues of f, so that ω_f is defined over $\mathbf{Q}(f)$. Finally, set $F' := F \cdot \mathbf{Q}(f)$.

Let v be a place of F' lying over the prime $\mathfrak{p}$.

Proposition 4.2.1. *For $\sigma \in \mathrm{Gal}(F/H_f)$, the value*

$$\beta_v(\Delta^\sigma_{r_1,r_2,\mathfrak{a}})[\omega_f \wedge \omega^j \eta^{r_2-j}] \tag{4.2.3}$$

is independent of the choice of σ.

Proof. By Proposition 4.1.2 and the fact that the correspondence P is defined over H, it suffices to show this in the special case $r_1 = r_2 = r$, say. Suppose that $\sigma \in \mathrm{Gal}(F/H)$ is the image of the element $a \in (\mathbf{Z}/N\mathbf{Z})^\times$. Then σ sends $\Delta_{r_1,r_2,\mathfrak{a}}$ to a cycle defined similarly, but with t (resp. $t_\mathfrak{a}$) replaced by at (resp. $at_\mathfrak{a}$). It follows from [3, Lemma 3.22 and Proposition 3.24] that

$$\beta_v(\Delta^{\sigma}_{r_1,r_2,\mathfrak{a}})[\omega_f \wedge \omega^j \eta^{r_2-j}] = \varepsilon_f(a) \cdot \beta_v(\Delta_{r_1,r_2,\mathfrak{a}})[\omega_f \wedge \omega^j \eta^{r_2-j}].$$

For $\sigma \in \mathrm{Gal}(F/H_f)$, we can choose a to lie in $\ker(\varepsilon_f)$, so the expression (4.2.3) is indeed independent of the choice of σ. □

Corollary 4.2.2. *We have*

$$\beta_v(\tilde{\Delta}_{r_1,r_2,\mathfrak{a}})[\omega_f \wedge \omega^j \eta^{r_2-j}] = \beta_v(\Delta_{r_1,r_2,\mathfrak{a}})[\omega_f \wedge \omega^j \eta^{r_2-j}].$$

We now pick a set S of coset representatives $\{\mathfrak{a}\}$ for $\mathrm{Pic}(\mathcal{O}_K)$ that are all prime to $\mathfrak{N}$ to make the following definition.

Definition 4.2.3. Let χ be a character of infinity type $(r_2 + 1 - j, 1 + j)$ and of finite type $(\mathfrak{N}, \varepsilon_f)$. Then set

$$\tilde{\Delta}^{\chi}_{r_1,r_2} := \sum_{\mathfrak{a} \in S} \chi^{-1}(\mathfrak{a}) \mathrm{N}\mathfrak{a} \cdot \tilde{\Delta}_{r_1,r_2,\mathfrak{a}} \in \mathrm{CH}^{s+1}(X_{r_1,r_2/H_f})_0 \otimes \mathbf{Q}(\chi),$$

where $\mathbf{Q}(\chi)$ is the (number) field generated over $\mathbf{Q}$ by the values of χ on S.

Remark 4.2.4. Note that the cycle $\tilde{\Delta}^{\chi}_{r_1,r_2}$ does depend on the choice of S. However, as follows from the discussion in [3, Section 5.1], the value $\beta_v(\tilde{\Delta}^{\chi}_{r_1,r_2})[\omega_f \wedge \omega_A^j \eta_A^{r_2-j}]$ is independent of this choice. This independence is a reflection of the fact that the "f" part of $\chi^{-1}(\mathfrak{a})\Delta_{r_1,r_2,\mathfrak{a}}$, i.e., the image of $\chi^{-1}(\mathfrak{a})\Delta_{r_1,r_2,\mathfrak{a}}$ under the projector defining the Grothendieck motive M_f (constructed in [35]), is independent of the choice of S, at least up to a cycle in the kernel of the Abel–Jacobi map. The last statement is an easy exercise if one recalls that χ is of finite type $(\mathfrak{N}, \varepsilon_f)$ and uses the following: viewing X_{r_1,r_2} as fibered over $X_1(N)$, the Abel–Jacobi image of a cycle supported on a fiber above a point $P_\mathfrak{a}$ and in the image of the projector $\epsilon_{X_{r_1,r_2}}$ only depends on the cohomology class of the cycle in this fiber.

Theorem 4.2.5. *For v any place of $H_f \cdot \mathbf{Q}(f)$ lying above $\mathfrak{p}$,*

$$\left(\beta_v(\tilde{\Delta}^{\chi}_{r_1,r_2})[\omega_f \wedge \omega_A^j \eta_A^{r_2-j}]\right)^2 = (-4D)^u \mathcal{E}_p(f, \chi \mathbf{N}_K^u)^{-1} \frac{L_p(f, \chi \mathbf{N}_K^u)}{\Omega_p^{r_2-2j}}.$$

Proof. This follows immediately from Theorem 4.1.3 and Corollary 4.2.2. □

4.3. Cycles associated with Grössencharacters. We now return to the situation of the introduction, namely we assume that K has odd discriminant $-D$, class number one, and that $w_K = 2$, and specialize the constructions from the previous sections to this case. Recall that $\mathfrak{d}_K = (\sqrt{-D})$ and ε_K is the quadratic Dirichlet character corresponding to the extension $K/\mathbf{Q}$. There is then a *canonical Hecke character* ψ of K of infinity type $(1, 0)$ and conductor $\mathfrak{d}_K$, whose square ψ^2 is the unique unramified Hecke character of K of infinity type $(2,0)$. To construct this character explicitly, we note that for all such K, the composite map $\mathbf{Z} \to \mathcal{O}_K \to \mathcal{O}_K/\mathfrak{d}_K$ induces an isomorphism

$$(\mathbf{Z}/D\mathbf{Z})^\times \simeq (\mathcal{O}_K/\mathfrak{d}_K)^\times. \tag{4.3.1}$$

Then

$$\psi((a)) = \varepsilon_K(a \bmod \mathfrak{d}_K) \cdot a$$

for any ideal (a) with $a \in \mathcal{O}_K$ prime to $\mathfrak{d}_K$. Here we view $a \bmod \mathfrak{d}_K$ as an element in $(\mathbf{Z}/D\mathbf{Z})^\times$ via (4.3.1).

The motive we are interested in is that attached to ψ^{2r+1} for some positive integer r. Let i be any integer satisfying

$$0 \le i \le r \quad \text{and} \quad i \equiv r-1 \bmod 2.$$

Let f be the modular form

$$f = \theta_{\psi^{r+i+1}} = \sum_{\mathfrak{a}} \psi^{r+i+1}(\mathfrak{a}) e^{2\pi i \mathrm{N}\mathfrak{a} z} \in S_{r+i+2}(\Gamma_0(D), \varepsilon_K).$$

Thus $N = D$, $\varepsilon_f = \varepsilon_K$, $\mathbf{Q}(f) = K$ and

$$\ker(\varepsilon_f)\{\pm 1\}/\{\pm 1\} = (\mathbf{Z}/D\mathbf{Z})^\times/\{\pm 1\},$$

so that $H_f = H = K$. Let χ the character $\psi^{*r-i}\mathbf{N}_K$. Picking $\mathfrak{N} = \mathfrak{d}_K$, one sees that χ is of finite type $(1, \mathfrak{N}, \varepsilon_f)$.

Consider the variety

$$X_{r+i,r-i} = W_{r+i} \times A^{r-i}$$

of dimension $2r+1$, where A is the unique elliptic curve over $\mathbf{Q}$ with CM by $\mathcal{O}_K$ and minimal conductor D^2. Note that $X_{r+i,r-i}$ is in fact defined over $\mathbf{Q}$. Since K has class number one, we may pick $S = \{1\}$ and we have

$$\tilde{\Delta}^{\chi}_{r+i,r-i} = \tilde{\Delta}_{r+i,r-i,\mathcal{O}_K} \in \mathrm{CH}^{r+1}(X_{r+i,r-i\,/K})_0 \otimes \mathbf{Q}. \tag{4.3.2}$$

For ease of notation, we will simply call this cycle $\tilde{\Delta}_{r+i,r-i}$.

Since $X_{r+i,r-i}$ is defined over $\mathbf{Q}$, there is an action of complex conjugation (denoted τ) on the Chow group in (4.3.2), and we consider its effect on the cycle $\tilde{\Delta}_{r+i,r-i}$. For any generator t' of $A[\mathfrak{d}_K]$, τ sends the point (A, t') on $X_1(D)$ to (A, t''), where t'' is some other generator of $A[\mathfrak{d}_K]$. Since τ preserves the graph of the identity on $A \times A$ but sends the graph of $\sqrt{-D}$ on $A \times A$ to the graph of $-\sqrt{-D}$, we see that $\tilde{\Delta}_{r+i,r-i}$ is in the $+$ or $-$ eigenspace for the action of τ depending on whether i is even or odd. However, there is an obvious action of $\mathrm{End}(A)$ on $\mathrm{CH}^{r+1}(X_{r+i,r-i})$, for instance via its action on any one of the elliptic curve factors of $X_{r+i,r-i}$ and the induced pullback or pushforward on cycles. (Note that there is at least one such factor since $i \equiv r-1 \bmod 2$.) Defining

$$\tilde{\Delta}'_{r+i,r-i} := [\sqrt{-D}]^* \cdot \tilde{\Delta}_{r+i,r-i},$$

we see that $\tilde{\Delta}'_{r+i,r-i}$ is in the opposite eigenspace for the action of τ as $\tilde{\Delta}_{r+i,r-i}$.

Let $p = \mathfrak{p}\bar{\mathfrak{p}}$ be a prime split in K, and let $\mathcal{L}_{p,\mathfrak{d}_K}$ denote the Katz p-adic L-function as in Section 3. We suppress $\mathfrak{d}_K$ from the notation and simply write $\mathcal{L}_p$ henceforth.

Theorem 4.3.1. *Let i be an integer in the range $0 \le i \le r$ satisfying $i \equiv r-1 \bmod 2$. Suppose that*

$$L(\psi^{2i+1}, i+1) \neq 0 \quad \text{and} \quad \mathcal{L}_p(\psi^{-r}\psi^{*r+1}) \neq 0.$$

Then the cycles $\tilde{\Delta}_{r+i,r-i}, \tilde{\Delta}'_{r+i,r-i} \in \mathrm{CH}^{r+1}(X_{r+i,r-i\,/K})_0 \otimes \mathbf{Q}$ are nontrivial in the top graded piece of the coniveau filtration, i.e., the classes

$$[\tilde{\Delta}_{r+i,r-i}], [\tilde{\Delta}'_{r+i,r-i}] \in N^0\mathrm{CH}^{r+1}(X_{r+i,r-i})_{\mathbf{Q}}/N^1\mathrm{CH}^{r+1}(X_{r+i,r-i})_{\mathbf{Q}}$$

are nonzero. In particular, they are nonzero in $\mathrm{Gr}^{r+1}(X_{r+i,r-i}) \otimes \mathbf{Q}$.

Proof. It clearly suffices to prove this for one of the cycles, say $\tilde{\Delta}_{r+i,r-i}$. We compute $\beta_{\mathfrak{p}}(\tilde{\Delta}_{r+i,r-i})[\omega_f \wedge \omega_A^{r-i}]$. By Theorem 4.2.5,

$$\left(\beta_{\mathfrak{p}}(\tilde{\Delta}_{r+i,r-i})[\omega_f \wedge \omega_A^{r-i}]\right)^2 \sim L_p(f, \chi\mathbf{N}_K^i),$$

where we use $\sim$ (in this proof) to denote equality up to multiplication by a nonzero factor. But by Proposition 3.3.1,

$$\begin{aligned} L_p(f, \chi\mathbf{N}_K^i) &\sim \mathcal{L}_p(\psi^{-(r+i+1)}\chi\mathbf{N}_K^i)\mathcal{L}_p(\psi^{*-(r+i+1)}\chi\mathbf{N}_K^i) \\ &= \mathcal{L}_p(\psi^{-(r+i+1)}\psi^{*r-i}\mathbf{N}_K^{i+1})\mathcal{L}_p(\psi^{*-(r+i+1)}\psi^{*r-i}\mathbf{N}_K^{i+1}) \\ &= \mathcal{L}_p(\psi^{-r}\psi^{*r+1})\mathcal{L}_p(\psi^{i+1}\psi^{*-i}). \end{aligned}$$

Now in the second term above the character $\psi^{i+1}\psi^{*-i}$ is in the range of interpolation for $\mathcal{L}_p$, and so up to a nonzero factor, it equals (the algebraic part of) the L-value

$$L(\psi^{-(i+1)}\psi^{*i}, 0) = L(\psi^{*2i+1}, i+1) = L(\psi^{2i+1}, i+1).$$

The result now follows by using Corollary 2.4.4. □

Let $\Omega(A)$ denote the real period of the elliptic curve A. This is related to the period Ω by

$$\Omega = \Omega(A)/\sqrt{-D}.$$

Let us define the algebraic part of the L-value $L(\psi^{2k+1}, k+1)$ by

$$L^{\mathrm{alg}}(\psi^{2k+1}, k+1) = 2 \cdot \frac{k!(2\pi\sqrt{D})^k}{\Omega(A)^{2k+1}} \cdot L(\psi^{2k+1}, k+1). \tag{4.3.3}$$

Proposition 4.3.2. *The algebraic part $L^{\mathrm{alg}}(\psi^{2k+1}, k+1)$ lies in $\mathbf{Q}$, and has denominator (multiplicatively) bounded above by $2^{k-1}D^{2k+1}$.*

Proof. That L^{alg} lies in $\mathbf{Q}$ follows from [15, Theorem 8.1]. That the denominator is bounded by $2^{k-1}D^{2k+1}$ follows from [15, Proposition 10.11 (a)]. □

In particular, the values of L^{alg} are p-adic integers for all primes p not dividing $2D$. In practice, we find that the values of L^{alg} are in fact integers and also perfect squares, as is seen in the tables in Appendix A. The values in those tables were computed using MAGMA.

In order to provably identify these values, one may use the following rapidly converging series (see [16, equation (1)] for example):

$$L(\psi^{2k+1}, k+1) = \sum_{\mathfrak{a}} \big(\psi^{2k+1}(\mathfrak{a}) + \psi^{2k+1}(\bar{\mathfrak{a}})\big) \mathrm{N}(\mathfrak{a})^{-(k+1)} \cdot e^{-2\pi \mathrm{N}(\mathfrak{a})/D^2} \sum_{j=0}^{k} \frac{(2\pi \mathrm{N}(\mathfrak{a})/D^2)^j}{j!}, \tag{4.3.4}$$

where the sum is over integral ideals $\mathfrak{a}$ of $\mathcal{O}_K$. By estimating the tail of the series (4.3.4) and using Proposition 4.3.2 above which bounds the denominator of the L-value, we can provably identify the L-value exactly.

Corollary 4.3.3. *Let i be an integer in the range $0 \le i \le r$ satisfying $i \equiv r - 1 \bmod 2$. Suppose that there are odd primes p_1, p_2 (not necessarily distinct) split in K, and positive integers k_1 and k_2 satisfying*

$$k_1 \equiv i \bmod (p_1 - 1), \quad k_2 \equiv -r - 1 \bmod (p_2 - 1)$$

such that

$$L^{\mathrm{alg}}(\psi^{2k_1+1}, k_1 + 1) \not\equiv 0 \bmod p_1, \quad L^{\mathrm{alg}}(\psi^{2k_2+1}, k_2 + 1) \not\equiv 0 \bmod p_2. \tag{4.3.5}$$

Then the conclusion of Theorem 4.3.1 holds for the pair (r, i).

Proof. Let (p, k) be the pair (p_1, k_1) or (p_2, k_2). By the interpolation property of $\mathcal{L}_p$, we have

$$\begin{aligned}\frac{\mathcal{L}_p(\psi^{k+1}\psi^{*-k})}{\Omega_p^{2k+1}} &= \frac{k!(2\pi/\sqrt{-D})^k}{\Omega^{2k+1}} \left(1 - \frac{\psi^{k+1}\psi^{*-k}(\mathfrak{p})}{p}\right)(1 - \psi^{-(k+1)}\psi^{*k}(\bar{\mathfrak{p}})) \\ &\quad \cdot L(\psi^{-(k+1)}\psi^{*k}, 0) \\ &\sim \left(1 - \frac{\psi^{k+1}\psi^{*-k}(\mathfrak{p})}{p}\right)(1 - \psi^{-(k+1)}\psi^{*k}(\bar{\mathfrak{p}})) L^{\mathrm{alg}}(\psi^{2k+1}, k+1),\end{aligned}$$

where we use $\sim$ to denote that LHS $=$ (a $\mathfrak{p}$-adic unit) $\times$ RHS. The Euler and Euler-like factor are both seen to be $\mathfrak{p}$-adic units, since $\psi(\mathfrak{p})/p$ and $\psi^*(\mathfrak{p})$ are both $\mathfrak{p}$-adic units. Since Ω_p is a unit in $\mathcal{O}_{\mathbf{C}_p}$, we conclude that $\mathcal{L}_p(\psi^{k+1}\psi^{*-k})$ is a unit in $\mathcal{O}_{\mathbf{C}_p}$. The congruence

$$\mathcal{L}_p(\psi^{k+1}(\psi^*)^{-k}) \equiv \mathcal{L}_p(\psi^{k+1}(\psi^*)^{-k} \cdot (\psi/\psi^*)^{(p-1)\ell}) \bmod \mathfrak{m}_{\mathcal{O}_{\mathbf{C}_p}}$$

implies that

$$\mathcal{L}_{p_1}(\psi^{i+1}\psi^{*-i}) \not\equiv 0 \bmod \mathfrak{m}_{\mathcal{O}_{\mathbf{C}_{p_1}}} \quad \text{and} \quad \mathcal{L}_{p_2}(\psi^{-r}\psi^{*r+1}) \not\equiv 0 \bmod \mathfrak{m}_{\mathcal{O}_{\mathbf{C}_{p_2}}}.$$

Thus, certainly

$$\mathcal{L}_{p_2}(\psi^{-r}\psi^{*r+1}) \neq 0,$$

and by the interpolation property for $\mathcal{L}_{p_1}$ we see that

$$L(\psi^{2i+1}, i+1) \neq 0.$$

The conclusion then follows by applying Theorem 4.3.1 with $p = p_2$. □

Corollary 4.3.4. *Let $\varepsilon = 0$ if $\operatorname{sign} L(\psi, s) = +1$ and $\varepsilon = 1$ if $\operatorname{sign} L(\psi, s) = -1$. Suppose that there is an odd prime p split in K such that*

$$L^{\mathrm{alg}}(\psi^{2k+1}, k+1) \neq 0 \bmod p \tag{4.3.6}$$

for

$$k \in S_{\varepsilon,p} := \{\varepsilon, \varepsilon + 2, \dots, \varepsilon + (p-3)\}.$$

Let r be a positive integer such that

$$r \equiv \varepsilon + 1 \bmod 2,$$

and i an integer in the range $0 \le i \le r$ such that

$$i \equiv \varepsilon \bmod 2.$$

Then the conclusion of Theorem 4.3.1 *holds for the pair* (r, i).

Proof. Since $i \equiv \varepsilon \bmod 2$ and $-r-1 \equiv \varepsilon \bmod 2$, there exist $k_1, k_2 \in S_{\varepsilon,p}$ such that

$$k_1 \equiv i \bmod (p-1) \quad \text{and} \quad k_2 \equiv -r-1 \bmod (p-1).$$

Now apply the previous corollary with $p_1 = p_2 = p$. □

We now define $\tilde{\Delta}^{\pm}_{r+i,r-i}$. Set

$$\tilde{\Delta}^{+}_{r+i,r-i} = \begin{cases} \tilde{\Delta}_{r+i,r-i}, & \text{if } i \text{ is even,} \\ \tilde{\Delta}'_{r+i,r-i}, & \text{if } i \text{ is odd.} \end{cases}$$

Likewise, set

$$\tilde{\Delta}^{-}_{r+i,r-i} = \begin{cases} \tilde{\Delta}_{r+i,r-i}, & \text{if } i \text{ is odd;} \\ \tilde{\Delta}'_{r+i,r-i}, & \text{if } i \text{ is even.} \end{cases}$$

Corollary 4.3.5. *Suppose that r is odd if $D = 7$ and is even if $D = 11, 19, 43, 67, 163$. Then for any positive integer i such that*

$$0 \le i \le r \quad \textit{and} \quad i \equiv r - 1 \bmod 2,$$

the cycles $\tilde{\Delta}^{+}_{r+i,r-i}$ and $\tilde{\Delta}^{-}_{r+i,r-i}$ are defined over K, are in the respective ± 1-eigenspaces for the action of complex conjugation, and are nonzero in the top graded piece of the coniveau filtration; i.e.,

$$[\Delta^{+}_{r+i,r-i}], [\Delta^{-}_{r+i,r-i}] \in N^0\mathrm{CH}^{r+1}(X_{r+i,r-i})_{\mathbf{Q}}/N^1\mathrm{CH}^{r+1}(X_{r+i,r-i})_{\mathbf{Q}}$$

are nonzero. In particular, these cycles are nonzero in the Griffiths group $\mathrm{Gr}^{r+1}(X_{r+i,r-i}) \otimes \mathbf{Q}$.

Proof. We verify that for each value of D, there is some odd prime p split in K such that the condition (4.3.6) is verified for all k in $S_{\varepsilon,p}$. The L-values in question were computed as explained earlier in this section and are listed in the tables in Appendix A. (Rather, what is listed is the prime factorization of the square-root of L^{alg}.) Indeed, we find the required non-divisibility holds for $D = 7$ with $p = 11$, for $D = 11, 19$ with $p = 5$, for $D = 43$ with $p = 13$, for $D = 67$ with $p = 23$ and for $D = 163$ with $p = 41$! □

5. Miscellany

5.1. A non-CM example. We consider a special case of the more general construction of Section 4.1. Take $N = 5$ and let $K = \mathbf{Q}(\sqrt{-11})$. Note that the prime 5 is split in K, so the constructions of Section 4.1 apply. Consider the Kuga–Sato variety W_2 over $X_1(5)$, which has dimension 3 and is defined over $\mathbf{Q}$. The middle-dimensional cohomology $H^3(W_2)$ is 1-dimensional, corresponding to the unique cusp form f on $\Gamma_1(5)$ of weight 4. The form f in fact has trivial central character, is not of CM type and has q-expansion:

$$f(q) = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 - 23q^9 + 20q^{10} + 32q^{11} + \cdots .$$

Theorem 5.1.1. *We have*

$$\operatorname{rank} \operatorname{Gr}^2(W_{2/K}) \geq 1.$$

Recall that $X_{2,2} = W_2 \times A^2$, where A is an elliptic curve over $\mathbf{Q}$ with CM by $\mathcal{O}_K$ and (minimal) conductor 121. We will simultaneously show the following result as well.

Theorem 5.1.2. *We have*

$$\dim \operatorname{CH}^3(X_{2,2})_{0,\mathbf{Q}}/N^1\operatorname{CH}^3(X_{2,2})_{\mathbf{Q}} \geq 1.$$

In particular,

$$\operatorname{rank} \operatorname{Gr}^3(X_{2,2}) \geq 1.$$

For the similar groups of cycles defined over K, the ranks of the ± 1-eigenspaces for the action of complex conjugation are each ≥ 1.

Remark 5.1.3. The L-function $L(f,s)$ has sign $+1$ and is in fact nonzero at the center. Thus one does not expect to be able to construct a nontorsion cycle in $\operatorname{Gr}^2(W_2)$ that is defined over $\mathbf{Q}$. However, the L-function $L_K(f,s)$ has sign -1 since K satisfies the Heegner hypothesis for f and so one does expect to get such a nontorsion cycle defined over K, as given by Theorem 5.1.1. On the other hand, for the variety $X_{2,2}$, the relevant cohomology is H^5, which contains the motive $M_f \otimes M_{\theta_{\psi^2}}$, with ψ the Hecke character of K corresponding to A. The corresponding L-function

$$L(s, M_f \otimes M_{\theta_{\psi^2}}) = L(f \otimes \theta_{\psi^2}, s)$$

has sign -1, again due to the Heegner condition, so that one does expect to construct a nontorsion cycle in Gr^3 that is defined over $\mathbf{Q}$ as well as one that is defined over K and is in the -1-eigenspace for complex conjugation, as is given by Theorem 5.1.2.

Proof of Theorems 5.1.1 *and* 5.1.2. Let χ_1 and χ_2 be the Hecke characters of K defined by

$$\chi_1 := \mathbf{N}_K, \quad \chi_2 := \psi^2 \cdot \mathbf{N}_K.$$

Both χ_1 and χ_2 are unramified characters, and are certainly of type $(1, \mathfrak{N}, \varepsilon_f)$ for any of the two choices of $\mathfrak{N}$. Fix an $\mathfrak{N}$ and consider the cycles

$$\tilde{\Delta}^{\chi_1}_{2,0} \in \operatorname{CH}^2(W_{2/K})_{0,\mathbf{Q}} \quad \text{and} \quad \tilde{\Delta}^{\chi_2}_{2,2} \in \operatorname{CH}^3(X_{2,2/K})_{0,\mathbf{Q}}.$$

We will show that these cycles are nontrivial in N^0/N^1. We first give the argument for $\tilde{\Delta}^{\chi_1}_{2,0}$ and then outline the changes necessary to deal with $\tilde{\Delta}^{\chi_2}_{2,2}$. By Corollary 2.4.4 and Theorem 4.2.5, to show that $\tilde{\Delta}^{\chi_1}_{2,0}$ is nontrivial in N^0/N^1, it suffices to show that

$$L_p(f, \chi_1) \neq 0 \tag{5.1.1}$$

for some prime p split in K. In fact, by [3, Theorem 5.9 and Proposition 5.10], to show (5.1.1) is equivalent to showing that

$$\theta^{-2} f^\flat(A, A[\mathfrak{N}], \omega_A) \neq 0, \tag{5.1.2}$$

where $\theta = q\frac{d}{dq}$ is the Atkin–Serre operator and $f^\flat$ is the modular form defined by

$$f^\flat := \sum_{(n,p)=1} a_n(f) q^n.$$

Recall that for any integer k, $\theta^k f^\flat$ is a p-adic modular form of weight $4 + 2k$ and satisfies the p-adic continuity property

$$\theta^{k_1} f^\flat(A, A[\mathfrak{N}], \omega_A) \equiv \theta^{k_2} f^\flat(A, A[\mathfrak{N}], \omega_A) \bmod \mathfrak{p}^n, \quad \text{if } k_1 \equiv k_2 \bmod (p-1)p^{n-1}.$$

Thus (5.1.2) would follow if one could show that for some choice of prime p split in K and some choice of *positive* integer k, one has

$$k \equiv -2 \bmod (p-1) \quad \text{and} \quad \theta^k f^\flat(A, A[\mathfrak{N}], \omega_A) \neq 0 \bmod \mathfrak{p}.$$

Now, for positive k, we have a congruence

$$\theta^k f^\flat(A, A[\mathfrak{N}], \omega_A) \equiv \theta^k f(A, A[\mathfrak{N}], \omega_A) \bmod \mathfrak{p},$$

since (by the proof of [3, Theorem 5.9]) these two values differ by an Euler-type factor which is seen to equal 1 mod $\mathfrak{p}$. In addition, by [23, Theorem 2.6.36], we have

$$\theta^k f(A, A[\mathfrak{N}], \omega_A) = \delta^k f(A, A[\mathfrak{N}], \omega_A),$$

where

$$\delta^k f := \delta_{4+2k-2} \circ \cdots \circ \delta_4 f, \quad \delta_j := \frac{1}{2\pi i}\left(\frac{d}{dz} + \frac{j}{2iy}\right).$$

The operator δ_j is the Shimura–Maass operator that takes nearly holomorphic modular forms of weight j to nearly holomorphic modular forms of weight $j+2$.

We now take $k = 0$ and $p = 3$; note that 3 is split in K. The value

$$\delta^k f(A, A[\mathfrak{N}], \omega_A) = f(A, A[\mathfrak{N}], \omega_A)$$

lies in K and is certainly integral outside of 5. Since its denominator at 5 can be explicitly bounded by [12, Chapter 7, Theorem 3.9 and Sections 3.17–3.18], one can provably identify this value by a finite computation. We pick an ideal $\mathfrak{N} = (\alpha)$, where $\alpha := (3 - \sqrt{-11})/2$, so that $\tau := (-3 + \sqrt{-11})/10$ is a representative on $\Gamma_0(5)\backslash\mathfrak{H}$ for the associated Heegner point $(\mathbf{C}/\mathfrak{N}^{-1} \to \mathbf{C}/\mathcal{O}_K)$. Then

$$\mathbf{Z}\tau + \mathbf{Z} = \frac{1}{5}\mathfrak{N},$$

while the period lattice of (A, ω_A) is $\frac{1}{\sqrt{-11}}\mathcal{O}_K \cdot \Omega_A$, where Ω_A is the real period of ω_A. Then one finds that

$$f(A, A[\mathfrak{N}], \omega_A) = \left(\frac{2\pi i}{\Omega_0}\right)^4 f(\tau),$$

where

$$\Omega_0 := \frac{5}{\alpha\sqrt{-11}} \cdot \Omega_A.$$

Computing this using MAGMA, we find that $f(A, A[\mathfrak{N}], \omega_A) = -\bar{\alpha}^{-2} \times 11$. Since this last value is a 3-adic unit, Theorem 5.1.1 follows.

For Theorem 5.1.2, one needs to show that $\theta^{-3} f^{\flat}(A, A[\mathfrak{N}], \omega_A)$ is not zero. Arguing as above, this value is congruent to $\delta^1 f(A, A[\mathfrak{N}], \omega_A)$ modulo a prime above 3, so it suffices to check that $\delta^1 f(A, A[\mathfrak{N}], \omega_A)$ is a 3-adic unit; indeed, we find computationally that $\delta^1 f(A, A[\mathfrak{N}], \omega_A) = -\bar{\alpha}^{-3} \times 2^2 \times 11$. (One needs to justify that this value can be provably identified by giving an argument analogous to that for $\delta^0 f(A, A[\mathfrak{N}], \omega_A)$ given above, that used the results of [12]. For this justification, see the remarks following the end of the proof; this rests on Appendix B.) This shows that the rank of the group $\mathrm{Gr}^3(X_{2,2})$ over K is at least 1, but now using the CM action of K on either of the elliptic curve factors of $W_{2,2}$ (as in Section 4.3), we see that the ranks of both the ± 1-eigenspaces for complex conjugation are at least 1. □

We now explain how one can provably compute the value $\delta^1 f(A, A[\mathfrak{N}], \omega_A)$ or equivalently $\delta_4 f(\tau)$. First we use the following idea due to Shimura (see [36, Section 1]). Pick an element $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Q})$ such that $\alpha \cdot \tau = \tau$ and $\alpha \neq \pm 1$. Let

$$h(z) := \frac{(f|_4\alpha)(z)}{f(z)}.$$

(Here as usual $F|_\ell\beta(z) := F(\beta \cdot z) \cdot \det(\beta)^{\ell/2} \cdot (cz + d)^{-\ell}$.) Then

$$h(\tau) = \lambda^{-4},$$

where $\lambda := c\tau + d$. Applying δ_4 to the relation

$$f|_4\alpha = fh$$

yields

$$(\delta_4 f)|_6\alpha = \delta_4(f|_4\alpha) = \delta_4 f \cdot h + f\delta_0 h.$$

Evaluating at τ yields

$$\delta_4 f(\tau) \cdot \lambda^{-6} = \delta_4 f(\tau) \cdot \lambda^{-4} + f(\tau)\delta_0 h(\tau),$$

that is,

$$\lambda^{-4}(\lambda^{-2} - 1) \cdot \delta_4 f(\tau) = f(\tau) \cdot \delta_0 h(\tau).$$

So to compute $\delta_4 f(\tau)$, given that we have already computed $f(\tau)$, it suffices to bound the denominator of the algebraic part of $\delta_0 h(\tau)$.

In our case, we can take α to correspond (for example) to the image of $\frac{1+\sqrt{-11}}{1-\sqrt{-11}}$ under the embedding $K \hookrightarrow M_2(\mathbf{Q})$ associated with the point τ. Then

$$\alpha = \begin{pmatrix} -\frac{4}{3} & -\frac{1}{3} \\ \frac{5}{3} & -\frac{1}{3} \end{pmatrix} = \frac{1}{3}\beta$$

with

$$\beta := \begin{pmatrix} -4 & -1 \\ 5 & -1 \end{pmatrix}.$$

Note that $\gamma := \left(\begin{smallmatrix} -1 & 1 \\ -5 & 4 \end{smallmatrix}\right) \in \Gamma_0(5)$ and

$$\gamma \cdot \beta = \begin{pmatrix} 1 & -2 \\ 0 & 9 \end{pmatrix},$$

so that

$$(f|_4\alpha)(z) = (f|_4\beta)(z) = (f|_4\gamma\beta)(z) = 9^{-2} \cdot f\left(\frac{z-2}{9}\right),$$

and

$$h(z) = 9^{-2} \cdot f\left(\frac{z-2}{9}\right) \cdot f(z)^{-1}.$$

Letting $g(z) := 9^2 \cdot h(9z+2)$, we get

$$g(z) = f(z) \cdot f(9z+2)^{-1} = f(z) f(9z)^{-1}.$$

Now,

$$\delta_0 h(\tau) = 9^{-3} \cdot \delta_0 g(\tau')$$

with $\tau' := (\tau - 2)/9$. But g is a meromorphic modular function on $\Gamma_0(9 \cdot 5)$, hence $\delta_0 g$ is a meromorphic modular form of weight 2 on $\Gamma_0(9 \cdot 5)$ and is given by the formula

$$\delta_0 g(z) = \frac{f(9z) f'(z) - 9 f'(9z) f(z)}{f(9z)^2}.$$

Thus

$$F(z) := f(9z)^2 \cdot \delta_0 g(z) = f(9z) f'(z) - 9 f'(9z) f(z)$$

is a *holomorphic* cusp form of weight 10 on $\Gamma_0(3^2 \cdot 5)$. The algebraic part of $F(\tau')$ is integral outside of the primes 3 and 5 and Theorem B.1.3.1 in Appendix B gives an effective lower bound on its denominator at 5. Tracing backwards through the computations above, we find that there is an effectively computable algebraic integer $c_1 \in \mathcal{O}_K$ such that $c_1 \delta^1 f(A, A[\mathfrak{N}], \omega_A)$ is integral outside of 3. The argument above can now be repeated with a different choice of α, say corresponding to the image of $(5 + 3\sqrt{-11})(5 - 3\sqrt{-11})^{-1}$. (Note that the element $(5 + 3\sqrt{-11})/2$ has norm 31 which is the smallest split prime in K after 3 and 5.) In exactly the same way, we find that there is an effectively computable algebraic integer $c_2 \in \mathcal{O}_K$ such that $c_2 \delta^1 f(A, A[\mathfrak{N}], \omega_A)$ is integral outside of 31. Then $c_1 c_2 \delta^1 f(A, A[\mathfrak{N}], \omega_A)$ is integral, and hence can be computed effectively, as desired.

Remark 5.1.4. The reader will find in Appendix A a table of values $\delta^k f(A, A[\mathfrak{N}], \omega_A)$ for k in the range 0 through 21, computed to many digits of accuracy. From this table, we see that instead of $p = 3$ and $k = 0, 1$, we could argue instead using the split prime $p = 23$ and the values corresponding to $k = 20$ and 19, *provided* we could show that these computed values are indeed correct. However, this seems to be a nontrivial issue since our method of proof above for $k = 0, 1$, namely translating the problem to computing the value of a holomorphic form with integral q-expansion and applying Theorem B.1.3.1 in Appendix B, does not seem to extend easily to larger values of k.

5.2. The Fermat quartic revisited. We end by using the methods of this paper to give another proof of Bloch's theorem (see Section 1) that rank $\mathrm{Gr}^2(X) \geq 1$ for X, the Jacobian of the Fermat quartic.

Recall that X is isogenous over $\mathbf{Q}$ to the product $A \times A \times A'$, where A and A' are the elliptic curves given by the equations

$$A : y^2 = x^3 - x, \quad A' : y^2 = x^3 + x.$$

The curves A and A' have conductors 32 and 64 respectively, and both have CM by the ring of integers of $K = \mathbf{Q}(i)$. We will need to identify explicitly the associated Grössencharacters ψ and ψ' of K. Let $\mathfrak{q} = (1+i)$ be the unique prime of $\mathcal{O}_K$ above 2, so that $\mathfrak{q}^2 = (2)$. Since the conductor $N_A = \mathrm{N}_{K/\mathbf{Q}}(\mathfrak{c}_\psi)D$ and likewise for A', the conductors of ψ and ψ' must be $\mathfrak{q}^3$ and $\mathfrak{q}^4$ respectively. Further, since ψ and ψ' are associated with elliptic curves, their restrictions to $\mathbf{Q}$ must equal ε_K. Since K has class number one, we have

$$\mathbf{A}_K^\times = K^\times \cdot \left(\prod_v \mathcal{O}_v^\times \times K_\infty^\times \right)$$

and to give a Hecke character $\eta = \prod_v \eta_v$ of conductor dividing $\mathfrak{q}^n$ of infinity type $(1,0)$ is equivalent to giving a finite order character $\eta_{\mathfrak{q}}$ of

$$U_{\mathfrak{q},n} := \mathcal{O}_{\mathfrak{q}}^\times / 1 + \mathfrak{q}^n \mathcal{O}_{\mathfrak{q}} = (1 + \mathfrak{q}\mathcal{O}_q)/(1 + \mathfrak{q}^n \mathcal{O}_q)$$

such that for any global unit $u \in \mathcal{O}_K$, we have

$$\eta_{\mathfrak{q}}(u) = u.$$

Now $U_{\mathfrak{q},3} \simeq \mathbf{Z}/4\mathbf{Z}$ is cyclic, generated by i, hence there is a unique character ψ of type $(1,0)$ and conductor $\mathfrak{q}^3$. Globally, this character is determined by

$$\psi((\alpha)) = \alpha, \quad \text{if } \alpha \in \mathcal{O}_K,\ \alpha \equiv 1 \bmod \mathfrak{q}^3.$$

Likewise $U_{\mathfrak{q},4} \simeq \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, where the first factor is generated by $a := 2+i$ and the second by $b := 1-2i$. It is easy to see then that the only characters of conductor dividing $\mathfrak{q}^4$ of infinity type $(1,0)$ are ψ and ψ', the latter given by

$$\psi'_{\mathfrak{q}}(a) = i, \quad \psi'_{\mathfrak{q}}(b) = 1.$$

In contrast, ψ satisfies

$$\psi_{\mathfrak{q}}(a) = -i, \quad \psi_{\mathfrak{q}}(b) = -1.$$

Thus $\psi' = \psi\mu$, where μ is the finite order (quadratic) character of K of conductor $\mathfrak{q}^4$, characterized by

$$\mu_{\mathfrak{q}}(a) = -1, \quad \mu_{\mathfrak{q}}(b) = -1,$$

so that μ is in fact the quadratic character associated with the extension $\mathbf{Q}(\zeta_8)/K$.

By the Künneth formula, the motive $H^3(X)$ splits as the sum $M_{\theta_\chi} \oplus M_{\theta_{\psi'}}(-1)^3$, where χ is the Hecke character

$$\chi := \psi^2\psi' = \psi^3\mu.$$

Let f be the modular form

$$f := \theta_{\psi^2} = \sum_{\mathfrak{a}} \psi^2(\mathfrak{a}) e^{2\pi i \mathrm{N}\mathfrak{a} z}.$$

Since ψ^2 has conductor $\mathfrak{q}^2$, we have $f \in S_3(\Gamma_0(16), \varepsilon_K)$. Let W be the elliptic modular surface over the modular curve X_Γ associated with the subgroup $\Gamma := \Gamma_0(16) \cap \Gamma_1(4) \subset \mathrm{SL}_2(\mathbf{Z})$. By Künneth, $H^3(W \times E')$ contains as a subfactor the motive

$$M_f \otimes M_{\theta_{\psi'}} = M_{\theta_{\psi^2}} \otimes M_{\theta_{\psi'}} = M_{\theta_\chi} \oplus M_{\theta_{\psi'}}(-1).$$

Note that the group Γ is conjugate in $\mathrm{SL}_2(\mathbf{Z})$ to the subgroup $\Gamma(4)$, so that W is *isogenous* to the (universal) elliptic modular surface W' over $X(4)$. (This isogeny will be described in moduli-theoretic terms below.)

We will show $\operatorname{rank} \mathrm{Gr}^2(X) \geq 1$ in two steps:

(i) Construct a generalized Heegner-type cycle $\tilde{\Delta}$ on $W' \times A''$, where A'' is a curve $\mathbf{Q}$-isogenous to A' and show that $\tilde{\Delta}$ is nontorsion in $\mathrm{Gr}^2(W' \times A'')$.

(ii) Construct a correspondence Π from X to the product $W' \times A''$, and verify that $\tilde{\Delta}$ transfers via Π to a cycle on X that is nontorsion in $\mathrm{Gr}^2(X)$.

Step (i). The curve $X(4)$ has a canonical model over K, given as the solution to a fine moduli problem, namely classifying triples (E, P, Q) where E is an elliptic curve, P and Q are points of order exactly 4, such that the Weil pairing $\langle P, Q\rangle = i$. In fact, $X(4)$ is isomorphic to $\mathbf{P}^1$, and an explicit equation for W' is the classical Jacobi quartic

$$Y^2 = (X^2 - \sigma^2)(X^2 - \sigma^{-2}), \qquad (5.2.1)$$

or in Weierstrass form:

$$Y^2 = X(X-1)(X-\lambda), \qquad (5.2.2)$$

where

$$\lambda = \frac{1}{4}(\sigma + \sigma^{-1})^2,$$

with σ being the parameter on $X(4) = \mathbf{P}^1$. Note that $X(4)$ and W' are in fact defined over $\mathbf{Q}$. A basis for the 4-torsion (in the Weierstrass form) is given by (P_σ, Q_σ), where

$$P_\sigma = \left(\frac{1}{2}(\sigma^2+1)\sigma^{-2}, -\frac{1}{4}(\sigma^4-1)\sigma^{-3}\right),$$
$$Q_\sigma = \left(-\frac{1}{2}(\sigma^2+1)\sigma^{-1}, \frac{1}{4}i(\sigma^4+2\sigma^3+2\sigma^2+2\sigma+1)\sigma^{-2}\right).$$

These satisfy $\langle P_\sigma, Q_\sigma\rangle = i$, with $\langle\cdot,\cdot\rangle$ being the Weil pairing.

Fix ζ a primitive 8th root of unity such that $\zeta^2 = i$. The fiber of W' over $\sigma = \zeta$ is an elliptic curve isomorphic to the curve

$$A'' : y^2 = x^3 - 4x$$

which is 2-isogenous to A'. On the variety $W' \times A''$, let $\Delta \in \mathrm{CH}^2(W' \times A'')_0$ be the cycle obtained by applying the projector $\epsilon_{W'}\epsilon_{A''}$ to the diagonal in $A'' \times A'' \subset W' \times A''$. This cycle is defined over $\mathbf{Q}(\zeta)$. Define

$$\tilde{\Delta} := \mathrm{Tr}_{\mathbf{Q}(\zeta)/K}(\Delta).$$

Then $\tilde{\Delta}$ is defined over K and we will show below that it is nontorsion in $\mathrm{Gr}^2(W' \times A'')$.

To verify the last claim, we use that there is an isogeny $W' \to W$ over $X(4) \simeq X_\Gamma$. Since

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(4) \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma,$$

the natural isomorphism

$$X(4) = \Gamma(4)\backslash\mathfrak{H} \simeq \Gamma\backslash\mathfrak{H} = X_\Gamma$$

is given by $z \mapsto \frac{z}{4}$. In terms of the moduli interpretations, this isomorphism sends a triple (E, P, Q) to the triple $(E', \varphi(P), \varphi(C))$, where

$$\varphi : E \to E' := E/\langle Q\rangle$$

is the natural isogeny, and C is the subgroup

$$C := \{R \in E : 4R \in \langle P\rangle\}.$$

Thus C is a subgroup of E of order 64 but $\varphi(C)$ is cyclic of order 16. The isogenies φ above patch together to give the desired isogeny from W' to W over $X(4) \simeq X_\Gamma$. Further, under this last isomorphism, the modular form f on Γ is identified with the modular form $\eta(z)^6$ on $\Gamma(4)$, i.e., one has the identity

$$f\left(\frac{z}{4}\right) = \eta(z)^6 \quad \text{or} \quad f(z) = \eta(4z)^6.$$

We can now show:

Proposition 5.2.1. *The cycle $\tilde{\Delta}$ is nontorsion in* $\mathrm{Gr}^2(W' \times A'')$.

Proof. It suffices to show that the pushforward $\varphi_*(\tilde{\Delta})$ of $\tilde{\Delta}$ under

$$(\varphi, 1) : W' \times A'' \to W \times A''$$

is nontorsion in Gr^2. Let x_ζ denote the point of $X(4)$ corresponding to $\sigma = \zeta$ and let y_ζ be its image in X_Γ. The point y_ζ then corresponds to a triple of the form (A'', P, C) with P a point of order 4 on A'' and C a cyclic subgroup of order 16 on A'' containing P, both P and C being defined over $\mathbf{Q}(\zeta)$. Let us fix a Néron differential ω on A'' defined over $\mathbf{Q}$. Using directly [3, Lemma 3.22–3.23 and Proposition 3.24], it suffices to show that

$$\theta^{-2} f^\flat(A'', \omega, P, C) + \theta^{-2} f^\flat(A'', \omega, P^\tau, C^\tau) \neq 0,$$

where τ is the nontrivial Galois automorphism of $\mathbf{Q}(\zeta)/K$. Again, as in the previous section, it suffices to find a prime p split in K, say $p = \mathfrak{p}\bar{\mathfrak{p}}$, and a positive integer $k \equiv -2 \bmod p-1$, such that

$$\delta^k f(A'', \omega, P, C) + \delta^k f(A'', \omega, P^\tau, C^\tau) \not\equiv 0 \bmod \mathfrak{p}. \tag{5.2.3}$$

Now, a simple computation shows that in the complex analytic uniformization, the point x_ζ corresponds to the class $[i] \in \Gamma(4)\backslash\mathfrak{H}$ while its conjugate by τ corresponds to the point $[\gamma i] \in \Gamma(4)\backslash\mathfrak{H}$, where

$$\gamma := \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}.$$

Let Ω be the real period of A''. The period lattice of A'' is $\Omega\cdot(\mathbf{Z}+\mathbf{Z}i)$ and $\gamma\cdot i = \frac{(i-8)}{5}$, so

$$\delta^k f(A'',\omega,P,C) = \left(\frac{2\pi i}{\Omega}\right)^{3+2k}\cdot\delta^k f\left(\frac{i}{4}\right) \tag{5.2.4}$$

and

$$\delta^k f(A'',\omega,P^\tau,C^\tau) = \left(\frac{2\pi i}{(2i+1)\cdot\Omega}\right)^{3+2k}\cdot\delta^k f\left(\frac{i-8}{20}\right). \tag{5.2.5}$$

These values can be computed, and we find that in fact the values of (5.2.4) and (5.2.5) are *equal* for all k and zero for odd values of k; for even k, a table of these values can be found in Appendix A. Taking either $k=2, p=5$ or $k=8, p=11$ or $k=14, p=17$, we see that (5.2.3) holds. □

As in the previous section, one needs to establish that the computed values of $\delta^k f$ used above are indeed correct. In this case, the method of the previous section seems difficult to apply (since we need $k\geq 2$), but one can argue differently as follows. Let

$$\Lambda^{\pm}(k) := \delta^k f(A'',\omega,P,C)\pm\delta^k f(A'',\omega,P^\tau,C^\tau).$$

Under the embedding of K into $\mathrm{M}_2(\mathbf{Q})$ corresponding to $z:=\frac{i}{4}$, the order

$$\left\{\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}\in\mathrm{M}_2(\mathbf{Z}) : c'\equiv 0 \bmod 16\right\}$$

pulls back to the order $\mathcal{O}_c$ in K of conductor $c:=4$. Since $K(\zeta)$ is the ring class field of K of conductor 4, by a formula of Waldspurger [37] one has (for $k\geq 0$)

$$\begin{aligned}\Lambda^+(k)^2 &= c_1(k)\cdot L^{\mathrm{alg}}(f,\psi'\psi^{2k+2},k+3)\\ &= c_1(k)\cdot L^{\mathrm{alg}}(\psi'\psi^{2k+4},k+2)\cdot L^{\mathrm{alg}}(\psi'\psi^{2k},k+1)\\ &= c_1(k)\cdot L^{\mathrm{alg}}(\psi^{2k+5}\mu,k+2)\cdot L^{\mathrm{alg}}(\psi^{2k+1}\mu,k+1)\end{aligned}$$

and

$$\begin{aligned}\Lambda^-(k)^2 &= c_2(k)\cdot L^{\mathrm{alg}}(f,\psi'\psi^{2k+2}\mu,k+3)\\ &= c_2(k)\cdot L^{\mathrm{alg}}(\psi'\psi^{2k+4}\mu,k+2)\cdot L^{\mathrm{alg}}(\psi'\psi^{2k}\mu,k+1)\\ &= c_2(k)\cdot L^{\mathrm{alg}}(\psi^{2k+5},k+2)\cdot L^{\mathrm{alg}}(\psi^{2k+1},k+1),\end{aligned}$$

where $c_1(k)$ and $c_2(k)$ are elements in K that are possibly zero. (There is an exact version of Waldspurger's formula worked out in [3, Section 4], which does not quite cover this case since we assumed in that article that the imaginary quadratic field has odd discriminant and the conductor of the embedding $K\hookrightarrow\mathrm{M}_2(\mathbf{Q})$ is also odd. However, for the current application, an exact formula is not needed.) The signs of the L-functions $L(\psi^{2k+1},s)$ and $L(\psi^{2k+1}\mu,s)$ are easy to compute; one finds that

$$\operatorname{sign} L(\psi^{2k+1},s) = \begin{cases}+1, & \text{if } 2k+1\equiv 1,3 \bmod 8,\\ -1, & \text{if } 2k+1\equiv 5,7 \bmod 8,\end{cases}$$

and

$$\operatorname{sign} L(\psi^{2k+1}\mu, s) = \begin{cases} +1, & \text{if } 2k+1 \equiv 1, 5 \bmod 8, \\ -1, & \text{if } 2k+1 \equiv 3, 7 \bmod 8. \end{cases}$$

It follows that simply for sign reasons, $\Lambda^-(k)$ is always zero while $\Lambda^+(k)$ is zero for odd values of k. For even k, at least in the range of computation of the table in Appendix A, one finds computationally that $\Lambda^+(k) \neq 0$, hence also $c_1(k) \neq 0$. Now $c_1(k)$ is a product of local factors $c_1(k) = \prod_v c_{1,v}(k)$, over all the places v of $\mathbf{Q}$. The factors $c_{1,v}(k)$ for $v \neq 2$ are computed explicitly in [3, Section 4], which as mentioned above does not cover the computation of $c_{1,2}(k)$. However, $c_{1,2}(k)$ is a local 2-adic integral and so one can effectively bound it from below in terms of the ramification data at 2, since one knows that is not zero. Then, as in Section 4.3, one can use [15, Proposition 10.11 (a)] to bound the denominator of $L^{\text{alg}}(\psi^{2k+1}\mu, k+1)$ and hence the denominator of $\delta^k f(A'', \omega, P, C)$.

Remark 5.2.2. The fact that $\Lambda^-(k) = 0$ for all k implies that the image of $\Delta - \Delta^\tau$ under the p-adic Abel–Jacobi map is 0, at least for all primes p split in K. This is consistent with the BB conjecture since $\Delta - \Delta^\tau$ lies in the -1-eigenspace of $\mathrm{CH}^2(W' \times A'')_0$ for the action of $\mathrm{Gal}(\mathbf{Q}(\zeta)/K)$. The dimension of this eigenspace should be controlled by the order of vanishing at the center of the twist by μ of the motive $M_f \otimes M_{\theta_{\psi'}}$, namely the L-function

$$L(\chi\mu, s) L(\psi'\mu, s-1) = L(\psi^3, s) \cdot L(\psi, s).$$

This last L-function can be checked to be nonvanishing at the center.

Remark 5.2.3. As in the previous sections, the CM action of K on A'' can be used to define cycles $\tilde{\Delta}^\pm$ in the respective ± 1-eigenspaces for the action of complex conjugation, that are both nontorsion in $\mathrm{Gr}^2(W' \times A'')$.

Step (ii). We show (ii) by constructing an explicit correspondence $\Pi_1 : A \times A \dashrightarrow W'$ defined over $\mathbf{Q}$. (This construction is likely classical, but we include it for want of a reference; see also [32, Remark 2.4.1].) First map $A \times A$ to $A_0 \times A_0$, where A_0 is the $\mathbf{Q}$-isogenous curve

$$A_0 : y^2 = x^4 - 1.$$

Taking two copies of A_0, say $y_1^2 = x_1^4 - 1$, $y_2^2 = x_2^4 - 1$, define a map

$$A_0 \times A_0 \to W',$$

the latter given by (5.2.1), by the formulas

$$(x_1, y_1, x_2, y_2) \mapsto (x, y, \sigma),$$

where $x = x_1 x_2$, $y = y_1 y_2$, $\sigma = x_1/x_2$. It is straightforward to check that the unique (up to scaling) nonzero holomorphic 2-form ω_f on W' pulls back under this map to a nonzero multiple of $p_1^*(\omega_{A_0}) \wedge p_2^*(\omega_{A_0})$, where ω_{A_0} is a nonzero holomorphic one form on A_0. The correspondence Π_1 induces a correspondence

$$\Pi : X \dashrightarrow A \times A \times A' \dashrightarrow W' \times A''$$

such that $\omega_f \wedge \omega_{A'}$ pulls back to a nonzero multiple of the unique holomorphic 3-form on X, from which it follows immediately that the cycles $\Pi^*(\tilde{\Delta}^\pm)$ are nontorsion in $\mathrm{Gr}^2(X)$.

A. Tables

Let $L(k) := L^{\mathrm{alg}}(\theta_{\psi^{2k+1}})$. The following tables list the L-values $L(k)^{1/2}$, for a range of values of k.

k	0	2	4	6	8
$L(k)^{1/2}$	1	2	2	6	14

Table 1. The values $L(k)^{1/2}$ for $D = 7, 0 \le k \le 8$, k even.

$k \backslash D$	11	19
1	1	1
3	2	$2 \cdot 3$
5	2^3	2^4
7	$2 \cdot 7$	$2 \cdot 3 \cdot 31$
9	$2^4 \cdot 19$	$2^4 \cdot 3^2 \cdot 29$
11	$2^5 \cdot 11$	$2^6 \cdot 3^2 \cdot 59$
13	$2^3 \cdot 11 \cdot 13 \cdot 67$	$2^3 \cdot 3 \cdot 13 \cdot 1747$
15	$2^4 \cdot 7 \cdot 11 \cdot 223$	$2^4 \cdot 3^2 \cdot 23 \cdot 83 \cdot 149$
17	$2^8 \cdot 11 \cdot 17^2$	$2^9 \cdot 3 \cdot 203911$
19	$2^6 \cdot 11 \cdot 19 \cdot 47 \cdot 109$	$2^6 \cdot 3^3 \cdot 19 \cdot 29 \cdot 179 \cdot 223$
21	$2^8 \cdot 7 \cdot 11 \cdot 317 \cdot 347$	$2^8 \cdot 3^3 \cdot 19 \cdot 2318077$

$k \backslash D$	43	67
1	1	1
3	$2^2 \cdot 3^2$	$2 \cdot 3 \cdot 19$
5	$2^3 \cdot 5 \cdot 11$	$2^3 \cdot 5 \cdot 173$
7	$2 \cdot 3^2 \cdot 7 \cdot 151$	$2 \cdot 3 \cdot 7 \cdot 61^2$
9	$2^5 \cdot 3^2 \cdot 29$	$2^4 \cdot 3^3 \cdot 17^2 \cdot 29^2$
11	$2^5 \cdot 3^6 \cdot 5 \cdot 17^2$	$2^5 \cdot 3^2 \cdot 5 \cdot 11 \cdot 251969$
13	$2^3 \cdot 3 \cdot 19 \cdot 653 \cdot 6967$	$2^3 \cdot 3 \cdot 13 \cdot 150044563$
15	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 191 \cdot 751$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19 \cdot 79 \cdot 181 \cdot 66359$
17	$2^8 \cdot 3 \cdot 5 \cdot 1877273779$	$2^8 \cdot 3 \cdot 5 \cdot 4639448759053$
19	$2^6 \cdot 3^4 \cdot 19 \cdot 29 \cdot 37 \cdot 7247281$	$2^6 \cdot 3^3 \cdot 690894919089413$
21	$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 127 \cdot 257 \cdot 10151899$	$2^8 \cdot 3^3 \cdot 5 \cdot 7 \cdot 19 \cdot 109 \cdot 793960993361$

Table 2. The values $L(k)^{1/2}$ for $D = 11, 19, 43, 67$, $1 \le k \le 21$, k odd.

$k \backslash D$	163
1	1
3	$2^2 \cdot 3 \cdot 181$
5	$2^3 \cdot 5^2 \cdot 17569$
7	$2 \cdot 3 \cdot 7 \cdot 523 \cdot 152389$
9	$2^5 \cdot 3^2 \cdot 829366883$
11	$2^5 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 1213 \cdot 1579 \cdot 25981$
13	$2^3 \cdot 3 \cdot 13 \cdot 17 \cdot 53 \cdot 6775046888887$
15	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 61 \cdot 28499 \cdot 472933737329$
17	$2^8 \cdot 3 \cdot 5^4 \cdot 17 \cdot 1012119331 \cdot 2184243869$
19	$2^6 \cdot 3^3 \cdot 19 \cdot 2347 \cdot 122795513 \cdot 235220529149$
21	$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 43 \cdot 701 \cdot 685301 \cdot 10549227661986929$
23	$2^9 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 23 \cdot 584359 \cdot 17638423193 \cdot 46435917679$
25	$2^7 \cdot 3^2 \cdot 5^2 \cdot 373 \cdot 17747 \cdot 11959219 \cdot 61407268956238912073$
27	$2^9 \cdot 3^5 \cdot 5^2 \cdot 13 \cdot 6551 \cdot 16995477783287 \cdot 510845704143990001$
29	$2^{10} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 29 \cdot 1907 \cdot 602217754893611 \cdot 4889245158810100367$
31	$2^8 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17 \cdot 31 \cdot 491 \cdot 4345577 \cdot 116423951774628163433706590548 97$
33	$2^{14} \cdot 3^5 \cdot 5 \cdot 11 \cdot 14895653 \cdot 76493417459931833536962636328539 79211$
35	$2^{14} \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 17 \cdot 233 \cdot 281 \cdot 1454839941553 \cdot 96916134714147846472190530 09$
37	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 37 \cdot 67 \cdot 199 \cdot 51239 \cdot 1327404593104322679691547262721973894957416 3$
39	$2^{14} \cdot 3^5 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 43 \cdot 53 \cdot 79 \cdot 1344250377771491844069374345645234060030577543203$
41	$2^{16} \cdot 3^4 \cdot 5^3 \cdot 13 \cdot 2027 \cdot 2197653300070968660189451708159964098498078082797912571$

Table 3. The values $L(k)^{1/2}$ for $D = 163$, $1 \leq k \leq 41$, k odd.

k	$2^{-(3+3k)} \cdot i^{-1} \cdot \delta^k f(A'', \omega, P, C)$
0	-1
2	$+1$
4	$+3$
6	$-3 \cdot 17$
8	$-3^2 \cdot 17$
10	$-3^3 \cdot 37$
12	$+3^2 \cdot 7 \cdot 37 \cdot 113$
14	$+3^2 \cdot 7 \cdot 43 \cdot 97 \cdot 113$
16	$-3^4 \cdot 43 \cdot 97 \cdot 3203$
18	$-3^5 \cdot 3203 \cdot 16231$
20	$-3^5 \cdot 7 \cdot 11 \cdot 53 \cdot 89 \cdot 16231$

Table 4. The values $2^{-(3+3k)} \cdot i^{-1} \cdot \delta^k f(A'', \omega, P, C)$ for $0 \leq k \leq 20$, k even, $f \in S_3(\Gamma_1(16))$, notations as in Section 5.2.

k	$\bar{\alpha}^{k+2} \cdot \delta^k f(A, A[\mathfrak{N}], \omega_A)$
0	-11
1	$-2^2 \cdot 11$
2	$-2^2 \cdot 5 \cdot 11$
3	$+2 \cdot 11 \cdot 61$
4	$+2^3 \cdot 11 \cdot 137$
5	$+2^5 \cdot 7^2 \cdot 11 \cdot 13$
6	$-2^4 \cdot 5 \cdot 11 \cdot 17 \cdot 137$
7	$+2^5 \cdot 7 \cdot 11 \cdot 5407$
8	$-2^6 \cdot 11^2 \cdot 31 \cdot 7639$
9	$+2^6 \cdot 11^2 \cdot 295459$
10	$-2^7 \cdot 5 \cdot 11^2 \cdot 1592281$
11	$+2^{10} \cdot 7 \cdot 11^2 \cdot 149 \cdot 199 \cdot 929$
12	$-2^6 \cdot 11^2 \cdot 61126101181$
13	$-2^{10} \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 151 \cdot 52609$
14	$-2^{10} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 10163 \cdot 1317713$
15	$+2^9 \cdot 7 \cdot 11^2 \cdot 23 \cdot 17293 \cdot 7958443$
16	$-2^{13} \cdot 11^2 \cdot 365267262447743$
17	$+2^{13} \cdot 11^2 \cdot 17 \cdot 547 \cdot 3491 \cdot 374761997$
18	$+2^{13} \cdot 5 \cdot 7 \cdot 11^2 \cdot 173 \cdot 1213 \cdot 103339254493$
19	$-2^{15} \cdot 7 \cdot 11^3 \cdot 13 \cdot 19 \cdot 289837 \cdot 788970227$
20	$-2^{13} \cdot 11^3 \cdot 141023 \cdot 1727597 \cdot 338696707$
21	$+2^{14} \cdot 7^2 \cdot 11^3 \cdot 67 \cdot 16427 \cdot 36677 \cdot 266115323$

Table 5. The values $\bar{\alpha}^{k+2} \cdot \delta^k f(A, A[\mathfrak{N}], \omega_A)$ for $0 \leq k \leq 21$, $f \in S_4(\Gamma_0(5))$; notations as in Section 5.1.

B. Integrality of modular forms via intersection theory

By *Brian Conrad* at Stanford[*)]

B.1. Introduction.

B.1.1. Integrality and q-expansions. Let $k, N \geq 1$ be integers, and $\Gamma = \Gamma_0(N)$. For a weight-k modular form $f \in M_k(\Gamma, \mathbf{C})$ on Γ, let $f_\infty \in \mathbf{C}[\![q]\!]$ denote its q-expansion at ∞. A general argument unrelated to modular curves (see (B.1.2.2) and Proposition B.2.1.1) ensures

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that for any number field $K \subset \mathbf{C}$, the $\mathscr{O}_K$-module

$$M_k(\Gamma, \mathscr{O}_K) := \{f \in M_k(\Gamma, \mathbf{C}) \mid f_\infty \in \mathscr{O}_K[\![q]\!]\}$$

is finitely generated and is an $\mathscr{O}_K$-structure on the $\mathbf{C}$-vector space $M_k(\Gamma, \mathbf{C})$ (in the sense that $\mathbf{C} \otimes_{\mathscr{O}_K} M_k(\Gamma, \mathscr{O}_K) \to M_k(\Gamma, \mathbf{C})$ is an isomorphism). This $\mathscr{O}_K$-structure is useful because membership in $M_k(\Gamma, \mathscr{O}_K)$ can be verified in practice via Riemann–Roch arguments by computing enough q-expansion coefficients. (The p-integrality aspects for primes $p \mid N$ rest on the geometry of the irreducible component through ∞ in the moduli stack $X_0(N)_{\mathbf{F}_p}$; see Definition B.1.2.1.)

This $\mathscr{O}_K$-structure suffers from a serious technical drawback, as follows. Let E be an elliptic curve over K with good reduction over $\mathscr{O}_K$ and Néron model $\mathscr{E}$ (the cases of most interest being CM elliptic curves), and let $G \subset E$ be a cyclic K-subgroup of order N. Choose a nonzero $\omega \in \Omega^1(E)$. For a modular form $f \in M_k(\Gamma, \mathscr{O}_K)$ and the point $x \in X_0(N)(K)$ arising from (E, G), the evaluation $f(x)$ lies in $\Omega^1(E)^{\otimes k}$ but may *fail* to lie in the $\mathscr{O}_K$-line $\Omega^1(\mathscr{E})^{\otimes k}$ (even though f has q-expansion coefficients in $\mathscr{O}_K$); we only know that $f(x) \in \Omega^1(\mathscr{E})^{\otimes k}[1/N]$. If we want to rigorously justify an evaluation of $f(x)/\omega^{\otimes k} \in K$ on a computer, then we need a theoretical multiplicative upper bound on the "denominator" of $f(x)$ with respect to the $\mathscr{O}_K$-line $\Omega^1(\mathscr{E})^{\otimes k}$. This is a question concerning powers of the primes $p \mid N$.

As is reviewed in Section B.1.2 (see (B.1.2.1)), we can use a suitable regular proper moduli stack over $\mathbf{Z}$ to define a finite-index $\mathscr{O}_K$-submodule $M_{k,\mathscr{O}_K} \subseteq M_k(\Gamma, \mathscr{O}_K)$ with the property that $f(x) \in \Omega^1(\mathscr{E})^{\otimes k}$ for any $f \in M_{k,\mathscr{O}_K}$. The construction provides compatible equalities

$$\mathscr{O}_K \otimes_{\mathbf{Z}} M_{k,\mathbf{Z}} = M_{k,\mathscr{O}_K}, \quad \mathscr{O}_K \otimes_{\mathbf{Z}} M_k(\Gamma, \mathbf{Z}) = M_k(\Gamma, \mathscr{O}_K),$$

so the exponent of the finite abelian group $M_k(\Gamma, \mathbf{Z})/M_{k,\mathbf{Z}}$ multiplies $M_k(\Gamma, \mathscr{O}_K)$ into $M_{k,\mathscr{O}_K}$ and hence bounds all failure of integrality when evaluating elements of $M_k(\Gamma, \mathscr{O}_K)$ at CM points over *any* number field K. This motivates our goal in this appendix: compute an explicit nonzero multiple of the exponent of $M_k(\Gamma, \mathbf{Z})/M_{k,\mathbf{Z}}$.

Remark B.1.1.1. The exponent of $M_k(\Gamma, \mathbf{Z})/M_{k,\mathbf{Z}}$ is divisible by all primes $p \mid N$ when k is sufficiently large and even; see Corollary B.3.1.5. More specifically, for even $k \gg_N 1$ there exists an $f \in M_k(\Gamma, \mathbf{Z})$ such that for each $p \mid N$ there is a cusp at which the q-expansion of f is not p-integral.

The case $\mathrm{ord}_p(N) = 1$ with $k = 2$ is thoroughly analyzed in [14, Section 2] via the geometry of Néron models of Jacobians, building on ideas introduced in the case $N = p$ for any k in [12, Chapter VII, Sections 3.19–3.20]. The technique used by Deligne and Rapoport rests on intersection theory on the regular proper Deligne–Mumford stack $X_0(p)$ over $\mathbf{Z}_{(p)}$, yielding that $p^{\lceil kp/(p-1)\rceil}$ is a multiple of the exponent in such cases. The arguments in [12, Chapter VII, Section 3.19] use that $X_0(p)_{\mathbf{F}_p}$ is reduced with exactly two irreducible components.

For general levels (divisible by arbitrarily high powers of p) we have to overcome the failure of reducedness and especially failure of the Deligne–Mumford condition for $X_0(N)_{\mathbf{F}_p}$ when $\mathrm{ord}_p(N) > 1$. This will be achieved via intersection theory on the regular proper Artin stack $X_0(N)$ over $\mathbf{Z}$ (see Definition B.1.2.1 below) that is *not* Deligne–Mumford when N is not squarefree.

There does not appear to be any literature for intersection theory on Artin stacks of finite type over a mixed-characteristic discrete valuation ring, and most of the general literature for intersection theory on stacks is limited to the Deligne–Mumford case. (See [21] for the case of Artin stacks over a field via K-theoretic methods.) The Artin stacks that arise for us have additional special properties (such as *finite* automorphism schemes at geometric points) that will enable us to develop the required intersection theory by bootstrapping from the scheme case.

B.1.2. Moduli stacks. To formulate our results in a precise form, we need to define the $\mathscr{O}_K$-structure $M_{k,\mathscr{O}_K}$ mentioned above. This rests on the following:

Definition B.1.2.1. Let $X = X_0(N)$ be the moduli stack over the base $\mathrm{Spec}(\mathbf{Z})$ that classifies generalized elliptic curves $E \to S$ (over arbitrary base schemes S) equipped with a $\Gamma_0(N)$-structure: a relatively ample finite locally free closed subgroup scheme $G \subset E^{\mathrm{sm}}$ that is "cyclic" in the sense that it admits a Drinfeld $\mathbf{Z}/N\mathbf{Z}$-generator fppf-locally on S.

See [11, Section 2.3] (especially Definition 2.3.4 there) for the notion of cyclicity in Definition B.1.2.1. Since we are interested in integrality properties of modular forms, we must work over $\mathbf{Z}$ and not just over $\mathbf{Z}[1/N]$. By [11, Theorem 1.2.1], X is a regular proper flat Artin stack over $\mathbf{Z}$ with geometrically connected fibers of pure dimension 1 and it is smooth over $\mathbf{Z}[1/N]$.

It is essential that we consider the moduli stack X rather than its associated coarse space, since modular forms for $\Gamma = \Gamma_0(N)$ correspond to sections of a line bundle on X that generally does not descend to the coarse space. The stack X is never a scheme, but it is Deligne–Mumford precisely on the open substack complementary to specific cusps that (always) exist in characteristics p for which $p^2 \mid N$ ([11, Theorem 3.1.7]).

Let $\pi : E \to X$ be the universal generalized elliptic curve over the moduli stack, and let $\omega_{E/X}$ be its relative dualizing sheaf, so

$$\omega := \pi_*(\omega_{E/X})$$

is a line bundle on X ([12, Chapter II, Section 1.6]). For $k \geq 1$, the torsion-free $\mathbf{Z}$-module of global sections $M_{k,\mathbf{Z}} := \mathrm{H}^0(X, \omega^{\otimes k})$ is finitely generated (by pullback to a $\mathbf{Z}$-flat finite scheme covering defined via auxiliary level structure, or by coherence of higher direct images for proper Artin stacks [30]). This $\mathbf{Z}$-module is a $\mathbf{Z}$-structure on the $\mathbf{C}$-vector space $M_k(\Gamma, \mathbf{C})$ of weight-k classical modular forms for Γ since $\mathrm{H}^0(X_{\mathbf{C}}, \omega^{\otimes k}) = M_k(\Gamma, \mathbf{C})$ (via GAGA applied to a higher-level moduli problem that is "rigid" and in fact represented by a scheme).

Since $\mathscr{O}_K$ is $\mathbf{Z}$-flat, clearly

$$M_{k,\mathscr{O}_K} := \mathrm{H}^0(X_{\mathscr{O}_K}, \omega^{\otimes k}) = \mathscr{O}_K \otimes_{\mathbf{Z}} M_{k,\mathbf{Z}}. \tag{B.1.2.1}$$

We want to relate $M_{k,\mathscr{O}_K}$ to the $\mathscr{O}_K$-submodule $M_k(\Gamma, \mathscr{O}_K) \subseteq M_k(\Gamma, \mathbf{C})$ defined in Section B.1.1 via the analytic theory of q-expansions. For this purpose, we now recall how q-expansions are defined algebraically in terms of the "completion" along a suitable $\mathbf{Z}$-point of the stack X.

Let $X^\infty \subset X$ be the closed cuspidal substack that is the "locus of non-smoothness" of the universal generalized elliptic curve $E \to X$ [11, Definition 2.1.8, Theorem 2.1.12]. This is a relative effective Cartier divisor in X [11, Theorem 4.1.1]. Define $\infty : \mathrm{Spec}(\mathbf{Z}) \to X$ to be the morphism arising from the standard 1-gon equipped with the canonical inclusion of μ_N into its smooth locus $\mathbf{G}_m$; this morphism is a degree-2 finite étale cover of a connected component ∞_X of the cuspidal substack of X, and ∞_X lies inside the open Deligne–Mumford locus in X (by [11, Theorem 3.1.7]).

For the morphism $h : \mathrm{Spec}(\mathbf{Z}[\![q]\!]) \to X$ corresponding to the Tate curve equipped with the standard μ_N subgroup inside its smooth locus, the induced morphisms

$$\mathrm{Spec}(\mathbf{Z}[q]/(q^{n+1})) \to X$$

are compatibly identified with étale double covers of the infinitesimal neighborhoods of ∞_X in X. Pullback along h defines q-expansions in $A \otimes_{\mathbf{Z}} \mathbf{Z}[\![q]\!] \subseteq A[\![q]\!]$ on $M_A := \mathrm{H}^0(X_A, \omega^{\otimes k})$ for any ring A, and for $A = \mathbf{C}$ this coincides with the analytic theory of q-expansions.

To understand the $\mathscr{O}_K$-module $M_k(\Gamma, \mathscr{O}_K)$ in more geometric terms, let $U \subset X$ be the open substack obtained by removing the fibral irreducible components of $X \to \mathrm{Spec}\,\mathbf{Z}$ disjoint from ∞_X, so $U[1/N] = X[1/N]$ and U is a $\mathbf{Z}$-smooth stack with *geometrically irreducible* fibers. The stack U is Deligne–Mumford [11, Theorem 3.1.7], so by using h and the completion of an étale scheme neighborhood of ∞_X we see that the proof of the algebraic q-expansion principle in the "scheme" case applies to the Artin stack U. Thus, the inclusion

$$\mathrm{H}^0(U_{\mathscr{O}_K}, \omega^{\otimes k}) \subset M_k(\Gamma, \mathscr{O}_K) \tag{B.1.2.2}$$

is an equality. In particular, since $\mathscr{O}_K \otimes_{\mathbf{Z}} \mathrm{H}^0(U, \omega^{\otimes k}) \to \mathrm{H}^0(U_{\mathscr{O}_K}, \omega^{\otimes k})$ is an isomorphism (by $\mathbf{Z}$-flatness of $\mathscr{O}_K$), we see that the natural map $\mathscr{O}_K \otimes_{\mathbf{Z}} M_k(\Gamma, \mathbf{Z}) \to M_k(\Gamma, \mathscr{O}_K)$ is an isomorphism. Thus, the inclusion of finitely generated $\mathscr{O}_K$-modules

$$\mathrm{H}^0(X_{\mathscr{O}_K}, \omega^{\otimes k}) \subset M_k(\Gamma, \mathscr{O}_K)$$

has finite index (as both sides are $\mathscr{O}_K$-structures on $\mathrm{H}^0(X_K, \omega^{\otimes k})$) and it is obtained by scalar extension along $\mathbf{Z} \to \mathscr{O}_K$ of the analogous inclusion over $\mathbf{Z}$. We conclude that the exponent of $M_k(\Gamma, \mathbf{Z})/M_{k,\mathbf{Z}}$ multiplies $M_k(\Gamma, \mathscr{O}_K)$ into $M_{k,\mathscr{O}_K}$.

B.1.3. Results. For the purpose of understanding the failure of integrality when evaluating elements of $M_k(\Gamma, \mathscr{O}_K)$ at CM-points, it suffices to solve:

Problem. Compute a nonzero multiple $m(k, N)$ of the exponent of

$$M_k(\Gamma_0(N), \mathbf{Z})/\mathrm{H}^0(X_0(N), \omega^{\otimes k}).$$

It is natural to consider additional moduli problems such as $\Gamma_1(N)$ and $\Gamma(N)$. We will focus on the above problem for $\Gamma = \Gamma_0(N)$ and leave it to the interested reader to consider other cases. Since the moduli stack for $\Gamma_1(Nq\ell)$ is a scheme for any two distinct primes $q, \ell \geq 5$ not dividing N, to merely find some theoretical multiplicative bound on the exponent for $\Gamma_0(N)$ it suffices to consider moduli *schemes* with extra level structure. Although this bypasses the intervention of Artin stacks, it appears to yield *much worse* bounds on the exponent. Thus, we prefer to work throughout with the $\mathbf{Z}$-stack $X_0(N)$ (and its intersection-theoretic properties). The regularity of $X_0(N)$ and the finiteness of the automorphism group schemes at its geometric points are crucial for the intersection theory arguments below. This regularity is generally lost upon passing to the coarse space.

The following theorem is a special case of our main result, generalizing the result in [12] for $\Gamma_0(p)$.

Theorem B.1.3.1. *For any integer $k \geq 1$ and number field K, if $r := \mathrm{ord}_p(N) \in \{1,2,3\}$ and $t := (p-1)^{-1}$, then*

$$p^{\lceil g(p,k,r)\rceil} M_k(\Gamma_0(N), \mathscr{O}_K) \subseteq \mathrm{H}^0(X_0(N)_{\mathscr{O}_K}, \omega^{\otimes k})$$

for $g(p,k,1) = k(1+t)$, $g(p,k,2) = 3kt(2t+1)$, and $g(p,k,3) = 2kt(t+1)(t+2)$.

In Theorem B.3.2.1 we address all $r > 3$ (but without explicit formulas for $g(p,k,r)$ for such r). Note that we need such bounds for cusp forms that are not U_p-eigenforms; e.g., $F \in S_{10}(\Gamma_0(45))$ in Section 5.1 does not appear to be an eigenform for any Hecke operator.

B.2. Intersection-theoretic bounds.

B.2.1. Elimination of q-expansions. The first step in the proof of Theorem B.1.3.1 and its generalizations is to reformulate the problem is purely geometric terms, without reference to modular curves or q-expansions. For the convenience of the reader, we first record a general geometric result that includes module-finiteness of $M_k(\Gamma, \mathscr{O}_K) = \mathscr{O}_K \otimes_{\mathbf{Z}} M_k(\Gamma, \mathbf{Z})$ as a special case. The localization $M_k(\Gamma, \mathbf{Z})[1/N] = \mathrm{H}^0(X_{\mathbf{Z}[1/N]}, \omega^{\otimes k})$ is $\mathbf{Z}[1/N]$-finite, so to prove that $M_k(\Gamma, \mathbf{Z})$ is $\mathbf{Z}$-finite it suffices to check $\mathbf{Z}_{(p)}$-finiteness of $M_k(\Gamma, \mathbf{Z})_{(p)}$ for each of the finitely many primes $p \mid N$. This latter finiteness follows by applying the following result to the open substack $U \subset X_0(N)$ defined above (B.1.2.2) and to the invertible sheaf $\omega^{\otimes k}$ on $X_0(N)_{\mathbf{Z}_{(p)}}$:

Proposition B.2.1.1. *Let $(R, \mathfrak{m})$ be a 1-dimensional noetherian local domain with fraction field F, and let X be a proper flat Artin stack over R with connected normal generic fiber. Let $U \subset X$ be an open substack such that $U_F = X_F$ and $U \to \mathrm{Spec}(R)$ is surjective. For any vector bundle $\mathscr{F}$ on X, $\mathscr{F}(U)$ is finitely generated over R.*

Proof. Since the inclusion $U \hookrightarrow X$ becomes an equality after localization at F, general "spreading out" principles provide a nonzero $a \in R$ such that $U_{R[1/a]} = X_{R[1/a]}$. Thus, $\mathscr{F}(U)_{R[1/a]} = \mathscr{F}(U_{R[1/a]}) = \mathscr{F}(X_{R[1/a]})$ is $R[1/a]$-finite, and R-flatness of $\mathscr{F}$ implies that the localization maps $\mathscr{F}(X) \to \mathscr{F}(X)_{R[1/a]}$ and $\mathscr{F}(U) \to \mathscr{F}(U)_{R[1/a]}$ are injective. Hence, we have inclusions of R-modules

$$\mathscr{F}(X) \subset \mathscr{F}(U) \subset \mathscr{F}(X_{R[1/a]}),$$

and it suffices to find an $n \geq 0$ such that $\mathscr{F}(U) \subset a^{-n}\mathscr{F}(X)$ inside $\mathscr{F}(X_{R[1/a]})$. We may assume $a \notin R^\times$, so $a \in \mathfrak{m}$.

For $m \geq 0$, define $\mathscr{F}(X)_m$ to be the R-module of elements $f \in \mathscr{F}(X)$ such that $f|_U$ has vanishing image in $\Gamma(U, \mathscr{F}/(a^m))$. We claim there exists an $n \geq 0$ so that $\mathscr{F}(X)_n \subseteq a\mathscr{F}(X)$. Granting this, let us see that $\mathscr{F}(U) \subset a^{-n}\mathscr{F}(X)$. Pick $f \in \mathscr{F}(U)$, so the equality

$$\mathscr{F}(U)_{R[1/a]} = \mathscr{F}(X)_{R[1/a]}$$

provides an exponent $\nu \geq 0$ such that $a^\nu f \in \mathscr{F}(X)$. If $\nu > n$, then $a^\nu f \in \mathscr{F}(X)_n \subseteq a\mathscr{F}(X)$, so $a^{\nu-1} f \in \mathscr{F}(X)$. Continuing by descending induction, $a^n f \in \mathscr{F}(X)$. This conclusion obviously also holds if $\nu \leq n$. Thus, $f \in a^{-n}\mathscr{F}(X)$ in all cases, or in other words

$$\mathscr{F}(U) \subset a^{-n}\mathscr{F}(X)$$

as desired.

It remains to find an n so that $\mathscr{F}(X)_n \subseteq a\mathscr{F}(X)$. It suffices to prove the same after the faithfully flat extension $R \to \widehat{R}$ to the completion of R (which may not be a domain if R is not normal, but in which a is not a zero-divisor), since

(i) the analogous submodule $\mathscr{F}(X)'_n$ over $\widehat{R}$ (i.e., the set of $f \in \mathscr{F}_{\widehat{R}}(X_{\widehat{R}})$ such that $f|_{U_{\widehat{R}}}$ has vanishing image in $\Gamma(U_{\widehat{R}}, \mathscr{F}_{\widehat{R}}/(a^m)))$ is equal to $\widehat{R} \otimes_R \mathscr{F}(X)_n$, and

(ii) $\widehat{R} \otimes_R \mathscr{F}(X) = \mathscr{F}'(X')$ for $X' := X_{\widehat{R}}$ and $\mathscr{F}' := \mathscr{F}_{\widehat{R}}$.

Consider the descending intersection $I = \bigcap_{m \geq 1} \mathscr{F}(X)'_m$. We shall prove that $I = 0$. Choose an element $f \in I$, so the restriction $f|_{U'}$ over $U' = U_{\widehat{R}}$ vanishes along the a-adic completion of $\mathscr{F}'$ over a smooth scheme cover of U'. Thus, $f|_{U'}$ vanishes over a Zariski-open neighborhood of the closed substack $U' \cap \{a = 0\}$ that contains the special fiber of U', so $f|_U$ vanishes over an open substack $V \subset U$ that contains the *nonempty* special fiber of U. It follows that V is nonempty, so V_F is a nonempty open locus inside the connected normal X_F. The locus of vanishing stalks for a global section of a vector bundle on a normal noetherian Artin stack is both open and closed, so by connectedness of X_F the global section $f_F \in \mathscr{F}_F(X_F)$ vanishes and hence $f = 0$ (since X is R-flat). This proves that $I = 0$.

Chevalley proved that a decreasing sequence of submodules $\{M_m\}$ of a finitely generated module M over a complete local noetherian ring B is cofinal for the max-adic topology on M if $\bigcap M_m = 0$. (This is proved for $M = B$ in [27, Exercise 8.7], and the proof there applies to any finitely generated module.) Since R/aR is an Artin local ring, there is an $e \geq 1$ such that $\mathfrak{m}^e \subseteq aR$. By Chevalley's result applied to $M = \mathscr{F}(X)'$ and $M_m := \mathscr{F}(X)'_m$ over $B = \widehat{R}$, there is an n such that $\mathscr{F}(X)'_n \subseteq \mathfrak{m}^e \mathscr{F}(X)' \subseteq a\mathscr{F}(X)'$, so $\mathscr{F}(X)_n \subseteq a\mathscr{F}(X)$. □

The inclusion $M_{k,\mathbf{Z}} := \mathrm{H}^0(X, \omega^{\otimes k}) \subseteq \mathrm{H}^0(U, \omega^{\otimes k}) = M_k(\Gamma, \mathbf{Z})$ between finite free $\mathbf{Z}$-modules becomes an equality after inverting N. Thus, to construct an explicit multiplicative upper bound on the exponent of $M_k(\Gamma, \mathbf{Z})/M_{k,\mathbf{Z}}$, for each prime $p \mid N$ we seek an explicit exponent $e_p \geq 0$ (depending on k, p, and N) so that $p^{e_p} M_k(\Gamma, \mathbf{Z})_{(p)} \subseteq (M_{k,\mathbf{Z}})_{(p)}$. Just as Deligne and Rapoport solved this problem for $\Gamma_0(p)$ by using intersection theory on the Deligne–Mumford stack $X_0(p)$, we will use intersection theory on the Artin stack $X_0(N)$. Thus, we now digress to develop a general formalism for intersection theory on a special class of regular proper flat Artin stacks over Dedekind domains (to be applied to $X_0(N)$).

B.2.2. Divisor and intersection theory formalism. Let X be a regular noetherian Artin stack that is proper and flat over a discrete valuation ring R with fraction field F, residue field k, and uniformizer π. Assume the following three properties:

(i) X is Deligne–Mumford away from a nowhere-dense closed substack of the special fiber X_0,

(ii) both fibers of $X \to \mathrm{Spec}(R)$ have pure dimension 1, and

(iii) there exists a regular finite flat scheme cover $h : X' \to X$ such that X' is also regular.

The existence of the regular finite flat scheme cover X' may appear to be restrictive (when X is not a scheme), but it is satisfied by $X_0(N)$; e.g., we can take X' to be the moduli stack corresponding to $\Gamma_1(Nq\ell)$ for two distinct primes $q, \ell \geq 5$ not dividing N, with h the "forgetful" map that involves a contraction over the cusps. (This X' is a regular scheme by [11, Theorem 4.1.1 (2) and Theorem 4.2.1 (2)], and the corresponding map h is finite flat by [11, Lemma 4.2.3 (2), (4)].) Since h is a finite flat surjection, X' has fibers of pure dimension 1 over $\mathrm{Spec}(R)$. Thus, the usual intersection theory of regular arithmetic surfaces applies to X'.

In this subsection, we will use h-pullback to define an intersection theory formalism on X by pulling down the usual theory on X' (and the resulting constructions will be independent of the choice of h). By applying this to $X_0(N)$, we shall use the descriptions of local deformation rings in [24, Chapter 13] to compute intersection numbers and multiplicities on $X_0(N)$ without any interference from auxiliary level structures or isotropy groups.

Remark B.2.2.1. Assume X and X' are connected. If $X \to \mathrm{Spec}(R)$ is its own Stein factorization, then it does not generally follow that $X' \to \mathrm{Spec}(R)$ is its own Stein factorization (though for $X = X_0(N)$ and $X' = X_1(Nq\ell)$ each has structural map to $\mathrm{Spec}(\mathbf{Z})$ that is its own Stein factorization). Thus, it is important to note at the outset that the development of intersection theory on X' as in [9, Sections 2–3] does not use Stein factorization, only the connectedness of X'. Since intersection theory on X' relative to the base $\mathrm{Spec}(R)$ "makes sense" whenever the regular arithmetic surface X' is merely connected, it is reasonable to use that theory to define an intersection theory on X.

In the scheme case, Stein factorization is essential for the proof that the intersection matrix among the irreducible components of the special fiber is negative semi-definite with 1-dimensional defect space [9, Lemma 7.1 (b)]. Thus, we will need to be attentive to Stein factorization issues when we analyze the dimension of the defect space for the intersection matrix associated to the special fiber of the stack X.

Our task is essentially one of using descent theory to make definitions on X using the known formalism on X'. The main complications are related to connectedness: X'_0 may have many more connected components than X_0, X_0 can be disconnected when X is connected, and $\mathscr{O}(X')$ may be larger than $\mathscr{O}(X)$ (so we cannot easily relate the Stein factorizations of X and X' over $\mathrm{Spec}(R)$). To handle issues related to Stein factorization later, we record:

Proposition B.2.2.2. *Let X' be a connected regular arithmetic surface over a henselian discrete valuation ring R. The intersection matrix associated to $X' \to \mathrm{Spec}(R)$ is negative semi-definite with a 1-dimensional defect space.*

Proof. Let k be the residue field of R, and let $X' \to \mathrm{Spec}(R')$ be the Stein factorization of X'. Here $R' = \mathscr{O}(X')$ has no nontrivial idempotents since X' is connected, yet it is a finite product of local rings since it is module-finite over the henselian local R. Thus, R' is local. The normality of X' implies that R' is normal [18, Chapter II, Section 8.8.6.1], so R' is a discrete valuation ring. Letting π and π' denote respective uniformizers of R and R', the special fibers X'_k and $X'_{k'}$ are identified with X' mod π and X' mod π' respectively. Thus, $X'_{k'} = X'_k \bmod \overline{\pi}'$ where $\overline{\pi}' := \pi' \bmod \pi R'$, so $X'_{k'}$ and X'_k have the same underlying reduced schemes.

It follows that the (reduced) irreducible components of $X'_{k'}$ and X'_k are the same closed subschemes of X', so any pair of distinct components has intersection scheme that is the same whether we view the base ring as R or R'. Hence, for such distinct components C_1 and C_2, the intersection pairings $C_1.C_2$ and $C_1 * C_2$ relative to R and R' respectively are

$$C_1.C_2 = \deg_k(C_1 \cap C_2) = [k' : k] \deg_{k'}(C_1 \cap C_2), \quad C_1 * C_2 = \deg_{k'}(C_1 \cap C_2).$$

The self-intersection numbers $C.C$ and $C * C$ for such components are determined by two data:

(i) the respective pairings $C_1.C_2$ and $C_1 * C_2$ for $C_1 \neq C_2$, and

(ii) the multiplicities μ_C and μ'_C of the special fiber along each component C.

But clearly $\mathrm{ord}_C(\pi) = e(R'|R)\mathrm{ord}_C(\pi')$ for every C, so the multiplicities all scale by the same factor $e(R'|R)$ when passing between the base rings R and R'. Hence, if $\{C_i\}$ is the common set of (reduced) irreducible components of the special fibers and $\mu_i := \mu_{C_i}$, then

$$\begin{aligned} C_i.C_i &= \sum_{j \neq i} \frac{\mu_j}{\mu_i}(C_j.C_i) \\ &= \sum_{j \neq i} \frac{e(R'|R)\mu'_j}{e(R'|R)\mu'_i}[k' : k](C_j * C_i) \\ &= [k' : k]C_i * C_i. \end{aligned}$$

To summarize, the intersection matrix associated to the special fiber of $X' \to \mathrm{Spec}(R)$ is $[k' : k]$ times the analogous such matrix associated to $X' \to \mathrm{Spec}(R')$. This latter matrix is negative semi-definite with a 1-dimensional defect space by [9, Lemma 7.1 (b)], and scaling by $[k' : k]$ preserves these properties. □

It is convenient at the start to define some notions for proper Artin stacks D of pure dimension 1 over an arbitrary field k (which can also be $F = \mathrm{Frac}(R)$ as above in subsequent applications), ignoring regularity and connectedness conditions. We assume that there is a finite flat scheme cover $q : D' \to D$. Note that for any $z \in D(k)$, the k-morphism $z : \mathrm{Spec}(k) \to D$ is finite because its q-pullback is $q^{-1}(z) \to D'$ (with $q^{-1}(z)$ a finite k-scheme). Thus, the automorphism group scheme $\mathrm{Aut}(z)$ at any k-point z of D is k-finite since it is the fiber product $\mathrm{Spec}(k) \times_{z,D,z} \mathrm{Spec}(k)$. For each $z \in D(k)$, we let $\# \mathrm{Aut}(z)$ denote the order of the finite k-group scheme $\mathrm{Aut}(z)$.

Every connected component C' of D' is finite flat over a unique connected component C of D, and conversely every connected component C of D is the image of any connected component C' of the open and closed subscheme $q^{-1}(C)$ of D'. For such C and C', let $q_{C'} : C' \to C$ be the finite flat map induced by q; it has *constant* fibral degree.

Lemma B.2.2.3. *Let $\mathscr{L}$ be an invertible sheaf on D, and let $Z \subset D$ be a nowhere-dense closed substack. Let C be a connected component of D. For all connected components C' of D', the ratios*

$$\frac{\deg_k(q^*_{C'}(\mathscr{L}|_C))}{\deg(q_{C'})}, \quad \frac{\deg_k(q^{-1}_{C'}(Z \cap C))}{\deg(q_{C'})}$$

are independent of C', and also are independent of the choice of D'.

We denote these ratios as $\deg_k(\mathscr{L}|_C)$ and $\deg_k(Z \cap C)$ respectively.

Proof. The problem is intrinsic to the finite flat covering map $q_{C'}$ (allowing variation in C'), so we may assume D and D' are connected and just have to check that the ratios $\deg_k(q^*(\mathscr{L}))/\deg(q)$ and $\deg_k(q^{-1}(Z))/\deg(q)$ are independent of the choice of $q : D' \to D$.

Let $D'' \to D$ be another connected finite flat scheme cover, so $D' \times_D D''$ is finite flat over D' and D'' (in particular, this fiber product is a scheme). Any connected component of $D' \times_D D''$ is a finite flat cover of D' and D'' (as D' and D'' are connected), so we may rename such a connected component as D'' to reduce to the case that $D'' \to D$ is the composition of q with a finite flat scheme surjection $q' : D'' \to D'$. Connectedness of D and D' ensures that the degrees of q and q' are constant and their product is equal to the constant degree of $q' \circ q$. Thus, it remains to note that under the finite flat surjection $q' : D'' \to D'$, the k-degree of the q'-pullback of a line bundle $\mathscr{L}'$ on D' (respectively of a 0-dimensional closed subscheme Z' of D') is $\deg(q')$ times the k-degree of $\mathscr{L}'$ on D' (respectively the k-degree of Z'). $\square$

It is now well-posed to define

$$\deg_k(\mathscr{L}) := \sum_C \deg_k(\mathscr{L}|_C),$$

$$\deg_k(Z) := \sum_C \deg_k(Z \cap C).$$

(We do not define $\deg_k(\mathscr{L})$ via cohomological methods on D because in positive characteristic the coherent cohomology of D may be nonzero in arbitrarily high degree.) The formation of these degrees is easily checked to be unaffected by any extension of the ground field, even though the formation of the set of connected components of D generally does not commute with such base change.

Lemma B.2.2.4. *If Z is the schematic image of $z \in D(k)$, then*

$$\deg_k(Z) = \frac{1}{\#\operatorname{Aut}(z)}.$$

Proof. We may and do replace D with its unique connected component containing Z, so D is connected. Choose a finite flat surjection $q : D' \to D$ from a scheme D'. We may and do assume D' is connected. The finite map $z : \operatorname{Spec}(k) \to D$ has schematic image Z that is reduced, so generic flatness is applicable to the finite surjection $z : \operatorname{Spec}(k) \to Z$. Openness on the source for the flat locus of any fppf morphism implies that $z : \operatorname{Spec}(k) \to Z$ is a finite flat covering, necessarily with a constant degree $d > 0$ since Z is connected. Thus, the projections $\operatorname{Aut}(z) = z \times_D z = z \times_Z z \rightrightarrows \operatorname{Spec}(k)$ are finite flat with degree d, so our problem is to show that $\deg_k(Z) = 1/d$.

By definition, $\deg_k(Z) = \deg_k(q^{-1}(Z))/\deg(q)$. Since $q^{-1}(Z) \to Z$ is finite flat with constant degree $\deg(q)$, the projection $\mathrm{pr}_2 : q^{-1}(Z) \times_{Z,z} \operatorname{Spec}(k) \to \operatorname{Spec}(k)$ is finite flat with constant degree $\deg(q)$. But $q^{-1}(Z) \times_{Z,z} \operatorname{Spec}(k)$ is a base change of z and hence is finite flat with constant degree d over the finite k-scheme $q^{-1}(Z)$ via pr_1. Thus, $d \mid \deg(q)$ and $q^{-1}(Z)$ has k-degree $\deg(q)/d$. We conclude that $\deg_k(q^{-1}(Z))/\deg(q) = 1/d$. $\square$

The variant below will be useful when working on the Deligne–Mumford stack $X_0(N)_{\mathbf{Q}}$.

Lemma B.2.2.5. *If D is Deligne–Mumford, then for any effective Cartier divisor Z on D the degree $\deg_k(Z)$ is equal to $\deg_k(\mathscr{O}_D(Z))$, where $\mathscr{O}_D(Z)$ denotes the inverse of the invertible coherent ideal of Z in $\mathscr{O}_D$.*

Proof. We may assume D is connected and choose a connected finite flat scheme cover $q : D' \to D$, so q has constant fiber degree d. Since $q^*(\mathscr{O}_D(Z)) \simeq \mathscr{O}_{D'}(q^{-1}(Z))$, multiplying both sides by d reduces the problem to its well-known analogue on the scheme D'. □

Now consider a proper flat Artin stack X with pure relative dimension 1 over R such that there exists a finite flat scheme cover $h : X' \to X$; we postpone the regularity hypotheses on X and X' until after we introduce several more definitions.

Since the map $h_0 : X'_0 \to X_0$ between special fibers is a finite flat scheme cover of X_0, the preceding considerations are applicable to X_0 as well as to any closed substack of X_0 with pure dimension 1. In particular, if Z_0 is a nowhere-dense closed substack of X_0 and if $\mathscr{L}_0$ is an invertible sheaf on X_0, then we may define $\deg_k(Z_0)$ and $\deg_k(\mathscr{L}_0)$ via the preceding formalism applied to X_0 over k (e.g., using connected components of X'_0 covering connected components of X_0); the same holds for the generic fiber X_F over the field F. We make two further definitions that imitate the case of arithmetic surfaces:

Definition B.2.2.6. For any invertible sheaf $\mathscr{L}$ on X and effective Cartier divisor $D \subset X$ that is a closed substack of X_0, we define

$$\mathscr{L}.D := \deg_k(\mathscr{L}|_D). \tag{B.2.2.1}$$

For effective Cartier divisors $D_1, D_2 \subset X$ with D_2 a closed substack of X_0, we define

$$D_1.D_2 := \mathscr{O}_X(D_1).D_2 = \deg_k(\mathscr{O}_X(D_1)|_{D_2}).$$

The preceding definitions are invariant under a local extension $R \to R'$ of discrete valuation rings. (Note that $X_{R'}$ may not be regular even if X is regular.) The next result uses regularity to establish familiar properties.

Proposition B.2.2.7. *Assume X is regular and that it admits a regular finite flat scheme cover $h : X' \to X$. The pairing $\mathscr{L}.D$ is additive in $\mathscr{L}$, and if $D = D' + D''$ for effective Cartier divisors D' and D'' on X (necessarily closed substacks of X_0), then $\mathscr{L}.D = \mathscr{L}.D' + \mathscr{L}.D''$. Moreover,* $\deg_k(\mathscr{L}_0) = \deg_F(\mathscr{L}_F)$ *for any invertible sheaf $\mathscr{L}$ on X.*

For effective Cartier divisors $D_1, D_2 \subset X$ that are closed substacks of X_0, the pairing $D_1.D_2$ is symmetric, and if D_1 and D_2 share no irreducible component of X_0 in their supports, then $D_1.D_2 =$ $\deg_k(D_1 \cap D_2)$.

Proof. We may extend scalars to make R strictly henselian without harming the regularity hypotheses, so we may and do assume k is separably closed. It also suffices to work separately over the distinct connected components of X and X', so we may and do assume X and X' are connected. In particular, h has constant fiber degree.

We claim that X'_0 is connected (so X_0 is also connected). Since X' is a connected normal scheme, it is also integral. Thus, the finite flat R-algebra $\mathscr{O}(X')$ is a normal domain ([18, Chapter II, Section 8.8.6.1]) and so it is Dedekind. By R-finiteness, $\mathscr{O}(X')$ must be semi-local. Thus, the henselian property of R forces $\mathscr{O}(X')$ to be a direct product of discrete valuation rings, so $\mathscr{O}(X')$ is a discrete valuation ring since it is a domain. The Stein factorization $f : X' \to \operatorname{Spec} \mathscr{O}(X')$ has (geometrically) connected fibers and the residue field of $\mathscr{O}(X')$ is purely inseparable over k (since $k = k_s$), so the special fiber X'_0 of $X' \to \operatorname{Spec} R$ inherits connectedness from that of the special fiber of f.

The finite flat $h_0 : X'_0 \to X_0$ has constant fiber degree equal to that of h. Thus, by the connectedness of X_0 and X'_0, the definitions over k using h_0 yield formulas

$$\mathscr{L}.D = \frac{h^*(\mathscr{L}).h^{-1}(D)}{\deg(h)},$$
$$D_1.D_2 = \frac{h^{-1}(D_1).h^{-1}(D_2)}{\deg(h)},$$
$$\deg_k(D_1 \cap D_2) = \frac{h^{-1}(D_1).h^{-1}(D_2)}{\deg(h)}.$$

The additivity of $\mathscr{L}.D$ in $\mathscr{L}$ and the identity $\mathscr{L}.D = \mathscr{L}.D_1 + \mathscr{L}.D_2$ for effective Cartier divisors D_1, D_2 satisfying $D_1 + D_2 = D$ therefore follow from the analogue for intersection theory on the connected regular arithmetic surface X' relative to $\mathrm{Spec}(R)$, and likewise for the symmetry of $D_1.D_2$ for $D_1, D_2 \subseteq X_0$ as well as for the equality $D_1.D_2 = \deg_k(D_1 \cap D_2)$ when such D_1 and D_2 share no common irreducible component of X_0 in their supports.

Finally, we prove the equality $\deg_k(\mathscr{L}_0) = \deg_F(\mathscr{L}_F)$. We have already seen that X'_0 and X_0 are connected, and X'_F inherits connectedness from the integral X' (so X_F is connected too). Clearly, h_0 and h_F have the same constant fiber degree (namely, that of h), so our problem is reduced to proving the equality of the numbers $\deg_k(h_0^*\mathscr{L}_0) = \deg_k((h^*\mathscr{L})_0)$ and $\deg_F((h^*\mathscr{L})_F)$. This equality follows from X' being proper and flat over R. □

Now we assume X is regular (as in the preceding proposition) and that it admits a regular finite flat scheme cover $h : X' \to X$. We have not yet used that X is Deligne–Mumford away from a nowhere-dense closed substack of X_0, but this property will now be used.

Lemma B.2.2.8. *Every 1-dimensional reduced and irreducible closed substack $D \subset X$ is Cartier. For the generic point η of D, $h^{-1}(\eta)$ consists of finitely many codimension-1 points of X'.*

Proof. Let $V \subseteq X$ be the maximal open substack that is Deligne–Mumford, so $X - V$ has preimage in X' that consists of finitely many closed points in X'_0. It follows that $D \mapsto D \cap V$ is a bijective correspondence between the sets of 1-dimensional reduced and irreducible closed substacks of X and of V, with inverse given by “schematic closure”. Using an étale scheme cover of V, we see that the reduced closed substack $D \cap V$ of V is Cartier. Thus, by computing with a smooth scheme cover of X it follows that the closure D of the Cartier locally closed substack $D \cap V$ in the regular X is also Cartier.

If η is the generic point of D, then its preimage $h^{-1}(\eta)$ in X' consists of the generic points of $h^{-1}(D)$ since h is finite flat. But $h^{-1}(D)$ must have all irreducible components of dimension 1 since D is irreducible of dimension 1, so $h^{-1}(\eta)$ consists of finitely many codimension-1 points of X'. □

Consider an invertible rational function f on X. By fppf descent from X', the maximal open substack $U \subset X$ on which f is a section of $\mathscr{O}_X^\times$ has closed complement D such that if $D \neq \emptyset$, then the irreducible components D_i of D are all 1-dimensional. In particular, if there are no irreducible components D_i (i.e., $U = X$), then f is a global unit on X. Each D_i has a unique generic point η_i. Since η_i lies in the Deligne–Mumford locus of X, there is a well-defined strictly henselian local ring $\mathscr{O}_{X,\eta_i}$ at η_i, and this is a discrete valuation ring. Thus, we can define $\mathrm{ord}_{\eta_i}(f) \in \mathbf{Z}$, and for any closed substack $D \subset X$ with irreducible components of dimension 1 we define the *generic multiplicity* $\mathrm{ord}_\eta(D)$ of D at a generic point η of D to be the length of the 0-dimensional strictly henselian local ring of D at the codimension-1 point η of X (which lies in the open Deligne–Mumford locus of D). For example, if η is a generic point of $X_0 = X \bmod \pi$, then the generic multiplicity of D is $\mathrm{ord}_\eta(\pi)$ for a uniformizer π of R.

For any effective Cartier divisor D in X with generic points η_i we define the associated Weil divisor

$$[D] := \sum_i \mathrm{ord}_{\eta_i}(D_i)[D_i]$$

in the free abelian group of Weil divisors of X (with $[C]$ denoting the Weil divisor associated to a reduced and irreducible closed substack $C \subset X$ of dimension 1). We also define the Weil divisor

$$\mathrm{div}(f) = \sum_i \mathrm{ord}_{\eta_i}(f)[D_i]$$

for invertible rational functions f on X. It is clear that $\mathrm{div}(f_1 f_2) = \mathrm{div}(f_1) + \mathrm{div}(f_2)$, and if $\mathrm{div}(f) = 0$, then f is a global unit on X.

For each irreducible component C_i of X_0 equipped with the reduced structure (so C_i is Cartier in X, by Lemma B.2.2.8) and for each invertible sheaf $\mathscr{L}$ on X, we use Definition B.2.2.6 to define

$$\mathscr{L}.[C_i] := \mathscr{L}.C_i = \deg_k(\mathscr{L}|_{C_i});$$

this is additive in $\mathscr{L}$ and we extend it by additivity in the second variable to define $\mathscr{L}.\Delta$ for any Weil divisor Δ of X supported in X_0. (We do not require Δ to have its multiplicities along the components of X_0 in its support to be bounded above by those of $[X_0]$.)

Proposition B.2.2.7 applied to an effective Cartier divisor D on X that is a closed substack of X_0 shows that

$$\mathscr{L}.D = \mathscr{L}.D' + \mathscr{L}.D''$$

when $D = D' + D''$ for effective Cartier divisors D' and D'' on X, so (by passage to the irreducible components C_i) we see that $\mathscr{L}.D = \mathscr{L}.[D]$ for such D. If Δ_1 and Δ_2 are Weil divisors on X with Δ_2 supported in X_0, then we may define $\Delta_1.\Delta_2$ to be biadditive in such Δ_1 and Δ_2 and to coincide with $\mathscr{O}_X(\Delta_1).\Delta_2$ when $\Delta_2 \leq [X_0]$; this recovers Definition B.2.2.6 for effective Cartier divisors that are closed substacks of X_0 when $\Delta_2 = [D_2]$ for a closed substack D_2 of X_0 that is Cartier in X. In particular,

$$\mathrm{div}(f).\Delta = 0$$

for any invertible rational function f on X and any Weil divisor Δ on X supported in X_0.

Corollary B.2.2.9. *Choose Weil divisors Δ_1 and Δ_2 on X supported inside X_0. The pairing $\Delta_1.\Delta_2$ is symmetric. If the supports of Δ_1 and Δ_2 do not share an irreducible component in common and Δ_i is the Weil divisor associated to an effective Cartier divisor D_i in X that is a closed substack of X_0, then*

$$\Delta_1.\Delta_2 = \deg_k(D_1 \cap D_2). \tag{B.2.2.2}$$

If $\{C_1, \ldots, C_n\}$ is the set of (reduced) irreducible components of X_0, then the symmetric intersection matrix $(C_i.C_j)$ is negative semi-definite, and its defect space is 1-dimensional if $X \to \mathrm{Spec}(R)$ is its own Stein factorization.

Proof. Biadditivity reduces the symmetry of $\Delta_1.\Delta_2$ to the case when each is an irreducible component of X_0 (equipped with the reduced structure). These correspond to closed substacks of X_0, so Proposition B.2.2.7 provides the symmetry. The identity (B.2.2.2) also follows from Proposition B.2.2.7.

It remains to prove that $(C_i.C_j)$ is negative semi-definite, with 1-dimensional defect space when $X \to \mathrm{Spec}(R)$ is its own Stein factorization. Negative semi-definiteness means $\Delta.\Delta \leq 0$ for all Weil divisors Δ generated by the divisors $[C_i]$. To prove such an inequality we may assume R is strictly henselian and pass to the case when X is connected (so X_0 is also connected). Thus, we can choose a connected regular finite flat scheme cover $h : X' \to X$ and X'_0 is also connected. The map h has constant degree, and the known intersection theory on X' relative to $\mathrm{Spec}(R)$ (see Proposition B.2.2.2) gives that

$$\Delta.\Delta = \frac{h^{-1}(\Delta).h^{-1}(\Delta)}{\deg(h)} \leq 0$$

with equality if and only if $h^{-1}(\Delta)$ is a $\mathbf{Q}$-multiple of $\mathrm{div}_{X'}(\pi) = h^{-1}(\mathrm{div}_X(\pi))$. Since h-pullback is injective on Weil divisors, it follows that $\Delta.\Delta = 0$ if and only if Δ is a $\mathbf{Q}$-multiple of $\mathrm{div}_X(\pi)$. $\square$

The final topic we address in this subsection is the definition of the Weil divisor $\mathrm{div}_{\mathscr{L}}(s)$ for an invertible sheaf $\mathscr{L}$ on X and a global section $s \in \mathscr{L}(X)$ that is nonzero on every connected component of X. For this purpose, we temporarily pass to the setting of an invertible sheaf $\mathscr{N}$ on an arbitrary Artin stack Y, and a global section $t \in \mathscr{N}(Y)$ that is nowhere a zero-divisor in $\mathscr{N}$. By computing on a smooth scheme cover of Y we see that the annihilator ideal $\mathscr{I}_{Y,\mathscr{N},s} = \mathrm{Ann}_{\mathscr{O}_Y}(\mathscr{N}/(s))$ is invertible on Y. In the setting of our regular X with equipped $(\mathscr{L}, s)$, we define the effective Weil divisor

$$\mathrm{div}_{\mathscr{L}}(s) = [\mathscr{I}^{-1}_{X,\mathscr{L},s}].$$

Clearly, $h^{-1}(\mathrm{div}_{\mathscr{L}}(s)) = \mathrm{div}_{h^*(\mathscr{L})}(h^*(s))$ as effective Weil divisors on X'. Also, by descent from a (necessarily normal) smooth scheme cover of X, the inclusion $s : \mathscr{O}_X \hookrightarrow \mathscr{L}$ uniquely extends to an isomorphism of invertible sheaves $\mathscr{O}_X(\mathrm{div}_{\mathscr{L}}(s)) \simeq \mathscr{L}$.

Proposition B.2.2.10. *Let C be an irreducible component of X_0. Equipping C with the reduced structure,*

$$\deg_k(\mathscr{L}|_C) = \mathrm{div}_{\mathscr{L}}(s).[C].$$

In this proposition, the left side is defined using that C is a proper Artin stack over k admitting a finite flat scheme cover.

Proof. Since $\mathscr{L} \simeq \mathscr{O}_X(\Delta)$ for the Weil divisor $\Delta = \mathrm{div}_{\mathscr{L}}(s)$, it suffices to prove more generally that $\deg_k(\mathscr{O}_X(\Delta)|_C) = \Delta.[C]$ for any Weil divisor Δ on X. The case of principal Δ is clear, since both sides vanish. In general, both sides are additive in Δ, so we may assume that Δ is a reduced and irreducible Weil divisor. Let D be the effective Cartier divisor in X satisfying $[D] = \Delta$ (see Lemma B.2.2.8).

The case $\Delta = [C]$ is reduced to the case $\Delta \neq [C]$ due to the established principal case applied to $\Delta = \mathrm{div}(\pi)$ (whose support contains $[C]$). Thus, we may assume $\Delta \neq [C]$, so $\Delta.[C] = \deg_k(D \cap C)$. Hence, it suffices to show that if C is a closed substack of X_0 that is Cartier in X and if D is an effective Cartier divisor on X, then the invertible sheaf $\mathscr{O}_X(D)|_C$ on C has k-degree $\deg_k(D \cap C)$. Passing to strictly henselian R and connected X (hence connected X_0), we may use h-pullback and h_0-pullback for a connected regular finite flat scheme cover $h : X' \to X$ to reduce the problem to its known analogue on the regular arithmetic surface X'. □

Since X is Deligne–Mumford away from a 0-dimensional closed substack of X_0, by Lemma B.2.2.8 it is now harmless to identify the notions of effective Cartier divisor and effective Weil divisor in X. Thus, for the remainder of this appendix we shall abuse terminology and notation by identifying the concepts of effective Weil divisor and effective Cartier divisor on X without comment. In particular, for an effective Cartier divisor D in X we now write "D" even where we should write "$[D]$".

Let us now return to the original motivating problem that is intrinsic to the regular proper flat Artin stack $X_p := X_0(N)_{\mathbf{Z}_{(p)}}$ over $\mathbf{Z}_{(p)}$ and its open substack $U_{\mathbf{Z}_{(p)}}$ for U as defined above (B.1.2.2). For a nonzero section $f \in \mathrm{H}^0(X_{\mathbf{Q}}, \omega^{\otimes k})$ such that the Weil divisor $\mathrm{div}_{\omega^{\otimes k}}(f)$ on X_p has multiplicity ≥ 0 along the irreducible component of the special fiber containing U mod p, we will show that its multiplicity on any other component of X mod p is bounded below by $-e_p$ for an explicit integer $e_p \geq 0$ depending only on k, p, and N; then $p^{e_p} f$ has all multiplicities ≥ 0 and so lies in $\mathrm{H}^0(X_p, \omega^{\otimes k})$ due to the normality of X_p.

To formulate our task in a more geometric manner, we introduce some notation. Let $\{C_1, \ldots, C_n\}$ be the set of (reduced) irreducible components of X mod p, and write

$$\mathrm{div}_{\mathscr{O}_X}(p) = \sum_j \mu_j C_j$$

(with $\mu_j \geq 1$ the generic multiplicity of C_j on X). For an arbitrary nonzero $f \in \mathrm{H}^0(X_{\mathbf{Q}}, \omega^{\otimes k})$, in the Weil divisor group of X_p we may write

$$\mathrm{div}_{\omega^{\otimes k}}(f) = D + \sum_j \nu_j C_j$$

for integers ν_j and an effective "horizontal" divisor D on X_p. Replacing f with $p^e f$ for $e \in \mathbf{Z}$ replaces ν_j with $\nu_j + e\mu_j$, so each $\mu_i \nu_j - \mu_j \nu_i$ is unaffected by p-power scaling of f.

In Theorem B.3.1.3, we will give explicit formulas for the multiplicities μ_i depending only on $p^{\mathrm{ord}_p(N)}$ (upon using an intrinsic labeling of the irreducible components C_i). Thus, to prove Theorem B.1.3.1 it will suffice to show that the absolute differences $|\mu_i \nu_j - \mu_j \nu_i|$ are bounded solely in terms of k, p, and $\mathrm{ord}_p(N)$, as then a lower bound $\nu_{i_0} \geq 0$ for some i_0 implies a negative lower bound on every ν_j depending only on k, p, and $\mathrm{ord}_p(N)$, as required. The invariance of $\mu_i \nu_j - \mu_j \nu_i$ under p-power scaling of f allows us to restrict attention to nonzero $f \in \mathrm{H}^0(X_p, \omega^{\otimes k})$ (rather than $f \in \mathrm{H}^0(X_{\mathbf{Q}}, \omega^{\otimes k})$) for the purpose of bounding the absolute differences $|\mu_i \nu_j - \mu_j \nu_i|$.

In Section B.2.3 we will bound every $|\mu_{i_0} \nu_{j_0} - \mu_{j_0} \nu_{i_0}|$ in terms of

- $\deg_{X_{\mathbf{Q}}}(\omega^{\otimes k}) = k \deg_{X_{\mathbf{Q}}}(\omega)$,
- the intersection multiplicities $C_i.C_j$, and
- the multiplicities μ_i.

These parameters all admit explicit formulas in terms of k, p, and N, as we shall see in Section B.3. From this we will obtain explicit bounds as in Theorem B.1.3.1 that depend on N only through $\mathrm{ord}_p(N)$.

B.2.3. Geometric calculations. We work in the same general setup as in Section B.2.2 after Lemma B.2.2.5, but we also assume X_F is smooth and geometrically connected (so the special fiber X_0 is geometrically connected). In particular, there is a regular finite flat scheme cover $X' \to X$ and the stack X is Deligne–Mumford away from a closed substack of dimension 0 in X_0. We assume that the special fiber X_0 is *reducible*.

Let $\{C_1, \ldots, C_n\}$ be the set of (reduced) irreducible components of X_0 (so $n \geq 2$); each C_i is Cartier in X by Lemma B.2.2.8. Let μ_j be the generic multiplicity of C_j, which is to say

$$\mathrm{div}_{\mathscr{O}_X}(\pi) = \sum_j \mu_j C_j.$$

Let $\mathscr{L}$ be a line bundle on X such that $\deg_k(\mathscr{L}|_{C_j}) \geq 0$ for all j, and let $s \in \mathscr{L}(X)$ be a nonzero global section, so

$$\mathrm{div}_{\mathscr{L}}(s) = D + \sum_j \nu_j C_j$$

for integers $\nu_j \geq 0$ and an effective horizontal divisor D. In this general setting, we claim that the absolute differences $|\mu_{i_0}\nu_{j_0} - \mu_{j_0}\mu_{i_0}|$ are all bounded by a universal formula in terms of

- $\deg_F(\mathscr{L}_F)$,
- the intersection numbers $C_i.C_j$, and
- the multiplicities μ_i.

Since

$$\mu_i\nu_j - \mu_j\nu_i = \frac{\mu_i}{\mu_1}(\mu_1\nu_j - \mu_j\nu_1) - \frac{\mu_j}{\mu_1}(\mu_1\nu_i - \mu_i\nu_1), \tag{B.2.3.1}$$

it suffices to restrict attention to bounding $|\mu_1\nu_j - \mu_j\nu_1|$ for each $j \neq 1$.

Since s is a nonzero section of $\mathscr{L}$, so $\mathscr{L} \simeq \mathscr{O}_X(\mathrm{div}_{\mathscr{L}}(s))$, we have

$$\deg_k(\mathscr{L}|_{C_j}) = \mathrm{div}_{\mathscr{L}}(s).C_j = D.C_j + \sum_i \nu_i(C_i.C_j).$$

Let M denote the symmetric negative semi-definite intersection matrix $(C_i.C_j)$ and define

$$\vec{\nu} := (\nu_1, \ldots, \nu_n),$$

so intersection theory on X ensures that $M : \mathbf{Q}^n \to \mathbf{Q}^n$ is negative semi-definite with kernel equal to the line spanned by $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ ($\sum_j \mu_j C_j = \mathrm{div}_{\mathscr{O}_X}(\pi)$ is principal). Thus, the image hyperplane of M is

$$H = \left\{\vec{a} \in \mathbf{Q}^n : \sum_i \mu_i a_i = 0\right\}$$

since for any $\vec{b} \in \mathbf{Q}^n$ and $a_i := \sum_j (C_i.C_j)b_j$ we have

$$\sum_i \mu_i a_i = \sum_j b_j \sum_i \mu_i(C_i.C_j) = \sum_j b_j(\mathrm{div}(\pi).C_j) = 0.$$

Remark B.2.3.1. For $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ as above and

$$\vec{a} := M(\vec{\nu})$$

we have

$$a_j = \deg_k(\mathscr{L}|_{C_j}) - D.C_j.$$

The identity $\sum_i \mu_i a_i = 0$ above also follows from the pair of identities

$$\deg_F(\mathrm{div}_{\mathscr{L}_F}(s_F)) = \deg_F \mathscr{L}_F = \deg_k \mathscr{L}_0 = \sum_j \mu_j \deg_k(\mathscr{L}|_{C_j})$$

and

$$\deg_F(\mathrm{div}_{\mathscr{L}_F}(s_F)) = \deg_F(D_F) = \deg_k(D_0) = D.\mathrm{div}(\pi) = \sum_j \mu_j(D.C_j).$$

Since $\deg_k(\mathscr{L}|_{C_j}) \geq 0$ for all j, we have

$$|\mu_i a_i| \leq \deg_F \mathscr{L}_F$$

for all i because $\mu_i a_i = \mu_i \deg_k(\mathscr{L}|_{C_i}) - \mu_i(D.C_j)$ is a difference between nonnegative integers that are each bounded above by $\deg_F \mathscr{L}_F$.

Rather generally, for any $\vec{a}$ such that $\sum_i \mu_i a_i = 0$, the $\vec{b} \in \mathbf{Q}^n$ such that $M(\vec{b}) = \vec{a}$ are determined up to adding a $\mathbf{Q}$-multiple of $\vec{\mu}$, so the differences $\mu_i b_j - \mu_j b_i$ are *independent* of the choice of such $\vec{b}$. Our aim is to give a formula for these differences entirely in terms of $\vec{a}$, $\vec{\mu}$, and the intersection numbers $C_{i'}.C_{i''}$ (to be applied for $\vec{a} = M(\vec{v})$ computed as in Remark B.2.3.1). We will focus on the case $i = 1$ (so $j > 1$), as this is sufficient by (B.2.3.1).

Let $W \subset \mathbf{Q}^n$ be the subspace of vectors with vanishing initial coordinate, so the natural projection $\mathrm{pr} : \mathbf{Q}^n \to W$ away from the initial coordinate restricts to an isomorphism on the hyperplane H (since $\mu_1 \neq 0$). Since the hyperplane H is the image of $M : \mathbf{Q}^n \to \mathbf{Q}^n$, and the subspace W is complementary to the line $\ker M$ (since $\mu_1 \neq 0$), the restriction of M to W is an isomorphism $W \simeq H$. Thus, we can define a composite linear automorphism

$$T : W \overset{M|_W}{\simeq} H \overset{\mathrm{pr}}{\simeq} W.$$

Via the evident isomorphism $W \simeq \mathbf{Q}^{n-1}$, the matrix for T is the lower-right $(n-1) \times (n-1)$ block in the symmetric intersection matrix defining M. The desired formula for the differences $\mu_1 b_j - \mu_j b_1$ ($j \geq 2$) is given by:

Proposition B.2.3.2. *For any $\vec{a} \in H$ and any vector $\vec{b} \in \mathbf{Q}^n$ such that $M(\vec{b}) = \vec{a}$, the point $T^{-1}(\mathrm{pr}(\vec{a})) \in W$ has jth coordinate $(\mu_1 b_j - \mu_j b_1)/\mu_1$ for $2 \leq j \leq n$; that is,*

$$\vec{b} - \frac{b_1}{\mu_1} \cdot \vec{\mu} = T^{-1}(\mathrm{pr}(\vec{a})).$$

Proof. Since $\vec{a} = M(\vec{b})$, in view of the definition of T we just need to observe that the vector $\vec{b} - (b_1/\mu_1)\vec{\mu}$ lies in W and so it is the unique point where W meets the congruence class of $\vec{b}$ modulo the kernel $\mathbf{Q}\vec{\mu}$ of M. □

By setting $\vec{b} = \vec{v}$ and $a_j = \deg_k(\mathscr{L}|_{C_j}) - D.C_j$ for all $j \geq 2$, Proposition B.2.3.2 gives a formula for $(\mu_1 v_j - \mu_j v_1)/\mu_1$ in terms of the intersection numbers $C_i.C_{i'}$ and the numbers $a_2, \ldots, a_n$. To make this formula explicit, we have to invert the $(n-1) \times (n-1)$ matrix for the linear automorphism T of $W = \mathbf{Q}^{n-1}$.

Example B.2.3.3. Suppose $n = 2$. In this case,

$$\frac{\mu_1\nu_2 - \mu_2\nu_1}{\mu_1} = \frac{a_2}{C_2.C_2} = \frac{\mu_2 a_2}{(\mu_2 C_2).C_2} = -\frac{\mu_2 a_2}{\mu_1(C_1.C_2)}$$

(the first equality by Proposition B.2.3.2) since the Weil divisor $\mu_1 C_1 + \mu_2 C_2 = \mathrm{div}_{\mathscr{O}_X}(\pi)$ has vanishing intersection pairing against any Weil divisor supported in X_0, so

$$\mu_1\nu_2 - \mu_2\nu_1 = -\frac{\mu_2 a_2}{C_1.C_2},$$

where $a_2 = \deg_k(\mathscr{L}|_{C_2}) - D.C_2$. By Remark B.2.3.1, we have

$$|\mu_2 a_2| \leq \deg_F(\mathscr{L}_F)$$

and hence

$$|\mu_1\nu_2 - \mu_2\nu_1| \leq \frac{\deg_F(\mathscr{L}_F)}{C_1.C_2}. \tag{B.2.3.2}$$

Remark B.2.3.4. Assume in the setting of Example B.2.3.3 that $\mu_1 = \mu_2 = 1$, so we use the upper bound $\deg_F(\mathscr{L}_F)$ on $\deg_k(\mathscr{L}|_{C_j})$. For example, if $X = X_0(p)$, C_1 is the component generically corresponding to the "multiplicative" level structure (so $C_1 \simeq X(1)$), and $\mathscr{L} = \omega^{\otimes k}$ for $k \geq 0$ with ω as in Section B.3.1 below, then this amounts to replacing $\deg(\omega^{\otimes k}|_{C_1}) = k/24$ with the upper bound $\deg(\omega^{\otimes k}) = k(p+1)/24$.

Thus, in this special case the use of (B.2.3.2) does not recover as good a bound as is obtained by Deligne and Rapoport in [12, Chapter VII, Sections 3.19–3.20], where the exact value $\deg(\omega^{\otimes k}|_{C_1}) = k/24$ is used. However, we can modify the computation in such cases (in a manner that is specific to $n = 2$) so that we recover the bound of Deligne and Rapoport. This requires the explicit determination of certain intersection pairings in Section B.3.1, so we postpone it until Theorem B.3.2.1 (with $m = 1$ there).

Example B.2.3.5. Suppose $n = 3$. We identify W with $\mathbf{Q}^2$ via pr_{23}, so the matrix for $T : W \simeq W$ is

$$[T] = \begin{pmatrix} C_2.C_2 & C_2.C_3 \\ C_2.C_3 & C_3.C_3 \end{pmatrix}.$$

Setting

$$a_j = \deg_k(\mathscr{L}|_{C_j}) - D.C_j$$

for $j = 2, 3$ (so $|a_j| \leq \deg_F(\mathscr{L}_F)/\mu_j$, as in Example B.2.3.3), we have

$$\mu_1^{-1} \cdot \begin{pmatrix} \mu_1\nu_2 - \mu_2\nu_1 \\ \mu_1\nu_3 - \mu_3\nu_1 \end{pmatrix} = \begin{pmatrix} C_2.C_2 & C_2.C_3 \\ C_2.C_3 & C_3.C_3 \end{pmatrix}^{-1} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}.$$

This yields upper bounds on $|\mu_1\nu_j - \mu_j\nu_1|$ in terms of the intersection numbers $C_i.C_{i'}$, $\vec{\mu}$, and $\deg_F(\mathscr{L}_F)$. Note that

$$C_2.C_2 = \mu_2^{-1}(\mu_1(C_1.C_2) + \mu_3(C_3.C_2))$$

and

$$C_3.C_3 = \mu_3^{-1}(\mu_1(C_1.C_3) + \mu_2(C_2.C_3)).$$

B.3. Explicit formulas.

B.3.1. Numerology for $X_0(N)$. The preceding considerations are applicable to the stack $X_0(N)$ ($N \geq 1$) that is Deligne–Mumford away from the cuspidal substack in characteristics p for which $p^2 \mid N$ ([11, Section 3.2.7]). In particular, this stack is Deligne–Mumford around supersingular geometric points in any positive characteristic.

Consider the line bundle $\mathscr{L} = \omega^{\otimes k}$ on $X = X_0(N)$, where $\omega = \omega_{\Gamma_0(N)}$ is the invertible pushforward of the relative dualizing sheaf of the universal generalized elliptic curve over $X_0(N)$. The degree $\deg_{\mathbf{F}_p}(\mathscr{L}_{\mathbf{F}_p})$ is equal to $\deg_{\mathbf{Q}}(\mathscr{L}_{\mathbf{Q}}) = k \deg_{\mathbf{Q}}(\omega_{\mathbf{Q}})$ for any prime p, so to compute the degree of $\mathscr{L}$ on the fibers of $X_0(N)$ over $\mathrm{Spec}(\mathbf{Z})$ we just need to compute the degree $\deg_{\mathbf{Q}}(\omega_{\mathbf{Q}})$.

The forgetful map of stacks $X_0(N) \to X(1)$ is proper, quasi-finite, and flat, but even over $\mathbf{Q}$ it is not finite (more specifically, not relatively representable in schemes) when N is not square-free. The problem is that if we choose $d \mid N$ such that $\gcd(d, N/d) > 1$, then for the standard d-gon E_d over $\overline{\mathbf{Q}}$ and the cyclic subgroup G of order N generated by the pair $(\zeta, 1 \bmod d) \in \mathbf{G}_m \times \mathbf{Z}/d\mathbf{Z}$ with a primitive Nth root of unity ζ, the nontrivial group of automorphisms $\mu_d \cap \mu_{N/d}$ of (E_d, G) induces the identity on the contraction $c(E_d) \simeq E_1$ away from the identity component.

It follows that the pullback of the map $X_0(N) \to X(1)$ along $\infty : \mathrm{Spec}(\mathbf{Q}) \to X(1)$ is not a scheme. Nonetheless, the degree of ω behaves as if $X_0(N) \to X(1)$ is finite flat:

Lemma B.3.1.1. *For any $N \geq 1$,*

$$\deg_{\mathbf{Q}}(\omega_{\Gamma_0(N)}) = [\Gamma(1) : \Gamma_0(N)] \deg_{\mathbf{Q}}(\omega_{\Gamma(1)}).$$

Proof. The forgetful map $X_1(N) \to X(1)$ is finite flat by [11, Theorem 4.1.1 (1)], and the map $X_1(N) \to X_0(N)$ is finite flat by [11, Lemma 4.2.3]. These maps are defined moduli-theoretically via contraction operations that have no effect on the relative identity component of the smooth locus of a generalized elliptic curve, and by computations over $\mathbf{Z}[1/N]$ away from cusps we see that their constant fiber degrees are $[\Gamma(1) : \Gamma_1(N)]$ and $[\Gamma_0(N) : \Gamma_1(N)]$ respectively.

For any generalized elliptic curve $f : E \to S$, the sheaf $f_*(\omega_{E/S})$ is naturally identified with $e^*(\omega_{E/S})$. Thus, $\omega_{\Gamma_1(N)}$ on $X_1(N)$ is identified with the pullbacks of both $\omega_{\Gamma_0(N)}$ and $\omega_{\Gamma(1)}$ under the forgetful maps $X_1(N) \rightrightarrows X_0(N), X(1)$. Hence,

$$[\Gamma_0(N) : \Gamma_1(N)] \deg_{\mathbf{Q}}(\omega_{\Gamma_0(N)}) = \deg_{\mathbf{Q}}(\omega_{\Gamma_1(N)}) = [\Gamma(1) : \Gamma_1(N)] \deg_{\mathbf{Q}}(\omega_{\Gamma(1)}).$$

Dividing throughout by $[\Gamma_0(N) : \Gamma_1(N)]$ yields the result. □

The line bundle $\omega_{\Gamma(1)}^{\otimes 12}$ over $X(1)_{\mathbf{Q}}$ admits a generating section away from ∞, namely Ramanujan's cusp form Δ, and

$$\mathrm{div}_{\omega_{\Gamma(1)}^{\otimes 12}}(\Delta) = \infty$$

by q-expansion considerations. But $\deg_{\mathbf{Q}}(\infty) = 1/2$ due to Lemma B.2.2.4 (since the standard 1-gon has automorphism group $\mathbf{Z}/(2)$), so the isomorphism $\omega_{\Gamma(1)}^{\otimes 12} \simeq \mathscr{O}_{X(1)}(\infty)$ implies that

$$12 \deg_{\mathbf{Q}}(\omega_{\Gamma(1)}) = \deg_{\mathbf{Q}}(\infty) = \frac{1}{2}.$$

Hence we obtain the following corollary.

Corollary B.3.1.2. *For all $k, N \geq 1$,*

$$\deg_{\mathbf{Q}}(\omega_{\Gamma_0(N)}^{\otimes k}) = \frac{k[\Gamma(1) : \Gamma_0(N)]}{24}.$$

Pick a prime $p \mid N$ and let $r = \mathrm{ord}_p(N) \geq 1$, so $N = p^r N'$ with $p \nmid N'$. The geometry of $X_0(N)$ mod p is worked out in [24, Chapter 13] except that the language of stacks is not used there: prime-to-p level structure is adjoined to create a "rigid" moduli problem represented by a scheme, and cusps are not treated moduli-theoretically. By using the agreement between the moduli-theoretic approach in [11] (which does incorporate the cusps) and the methods in [24] for cases with "enough" level structure (such as $\Gamma_1(Nq\ell)$ for distinct primes $q, \ell \geq 5$ not dividing N), we get most of:

Theorem B.3.1.3. *The stack $X_0(N)_{\mathbf{F}_p}$ has $r+1$ irreducible components $\{C_a\}_{0 \leq a \leq r}$ where C_a classifies objects for which the p-part of the level structure is an $(a, r-a)$-subgroup in the sense of* [24, Definition 13.4.1] *(so away from the supersingular locus, C_r classifies multiplicative-type level structures and C_0 classifies étale level structures). Equipping each C_a with the reduced structure, each is smooth and geometrically irreducible with*

$$\deg_{\mathbf{F}_p}(\omega|_{C_a}) > 0$$

for all a.

Any two such distinct components meet at precisely the supersingular points, and at any supersingular point ξ of $X_0(N)_{\overline{\mathbf{F}}_p}$ the completed local ring is isomorphic to

$$\overline{\mathbf{F}}_p[\![u, v]\!]/\Big((u - v^{p^r})(v - u^{p^r}) \prod_{1 \leq a \leq r-1} (u^{p^a} - v^{p^{r-a}})^{p-1}\Big)$$

with its quotient $\widehat{\mathscr{O}}_{C_a,\xi}$ defined by the radical of the equation $u^{p^a} = v^{p^{r-a}}$. In particular, the generic multiplicity of C_a is given by $\mu_0 = 1$, $\mu_r = 1$, and $\mu_a = p^{\min(a, r-a)}(p-1)$ for $0 < a < r$, and for $0 \leq a < b \leq m$ the 0-dimensional intersection stack $C_a \cap C_b$ is reduced if $a \leq r/2 \leq b$ whereas otherwise each of its points has multiplicity

$$m_{a,b} = \begin{cases} p^{r-2b}, & a < b < \frac{r}{2}, \\ p^{2a-m}, & \frac{r}{2} < a < b. \end{cases}$$

In particular, $m_{a,b} = m_{r-b,r-a}$.

Proof. This is [24, Theorem 13.4.7] except for the inequalities $\deg_{\mathbf{F}_p}(\omega|_{C_a}) > 0$ for all a. To prove this positivity we may instead work with $\omega^{\otimes 12}$, which is the pullback of $\mathscr{O}(\infty)$ on $X(1)$ along the quasi-finite flat proper forgetful map $X_0(N)_{\mathbf{F}_p} \to X(1)_{\mathbf{F}_p}$. This latter map is not finite (nor even relatively representable in schemes) when $p^2 \mid N$, and the same holds for each $C_a \to X(1)_{\mathbf{F}_p}$. Nonetheless, the effect on degree under such pullbacks does behave as if it is finite flat, exactly as in Lemma B.3.1.1 ff., so $\omega|_{C_a}$ has positive $\mathbf{F}_p$-degree for every a. □

Corollary B.3.1.4. *Let C, C' be two distinct irreducible components of $X_0(N)$* mod *p. Then*

$$C.C' \geq [\Gamma(1) : \Gamma_0(N')] \cdot \sum_{\xi} \frac{1}{\#\,\mathrm{Aut}(\xi)} = [\Gamma(1) : \Gamma_0(N')] \cdot \frac{p-1}{24},$$

where the sum is over all isomorphism classes of supersingular elliptic curves over $\overline{\mathbf{F}}_p$.

Proof. Since intersection numbers can be computed over $\overline{\mathbf{F}}_p$, we may and do work with $\mathcal{X} := X_0(N)_{\overline{\mathbf{F}}_p}$. By Theorem B.3.1.3,

$$\mathcal{C} := C_{\overline{\mathbf{F}}_p} \quad \text{and} \quad \mathcal{C}' := C'_{\overline{\mathbf{F}}_p}$$

are irreducible components of $\mathcal{X}$ with the reduced structure and $\mathcal{C} \cap \mathcal{C}'$ is a closed substack whose support consists of precisely the supersingular $\overline{\mathbf{F}}_p$-points with each such point having a common multiplicity (depending on C and C').

Any supersingular elliptic curve E over $\overline{\mathbf{F}}_p$ admits a unique Drinfeld cyclic subgroup scheme of order p^r, namely the kernel of the r-fold relative Frobenius morphism (for which a Drinfeld generator is given by 0). Thus, if $N = p^r$ (i.e., $N' = 1$), then Lemma B.2.2.4 and Proposition B.2.2.7 give that

$$\mathcal{C}.\mathcal{C}' \geq \sum_{\xi} \frac{1}{\#\,\mathrm{Aut}(\xi)} = \frac{p-1}{24}$$

(with equality precisely when C and C' meet transversally). More generally, consider the map $h : X_0(N) \to X_0(p^r)$ that “forgets” the N'-part of the level structure. This generically étale proper map generally is not relatively representable near cusps but is finite étale of constant degree $\deg(h) := [\Gamma_0(p^r) : \Gamma_0(N)] = [\Gamma(1) : \Gamma_0(N')]$ away from the cusps. Also, the pullback operation $Y \rightsquigarrow h^{-1}(Y)$ defines a bijection between the sets of irreducible components of $X_0(p^r)_{\mathbf{F}_p}$ and $X_0(N)_{\mathbf{F}_p}$.

Let D and D' be the two irreducible components of $X_0(p^r)_{\mathbf{F}_p}$ that satisfy $C = h^{-1}(D)$ and $C' = h^{-1}(D')$, so by using a finite flat scheme cover of $X_0(N)$ that is also finite flat over $X_0(p^r)$ (such as by adjoining suitable auxiliary level structure) we see that

$$C.C' = \deg(h) D.D' = [\Gamma(1) : \Gamma_0(N')] D.D'.$$

Since $D.D' \geq (p-1)/24$, we are done. □

Corollary B.3.1.5. *Choose a prime $p \mid N$. For sufficiently large even k, the p-torsion subgroup of the finite abelian group $M_k(\Gamma_0(N), \mathbf{Z})/M_{k,\mathbf{Z}}$ is nontrivial.*

Since $M_k(\Gamma_0(N), \mathbf{C}) = 0$ for odd k, the parity condition on k cannot be removed. In the special case $k = 4$ and $N = p = 5$, f in Section 5.1 represents a nontrivial class.

Proof. By flat base change considerations, it suffices to show that $M_k(\Gamma, R)/M_{k,R} \neq 0$ for the strict henselization $R = \mathbf{Z}_{(p)}^{\mathrm{sh}}$. Let κ be the residue field of R, and let D be the reduced effective Cartier divisor on the regular stack X_R such that the support of D is the unique irreducible component of the special fiber X_κ passing through ∞. Finally, let $[x_0]$ be the 0-dimensional closed substack of X_κ that is the schematic image of a non-cuspidal non-CM point $x_0 : \operatorname{Spec} \kappa \to X_\kappa$ away from D and let $j : [x_0] \to X_R$ be the canonical closed immersion.

The non-cuspidality ensures that $[x_0]$ is Deligne–Mumford, so the map

$$x_0 : \operatorname{Spec} \kappa \to [x_0]$$

is finite since it is a proper quasi-finite map from a scheme to a Deligne–Mumford stack. By the structure of DM-stacky points, x_0 identifies $[x_0]$ with the quotient stack $[\operatorname{Spec}(\kappa)/G_0]$ for the finite constant group $G_0 = \operatorname{Spec}(\kappa) \times_{x_0, X_\kappa, x_0} \operatorname{Spec}(\kappa)$ that is the automorphism group of the object over $\kappa = \overline{\kappa}$ classified by x_0. Since x_0 is non-cuspidal and non-CM, clearly $G_0 = \mathbf{Z}/2\mathbf{Z}$.

Thus, $\mathrm{Pic}([x_0])$ consists of isomorphism classes of 1-dimensional κ-vector spaces L equipped with an action by $G_0 = \mathbf{Z}/2\mathbf{Z}$, so the G_0-action on $L^{\otimes 2}$ is trivial. Hence, $j^*(\omega_R)^{\otimes k} \simeq \mathscr{O}_{[x_0]}$ for any *even* k.

Choose a prime $\ell \geq 5$ not dividing Np, and let p^a denote the p-part of the degree of the finite flat map $q : X_1(N\ell) \to X_0(N) = X$ with $X_1(N\ell)$ a *scheme* over $\mathbf{Z}[1/\ell]$ (and hence over $\mathbf{Z}_{(p)}$). For any coherent sheaf $\mathscr{G}$ on X_R we claim that if k is sufficiently large (without a parity constraint), then $\mathrm{H}^1(X_R, \omega^{\otimes k} \otimes \mathscr{G})$ is killed by p^a. (There is no finite flat scheme cover of $X = X_0(N)_\kappa$ with degree prime to p when $\mathrm{ord}_p(N) > 1$, due to degree considerations at the cusps in characteristic p whose automorphism scheme has order divisible by p.) Granting this, consider the twist of $\mathscr{O}_{X_R} \twoheadrightarrow j_*(\mathscr{O}_{[x_0]})$ by the invertible ideal sheaf $\mathscr{O}_{X_R}(-(a+1)D)$ (which is trivial near $[x_0]$); this twist is a surjection

$$\varphi : \mathscr{O}_{X_R}(-(a+1)D) \to j_*(\mathscr{O}_{[x_0]}).$$

Let $\mathscr{I}$ denote its coherent kernel, so $\omega_R^{\otimes k} \otimes \mathscr{I}$ is the kernel of the natural surjection

$$\omega_R^{\otimes k}(-(a+1)D) \to j_*(j^*(\omega_R^{\otimes k}))$$

obtained by twisting the surjection φ by $\omega_R^{\otimes k}$. For *even* k the line bundle $j^*(\omega_R^{\otimes k})$ on $[x_0]$ is trivial, and the obstruction to lifting a global section of this line bundle to a global section f of $\omega_R^{\otimes k}(-(a+1)D)$ lies in the cohomology group $\mathrm{H}^1(X_R, \omega_R^{\otimes k} \otimes \mathscr{I})$ that is killed by p^a. Thus, there exists an $f \in M_{k,R}$ generating $p^a \omega_R^{\otimes k}$ near $[x_0]$ and vanishing to order at least $a+1$ along X_κ near ∞_κ. Clearly, f has q-expansion that vanishes modulo p^{a+1}, so $(1/p^{a+1})f$ represents a nontrivial class in $M_k(\Gamma, R)/M_{k,R}$.

It remains to show that for any coherent sheaf $\mathscr{G}$ on X_R, $\mathrm{H}^1(X_R, \omega_R^{\otimes k} \otimes \mathscr{G})$ is killed by p^a for all sufficiently large k (without a parity constraint). Since $q : X_1(N\ell) \to X_0(N)$ is a finite flat scheme cover whose degree has p-part p^a, by a trace argument it suffices to prove that for any coherent sheaf $\mathscr{F}$ on $X_1(N)_R$ (such as $q^*\mathscr{G}$) if k is sufficiently large, then $\mathrm{H}^1(X_1(N\ell)_R, q^*(\omega)_R^{\otimes k} \otimes \mathscr{F}) = 0$. By the standard base change theorems for coherent cohomology on schemes, it suffices to prove the analogous vanishing on the special fiber X_κ, so it is enough to show that the line bundle $q^*(\omega)_\kappa$ on $X_1(N\ell)_\kappa$ is ample.

As C varies through the irreducible components of the stacky curve $X_0(N)_\kappa$ (giving C the reduced structure), the pullbacks $q^{-1}(C)$ vary through the irreducible components of $X_1(N\ell)_\kappa$ (equipped with possibly non-reduced scheme structure). For each such C the rational number $\deg_\kappa(\omega_\kappa|_C)$ is positive by Theorem B.3.1.3, so $q^*(\omega)_\kappa$ has positive degree on the irreducible 1-dimensional proper κ-scheme $q^{-1}(C)$. Hence, $q^*(\omega)_\kappa|_{q^{-1}(C)}$ is ample for all such C, so $q^*(\omega)_\kappa$ is ample as desired. □

The interested reader will easily check that Corollary B.3.1.5 remains true with the same proof when using cusp forms in place of modular forms: one just has to introduce a twist by the invertible ideal sheaf defining the Cartier closed substack of cusps in some of the sheaf-theoretic calculations.

B.3.2. Multiplicity bounds for $\Gamma_0(N)$. Consider a prime $p \mid N$ and let $r := \mathrm{ord}_p(N)$, so $N = p^r N'$ with $p \nmid N'$. We seek an $e = e(p,k,N)$ so that

$$p^e M_k(\Gamma_0(N), \mathbf{Z}_{(p)}) \subseteq \mathrm{H}^0(X_0(N)_{\mathbf{Z}_{(p)}}, \omega^{\otimes k}).$$

Theorem B.3.2.1. *We can choose the exponent $e(p,k,N)$ to depend on N only through $r := \mathrm{ord}_p(N) > 0$. For $r = 1,2,3$ and $t = (p-1)^{-1}$ valid choices are $\lceil k(1+t)\rceil$, $\lceil 3kt(2t+1)\rceil$, $\lceil 2kt(t+1)(t+2)\rceil$ respectively.*

Finding an explicit formula for $e(p,k,N)$ via our methods requires understanding the inverse of an $r \times r$ matrix of intersection numbers, so it is not clear how to give a clean general formula for all $r > 3$.

Proof. We may assume that there exists a nonzero $s \in M_k(\Gamma_0(N), \mathbf{Z}_{(p)})$ (or else we can take $e(k,p,N)$ to be 0), so s is an invertible rational section s of $\omega_{\Gamma_0(N)}^{\otimes k}$ and we write its divisor as

$$D + \sum_{i=0}^{r} \nu_i C_i$$

with effective horizontal D. By effectivity of D and normality of the "stacky arithmetic curve" $X_0(N)$, the necessary and sufficient condition on $e \geq 0$ to ensure that

$$p^e s \in \mathrm{H}^0(X_0(N)_{\mathbf{Z}_{(p)}}, \omega^{\otimes k})$$

is that $e \cdot \mathrm{ord}_{C_i}(p) + \nu_i \geq 0$ for all i, which is to say $e \geq -\nu_i/\mu_i$ for all i. But $\nu_r \geq 0$ because q-expansions are computed at the cusp ∞ corresponding to a multiplicative level structure, so we seek a uniform upper bound $e(p,k,N)$ on $\max_{0 \leq i < r}(-\nu_i/\mu_i)$ (independent of such s).

The case $r = 1$ requires special care (relative to arguments that work for $r > 1$), so we first assume $r = 1$ and adapt the argument in [12, Chapter VII, Sections 3.19–3.20] to show that $|\nu_2 - \nu_1| \leq kp/(p-1)$ (regardless of the prime-to-p part of N).

Consider Example B.2.3.3 with C_1 equal to the component $X_0(N')_{\mathbf{F}_p}$ of $X_0(N)_{\mathbf{F}_p}$ corresponding to multiplicative p-part in the level structure (away from the supersingular locus) and C_0 equal to the other component (corresponding to étale p-part in the level structure, away from the supersingular points). This yields the equality

$$(\nu_1 - \nu_0)(C_0.C_1) = D.C_1 - \deg_{\mathbf{F}_p}(\omega^{\otimes k}|_{C_1}) = D.C_1 - \frac{k[\Gamma(1) : \Gamma_0(N')]}{24}$$

(using Corollary B.3.1.2 for $\Gamma_0(N')$ over the characteristic-p point of $\mathrm{Spec}(\mathbf{Z}[1/N'])$).

The key trick is to observe that

$$D.C_1 = \frac{1}{2}D.(C_1 + C_0) + \frac{1}{2}D.(C_1 - C_0),$$

and that the Weil divisor $C_1 + C_0 = \mathrm{div}_{X_0(N)}(p)$ has vanishing pairing against any vertical divisor in the mod-p fiber. Thus,

$$\begin{aligned} D.(C_0 + C_1) &= (D + \nu_0 C_0 + \nu_1 C_1).(C_0 + C_1) \\ &= \deg_{\mathbf{F}_p}(\omega_{\Gamma_0(N)}^{\otimes k}) \\ &= \deg_{\mathbf{Q}}(\omega_{\Gamma_0(N)}^{\otimes k}) \\ &= \frac{k[\Gamma(1) : \Gamma_0(N)]}{24}. \end{aligned}$$

Since $N = pN'$ with $p \nmid N'$, we have

$$[\Gamma(1) : \Gamma_0(N)] = [\Gamma(1) : \Gamma_0(N')][\Gamma(1) : \Gamma_0(p)] = (p+1)[\Gamma(1) : \Gamma_0(N')].$$

Hence,

$$\begin{aligned}(\nu_1-\nu_0)(C_0.C_1) &= k[\Gamma(1):\Gamma_0(N')]\left(-\frac{1}{24}+\frac{1}{2}\frac{p+1}{24}\right)+\frac{1}{2}D.(C_1-C_0)\\ &= \frac{k}{2}\frac{[\Gamma(1):\Gamma_0(N')](p-1)}{24}+\frac{1}{2}D.(C_1-C_0).\end{aligned}$$

The horizontality of D implies that

$$|D.(C_1-C_0)|\le D.(C_1+C_0),$$

and by Corollary B.3.1.4,

$$C_1.C_0 \ge \frac{[\Gamma(1):\Gamma_0(N')](p-1)}{24}.$$

By cancelling $[\Gamma(1):\Gamma_0(N')](p-1)/24$ throughout, we conclude that

$$\left|\nu_1-\nu_0-\frac{k}{2}\right| \le \frac{k}{2}\frac{p+1}{p-1} = \frac{k}{2}\left(1+\frac{2}{p-1}\right),$$

so

$$|\nu_1-\nu_0|\le k+\frac{k}{p-1}.$$

Since $\nu_1\ge 0$ by design and $\mu_0=1$, we get

$$-\frac{\nu_0}{\mu_0}\le k\left(1+\frac{1}{p-1}\right),$$

providing the desired uniform upper bound.

Now suppose $r=2$, so for the three irreducible components C_0, C_1, C_2 of $X_0(N)$ mod p the respective multiplicities are $\mu_0=\mu_2=1$ and $\mu_1=p(p-1)$ by Theorem B.3.1.3. Hence, we seek a uniform upper bound on $\max(-\nu_0, -\nu_1/p(p-1))$ (using that $\nu_2\ge 0$). Since there are three irreducible components C_0, C_1, C_2, we obtain from Example B.2.3.5 pivoting on C_2 (with $\mu_2=1$) that

(B.3.2.1) $$\begin{pmatrix}\nu_0-\nu_2\\ \nu_1-p(p-1)\nu_2\end{pmatrix}=\begin{pmatrix}C_0.C_0 & C_0.C_1\\ C_0.C_1 & C_1.C_1\end{pmatrix}^{-1}\begin{pmatrix}a_0\\ a_1\end{pmatrix},$$

and

(B.3.2.2) $$|\mu_j a_j|\le \deg_{\mathbf{Q}}(\omega^{\otimes k})=\frac{kp(p+1)[\Gamma(1):\Gamma_0(N')]}{24}$$

by Remark B.2.3.1 and Corollary B.3.1.2. The description of the completed local rings at geometric supersingular points in Theorem B.3.1.3 gives (via Lemma B.2.2.4 with $m=2$) that C_i meets C_j transversally whenever $i\ne j$, so

$$C_0.C_2=C_1.C_2=C_0.C_1=\sum_{(E,G)}\frac{1}{\#\operatorname{Aut}(E,G)}=:I,$$

where the sum is taken over all $\overline{\mathbf{F}}_p$-isomorphism classes of pairs consisting of a supersingular elliptic curve E equipped with a subgroup scheme G of order N that is cyclic in the Drinfeld sense. Also, $\operatorname{div}(p)=C_0+p(p-1)C_1+C_2$, so

$$\begin{aligned}C_0.C_0 &= -(p(p-1)C_0.C_1+C_0.C_2)=-(p(p-1)+1)I,\\ p(p-1)C_1.C_1 &= -(C_0.C_1+C_2.C_1)=-2I,\end{aligned}$$

so

$$\begin{pmatrix} C_0.C_0 & C_0.C_1 \\ C_0.C_1 & C_1.C_1 \end{pmatrix}^{-1} = I^{-1}\begin{pmatrix} -(p(p-1)+1) & 1 \\ 1 & -2/p(p-1) \end{pmatrix}^{-1}$$
$$= \frac{1}{I(1+2/p(p-1))}\begin{pmatrix} -2/p(p-1) & -1 \\ -1 & -(p(p-1)+1) \end{pmatrix}.$$

By (B.3.2.1) we conclude that

$$|\nu_0 - \nu_2| \leq \frac{|a_1| + 2|a_0|/p(p-1)}{I(1+2/p(p-1))},$$
$$|\nu_1 - p(p-1)\nu_2| \leq \frac{(p(p-1)+1)|a_1| + |a_0|}{I(1+2/p(p-1))}.$$

Moreover,

$$I \geq \frac{[\Gamma(1):\Gamma_0(N')](p-1)}{24}$$

by Corollary B.3.1.4, and the upper bound (B.3.2.2) says

$$|a_0|,\, p(p-1)|a_1| \leq \frac{kp(p+1)[\Gamma(1):\Gamma_0(N')]}{24}.$$

Thus, after cancellation of $[\Gamma(1):\Gamma_0(N')]/24$ we arrive at

$$|\nu_0 - \nu_2| \leq \frac{3}{1+2/p(p-1)} \cdot \frac{p+1}{(p-1)^2} \cdot k,$$
$$|\nu_1 - p(p-1)\nu_2| \leq \frac{2+1/p(p-1)}{1+2/p(p-1)} \cdot \frac{p+1}{p-1} \cdot kp.$$

Dividing through the second inequality by $\mu_1 = p(p-1)$ gives

$$|\nu_1/\mu_1 - \nu_2| \leq \frac{2+1/p(p-1)}{1+2/p(p-1)} \cdot \frac{p+1}{(p-1)^2} \cdot k.$$

Using that $\nu_2 \geq 0$, these upper bounds on $|\nu_0 - \nu_2|$ and $|\nu_1/\mu_1 - \nu_2|$ are respective upper bounds on $-\nu_0/\mu_0 = -\nu_0$ and $-\nu_1/\mu_1$, so since $2 + 1/p(p-1) < 3$ we conclude that when $\mathrm{ord}_p(N) = 2$ a uniform upper bound is provided by

$$\frac{3k}{p-1} \cdot \frac{p+1}{p-1+2/p} < \frac{3k/(p-1)}{1-2/(p+1)} = \frac{3k}{p-1} \cdot \left(1 + \frac{2}{p-1}\right),$$

giving the desired bound when $\mathrm{ord}_p(N) = 2$.

Finally, we adapt the treatment of the case $r = 2$ to handle general $r > 1$. Let $C_0, \ldots, C_r$ be as in Theorem B.3.1.3, and let T be the $r \times r$ matrix whose ij-entry $(0 \leq i, j \leq r-1)$ is $C_i.C_j$. If $0 \leq i < j \leq r$, then $C_i.C_j = C_j.C_i = m_{i,j} I$, where

$$I := \sum_{(E,G)} \frac{1}{\#\,\mathrm{Aut}(E,G)} \geq [\Gamma(1):\Gamma_0(N')] \cdot \frac{p-1}{24}$$

and $m_{i,j} = 1$ except when $i < j < r/2$ or $r/2 < i < j$, in which case $m_{i,j} = p^{r-2j}$ or $m_{i,j} = p^{2i-r}$ respectively. For ease of notation, we define $m_{j,i} = m_{i,j}$ when $j > i$.

For $i \geq 0$, the self-intersection $C_i.C_i$ satisfies

$$\mu_i C_i.C_i = -\sum_{h \neq i}(\mu_h C_h.C_i) = -I \cdot \sum_{h \neq i} m_{h,i}\mu_h$$

(where the sums include $h = r$, with $\mu_r = 1$). In other words,

$$C_i.C_i = -I \cdot \sum_{h \neq i} \frac{m_{h,i}\mu_h}{\mu_i},$$

so $T^{-1} = (1/I)B^{-1}$, where the matrix $B = (b_{ij})$ with $0 \leq i, j \leq r-1$ has entries $b_{ij} = m_{i,j}$ for $i \neq j$ and $b_{ii} = -\sum_{h \neq i} m_{h,i}\mu_h/\mu_i$ (including $h = r$). Each b_{ij} depends only on p^r (not on the prime-to-p part N' of N) because the same holds for the multiplicities μ_i of the C_i in $X_0(N)_{\mathbf{F}_p}$ by Theorem B.3.1.3. (Explicitly, $\mu_0 = 1$ and $\mu_i = p^{\min(i,r-i)}(p-1)$ for $1 \leq i \leq r-1$.) Note in particular that $b_{ii} = b_{r-i,r-i}$ since $m_{a,b} = m_{r-a,r-b}$ and $\mu_h = \mu_{r-h}$.

Define $\vec{\nu} = (\nu_0, \ldots, \nu_{r-1})$ and $\vec{\mu} = (\mu_0, \ldots, \mu_{r-1})$ in $\mathbf{Q}^r$, and let Δ be the diagonal $r \times r$ matrix whose ith diagonal entry is μ_i ($0 \leq i \leq r-1$). Since $\mu_r = 1$, applying Proposition B.2.3.2 (to $C_0, \ldots, C_r$) gives that

(B.3.2.3) $$\vec{\nu} - \nu_r \cdot \vec{\mu} = \frac{1}{I}(b_{ij})^{-1}\vec{a},$$

where

$$|\mu_j a_j| \leq \frac{kp^{r-1}(p+1)[\Gamma(1) : \Gamma_0(N')]}{24}.$$

Let $\vec{w} = (w_0, \ldots, w_{r-1}) \in \mathbf{Q}^r$ with $w_j := \mu_j a_j$. Clearly,

$$\left|\frac{w_j}{I}\right| = \left|\frac{\mu_j a_j}{I}\right| \leq \frac{kp^{r-1}(p+1)}{p-1},$$

so inserting the trivial identity $\vec{a} = \Delta^{-1}\vec{w}$ into (B.3.2.3) gives that

$$|\nu_i - \mu_i \nu_r| \leq \frac{kp^{r-1}(p+1)}{p-1} \cdot \sum_j |c^{ij}|,$$

where (c^{ij}) is the inverse of the matrix

$$(c_{ij}) := \Delta \cdot B$$

with $c_{ij} = \mu_i m_{i,j}$ if $j \neq i$ and $c_{ii} = -\sum_{h \neq i} m_{h,i}\mu_h$ (including $h = r$). Since $\nu_r \geq 0$, we get

$$-\frac{\nu_i}{\mu_i} \leq \frac{kp^{r-1}(p+1)}{\mu_i(p-1)} \cdot \sum_{j=0}^{r-1} |c^{ij}|,$$

where $\mu_0 = 1$ and $\mu_i = p^{\min(i,r-i)}(p-1)$ for $1 \leq i \leq r-1$, so $e(p,k,N)$ may be chosen to be the least integer greater than or equal to

(B.3.2.4) $$kp^{r-1} \cdot \frac{p+1}{p-1} \max_{0 \leq i < r} \frac{1}{\mu_i} \cdot \sum_{j=0}^{r-1} |c^{ij}|.$$

This bound visibly depends on N only through its p-part p^r.

If $r = 3$, then $(c_{ij}) = \Delta \cdot B$ is equal to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p(p-1) & 0 \\ 0 & 0 & p(p-1) \end{pmatrix} \begin{pmatrix} b_{00} & p & 1 \\ p & b_{11} & 1 \\ 1 & 1 & b_{22} \end{pmatrix}$$

with

$$b_{00} = -\sum_{h \neq 0} m_{h,0}\mu_h = -(p(p-1)(p+1)+1) = -p^3 + p - 1,$$

$$b_{11} = b_{22} = \frac{-1}{p(p-1)} \sum_{h \neq 2} m_{h,2}\mu_h = \frac{-p^2-1}{p(p-1)},$$

so

$$(c_{ij}) = \begin{pmatrix} -p^3+p-1 & p & 1 \\ p^2(p-1) & -p^2-1 & p(p-1) \\ p(p-1) & p(p-1) & -p^2-1 \end{pmatrix}.$$

This has determinant $\delta(p) = -(p^3+1)^2$, and explicit computation gives the inverse

$$(c^{ij}) = \frac{-1}{(p^3+1)^2} \begin{pmatrix} (p+1)(2p^2-p+1) & p^3+p^2 & p^3+1 \\ p^3(p^2-1) & p^5+1 & p(p^3+1)(p-1) \\ p(p^3+1)(p-1) & p(p^3+1)(p-1) & p^5-p^4+p^3+p^2-p+1 \end{pmatrix}.$$

By using the explicit determination of the entries c^{ij}, the ith term

$$h_i(p) := \frac{1}{\mu_i} \sum_{0 \leq j < r} |c^{ij}|$$

inside the maximum in (B.3.2.4) for $r = 3$ is seen to be a rational function in p with degree 3 and $h_i(p)/p^3$ is given by the following respective expressions in terms of $x = 1/p \in (0, 1/2]$:

$$2(1+x)(x^2-x+2), \ \frac{(1+x)(x^5-2x^4+3x^2-4x+3)}{1-x}, \ \frac{(1+x)(x^5-4x^4+7x^2-6x+3)}{1-x}.$$

It follows that $h_0(p) > h_1(p) > h_2(p)$, so when $\mathrm{ord}_p(N) = 3$, we can take $e(p,k,N)$ to be the least integer greater than or equal to

$$\begin{aligned} \frac{2kp^2(p+1)^2(2p^2-p+1)}{(p-1)(p^3+1)^2} &= \frac{2k}{p-1} \cdot \frac{1}{1-(x-x^2)} \cdot \left(1 + \frac{1}{1-(x-x^2)}\right) \\ &< \frac{2k}{p-1} \cdot \frac{1}{1-x} \cdot \left(1 + \frac{1}{1-x}\right) \end{aligned}$$

for $x = 1/p$. Thus, by substituting $p/(p-1)$ for $1/(1-x)$ we may use

$$e(p,k,N) = \left\lceil \frac{2kp(2p-1)}{(p-1)^3} \right\rceil.$$

Writing this in terms of $t = 1/(p-1)$ then gives $\lceil 2kt(t+1)(t+2) \rceil$ as desired. $\square$

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Massimo Bertolini, Fakultät für Mathematik, Universität Duisburg-Essen,
45117 Essen, Germany

Henri Darmon, Department of Mathematics, McGill University,
805 Sherbrooke Street West, Montreal, Quebec H3A2K6, Canada
e-mail: darmon@math.mcgill.ca

Kartik Prasanna, Department of Mathematics, University of Michigan,
2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA

Brian Conrad, Department of Mathematics, Stanford University,
Building 380, Sloan Hall, Stanford, CA 94305, USA
e-mail: conrad@math.stanford.edu

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